Approximating Stochastic Programs

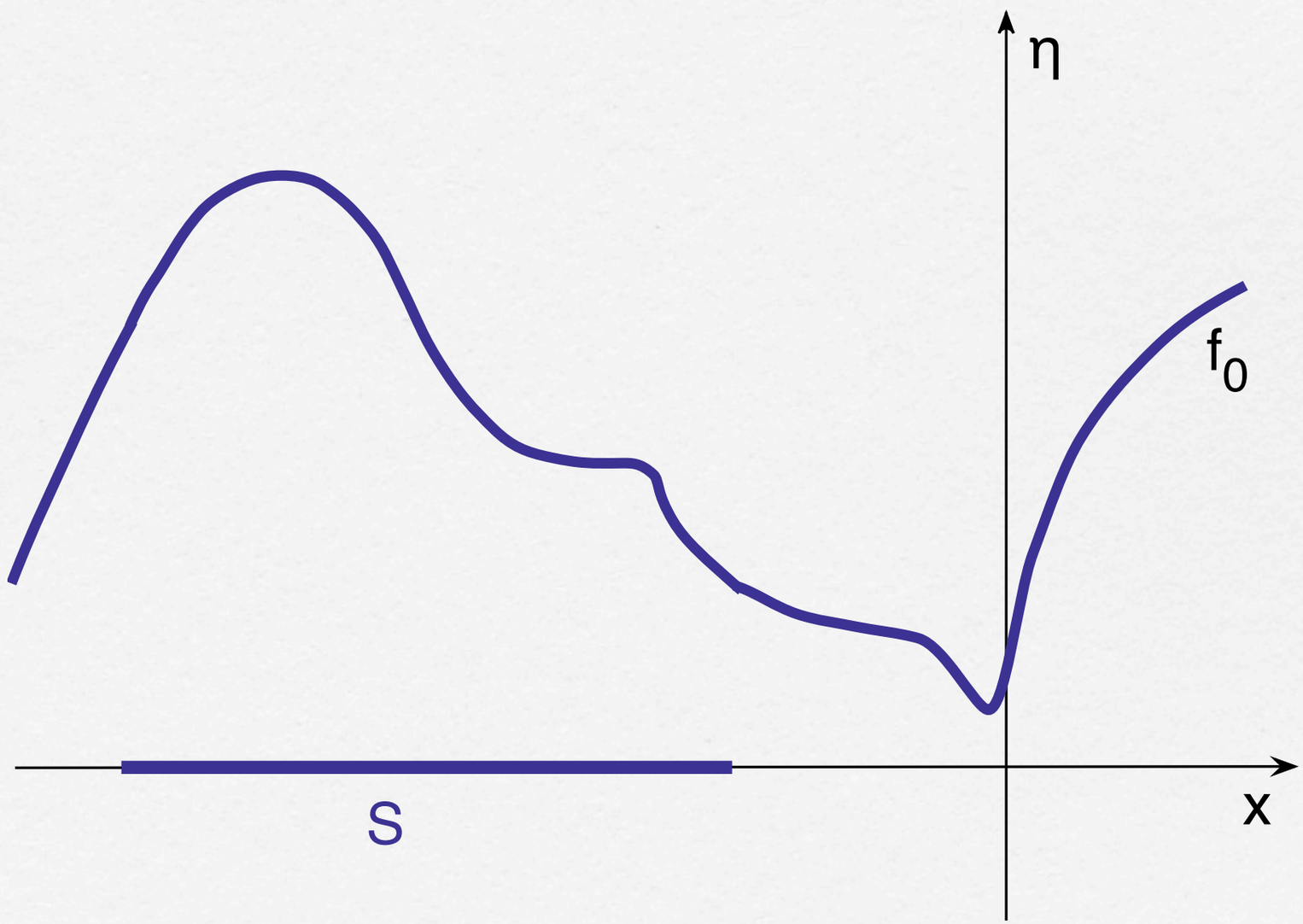
Roger J-B Wets
Mathematics, University of California, Davis

Preliminaries (unavoidable)

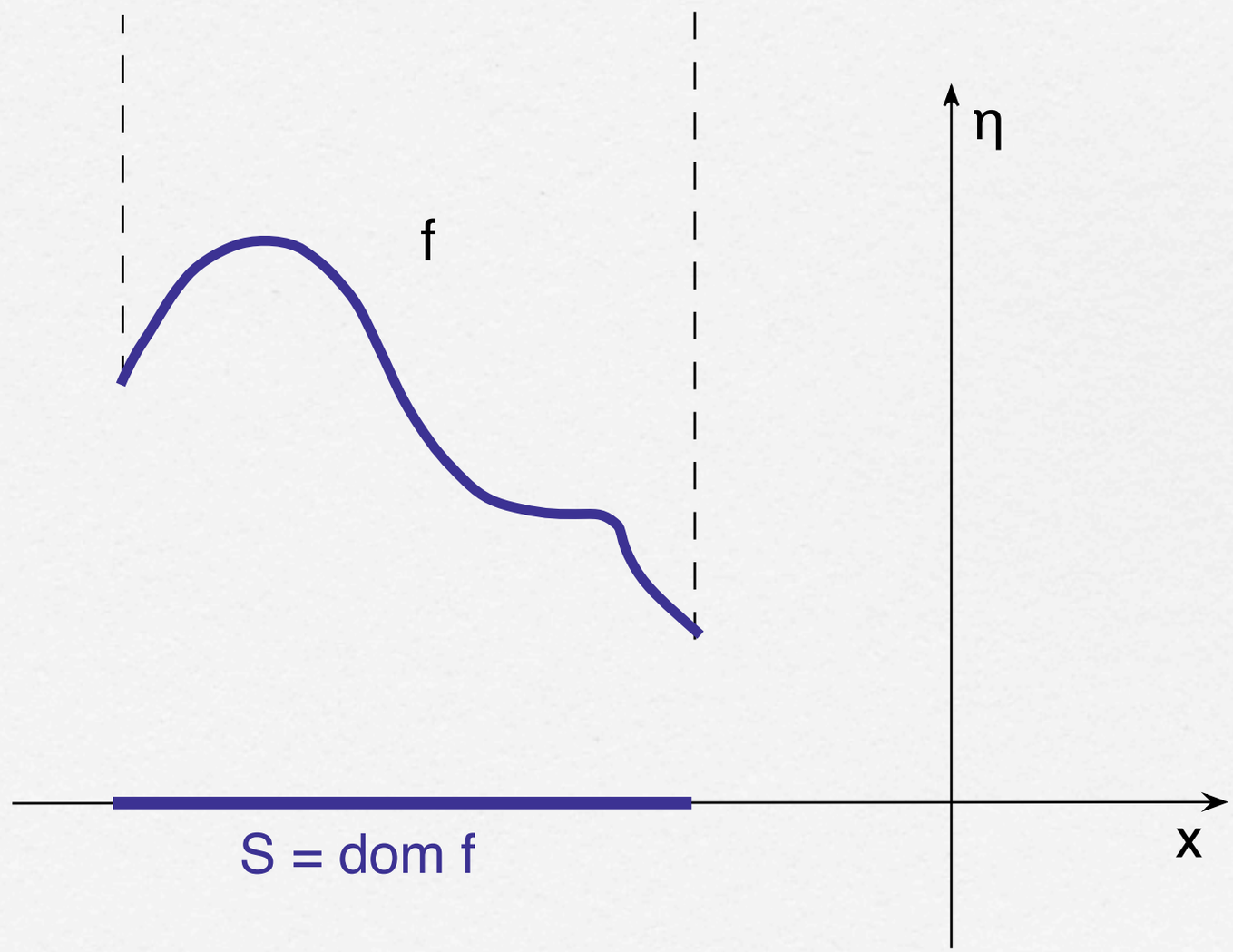
Optimization problem

$$\min f_0(x),\ x \in S,$$

$$S = \left\{ x \in \mathbb{R}^n \middle| f_i(x) \le 0,\ i = 1 \to s,\ f_i(x) = 0,\ i = s+1 \to m \right\}$$

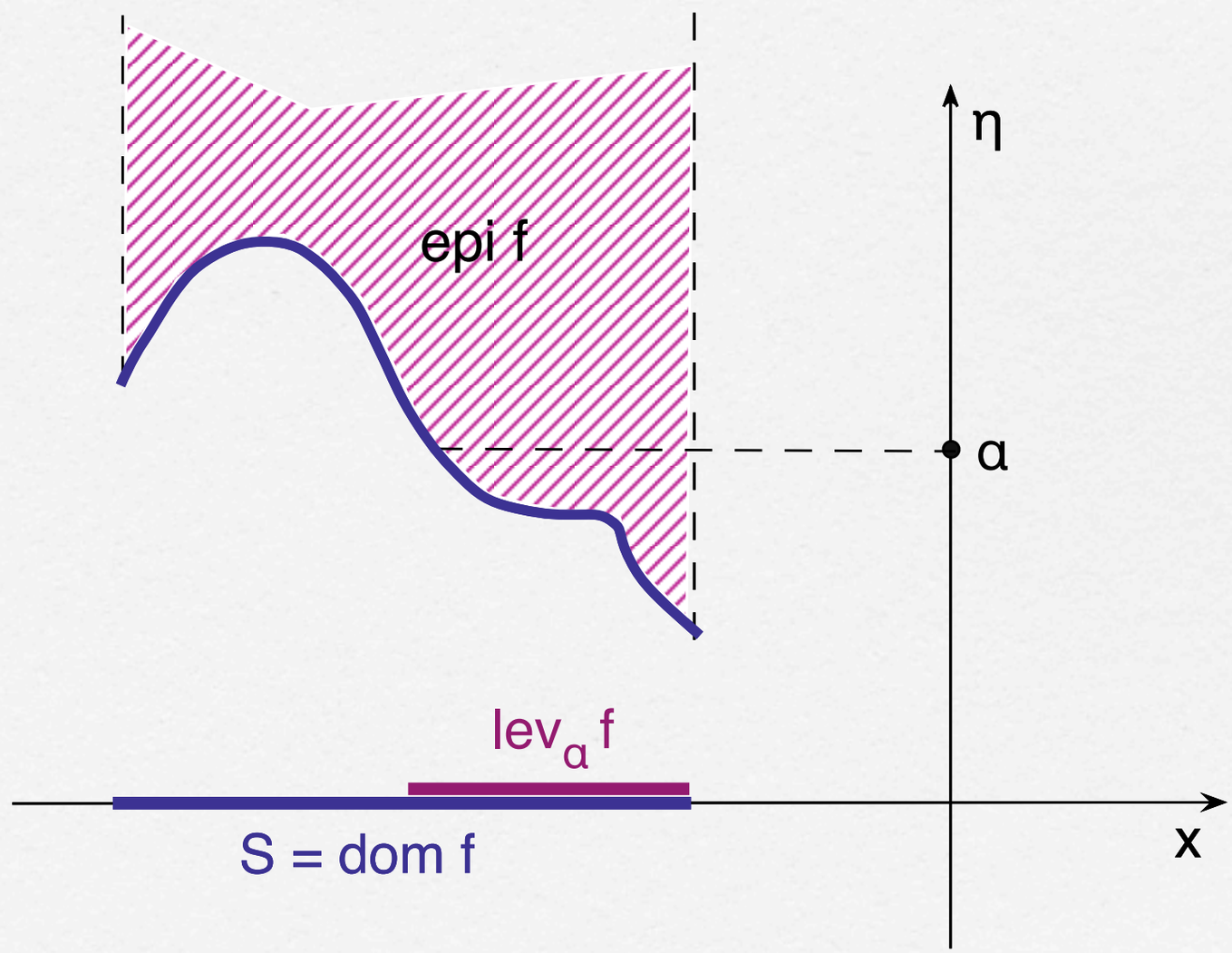


$$\min f \text{ on } E, \quad f = f_0 + \iota_S(x), \quad \iota_S \text{ indicator function of } S$$



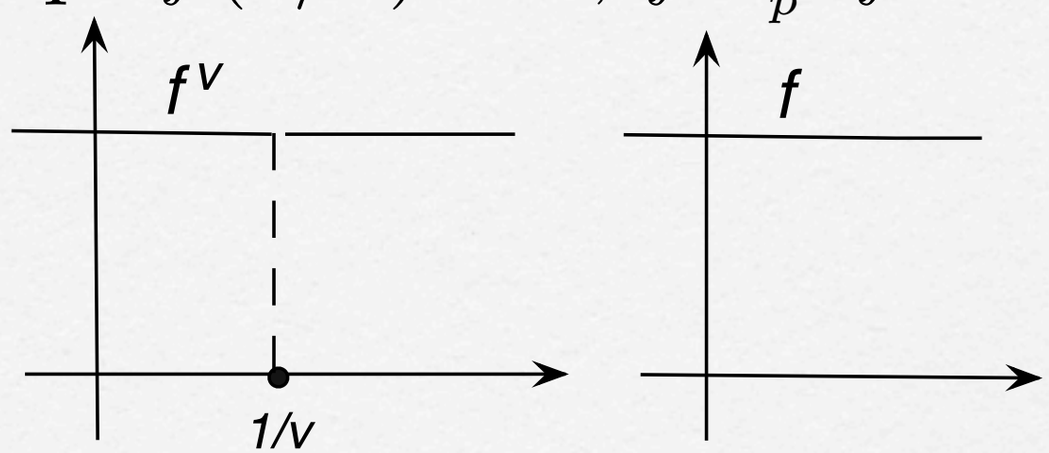
$$\min f \text{ on } E, \quad f = f_0 + \iota_S(x), \quad \iota_S \text{ indicator function of } S$$

$$\operatorname{epi} f = \left\{(x,\alpha) \in E \times R \,\middle|\, f(x) \le \alpha\right\}, \quad \operatorname{lev}_\alpha f = \left\{x \in E \,\middle|\, f(x) \le \alpha\right\}$$

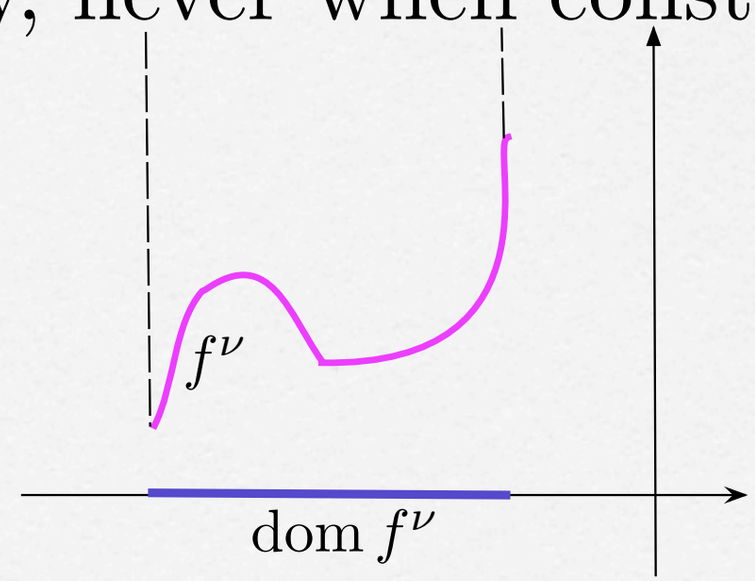


1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

$f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$

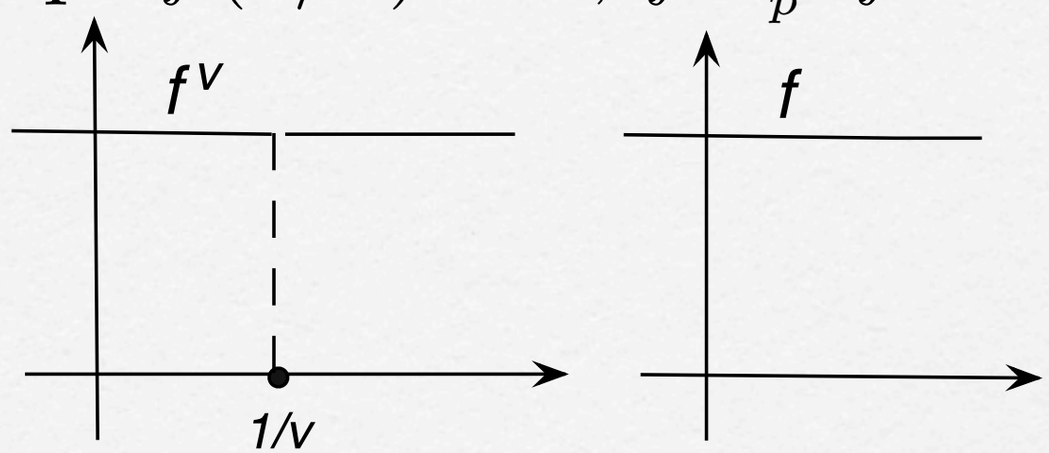


2. uniform convergence implies convergence of minimizers but applies rarely, never when constraints depend on ν

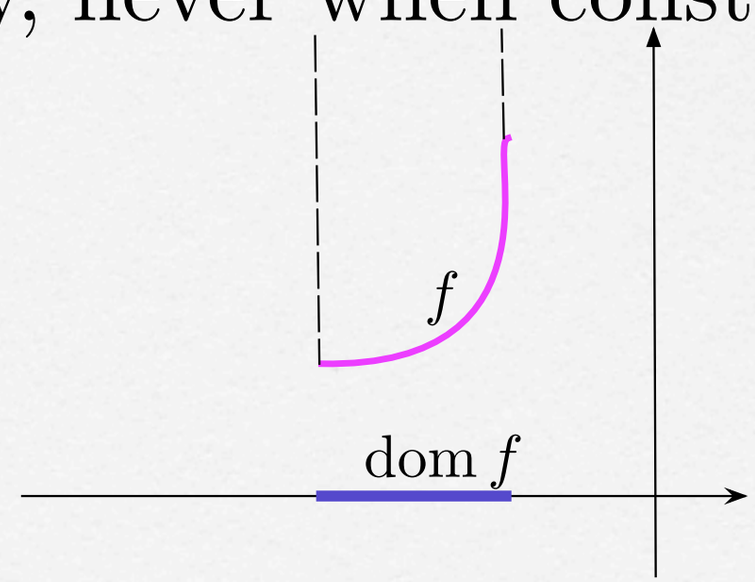


1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

$f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$

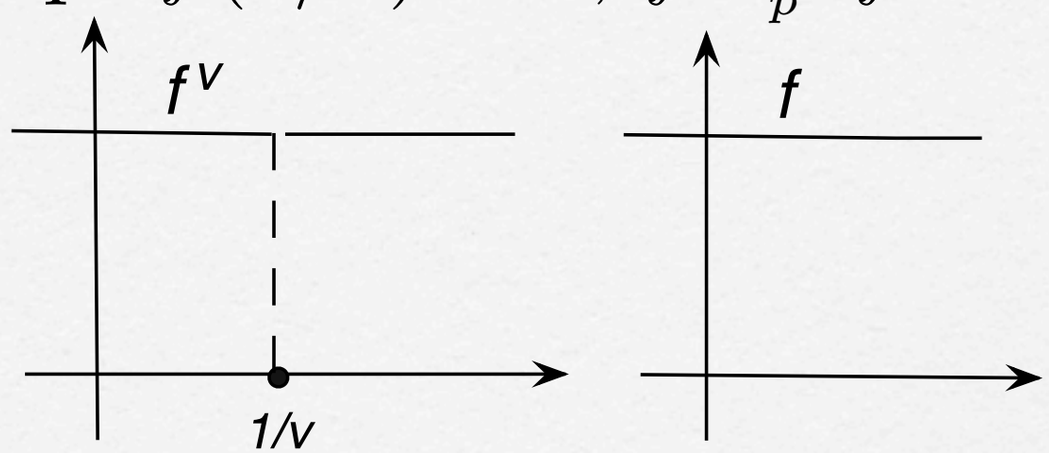


2. uniform convergence implies convergence of minimizers but applies rarely, never when constraints depend on ν

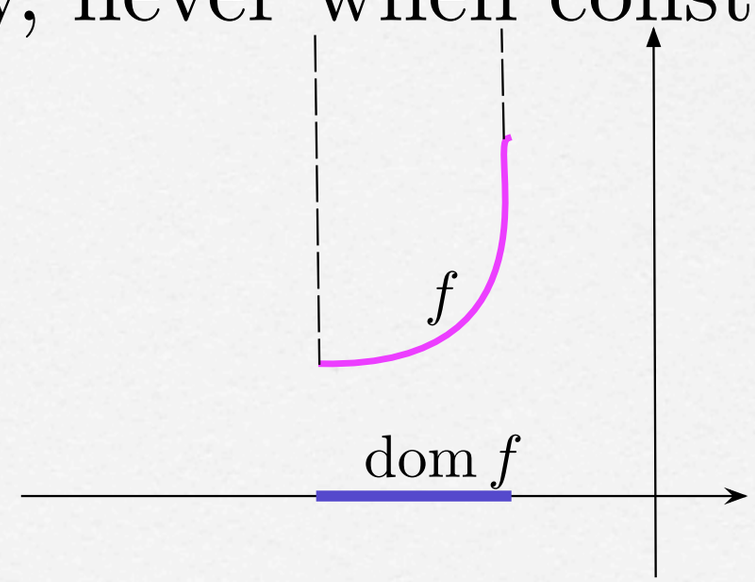


1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

$f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$



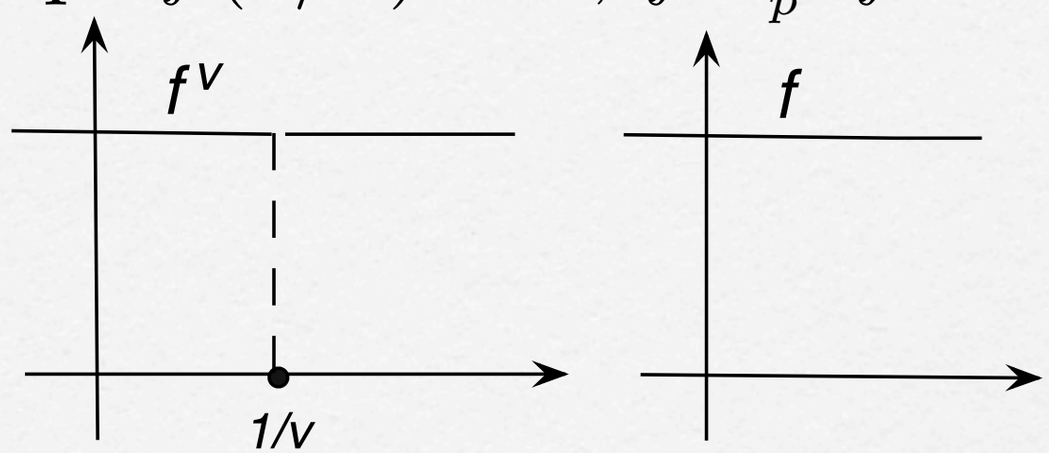
2. uniform convergence implies convergence of minimizers but applies rarely, never when constraints depend on ν



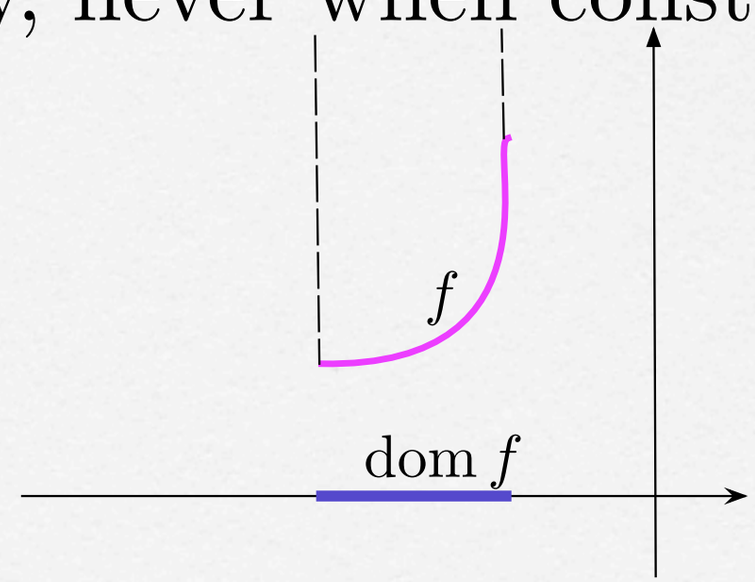
One-sided uniform convergence

1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

$f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$



2. uniform convergence implies convergence of minimizers but applies rarely, never when constraints depend on ν



variational epi-convergence

Epi-Convergence

$f^\nu \xrightarrow{e} f$ if for all $x \in E$,

1. $\forall\, x^\nu \to x,\ \liminf_\nu f^\nu(x^\nu) \geq f(x)$
2. $\exists\, x^\nu \to x,\ \limsup_\nu f^\nu(x^\nu) \leq f(x)$

"Geometrically": $\operatorname{epi} f^\nu \to \operatorname{epi} f$ (later)

Pointwise:

$$\liminf_\nu f^\nu(x) \geq f(x), \quad \limsup_\nu f^\nu(x) \leq f(x)$$

Continuous: $\forall x^\nu \to x$,

$$\liminf_\nu f^\nu(x^\nu) \geq f(x), \quad \limsup_\nu f^\nu(x^\nu) \leq f(x)$$

Epi-Convergence $\Rightarrow$

$A^\nu = \arg\min f^\nu$, ε-A^ν : $\varepsilon > 0$ approximate minimizers,

$A = \arg\min f$ of limit problem, ε-A approx. minimizers

A^ν v-converges to A, written $A^\nu \Rightarrow_\nu A$, if

a) $\bar{x} \in$ cluster-points$\left\{x^\nu \in A^\nu\right\} \Rightarrow \bar{x} \in A$

b) $\bar{x} \in A \Rightarrow \exists\ \varepsilon_\nu \searrow 0, \mathrm{x}^\nu \in \varepsilon_\nu\text{-}A^\nu \to \bar{x}$

Epi-Convergence $\Rightarrow$

$A^\nu = \arg\min f^\nu$, ε-A^ν : $\varepsilon > 0$ approximate minimizers,

$A = \arg\min f$ of limit problem, ε-A approx. minimizers

A^ν v-converges to A, written $A^\nu \Rightarrow_v A$, if

a) $\bar{x} \in$ cluster-points$\left\{x^\nu \in A^\nu\right\} \Rightarrow \bar{x} \in A$

b) $\bar{x} \in A \Rightarrow \exists\ \varepsilon_\nu \searrow 0, \mathrm{x}^\nu \in \varepsilon_\nu\text{-}A^\nu \to \bar{x}$

$$\boxed{\begin{array}{l} f^\nu \xrightarrow{e} f \text{ implies } \varepsilon\text{-}A^\nu \Rightarrow_v \varepsilon\text{-}A,\ \forall\, \varepsilon \geq 0 \\ A \text{ unique minimizer, } \varepsilon^\nu\text{-}A^\nu \rightrightarrows A \text{ as } \varepsilon^\nu \searrow 0. \end{array}}$$

$(\inf f > -\infty)$

Mathematical Framework:
Random lsc functions &

Expectation Functionals

$$\widehat{Ef} = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot)\}$$

$f : \Xi \times E \to \overline{\mathbb{R}}$, random lsc function, $f(\xi, x) = f_0(\xi, x)$ when $x \in C(\xi)$

$E \subset \mathcal{M}(\Xi, \mathcal{A}; \mathbb{R}^n) : \mathcal{L}^p(\Xi, \mathcal{A}, P; \mathbb{R}^n), \ldots$

others: $C\big((\Xi, \tau); \mathbb{R}^n\big)$, Orlicz, Sobolev, lsc-fcns(E)

$$Ef(x) = \int_\Xi f(\xi, x(\xi))\, P(d\xi) = \mathbb{E}\big\{f(\boldsymbol{\xi}, x(\boldsymbol{\xi}))\big\}$$

$$= \infty \text{ whenever } \int_\Xi f_+(\xi, x(\xi))\, P(d\xi) = \infty$$

$Ef : E \to \overline{\mathbb{R}}$ always defined

Regression: (E is not a linear space)

$$\min\left\{ \int_{y \in \mathbb{R}} \int_{x \in [0,1]^n} \phi(y - h(x))\, P(dx, dy) \,\middle|\, h \in \text{ lsc-fcns}(\mathbb{R}^n) \cap \mathcal{H} \right\}$$

$\mathcal{H}$ shape restrictions (convex, unimodal, ...)

Random lsc functions

$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in x, $(\forall \xi \in \Xi)$; x decision variable

(b) (ξ, x)-measurable $\qquad (\mathcal{A} \times B_E)$-measurable

recall: $f(\xi, x) = f_0(\xi, x)$ when $x \in C(\xi)$ -- stochastic constraints

$$f^\nu(\xi, x) = \begin{cases} \dfrac{1}{\nu} \sum_{l=1}^{\nu} \left(f(\xi^l, x) \text{ if } \mathrm{x} \in C(\xi^l) \right) & \text{(typically)} \\ \infty \text{ otherwise} & (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi, \cdot)$ epi-converge to $\mathbb{E}\{f(\xi, \bullet)\}$ P-a.s.?

does $x^\nu \in \arg\min f^\nu \Rightarrow_\nu x^* \in \arg\min \mathbb{E}\{f(\xi, x)\}$ P-a.s.?

Random lsc functions

$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in x, $(\forall \xi \in \Xi)$; x decision variable

(b) (ξ, x)-measurable $\qquad (\mathcal{A} \times B_E)$-measurable

recall: $f(\xi, x) = f_0(\xi, x)$ when $x \in C(\xi)$ -- stochastic constraints

$$f^\nu(\xi, x) = \begin{cases} \frac{1}{\nu} \sum_{l=1}^{\nu} \left(f(\xi^l, x) \text{ if } \mathrm{x} \in C(\xi^l) \right) & \text{(typically)} \\ \infty \text{ otherwise} & (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi, \cdot)$ epi-converge to $\mathbb{E}\{f(\xi, \bullet)\}$ P-a.s.?

does $x^\nu \in \arg\min f^\nu \Rightarrow_\nu x^* \in \arg\min \mathbb{E}\{f(\xi, x)\}$ P-a.s.?

Law of Large Numbers for random lsc functions
$\sim$ LLN for Stochastic Optimization Problems.

Random lsc functions

$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in x, $(\forall \xi \in \Xi)$; x decision variable

(b) (ξ, x)-measurable $\qquad (\mathcal{A} \times B_E)$-measurable

recall: $f(\xi, x) = f_0(\xi, x)$ when $x \in C(\xi)$ -- stochastic constraints

$$f^\nu(\xi, x) = \begin{cases} \frac{1}{\nu} \sum_{l=1}^{\nu} \left(f(\xi^l, x) \text{ if } \mathrm{x} \in C(\xi^l) \right) & \text{(typically)} \\ \infty \text{ otherwise} & (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi, \cdot)$ epi-converge to $\mathbb{E}\{f(\xi, \bullet)\}$ P-a.s.?

does $x^\nu \in \arg\min f^\nu \Rightarrow_\nu x^* \in \arg\min \mathbb{E}\{f(\xi, x)\}$ P-a.s.?

$$E^\nu f \xrightarrow{e} Ef \ a.s., \quad E^\nu f(x) = \frac{1}{\nu} \sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, x)$$

LLN Theorem

$f : \Xi \times E \to \overline{\mathbb{R}}$, locally inf-integrable random lsc function $\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \ldots, \}$ are iid Ξ-valued random variables. Then,

$$E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot) = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\}$$

which means ε-argmin $E^\nu f \Rightarrow_v \varepsilon$-argmin Ef, $\forall\, \varepsilon \geq 0$

Ef unique minimizer, ε^ν-argmin $E^\nu f \rightrightarrows$ argmin Ef as $\varepsilon^\nu \searrow 0$.

SAA-applies without 'any' restrictions

loc.inf-integrable: $\int \inf\{f(\xi, \cdot) \,|\, \mathbb{B}(x, \delta)\} > -\infty$ for some $\delta > 0$,
irrelevant in applications

Ergodic Theorem

(E, d) Polish, $(\Xi, \mathcal{A}, P)$ & $\mathcal{A}$ P-complete
$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, locally inf-integrable
$\varphi : \Xi \to \Xi$ ergodic measure preserving transformation. Then,

$$\frac{1}{\nu} \sum_{l=1}^{\nu} f\big(\varphi^l(\boldsymbol{\xi}, \cdot)\big) \xrightarrow{e} Ef \quad a.s.$$

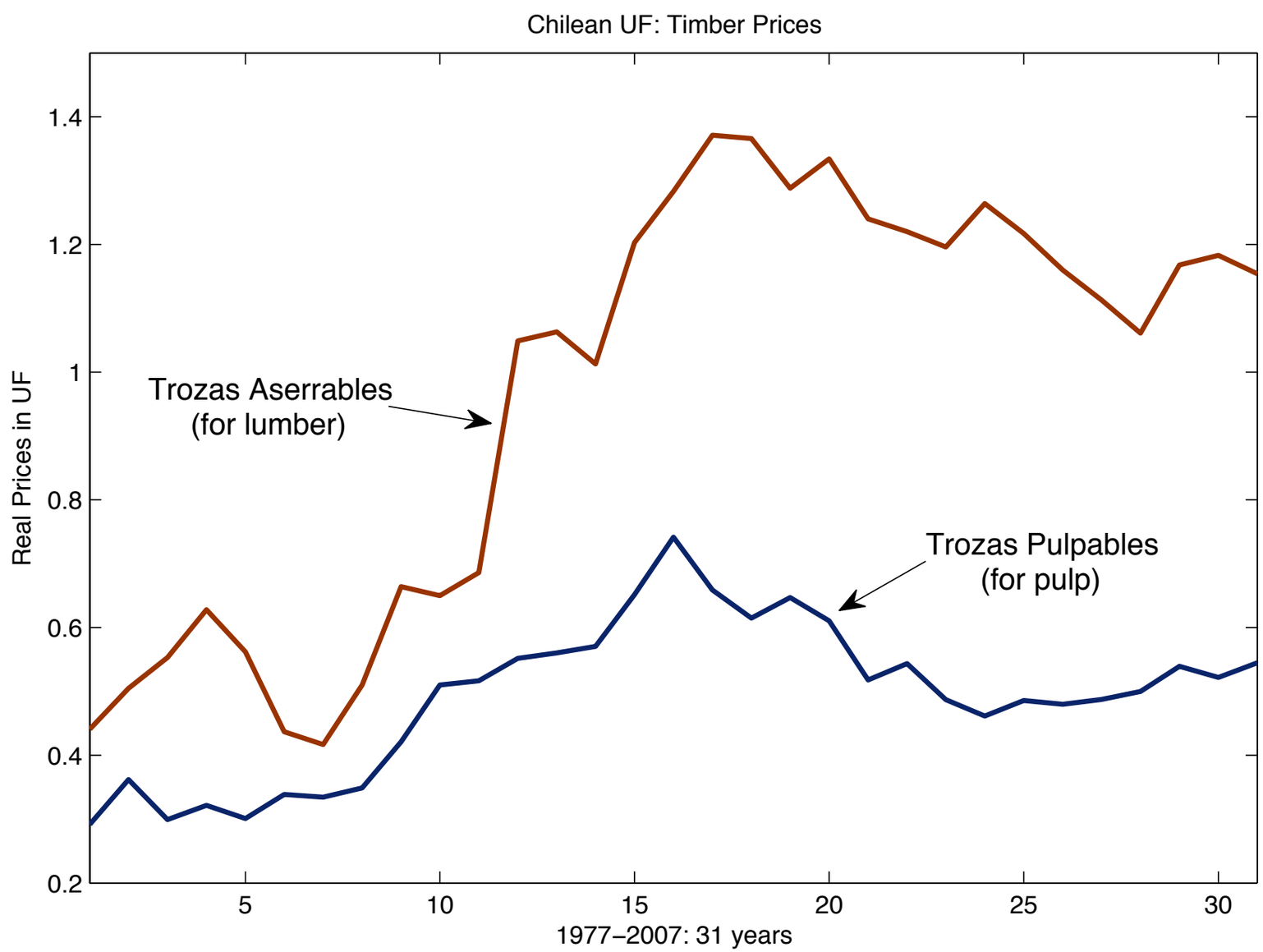
allows for stationary rather than iid samples.

Application: "samples" coming from dynamical systems,
time series, SDE, etc.

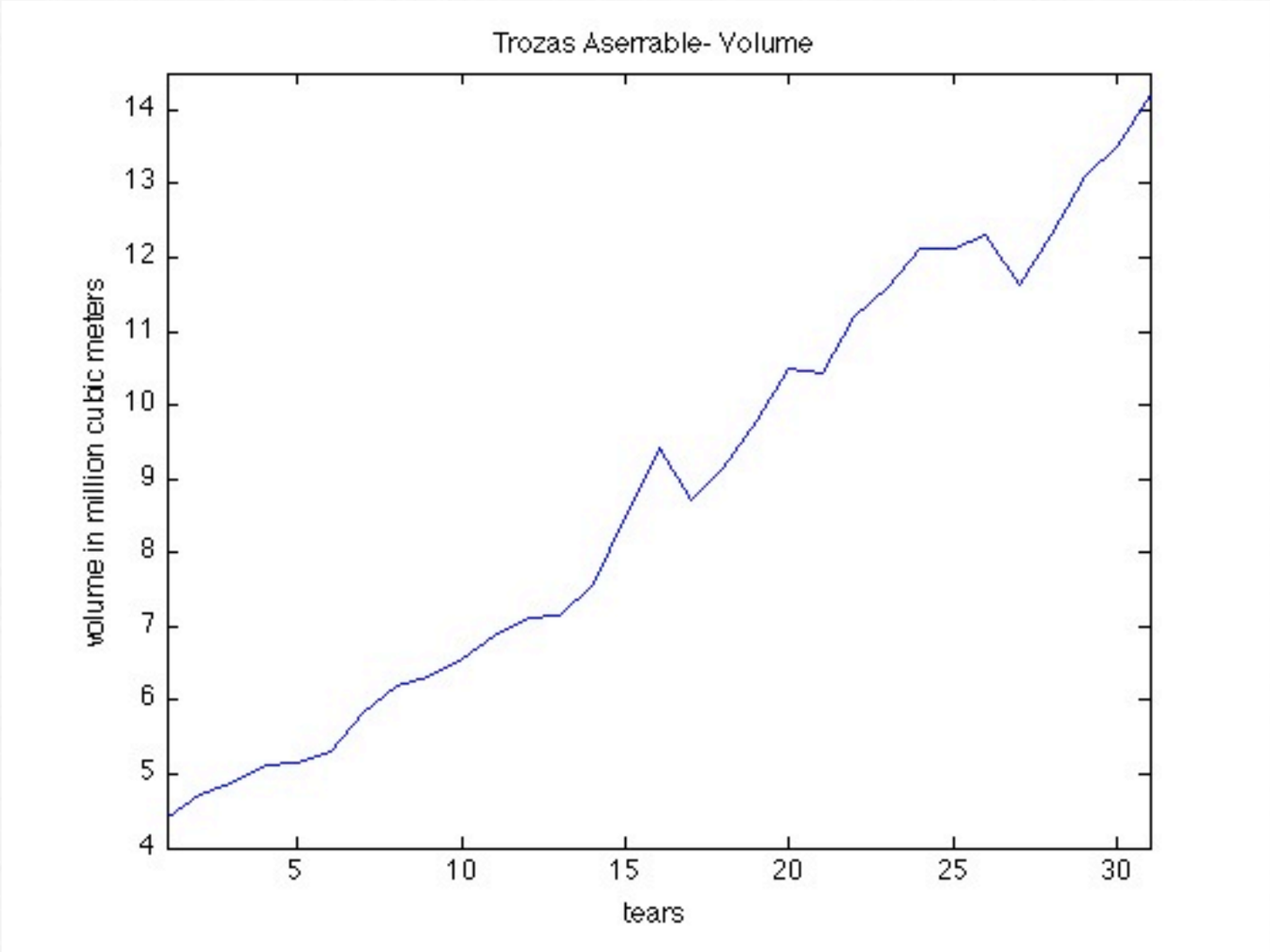
beyond LLN

Wood Prices Chile
Building Scenario Tree

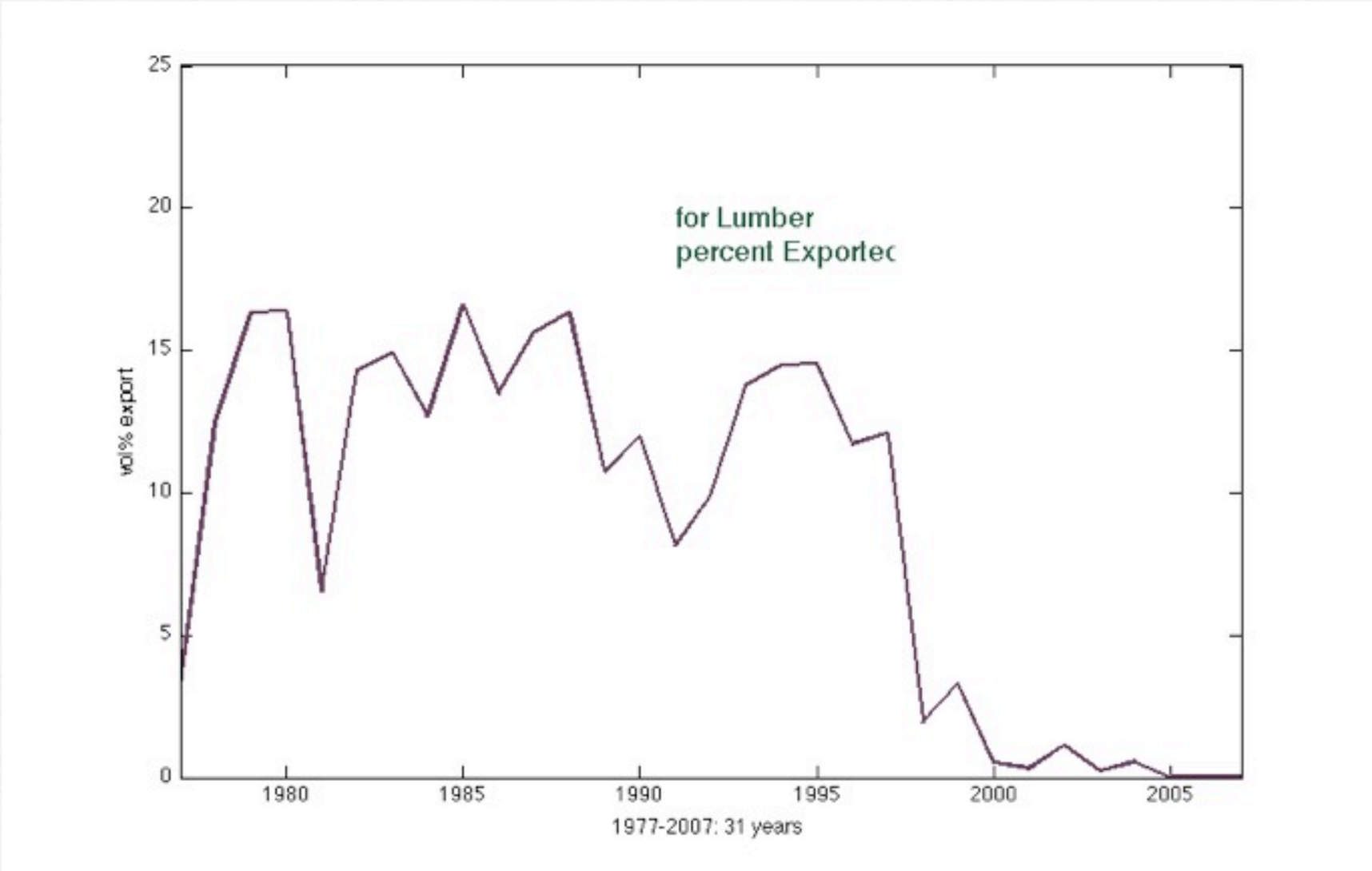
Lumber & Pulp Prices



Volume: Lumber Prices



Volume: Lumber Prices



Modeling the price process

~ geometric Brownian motion

$$dp(t) = \mu(\upsilon - p(t))dt + \sigma\, dw(t)p(t), \quad p(0) = p_0, \quad t \geq 0,$$

mean reversion

$$p(t) = p_0 \exp\left[-\left(\mu + \tfrac{1}{2}\sigma^2\right)t + \sigma w(t)\right] + \mu\upsilon \int_0^t e^{r(t,s)}\, ds$$

with

$$r(t,s) = -\left[\mu + \tfrac{1}{2}\sigma^2\right](t-s) + \sigma\left(w(t) - w(s)\right)$$

Approximation: $E\left\{\mu \int_0^t e^{r(t,s)}\, ds\right\} = 1 - e^{-\mu t}$ (small)

$$p(t) = \upsilon\left(1 - e^{-\mu t}\right) + p_0 \exp\left[(-\mu - \tfrac{1}{2}\sigma^2)t + \sigma w(t)\right], \; t \geq 0$$

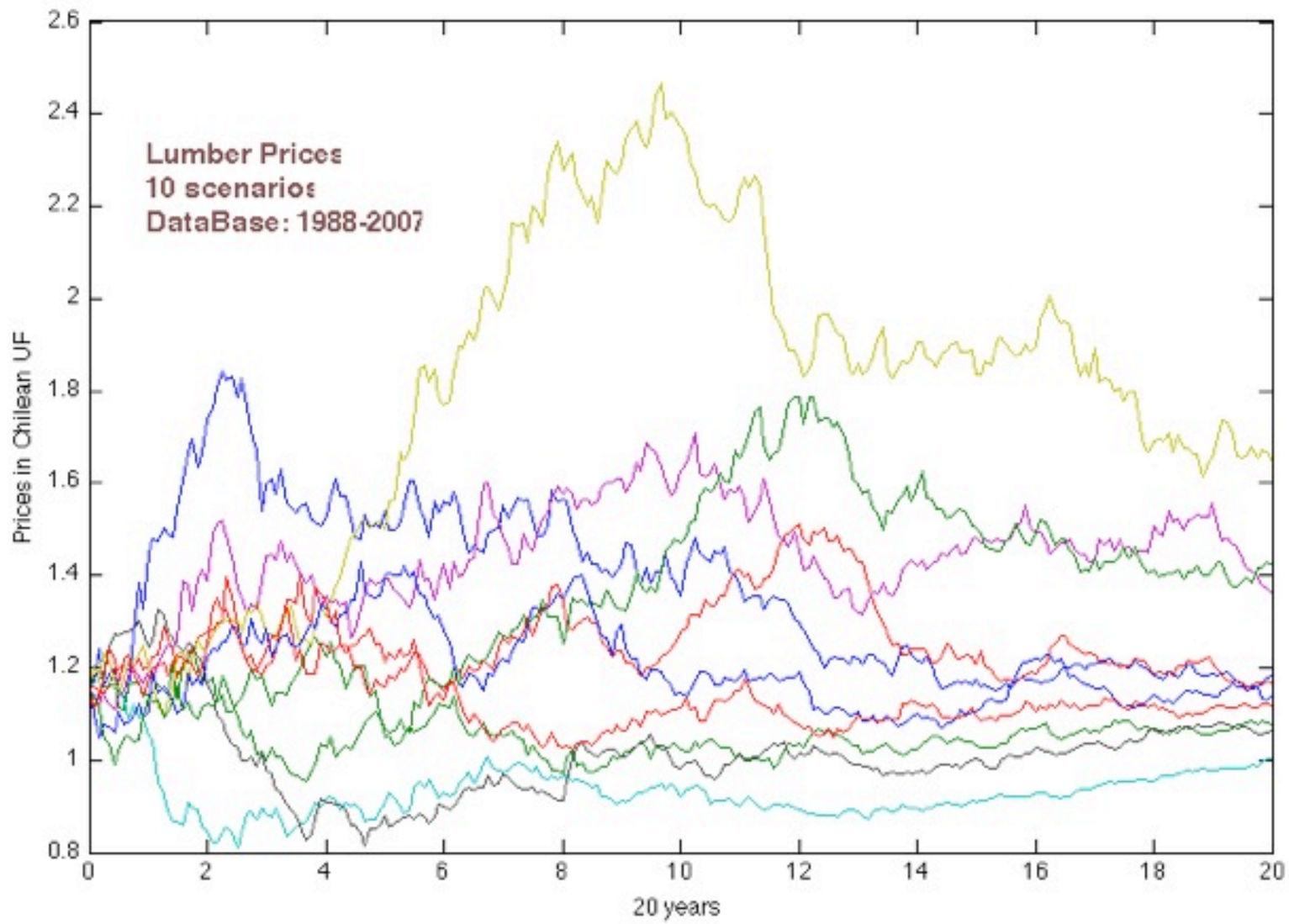
Estimating: coefficients

lumber and pulp prices

- use only data info 1988-2009(7), price at time 0: now
- mean reversion: υ=average 1988-now, μ=drift: 45 years
- estimating variance: σ, based on deviation from the historical data from "expected (solution) path"

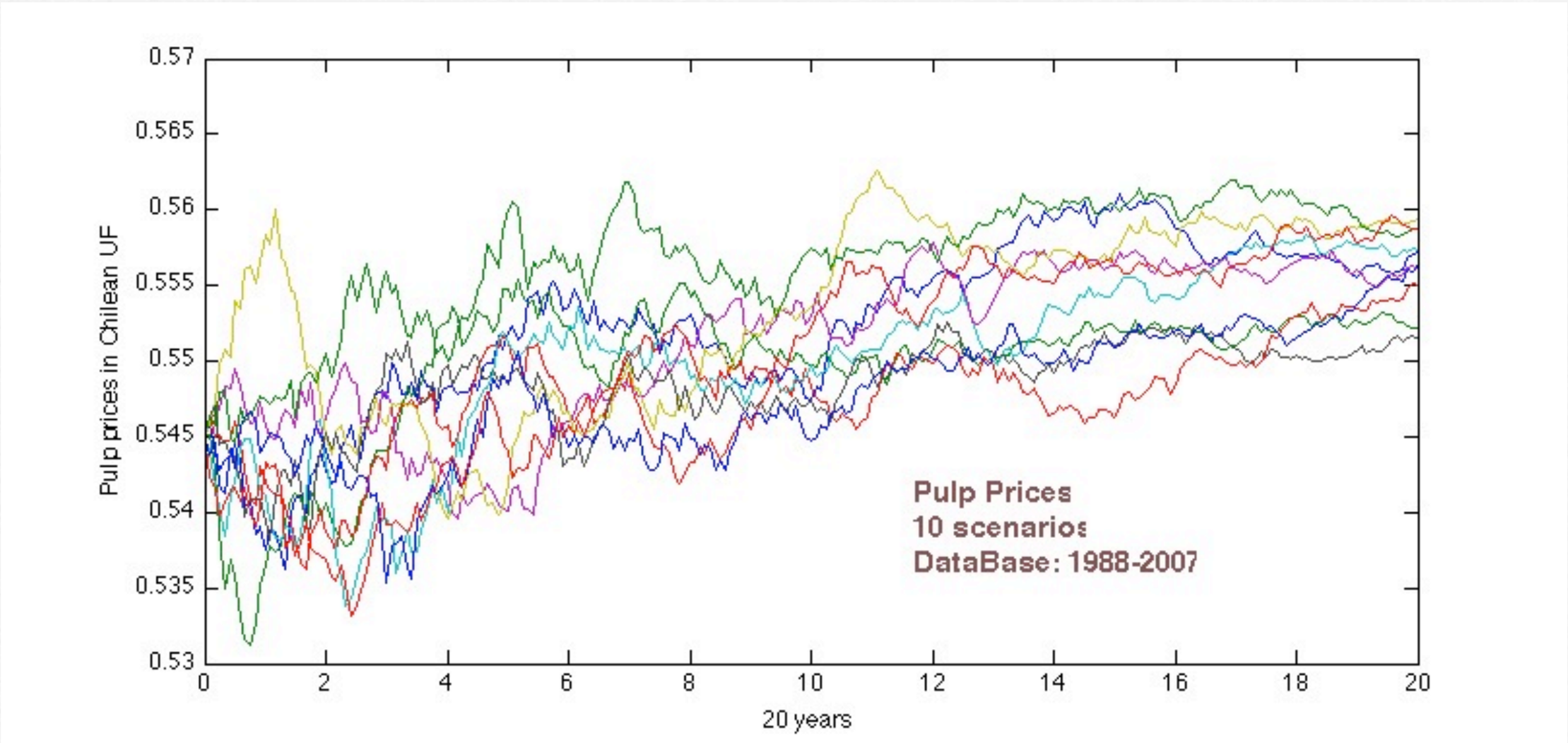
Lumber Price Process

$$E\{p(t)\} = \upsilon + (p^0 - \upsilon)e^{-\mu t}, \qquad \mathrm{Var}\{p(t)\} = \left(p_0 e^{-\mu t}\right)^2 \left(e^{\sigma^2 t} - 1\right).$$



Lumber Price Process

$$E\{p(t)\} = \upsilon + (p^0 - \upsilon)e^{-\mu t}, \qquad \mathrm{Var}\{p(t)\} = \left(p_0 e^{-\mu t}\right)^2 \left(e^{\sigma^2 t} - 1\right).$$



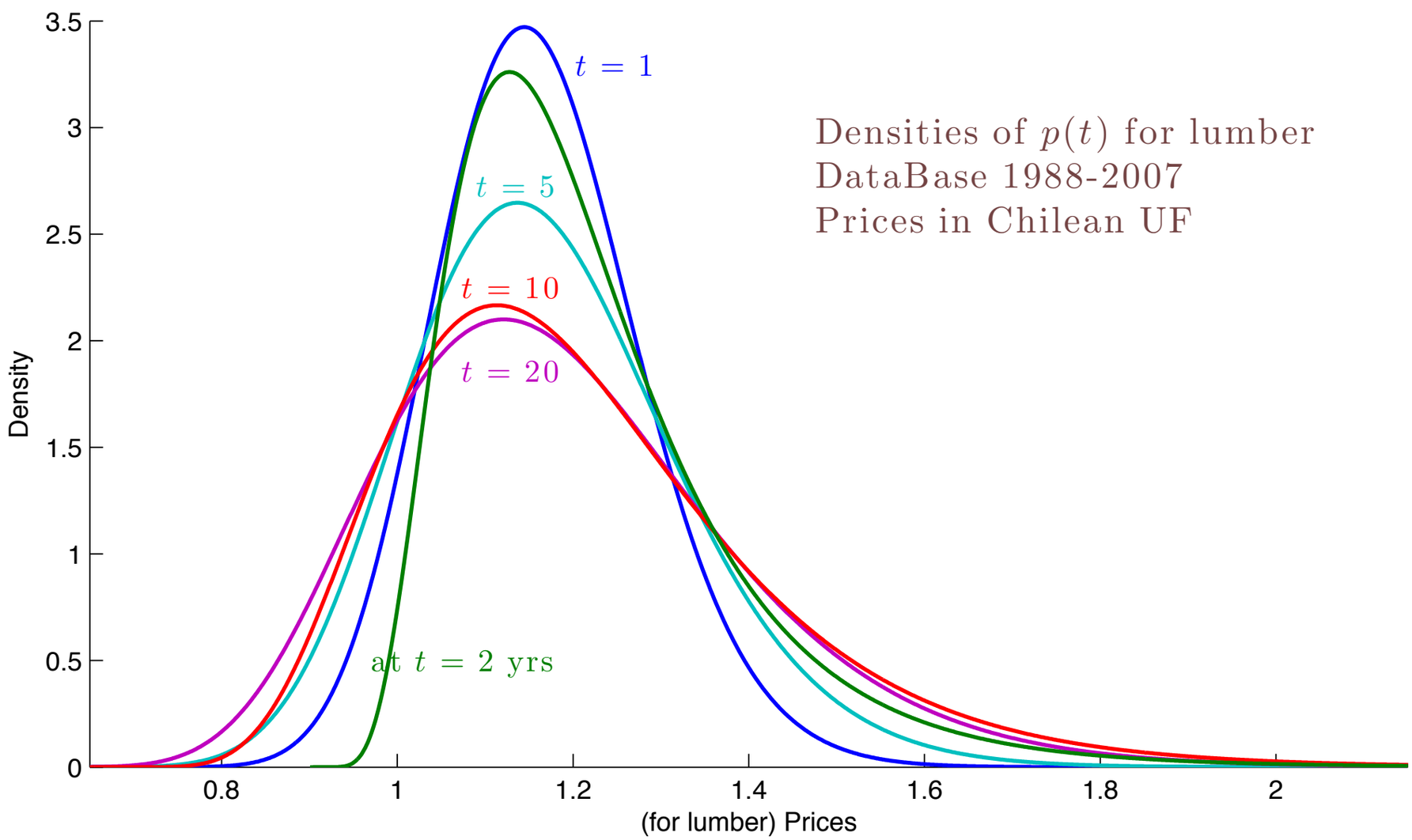
Distribution of *p(t)*

- Since $E\{p(t)\} = \upsilon + (p^0 - \upsilon)e^{-\mu t}$, $\mathrm{Var}\{p(t)\} = \left(p_0 e^{-\mu t}\right)^2 \left(e^{\sigma^2 t} - 1\right)$
- p(t) is "displaced" log-gaussian, displacement: $\upsilon(1 - e^{-\mu t})$
- $p(t) = Z_t + \upsilon\left(1 - e^{-\mu t}\right)$

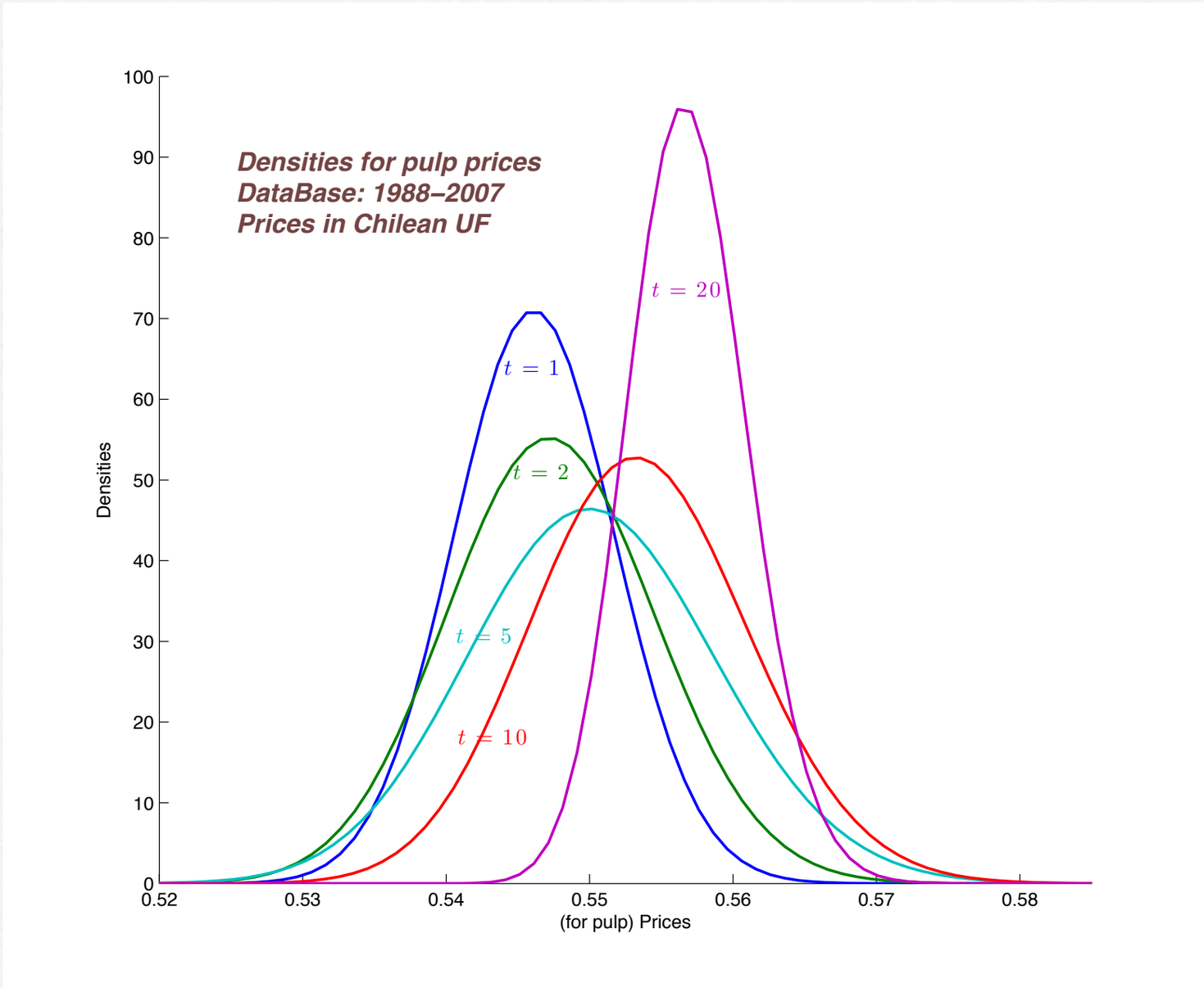
$$d_{Z_t}(s) = \left(s\tau\sqrt{2\pi}\right)^{-1} e^{-(\ln s - \theta)^2 / 2\tau^2}, \quad s \in (0, \infty),$$

$$\theta = \ln p_0 - \mu t, \quad \tau = \sigma\sqrt{t}.$$

Density $p(t)$, $t = 1,\ldots,20$

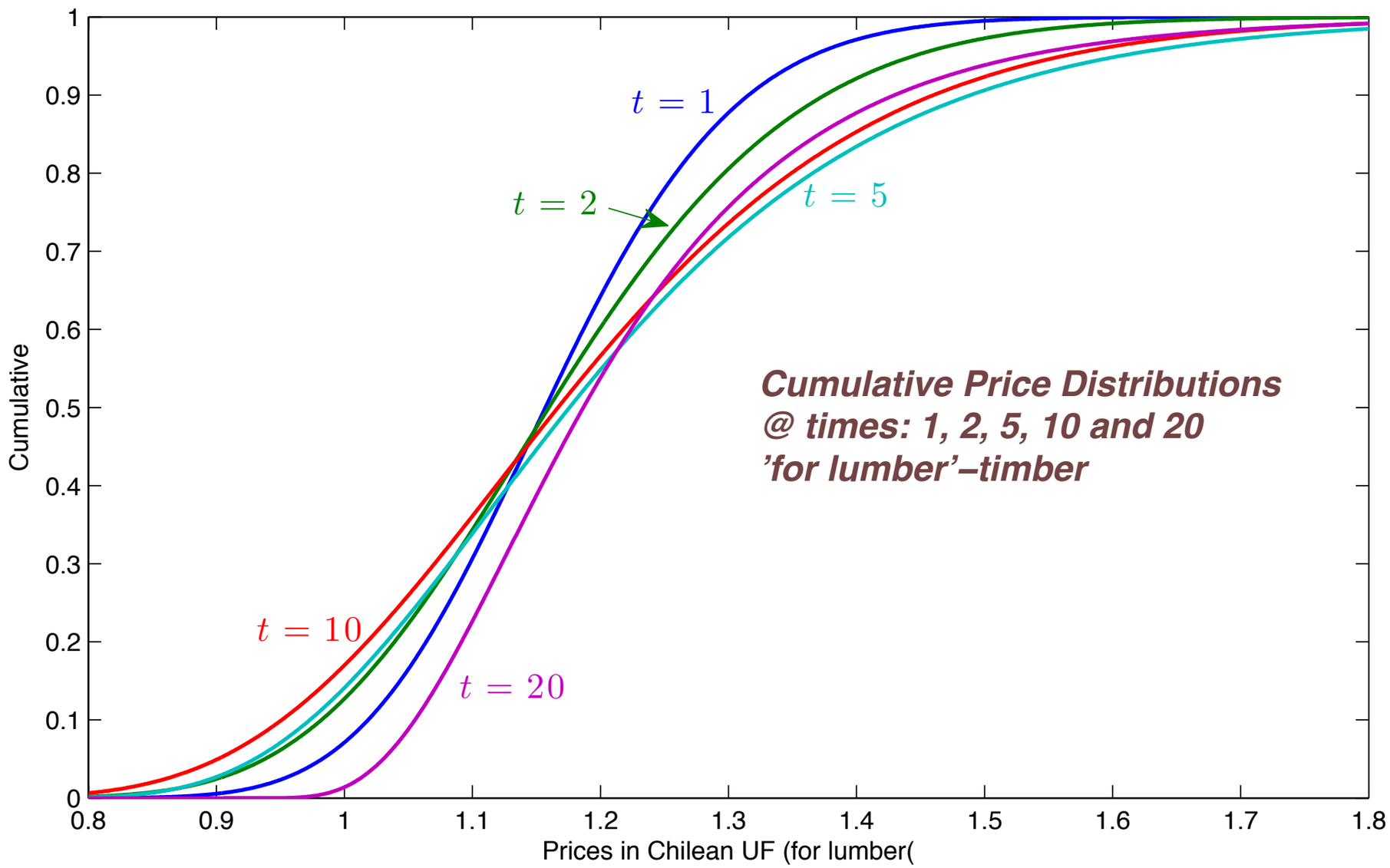


Density $p(t)$, $t = 1,\ldots,20$

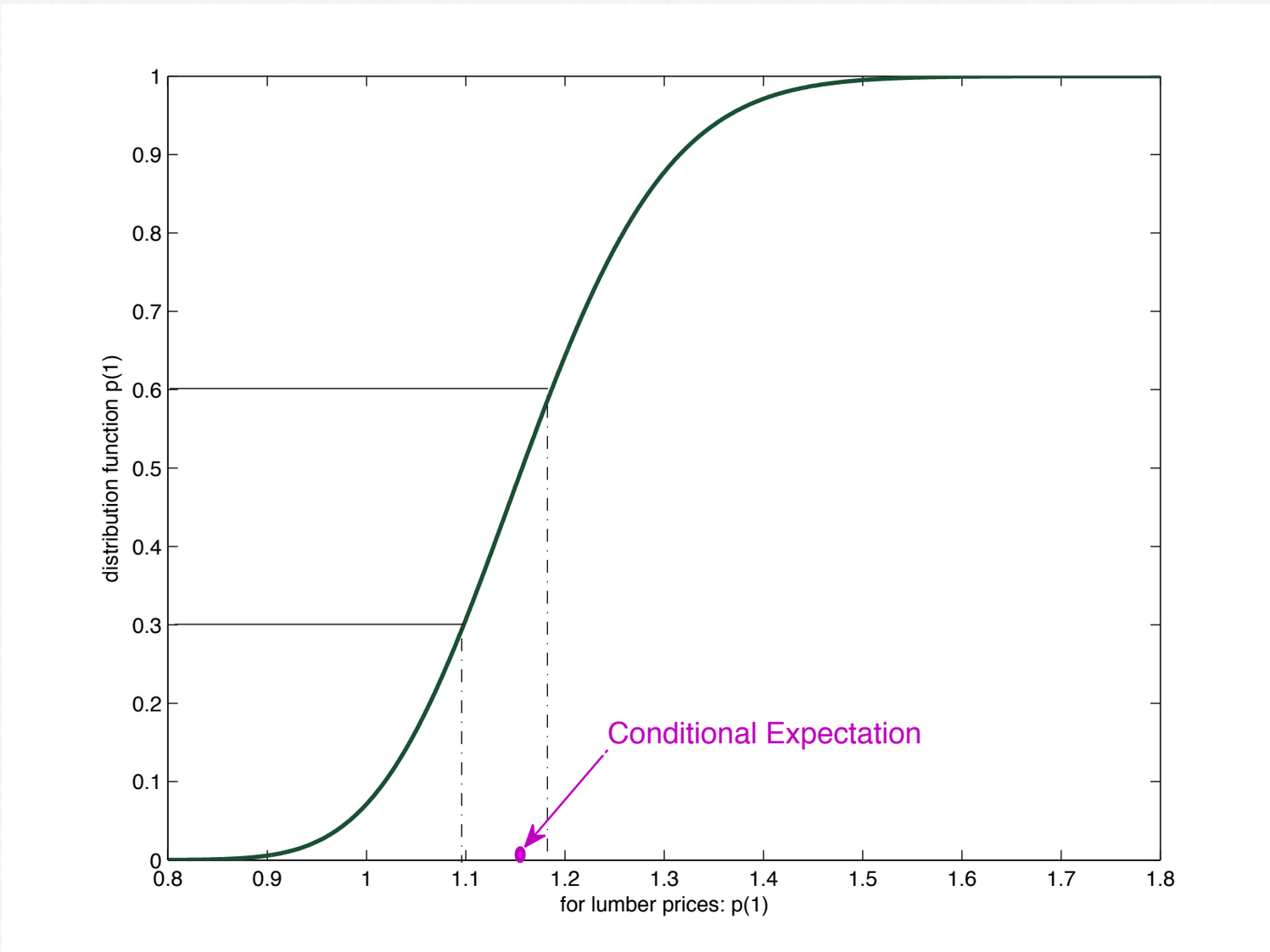


Cumulative *p(t)* - lumber

numerical integration

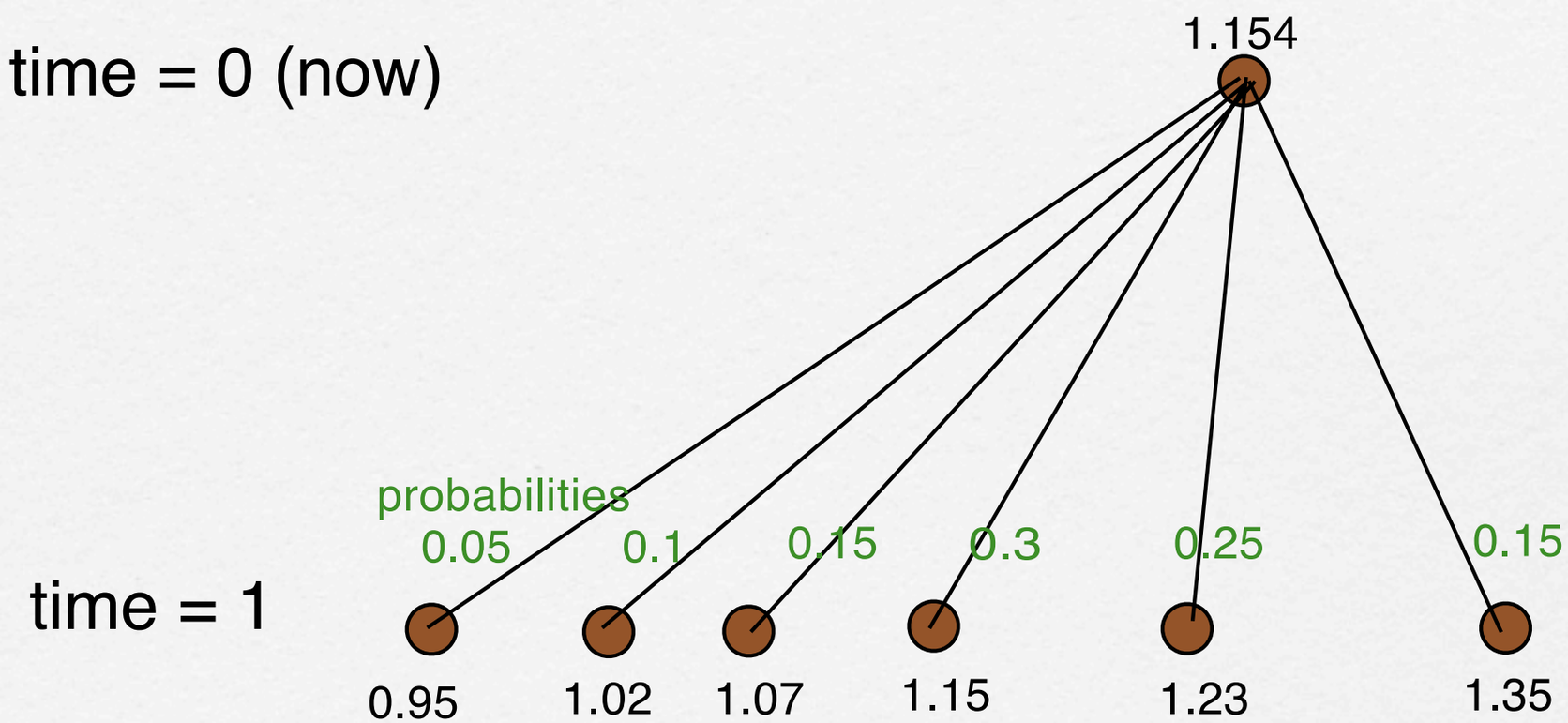


Building Scenario Tree



Percentiles: 0, 0.05, 0.15, 0.3, 0.6, 0.85, 1

Building Scenario Tree



Breakpoints: [0 0.05 0.15 0.3 0.6 0.85 1]

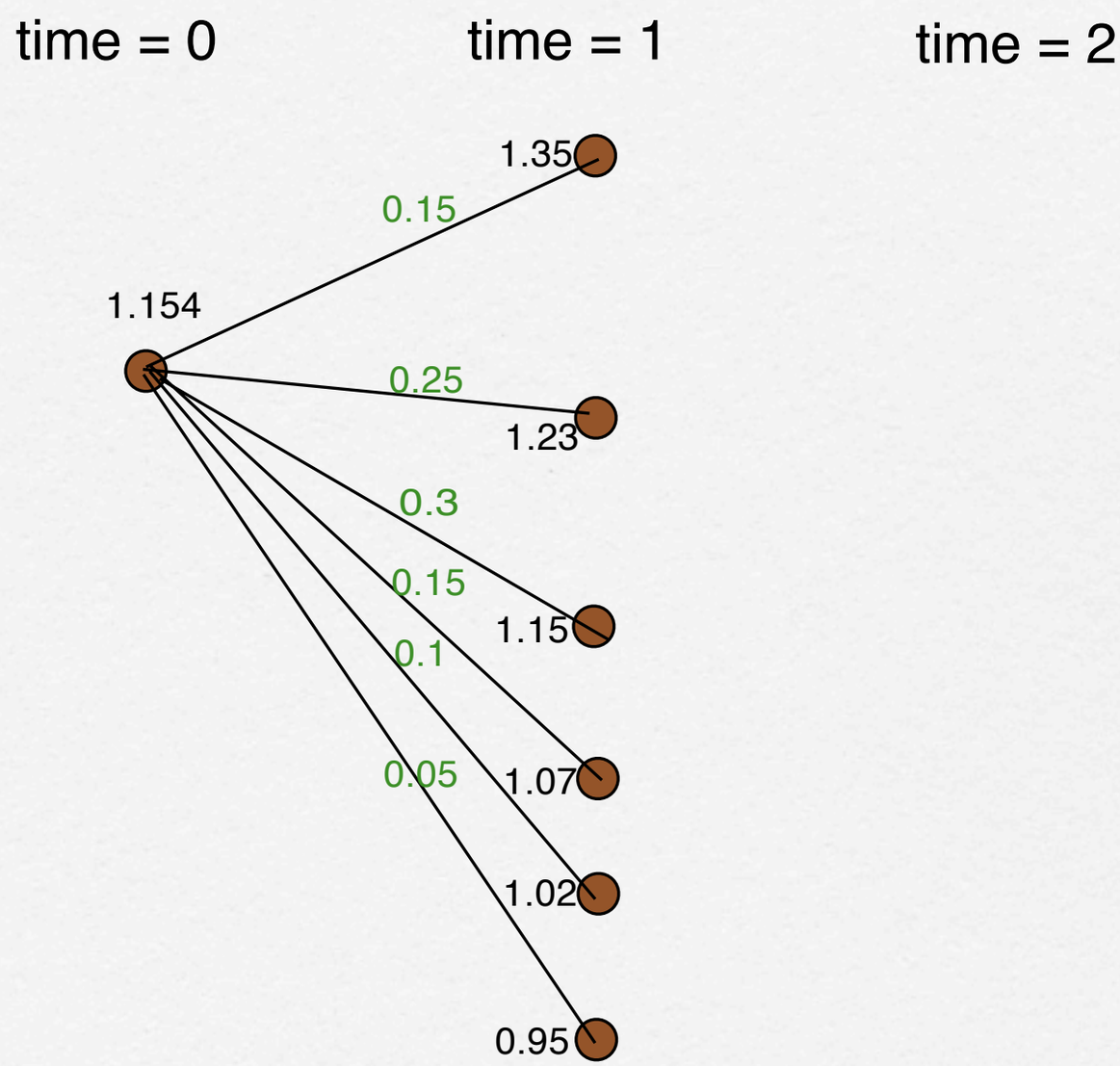
from Stage 1 to Stage 2

- leading dynamics (solution of SDE equation)
- $p(t) = \upsilon\left(1 - e^{-\mu(t-t_i)}\right) + p(t_i)\exp\left[(-\mu - \frac{1}{2}\sigma^2)(t - t_i) + \sigma w(t - t_i)\right],\ t \geq t_i$

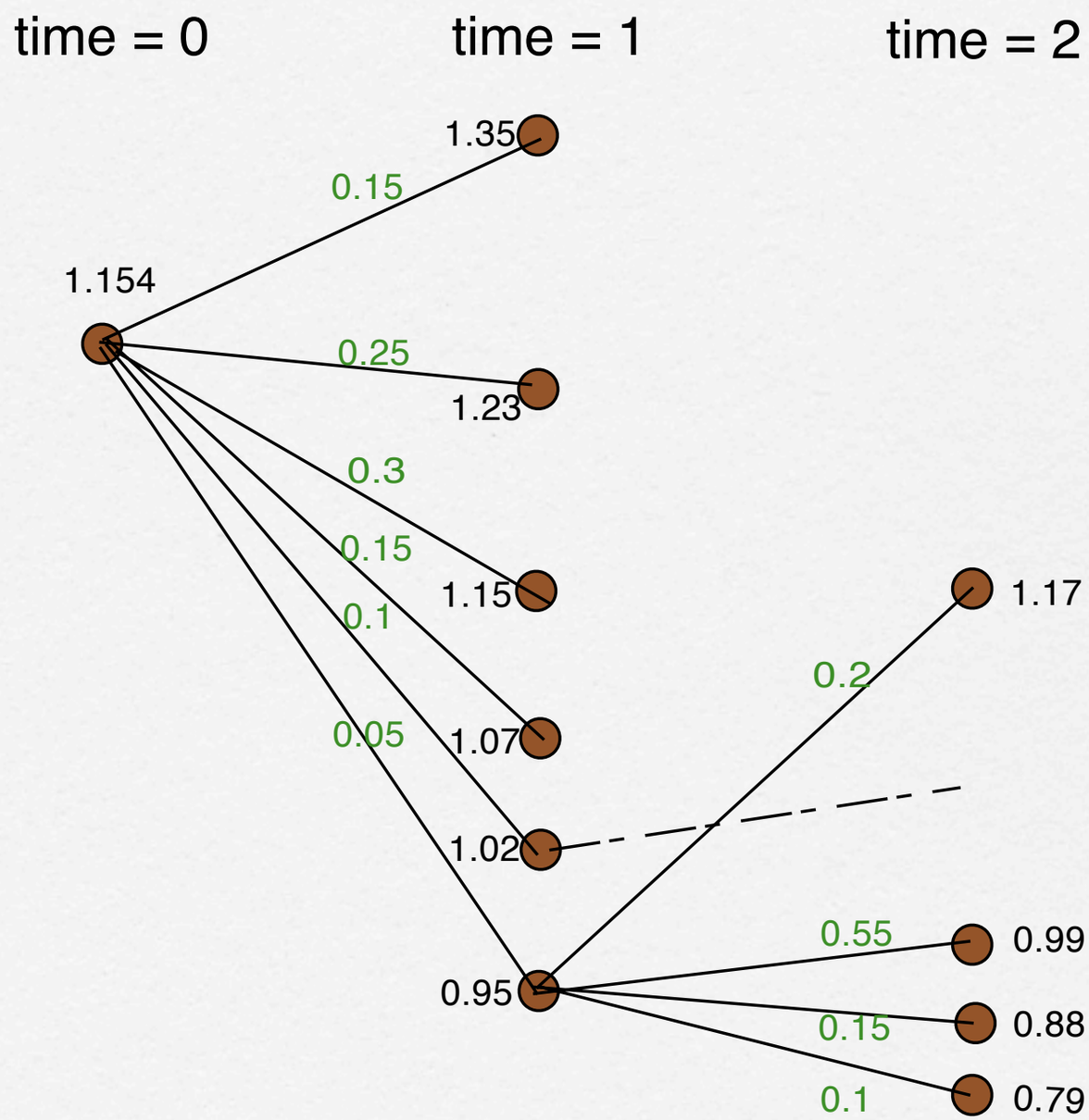
  where t_i = time at stage 1, $p(t_i) =$ price at one of the (stage 1)-nodes
- $t_e =$ time at stage 2 (end point) and fix percentiles breaks for the cumulative distribution of $p(t_e)$
- for example, [0 0.1 0.25 0.8 1] ⇒

  4 scenarios points

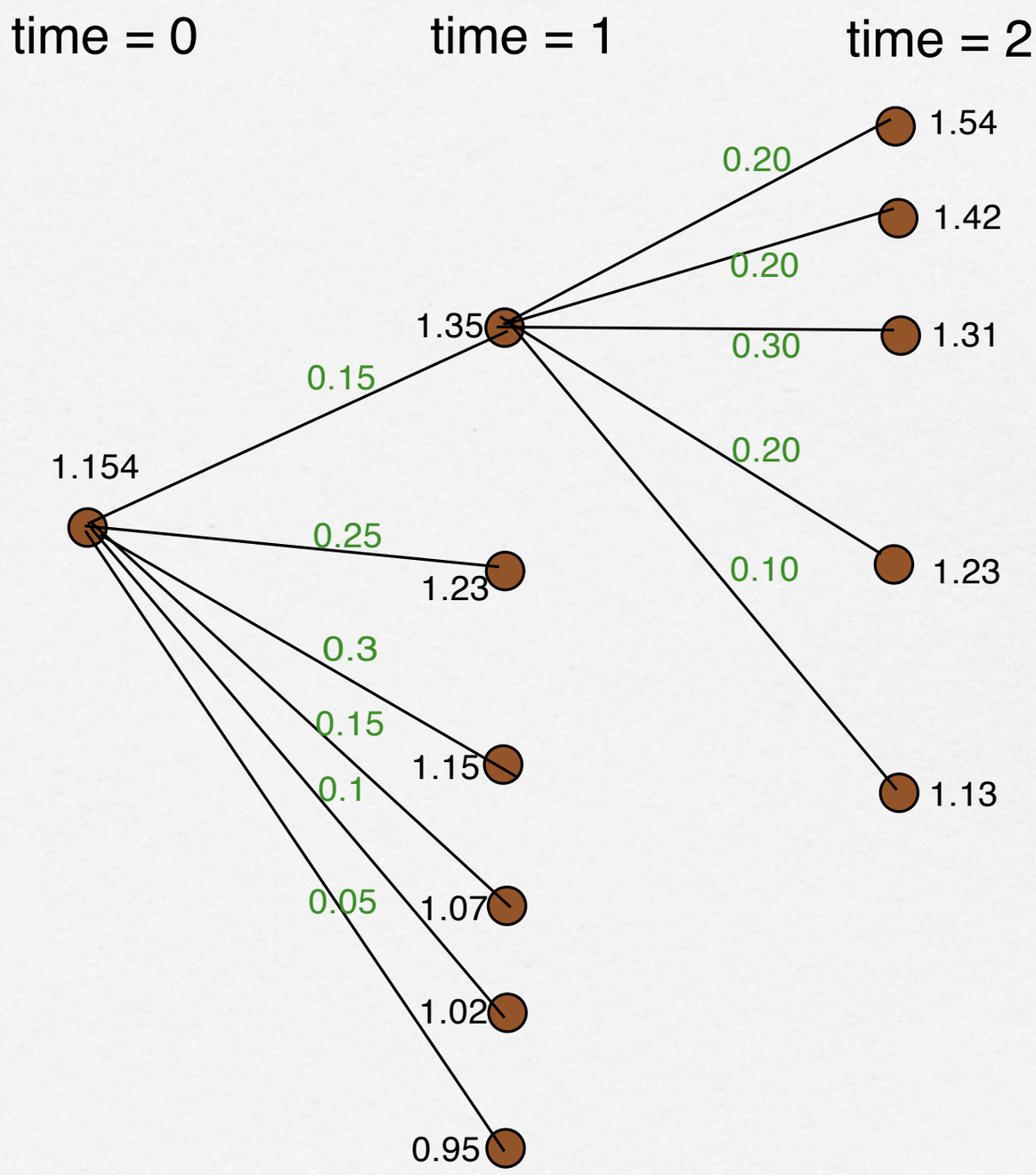
Scenario tree: Extended



Scenario tree: Extended



Scenario tree: Extended



Duality Theory Stochastic Programs

Linear ... with Recourse

Recourse: Two-stage, random RHS

$$\min \langle c, x\rangle + \mathbb{E}\{\langle q, y_\xi\rangle\}$$

$$\begin{aligned} \text{such that } Ax &= b \\ Tx + Wy_\xi &= d_\xi, \quad \forall \xi \in \Xi \\ x \geq 0, \quad y_\xi &\geq 0, \quad \forall \xi \in \Xi \end{aligned}$$

KKT-mutipliers: u and $\tilde{v}_\xi$, $\forall \xi \in \Xi$
Discrete case: $|\Xi|$ finite, (large scale) linear program

.

$\min \langle c, x\rangle + \sum_{\xi\in\Xi} \langle p_\xi, y_\xi\rangle$
KKT-mutipliers: u and $\tilde{v}_\xi$, $\forall \xi \in \Xi$ (finite number)

.

Discrete distribution
Extensive Formulation

c	$p_1 q$	$p_2 q$			$p\ q$	$p_L q$	
A							b
T_1	W_1						d_1
T_1		W_2					d_2
T	W_1				W		d
T_L		W				W_L	d_L

Dual Recourse Problem

discrete distribution

Dual I: Two-stage, random RHS

$$\max \langle b, u\rangle + \sum_{\xi\in\Xi} \langle d_\xi, \tilde{v}_\xi\rangle$$

$$\text{such that } A^\top u + \sum_{\xi\in\Xi} T^\top \tilde{v}_\xi \le c$$

$$W^\top \tilde{v}_\xi \le q, \quad \forall \xi \in \Xi$$

Dual II: Two-stage, random RHS program

"normalization" of dual variables: $\tilde{v}_\xi = p_\xi v_\xi$

$\boldsymbol{d} = (d_\xi, \xi \in \Xi), \quad \boldsymbol{v} = (v_\xi, \xi \in \Xi)$

$$\max \langle b, u\rangle + \mathbb{E}\{\langle \boldsymbol{d}, \boldsymbol{v}\rangle\}$$

$$\text{such that } A^\top u + T^\top \mathbb{E}\{\boldsymbol{v}\} \le c$$

$$W^\top \boldsymbol{v} \le q$$

Duality: Arbitrary Distribution

just RHS

Guessing ... intelligently(?)

$$\boldsymbol{d} = (d_\xi, \xi \in \Xi), \quad \boldsymbol{v} = (v_\xi, \xi \in \Xi)$$

$$\max \langle b, u\rangle + \mathbb{E}\{\langle \boldsymbol{d}, \boldsymbol{v}\rangle\}$$

$$\text{such that } A^\top u + T^\top \mathbb{E}\{\boldsymbol{v}\} \leq c$$

$$W^\top \boldsymbol{v} \leq q \ \ (\sim W^\top v_\xi \leq q, \ \forall \xi \in \Xi)$$

Duality: Arbitrary Distribution

just RHS

Guessing ... intelligently(?)

$$\boldsymbol{d} = (d_\xi, \xi \in \Xi), \quad \boldsymbol{v} = (v_\xi, \xi \in \Xi)$$

$$\begin{aligned} &\max \langle b, u\rangle + \mathbb{E}\{\langle \boldsymbol{d}, \boldsymbol{v}\rangle\} \\ \text{such that } &A^\top u + T^\top \mathbb{E}\{\boldsymbol{v}\} \le c \\ &W^\top \boldsymbol{v} \le q \;\; (\sim W^\top v_\xi \le q, \; \forall \xi \in \Xi) \end{aligned}$$

If correct, approximation via discretization,
yields approximation solution, epi-convergence?
yields (approximating) multipliers $\to$ correct multipliers.

.

A simple example

$\min x$ such that $x \geq 1$

$x - y_\xi \geq \xi,\ y_\xi \geq 0, \quad \boldsymbol{\xi}$ uniform on $[1, 2]$

Approximation of $\boldsymbol{\xi}$: split $[0, 1]$ in ν intervals, length $1/\nu$
and pick in each interval the mid point (= conditional expectation)
with probability $1/\nu = p_k^\nu$ for $\xi_k = 1 + (2k - 1)/2\nu$

Approximating l.p.: $\min x$ such that $x \geq 1,\ \ y_k \geq 0, \forall\, k$

$$x - y_k \geq \xi_k,\ k = 1, \ldots, \nu$$

.

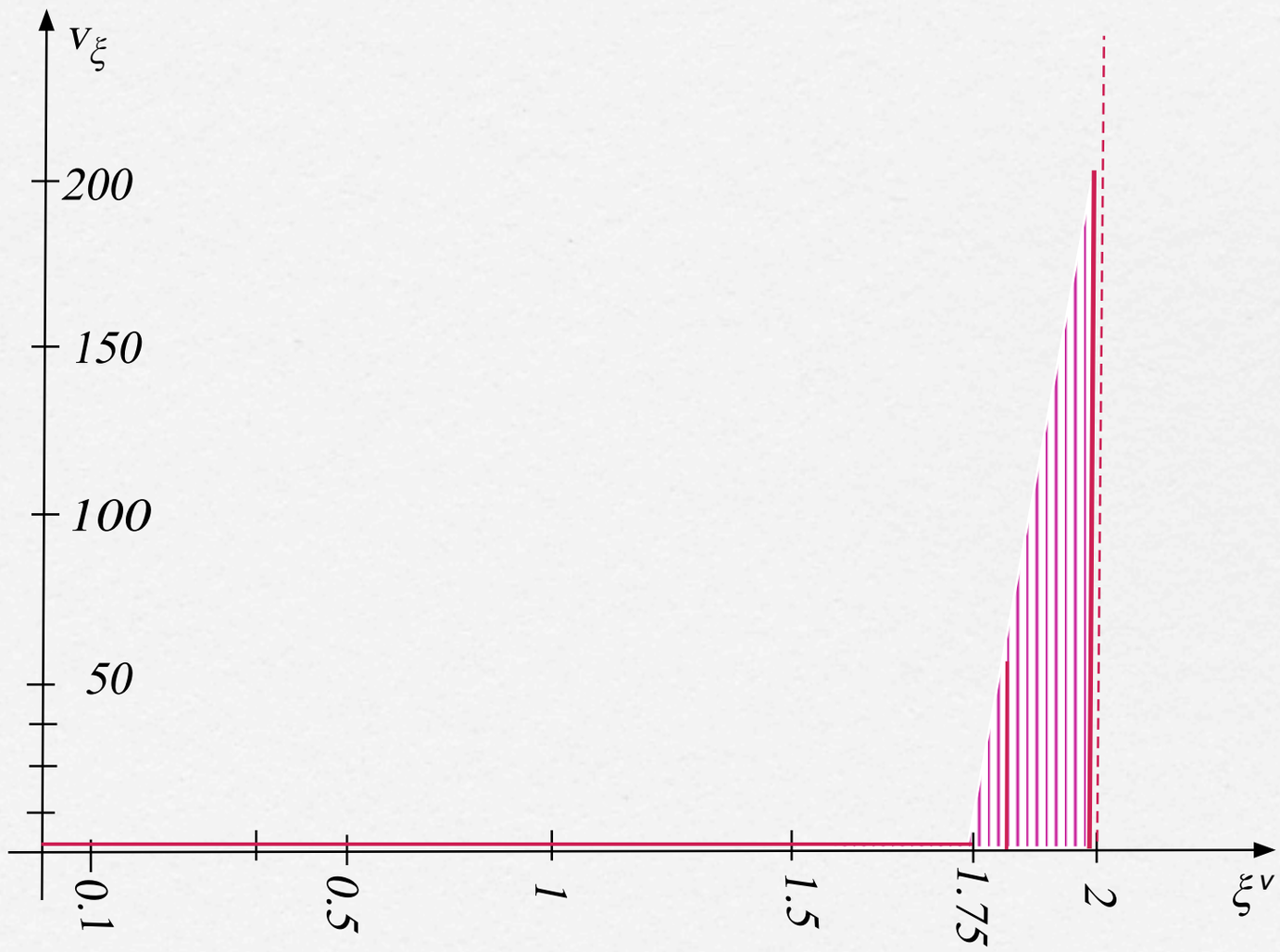
Dual Variables:

$$u^* = 0, \quad (v_k^\nu)^* = 0,\ k = 1, \ldots, \nu - 1, \quad (v_\nu^\nu)^* = \nu$$

.

Optimal Solution: $\quad x* = 2 - 1/\nu \quad \Longrightarrow \quad$ infeasible!

Refined discretization: associated KKT-multipliers



Duality Scheme(s)!

Convex duality scheme

$\min f, \quad f : E \to \overline{\mathbb{R}}$

embeded in a *family of perturbed problems*:

$$\left\{ \min_x F(u,x),\ u \in U \right\} \text{ with } F(0,x) = f(x)$$

Example: $f = \mathbb{E}\{f_0(\boldsymbol{\xi}, x, y_{\boldsymbol{\xi}})\}$

such that $f_{1i}(x) \le u_{1i},\ f_{2i}(\boldsymbol{\xi}, x, y_\xi) \le u_{2i}(\boldsymbol{\xi}),\ \ i \in I_1 \cup I_2$

1. $u_{1i} \in \mathbb{R},\ u_{2i} \in \mathbb{R} \quad \Longrightarrow \quad$ distribution contamination
2. $u_{1i} \in \mathbb{R},\ u_{2i} \in \mathcal{L}^\infty(\Xi, \mathcal{A}, P)$, bounded fcns
3. $u_{1i} \in \mathbb{R},\ u_{2i} \in \mathcal{D}(\Xi)$, space of "distributions"

Lagrangian: $\quad L\big((x,y),(u_1,u_2)\big) =$

$$\mathbb{E}\left\{ f_0(\boldsymbol{\xi}, x, y_{\boldsymbol{\xi}}) + \sum_{i \in I_1} u_{1i} f_i(x) + \sum_{i \in I_2} \langle u_{2i}(\boldsymbol{\xi}), f_{2i}(\boldsymbol{\xi}, x, y_{\boldsymbol{\xi}}) \rangle \right\}$$

Constraint Nonanticipativity

$u_{1i} \in \mathbb{R}$, $u_{ti} \in \mathcal{L}^\infty(\Xi, \mathcal{A}, P)$, bounded $\mathcal{A}_t$-measurable fcns

$$K_2 = \left\{ x \,\middle|\, \forall\, \xi \in \Xi,\ \exists\, y_\xi \text{ such that } f_{2i}(\xi, x, y_\xi) \leq 0,\ i \in I_2 \right\}$$

relatively complete recourse: $K_2 \supset K_1 = \left\{ x \,\middle|\, f_{1i}(x) \leq 0, i \in I_1 \right\}$

Constraint Nonanticipativity

$u_{1i} \in \mathbb{R}$, $u_{ti} \in \mathcal{L}^\infty(\Xi, \mathcal{A}, P)$, bounded $\mathcal{A}_t$-measurable fcns

$$K_2 = \left\{ x \,\middle|\, \forall\, \xi \in \Xi,\ \exists\, y_\xi \text{ such that } f_{2i}(\xi, x, y_\xi) \le 0,\ i \in I_2 \right\}$$

relatively complete recourse: $K_2 \supset K_1 = \left\{ x \,\middle|\, f_{1i}(x) \le 0, i \in I_1 \right\}$

filtration $\{\mathcal{A}_t\}_{t=1}^T$ (T-stage program)

$$\xi \mapsto K(\xi) = \left\{ x = (x_1, \ldots, x_T) \,\middle|\, f_{ti}(\xi, x) \le 0,\ i \in I_t,\ t = 1, \ldots, T \right\}$$

Nonanticipativity feasibility: for all t,

$$\xi \mapsto K_t(\xi) = \left\{ \vec{x}^{\,t} \,\middle|\, \exists\, x = (\vec{x}^{\,t}, x_{t+1}, \ldots, x_T) \in K(\xi) \right\} \text{ is } \mathcal{A}_t\text{-measurable}$$

Constraint Nonanticipativity

$u_{1i} \in \mathbb{R}$, $u_{ti} \in \mathcal{L}^\infty(\Xi, \mathcal{A}, P)$, bounded $\mathcal{A}_t$-measurable fcns

$$K_2 = \left\{ x \,\middle|\, \forall\, \xi \in \Xi,\ \exists\, y_\xi \text{ such that } f_{2i}(\xi, x, y_\xi) \le 0,\ i \in I_2 \right\}$$

relatively complete recourse: $K_2 \supset K_1 = \left\{ x \,\middle|\, f_{1i}(x) \le 0, i \in I_1 \right\}$

filtration $\{\mathcal{A}_t\}_{t=1}^T$ (T-stage program)

$$\xi \mapsto K(\xi) = \left\{ x = (x_1, \ldots, x_T) \,\middle|\, f_{ti}(\xi, x) \le 0,\ i \in I_t,\ t = 1, \ldots, T \right\}$$

Nonanticipativity feasibility: for all t,

$$\xi \mapsto K_t(\xi) = \left\{ \vec{x}^{\,t} \,\middle|\, \exists\, x = (\vec{x}^{\,t}, x_{t+1}, \ldots, x_T) \in K(\xi) \right\} \text{ is } \mathcal{A}_t\text{-measurable}$$

two-stage: $\forall\, \xi \in \Xi$, $\forall\, x \in K_1$,

$\exists\, y_\xi$ such that $f_{2,i}(\xi, x, y_\xi) \le 0,\ i \in I_2$

$K_1(\xi) = K_1$ is $\mathcal{A}_0 = \{0, \Xi\}$-measurable.

Duality Theorem

$$\min Ef(x) = \mathbb{E}\{f(\xi, x)\},\ x \in \mathcal{N}_a = \left\{x \,\middle|\, x_t : \xi \to \mathbb{R}^{n_y},\ \mathcal{A}_t\text{-measurable } \right\},$$
$$f : \Xi \times \mathbb{R}^N \to \overline{\mathbb{R}},\ N = n_1 + \cdots + n_T, \text{ random convex lsc function}$$

Theorem. Under 'classical' $\mathbb{C}.\mathbb{Q}.$ **and** nonanticipative feasibility, there exist multipliers $w \in \mathcal{L}^1(\Xi, \mathcal{A}, P; \mathbb{R}^N)$ with $\mathbb{E}\{w_t \,|\, \mathcal{A}_t\} = 0$ for all t such that x^* is optimal $\iff$

$$P\text{-almost surely } x^*(\boldsymbol{\xi}) \in \operatorname{argmin}_x f(\boldsymbol{\xi}, x)$$

Duality Theorem

$$\min Ef(x) = \mathbb{E}\{f(\xi, x)\},\ x \in \mathcal{N}_a = \left\{x \,\middle|\, x_t : \xi \to \mathbb{R}^{n_y},\ \mathcal{A}_t\text{-measurable } \right\},$$
$$f : \Xi \times \mathbb{R}^N \to \overline{\mathbb{R}},\ N = n_1 + \cdots + n_T, \text{ random convex lsc function}$$

Theorem. Under 'classical' $\mathbb{C}.\mathbb{Q}.$ **and** nonanticipative feasibility, there exist multipliers $w \in \mathcal{L}^1(\Xi, \mathcal{A}, P; \mathbb{R}^N)$ with $\mathbb{E}\{w_t \,|\, \mathcal{A}_t\} = 0$ for all t such that x^* is optimal $\iff$

$$P\text{-almost surely } x^*(\boldsymbol{\xi}) \in \operatorname{argmin}_x f(\boldsymbol{\xi}, x)$$

Theorem 2. Under 'classical' $\mathbb{C}.\mathbb{Q}.$ there exist multipliers $w \in \mathcal{L}^1(\Xi, \mathcal{A}, P; \mathbb{R}^N) \otimes \mathcal{S}(\Xi, \mathcal{A}, P; \mathbb{R}^N)$ with "$\mathbb{E}$"$\{w_t \,|\, \mathcal{A}_t\} = 0$ for all t such that x^* is optimal $\iff$ P-almost surely $x^*(\boldsymbol{\xi}) \in \operatorname{argmin}_x f(\boldsymbol{\xi}, x)$

induced constraints