(Stochastic) Optimal Control : State x, Control u (Dynamic Programming)
Stochastic Programming: Variables (x, u), State: $d_\xi - T_\xi x$

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STOCHASTIC DYNAMIC OPTIMIZATION: APPROACHES AND COMPUTATION
Pravin Varaiya and Roger J-B Wets
University of California, Berkeley-Davis
In: ``Mathematical Programming, Recent Developments and Applications",
M. Iri & K. Tanabe (eds), Kluwer Academic Publisher, 1989. pp. 309-332.

Lectures Plan

- *Why and how to deal with 'uncertainty' (1)*
- *Recourse Models (2 & 3)*
- *Aggregation Principle (4)*
- *Approximations (5)*
- *Duality Theory (6)*
- *Dispatching Energy - ISO (7)*

Stochastic Programs with Recourse

Roger J-B Wets
Mathematics, University of California, Davis

.. with Simple Recourse

decision: $x \rightsquigarrow$ observation: $\xi \rightsquigarrow$ recourse cost evaluation.

cost evaluation 'simple' $\Rightarrow$ simple recourse, i.e.,

$\min_{x \in S \subset \mathbb{R}^n} f_0(x) + \mathbb{E}\{Q(\xi, x)\}$ $\quad Q$ 'simple'

Product mix problem. With $\xi = (T, d)$,

$f_0(x) = \langle c, x \rangle, \;\; S = \mathbb{R}^4_+, \;\; Q(\xi, x) = \sum_{i=c,f} \max\left[0, \gamma_i(\langle T_i, x \rangle - d_i)\right]$

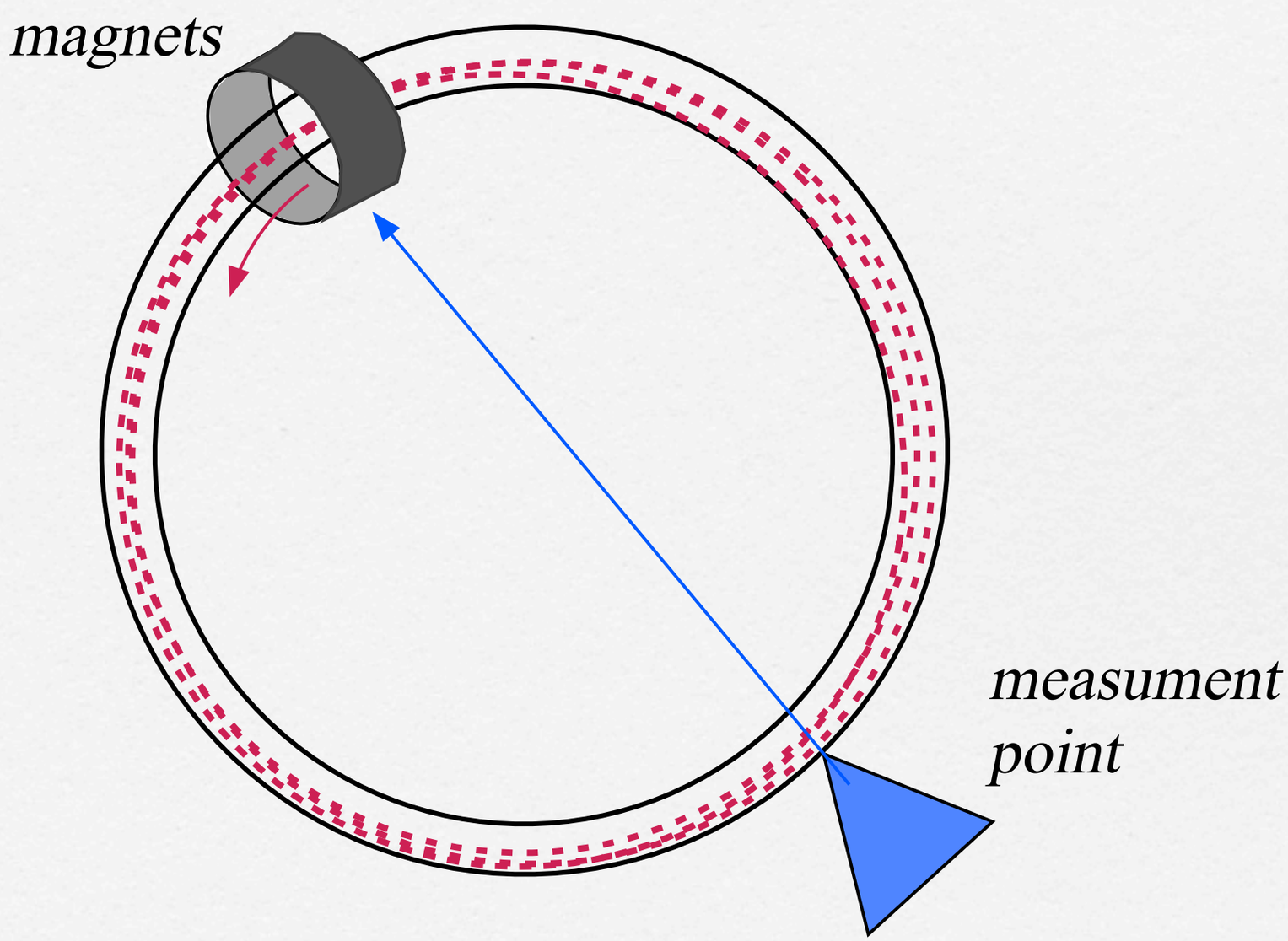
NewsVendor: cost: γ, sale price δ,

$\boldsymbol{\xi}$, demand distribution P, order x, $\Rightarrow$ explicit sol'n

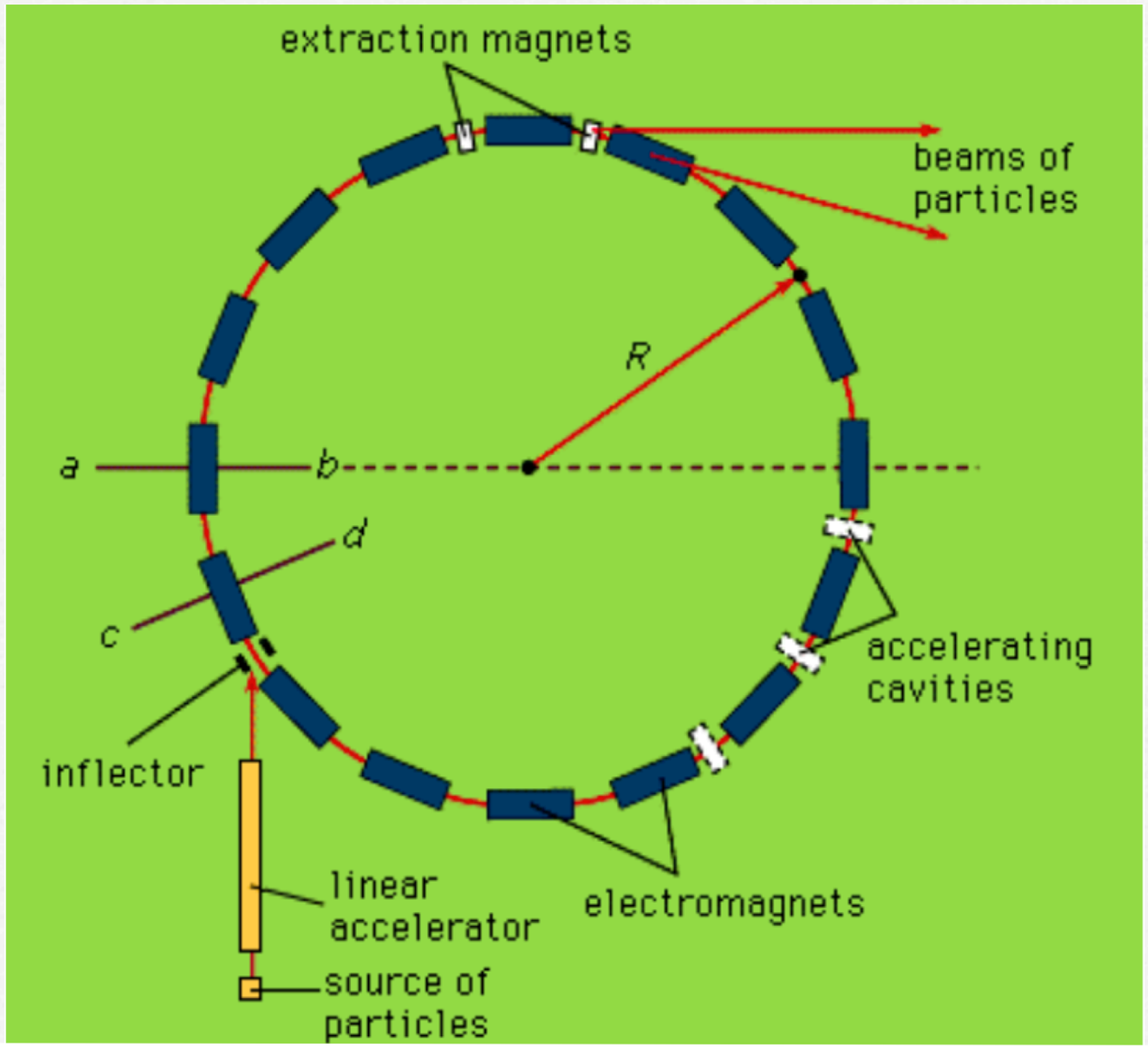
expected "loss": $\gamma x + \mathbb{E}\{Q(\boldsymbol{\xi}, x)\}$

$Q(\xi, x) = -\delta \cdot \min\{x, \xi\}$

Magnet Alignment I



Electronic generator diagram



Extensive formulation

EXF

$$
\begin{array}{llllllll}
\min & \langle -c, x\rangle & +p_1\langle q, y^1\rangle & +p_2\langle q, y^2\rangle & \cdots + & p_L\langle q, y^L\rangle & & \\
\text{s.t.} & T^1 x & -y^1 & & & & \leq & d^1 \\
 & T^2 x & & -y^2 & & & \leq & d^2 \\
 & \vdots & & & \ddots & & & \vdots \\
 & T^L x & & & & -y^L & \leq & d^L \\
 & x \geq 0, & y^1 \geq 0, & y^2 \geq 0, & \cdots & y^L \geq 0. & &
\end{array}
$$

Deterministic Equivalent Problem

$$Q(\xi, x) = \min\left\{\langle q, y\rangle \,\middle|\, T_\xi x + y \geq d_\xi,\ y \geq 0\right\}$$

$$EQ(x) = \mathbb{E}\{Q(\boldsymbol{\xi}, x)\} = \sum_{\xi\in\Xi} p_\xi Q(\xi, x)$$

the equivalent deterministic program:

DEP

$$\min\langle -c, x\rangle + EQ(x) \quad \text{such that } x \in \mathbb{R}^n_+$$

product mix problem

Closed Loop vs Open Loop

Closed Loop:
Observation of state $s \to$ control function $u(s)$ (optimal $\pm$)$\forall s$.
Stochastic Optimal Control: state $s(\xi_t)$, control function $u(s(\underset{\to t}{\xi}))$

Hopefully, $u(\,\cdot\,)$ is simple, manageable, ..., (maybe) time continuous

Open Loop:
Observation of state $s \to$ control function $u(s)$ (optimal $\pm$)$\forall s$.
Decision Process: from state $s(\xi_t)$ to decision $u(s(\underset{\to t}{\xi}))$

usually no 'closed form' expression
- may involve solving an optim. problem
- essentially 'closed loop' if decision occurs at $t + \Delta t$, Δt small

RHS: random right-hand sides

deterministic version:

$$\min \langle c, x\rangle \quad \text{such that} \quad Ax = b,\ Tx = \hat{\xi},\ x \geq 0$$

stochastic program with simple recourse RHS:

$$\min_x \langle c, x\rangle + \mathbb{E}\{Q(\boldsymbol{\xi}, x)\} \quad \text{such that} \quad Ax = b,\ x \geq 0$$

Deterministic Equivalent Problem: (SPWSR)

$$\min_x \langle c, x\rangle + EQ(x) \quad \text{such that} \quad Ax = b,\ x \geq 0$$

- $f_0 = \langle c, x\rangle$ is linear;
- $S = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$ is a polyhedral set, A is $m_1 \times n$;
- $Q(\xi, x) = q(\xi - Tx)$, T a non-random $m_2 \times n$ matrix;
- the *recourse cost function* $q\colon \mathbb{R}^{m_2} \to \mathbb{R}$ is convex;
- the *expectation functional* $EQ(x) = \int_\Xi q(\xi - Tx)\, P(d\xi)$.

Convex optim., linear constraints

$\mathcal{P}$: $\quad f_0 \colon \mathbb{R}^n \to \mathbb{R}$ convex, $\quad X$ polyhedral,

$$\begin{aligned} \min \quad & f_0(x), \qquad x \in X \subset \mathbb{R}^n \\ \text{such that} \quad & \langle A_i, x \rangle \geq b_i, \quad i = 1, \ldots, s, \\ & \langle A_i, x \rangle = b_i, \quad i = s+1, \ldots, m, \end{aligned}$$

x^* is an optimal solution of $(\mathcal{P}) \iff \exists$, KKT-multipliers $y \in \mathbb{R}^m$:

(a) $\langle A_i, x^* \rangle \geq b_i,\ i = 1, \ldots, s, \quad \langle A_i, x^* \rangle = b_i,\ i = s+1, \ldots, m,$

(b) for $i = 1, \ldots, s$: $\quad y_i \geq 0,\ y_i(\langle A_i, x^* \rangle - b_i) = 0,$

(c) $x^* \in \operatorname{argmin} \left\{ f_0(x) - \langle A^\top y, x \rangle \,\middle|\, x \in X \right\}.$

$\sim$ version of (c): $\exists\ -v \in N_X(x^*) = \left\{ u \,\middle|\, \langle u, x - x^* \rangle \leq 0, \forall\, x \in X \right\}$

Optimality: Simple Recourse

(SPWSR)

Suppose EQ finite-valued, x^* solves SP-simple recourse

$\iff$ one can find KKT-multipliers $u \in \mathbb{R}^{m_1}$

& summable KKT-multipliers $v : \Xi \to \mathbb{R}^{m_2}$:

1. $x^* \geq 0, \quad Ax^* = b;$
2. for all $\xi \in \Xi$: $\quad v(\xi) \in \partial q(\xi - Tx^*);$
3. $x^* \in \operatorname{argmin} \left\{ \langle c - A^\top u - T^\top \bar{v}, x \rangle \,\middle|\, x \in \mathbb{R}^n_+ \right\}$

where $\bar{v} = E\{v(\boldsymbol{\xi})\}$.

Reformulation: ... with Tenders

Change of variables (tenders): $\chi = Tx$

$$\min_{x,\chi} \langle c, x\rangle + \mathbb{E}\{\psi(\boldsymbol{\xi}, \chi)\} \quad \text{s.t.} \quad Ax = b,\ Tx = \chi,\ x \geq 0$$

$$\psi(\xi, \chi) = q(\xi - \chi), \quad \text{with} \quad E\psi(\chi) = \int_{\Xi} \psi(\xi, \chi)\, P(d\xi)$$

Deterministic equivalent problem:

$$\min_{x,\chi} \langle c, x\rangle + E\psi(\chi) \quad \text{s.t.} \quad Ax = b,\ Tx - \chi = 0,\ x \geq 0$$

$E\psi$ & EQ same properties & $\partial E\psi(\chi) = -\mathbb{E}\{\partial q(\boldsymbol{\xi} - \chi)\}$

Usually, ψ, $E\psi$, are *separable* while EQ <u>is not</u>,

$$\psi(\xi, \chi) = \sum_{i=1}^{m_2} \psi_i(\xi_i, \chi_i), = \sum_{i=1}^{m_2} q_i(\xi_i - \chi_i)$$

$$E\psi(\chi) = \sum_{i=1}^{m_2} E\psi_i(\chi_i)$$

Separable Simple Recourse (rhs)

$\min_{x,\chi} \langle c, x\rangle + \mathbb{E}\left\{\sum_{i=1}^{m_2} \psi_i(\boldsymbol{\xi}_i, \chi_i)\right\}$ such that $\quad Ax = b,\ Tx = \chi,\ x \geq 0$

$\psi_i(\xi_i, \chi_i) = q_i(\xi_i - \chi_i)$, q_i is convex, P_i marginal of $\boldsymbol{\xi}_i$, support $\Xi_i \subset \mathbb{R}$.

$$E\psi_i(\chi_i) = \mathbb{E}\{\psi_i(\boldsymbol{\xi}_i, \chi_i)\} = \int_{\Xi_i} \psi_i(\zeta, \chi_i)\, P_i(d\zeta),$$

the *deterministic equivalent problem* is then,

$$\min_{x,\chi} \langle c, x\rangle + \sum_{i=1}^{m_2} E\psi_i(\chi_i) \quad \text{such that} \quad Ax = b,\ Tx - \chi = 0,\ x \geq 0.$$

Linearly constrained convex program, but also *separable*.

$$\partial E\psi_i(\chi_i) = \left\{ -\int_{\Xi_i} v_i(\zeta)\, P_i(d\zeta) \,\middle|\, \forall \zeta \in \Xi_i, v_i(\zeta) \in \partial q_i(\zeta - \chi_i),\ \ v_i \text{ summable} \right\}.$$

Properties: Separable Simple Recourse

(x^*, χ^*) optimal $\iff$ $\exists$ KKT-multipliers $u \in \mathbb{R}^{m_1}$ and
for $i = 1, \ldots, m_2$, summable KKT-multipliers $v_i : \Xi_i \to \mathbb{R}$ such that

(a) $Ax^* = b, \quad Tx^* = \chi^*;$

(b) for $i = 1, \ldots, m_2$: $\quad \forall\, \zeta \in \Xi_i, \quad v_i(\zeta) \in \partial q_i(\zeta - \chi_i^*);$

(c) $x^* \in \operatorname{argmin} \left\{ \langle c - A^\top u - T^\top \bar{v}, x \rangle \,\middle|\, x \in \mathbb{R}^n_+ \right\}. \quad \bar{v}_i = \mathbb{E}\{v_i(\boldsymbol{\xi}_i)\}$

When $q_i(y) = \max[\,\gamma_i y,\, \delta_i y\,]$ with $\gamma_i \le \delta_i, \quad E\{\boldsymbol{\xi}_i\}$ finite
(x^*, χ^*) optimal solution $\iff$ $\exists$ KKT-multipliers $u \in \mathbb{R}^{m_1}$ and $\bar{v} \in \mathbb{R}^{m_2}$:

(a) $Ax^* = b, \quad Tx^* = \chi^*;$

(b) for $i = 1, \ldots, m_2, \qquad P_i(z) = \text{prob.}\,[\boldsymbol{\xi}_i \le z], \;\; p_z = \text{prob.}\,[\boldsymbol{\xi}_i = z]$

$$P_i(\chi_i^*) - p_{\chi_i^*} \le \frac{\delta_i - \bar{v}_i}{\delta_i - \gamma_i} \le P_i(\chi_i^*),$$

→ linear constraints, ξ discrete

(c) $x^* \ge 0, \quad A^\top u + T^\top \bar{v} \le c,$ and $\langle c - A^\top u - T^\top \bar{v}, x^* \rangle = 0.$

Approximation

Let $P : \mathbb{R} \to [0,1]$ continuous, increasing on interval Ξ,

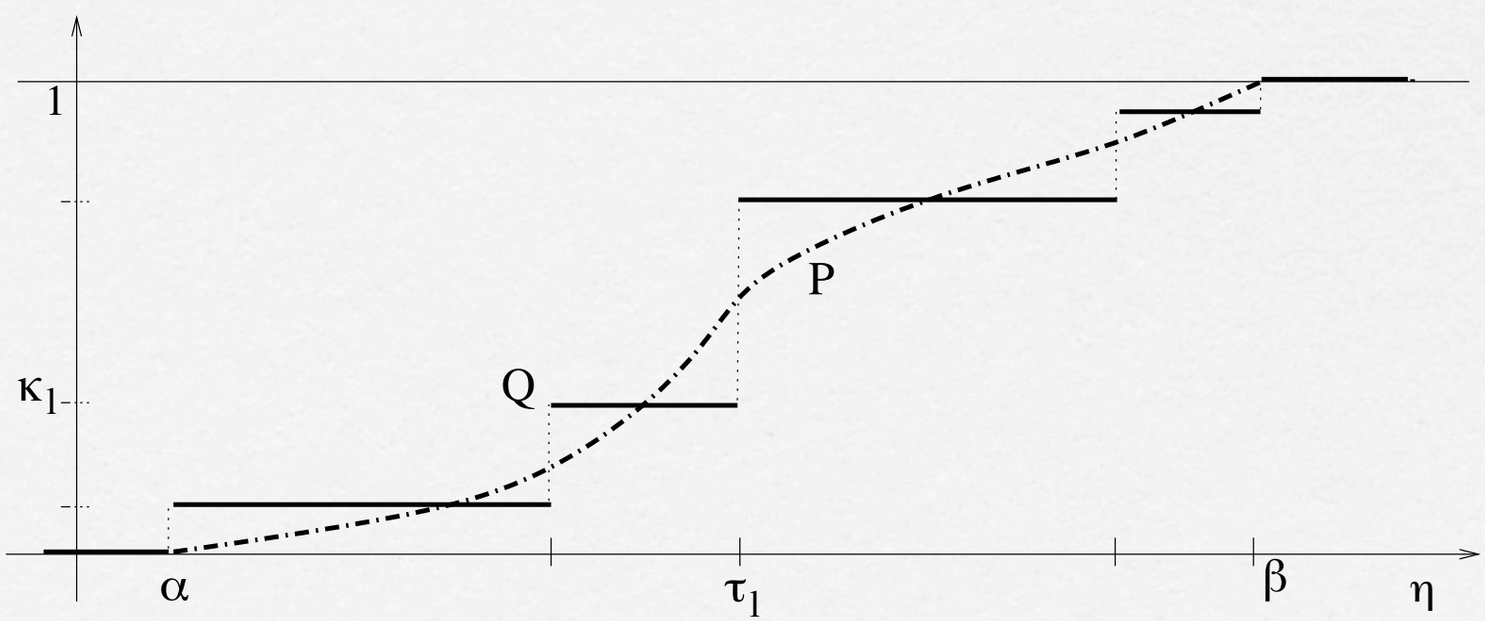
$$E\psi(\chi) = \delta(\bar{\xi} - \chi) + (\delta - \gamma)\big[\, \chi P(\chi) - \int_{-\infty}^{\chi} \zeta \, P(d\zeta) \,\big];$$

Hence,

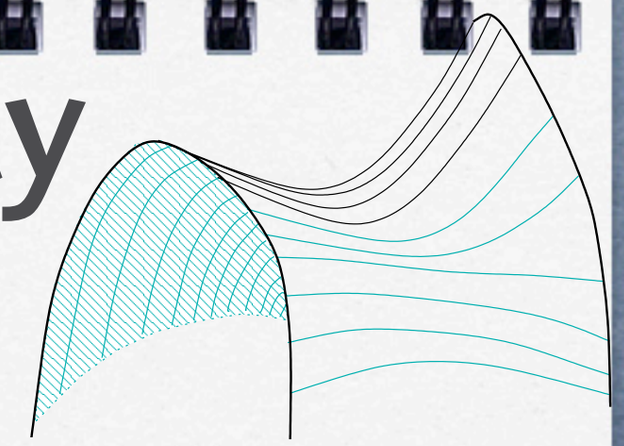
$$P(\chi^*) = \frac{\delta - \bar{v}}{\delta - \gamma} =: \kappa, \qquad \boxed{\chi^* = P^{-1}(\kappa).}$$

When Q has the same κ-quantile, $Q^{-1}(\kappa) = P^{-1}(\kappa)$, <u>same</u> optimal sol'n.

$\Longrightarrow$ choose Q is 'quantile close' to P



Lagrangian Duality



$$\min \quad f_0(x),\ x \in X \subset \mathbb{R}^n \quad \text{polyhedral}$$
$$\langle A_i, x\rangle \geq b_i,\ i = 1, \ldots, s, \quad \langle A_i, x\rangle = b_i,\ i = s+1, \ldots, m$$

The Lagrangian:

$$L(x, y) = f_0(x) + \langle y, b - Ax\rangle \quad \text{on} \quad X \times Y,\ Y = \mathbb{R}^s_+ \times \mathbb{R}^{m-s}.$$

x^*optimal $\iff$ $\exists$ a pair (x^*, y^*) that satisfies:

$$x^* \in \operatorname{argmin}_{x \in \mathbb{R}^n} L(x, y^*), \qquad y^* \in \operatorname{argmax}_{y \in Y} L(x^*, y).$$

primal & dual problems:

linear programs: $\min\langle c, x\rangle,\ Ax = b,\ x \geq 0$ & $\max\langle b, y\rangle,\ A^\top y \leq 0$

quadratic programs: $\min\langle c, x\rangle + \frac{1}{2}\langle x, Qx\rangle,\ Ax \geq b,\ x \in \mathbb{R}^n$

$$\max \alpha + \langle d, y\rangle - \tfrac{1}{2}\langle y, Py\rangle, y \in \mathbb{R}^m_+$$

$$\alpha = -\tfrac{1}{2}\langle c,\, Q^{-1}c\rangle,\ d = b + AQ^{-1}c \text{ and } P = AQ^{-1}A^\top.$$

Monitoring Functions

penalty substitutes

A *monitoring function* $\theta_{P,Q}(x) = \sup_{v \in \mathbb{R}^n} \left\{ \langle x, v \rangle - \frac{1}{2}\langle v, Qv \rangle \,\middle|\, v \in P \right\}$, convex *linear-quadratic function* for P polyhedral, Q psd. On $\mathbb{R}$

$$\theta_{r,\beta} = \theta_{I,\beta}(x) = \sup \left\{ xv - \frac{\beta}{2} v^2 \,\middle|\, v \in I \subset \mathbb{R} \right\}, \quad \beta \geq 0,\ P = [0, r]\ (r \leq \infty)$$

.

Monitoring Functions

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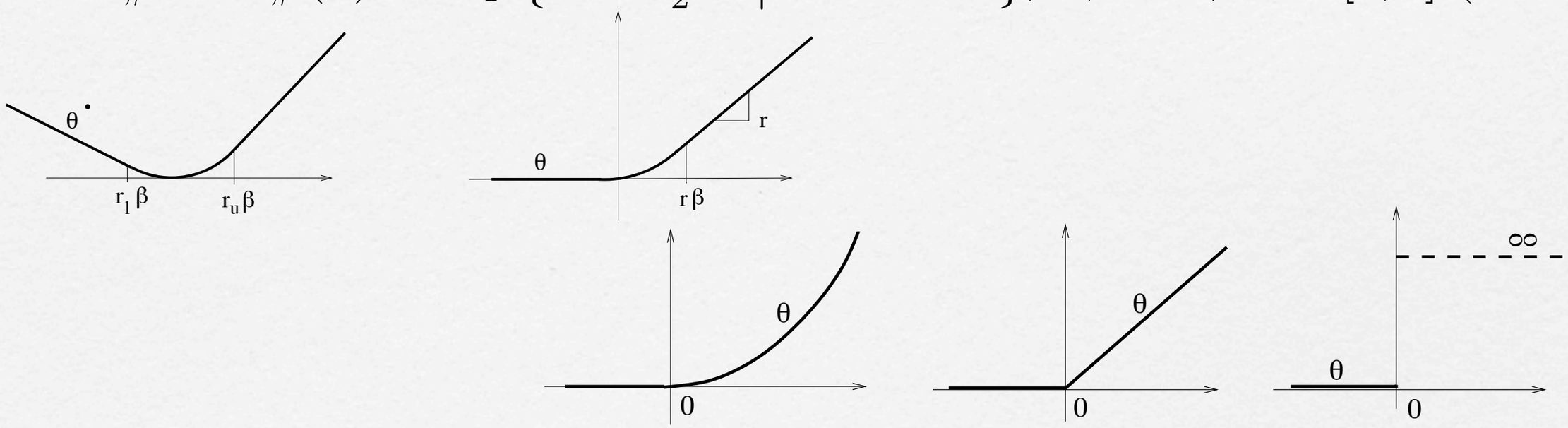

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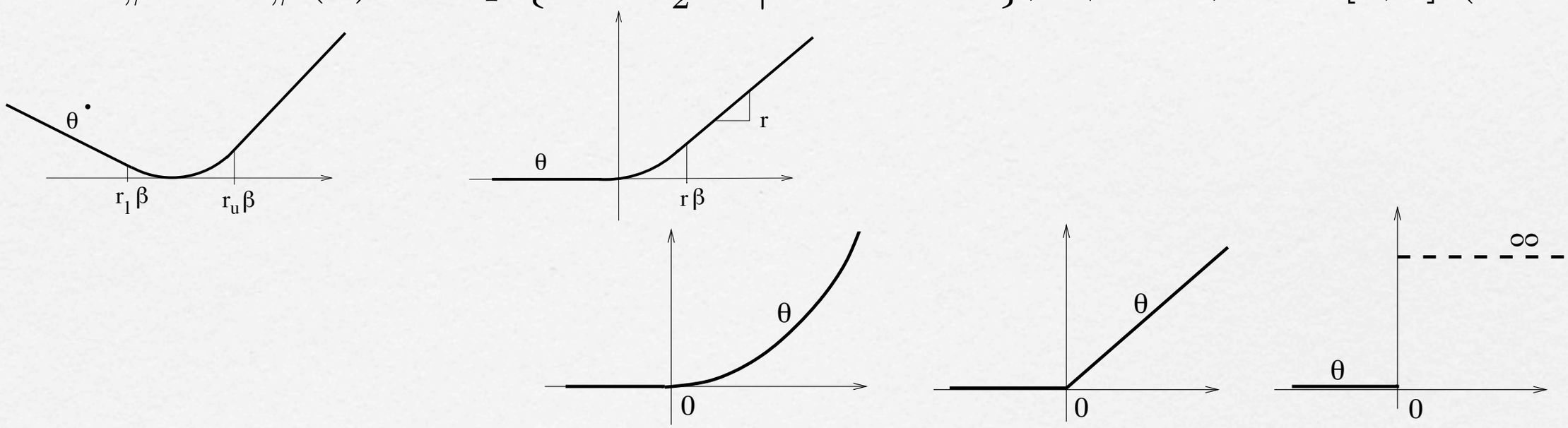


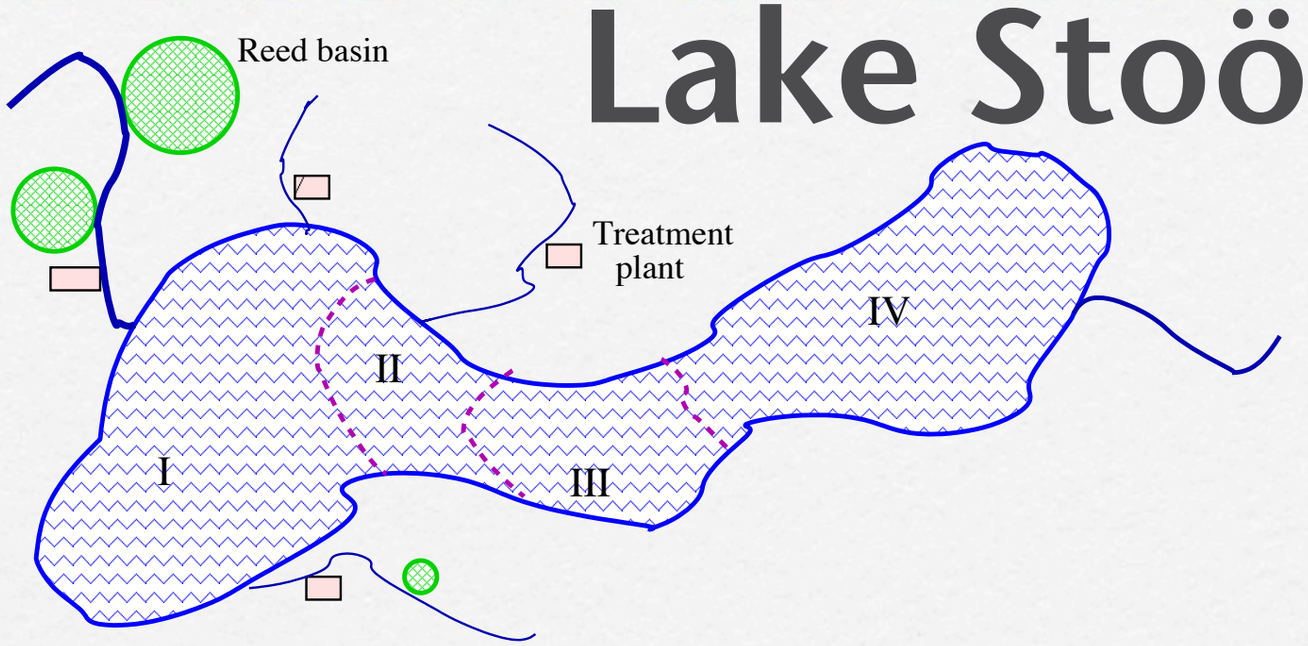
$$\min_{0\leq x\leq s}\ \langle c, x\rangle + \frac{1}{2}\sum_{j=1}^{n}\delta_j x_j^2 + \sum_{i=1}^{m}\theta_{r_i,\beta_i}(d_i - \langle T_i, x\rangle)$$

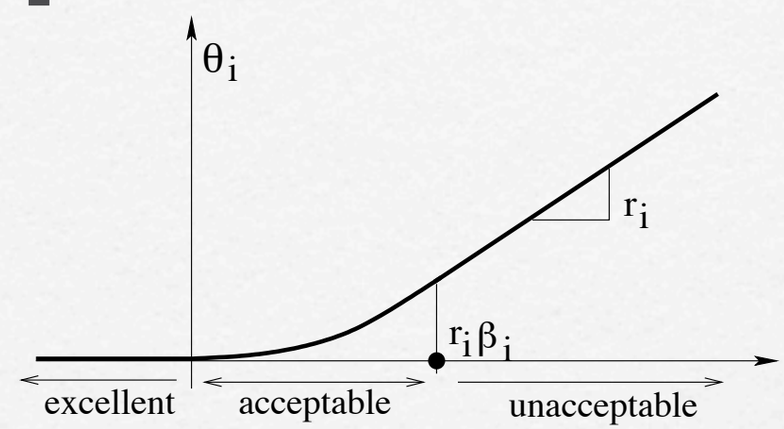
and

$$\max_{0\leq v\leq r}\ \langle d, v\rangle - \frac{1}{2}\sum_{i=1}^{m}\beta_i v_i^2 - \sum_{j=1}^{n}\theta_{s_j,\delta_j}(\langle T^j, v\rangle - c_j)$$

are a dual pair when $\delta_j, r_j, \beta_i, s_i$ strictly positive.

Lake Stoöpt





Water quality: $\boldsymbol{z}_i = T_i(\boldsymbol{\xi})x - d(\boldsymbol{\xi})$, $i = I, \ldots, IV$, technical, $\ldots$: $Ax \leq b$

Monitoring deviations from the desired quality: for $i = I, \ldots, , IV$,

$$\theta_{r_i,\beta_i}(\tau) = \begin{cases} 0 & \text{if } \tau < 0, \quad \text{(excellent)} \\ \tau^2/2\beta_i & \text{if } \tau \in [0, r_i\beta_i] \quad \text{(acceptable)} \\ r_i\tau - r_i^2\beta_i/2 & \text{if } \tau > r_i\beta_i \quad \text{(unacceptable)} \end{cases}$$

+ direct costs with building treatment plants and reed basins,

$\sum_{j=1}^n c_j x_j + \frac{1}{2}\sum_{j=1}^n \delta_j x_j^2$, $D = \mathrm{diag}(\delta_1, \ldots, \delta_n)$: leads to

$$\min \langle c, x\rangle + \tfrac{1}{2}\langle x, Dx\rangle + \mathbb{E}\Big\{ \textstyle\sum_{i=I}^{IV} \theta_{r_i,\beta_i}(d_i(\boldsymbol{\xi}) - \langle T_i(\boldsymbol{\xi}), x\rangle)\Big\}, \; Ax \leq b, 0 \leq x \leq$$

with random rhs & technology T-matrix (hydro-dynamics, atmospheric).

Exploiting duality

$$\min \langle c, x\rangle + \tfrac{1}{2}\langle x, Dx\rangle + \sum_{i=1}^{m_2} \mathbb{E}^i\{\theta_{r_i,\beta_i}(\boldsymbol{w}_i)\},\ Ax \geq b, \quad \boldsymbol{w} = \boldsymbol{d} - \boldsymbol{T}x,\ 0 \leq x \leq s$$

for all i, $\boldsymbol{\xi}_i = (\boldsymbol{r}_i, \boldsymbol{\beta}_i, \boldsymbol{d}_i, \boldsymbol{t}_{i1}, \ldots, \boldsymbol{t}_{in})$, $\boldsymbol{w}$ and, later, $\boldsymbol{v}$ as *very* long vectors.
The dual takes on the form, with $u \geq 0$, $\quad 0 \leq \boldsymbol{v}_i \leq \boldsymbol{r}_i,\ \forall\, i$

$$\max\ \langle b, u\rangle - \sum_{j=1}^{n} \theta_{s_j,\delta_j}(z_j) + \sum_{i=1}^{m_2} E^i\{\boldsymbol{d}_i\boldsymbol{v}_i - \tfrac{1}{2}\boldsymbol{\beta}_i\boldsymbol{v}_i^2\}$$

$$\text{such that}\ \ z_j = \langle A^j, u\rangle + \sum_{i=1}^{m_2} E^i\{\boldsymbol{t}_{ij}\boldsymbol{v}_i\} - c_j,\ \ j = 1, \ldots, n,$$

Only "simple" stochastic box constraints: $0 \leq \boldsymbol{v}_i \leq \boldsymbol{r}_i, \quad i = 1, \ldots, m_2$.
Lake Stoöpt: r and β are non-random.

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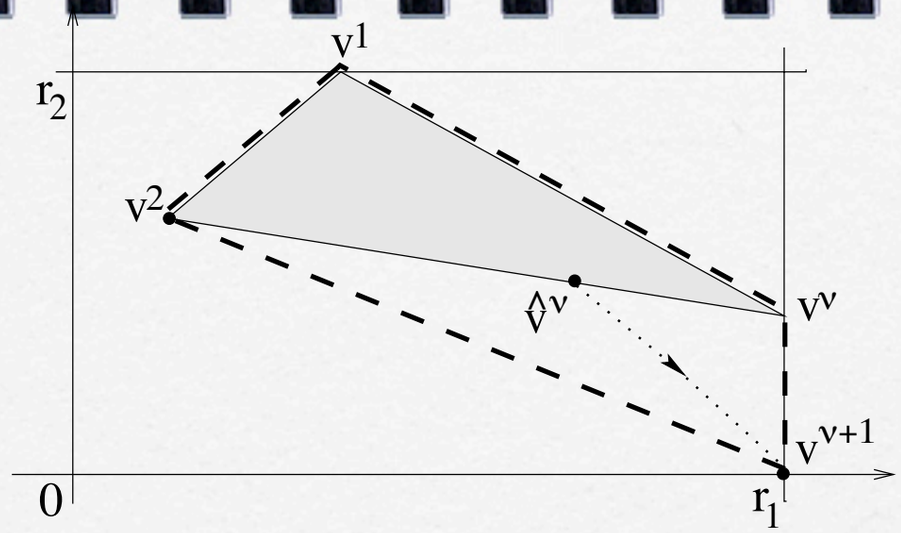
Lagrangian Finite Generation



Step 0. $\boldsymbol{V}_\nu = [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^\nu]$, $\boldsymbol{v}_i^k : \Xi_i \to [0, \boldsymbol{r}_i]$, $\forall i$

Step 1. Compute $d^\nu = E\{\boldsymbol{V}_\nu^\top \boldsymbol{d}\}$, $\quad B^\nu = E\{\boldsymbol{V}_\nu^\top B \boldsymbol{V}_\nu\}$, $\quad T_\nu^\top = E\{\boldsymbol{T}^\top \boldsymbol{V}_\nu\}$

Step 2. Solve the (deterministic) approximating dual program:

$$\max \langle b, u\rangle + \langle d^\nu, \lambda\rangle - \tfrac{1}{2}\langle \lambda, B^\nu \lambda\rangle - \textstyle\sum_{j=1}^n \theta_{s_j,\delta_j}(z_j)$$

$$\text{s.t} \;\; A^\top u + T_\nu^\top \lambda - z = c, \quad \textstyle\sum_{k=1}^\nu \lambda_k = 1, \quad u \geq 0, \;\; \lambda \geq 0$$

$(u^\nu, \lambda^\nu, z^\nu)$ optimal, and x^ν KKT-multipliers of equality constraints.

set $\hat{\boldsymbol{v}}^\nu = \boldsymbol{V}_\nu \lambda^\nu$, $\quad \boldsymbol{w}_i^\nu = \boldsymbol{d}_i - \langle \boldsymbol{T}_i, x^\nu\rangle$, $i = 1, \ldots, m_2$

Step 3 (*saddle point check*) Stop, if for each i,

$\hat{\boldsymbol{v}}_i^\nu \in \operatorname{argmax}_{\boldsymbol{v}_i \in [0, r_i]} \langle \boldsymbol{w}_i^\nu, \boldsymbol{v}_i\rangle - \frac{1}{2}\beta_i \boldsymbol{v}_i^2$,

otherwise, for $i = 1, \ldots, m_2$ and every $\zeta \in \Xi_i$, define

$v_i^{\nu+1}(\zeta) \in \operatorname{argmax}_{0 \leq v \leq r_i} [w_i^\nu(\zeta) v - \frac{1}{2}\beta_i v^2]$ with value $\theta_{r_i,\beta_i}(w_i^\nu(\zeta))$

Augment $\boldsymbol{V}_{\nu+1} = [\boldsymbol{V} \;\; \boldsymbol{v}^{\nu+1}]$, set $\nu \leftarrow \nu + 1$, return to Step 1.