Tropical methods in Dynamic Programming

Stephane.Gaubert@inria.fr

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Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $=$

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$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} =$$

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$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{''} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

The sister algebra: min-plus

$$\text{“}a + b\text{”} = \min(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

- $\text{“}2 + 3\text{”} = 2$
- $\text{“}2 \times 3\text{”} = 5$

The term “tropical” refers to Imre Simon, 1943 - 2009

who lived in São Paulo (south tropic).

These structures were invented by several schools in the world.

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ∼65 ... Zimmerman, Butkovic; Optimization
- Maslov ∼ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon ∼ 78- ... Hashiguchi, Leung, Pin, Krob, ... Automata theory
- Gondran, Minoux ∼ 77 Operations research
- Cohen, Quadrat, Viot ∼ 83- ... Olsder, Baccelli, S.G., Akian initially discrete event systems, then optimal control, idempotent probabilities, combinatorial linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ∼00- max-plus approximation of HJB
- Salut, Del Moral ∼95 Puhalskii ∼99, idempotent probabilities.

now in **tropical geometry**, after Viro, Mikhalkin, Passare, Sturmfels and many.

Part I

tropical approximation of the value function.

Lagrange problem / Lax-Oleinik semigroup

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall\lambda\in\mathbb{R}$, $\forall\phi,\psi$,

$$\begin{aligned} S^t(\sup(\phi,\psi)) &= \sup(S^t\phi, S^t\psi) \\ S^t(\lambda+\phi) &= \lambda + S^t\phi \end{aligned}$$

So S^t is max-plus linear.

Lagrange problem / Lax-Oleinik semigroup

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Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi, \psi$,

$$\begin{aligned} S^t(\text{“}\phi + \psi\text{''}) &= \text{“}S^t\phi + S^t\psi\text{''} \\ S^t(\text{“}\lambda\phi\text{''}) &= \text{“}\lambda S^t\phi\text{''} \end{aligned}$$

So S^t is max-plus linear.

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0, \cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x, p) = \sup_u (L(x, u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x - y}{t}) + \phi(y) \ .$$

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Hopf formula, when $L = L(u)$ concave:

$$v(t, x) = \text{“}\int G(x - y)\phi(y)dy\text{”} \quad .$$

Classical	Maxplus
Expectation	sup
Brownian motion	$L(\dot{\mathbf{x}}(s)) = (\dot{\mathbf{x}}(s))^2/2$
Heat equation:	Hamilton-Jacobi equation:
$\frac{\partial v}{\partial t} = -\frac{1}{2}\Delta v$	$\frac{\partial v}{\partial t} = \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$
$\exp(-\frac{1}{2}\|x\|^2)$	$-\frac{1}{2}\|x\|^2$
Fourier transform:	Fenchel transform:
$\int \exp(i\langle x, y\rangle) f(x) dx$	$\sup_x \langle x, y\rangle - f(x)$
convolution	inf or sup-convolution

See Akian, Quadrat, Viot 97 Duality & Opt. ...

Max-plus basis / finite-element method

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function by a "linear comb." of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

$$v(t,\cdot) \simeq \text{``}\sum_{i\in[p]} \lambda_i(t) w_i\text{''}$$

The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

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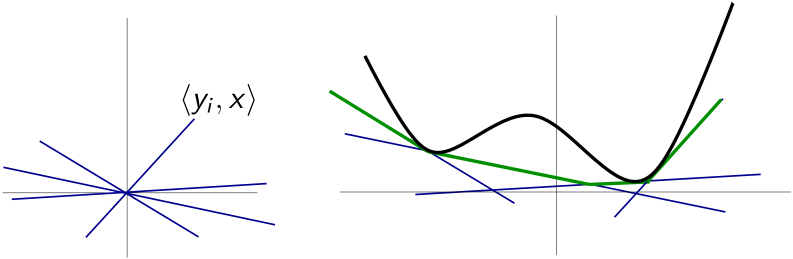
The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

Best max-plus approximation

$$P(f) := \max\{g \leq f \mid g \text{ “linear comb.” of } w_i\}$$

linear forms $w_i : x \mapsto \langle y_i, x \rangle$



adapted if v is convex

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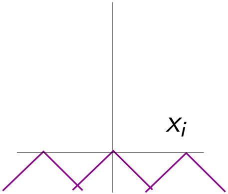
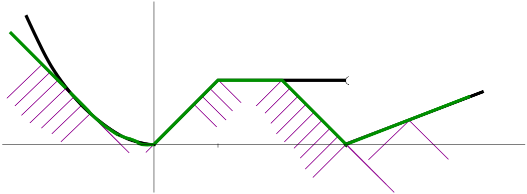
cone like functions $w_i : x \mapsto -C\|x - x_i\|$





adapted if v is C-Lip

Use max-plus linearity of S^h:

$$v^t = \text{``}\sum_{i\in[p]} \lambda_i(t) w_i\text{''}$$

and look for new coefficients $\lambda_i(t+h)$ such that

$$v^{t+h} \simeq \text{``}\sum_{i\in[p]} \lambda_i(t+h) w_i\text{''}$$

Use max-plus linearity of S^h:

$$v^{t+h} = S^h v^t \simeq \text{``}\sum_{i\in[p]} \lambda_i(t) S^h w_i\text{''}$$

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Max-plus variational approach

Max-plus scalar product

$$\text{“}\langle w, z\rangle\text{”} := \text{“}\int w(x)z(x)dx\text{”}$$

For all test functions z_j, $j \in [q]$

$$\begin{aligned}\text{“}\langle v^{t+h}, z_j\rangle\text{”} &= \text{“}\sum_{i\in[p]} \lambda_i(t+h)\text{” “}\langle w_i, z_j\rangle \\ &= \text{“}\sum_{k\in[p]} \lambda_k(t)\langle S^h w_k, z_j\rangle\text{”}\end{aligned}$$

Max-plus variational approach

Max-plus scalar product

$$\text{“}\langle w, z\rangle\text{”} := \sup_x w(x) + z(x)$$

For all test functions z_j, $j \in [q]$

$$\begin{aligned} &\sup_{i\in[p]} \lambda_i(t+h) + \text{“}\langle w_i, z_j\rangle \\ = &\sup_{k\in[p]} \lambda_k(t) + \text{“}\langle S^h w_k, z_j\rangle \end{aligned}$$

This is of the form

$$A\lambda(t+h) = B\lambda(t), \qquad A, B \in \mathbb{R}_{\max}^{q\times p}$$

The linear system $A\mu = b$ generically has no solution $\mu \in \mathbb{R}^p$, however, $A\mu \leq b$ has a maximal solution $A^\sharp b$ given by

$$(A^\sharp b)_j := \min_{i\in[q]} -A_{ij} + b_i \ .$$

Cohen, SG, Quadrat, LAA 04, Akian, SG, Kolokoltsov: Moreau conjugacies

So, the coeffs of $v(t+h)$ are recursively given:

$$\lambda(t+h) = A^\sharp B \lambda(t) \ .$$

The global error is controlled by the projection errors of all the $v(t,\cdot)$. The method is efficient if $S^h w_i$ is evaluated by a high order scheme. Then, $A^\sharp B$ glues the characteristics in time h.

McEneaney's curse of dimensionality reduction

Suppose the Hamiltonian is a finite max of Hamiltonians arising from LQ problems

$$H = \sup_{i \in [r]} H_i,\ \ H_i = -(\frac{1}{2}x^* D_i x + x^* A_i^* p + \frac{1}{2} p^* \Sigma_i p)$$

(=LQ with switching). Let S^t and S_i^t denote the corresponding Lax-Oleinik semigroups, S_i^t is exactly known (Riccati!)

Want to solve $v = S^t v, \forall t \geq 0$

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

$$v \simeq (S^h)^k \phi \ .$$

This is a sup of quadratic forms. Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\to$ solution of a typical instance in dim 6 on a single processor

McEneaney, Desphande, SG; ACC 08 SG, McEneaney, Qu; CDC 11

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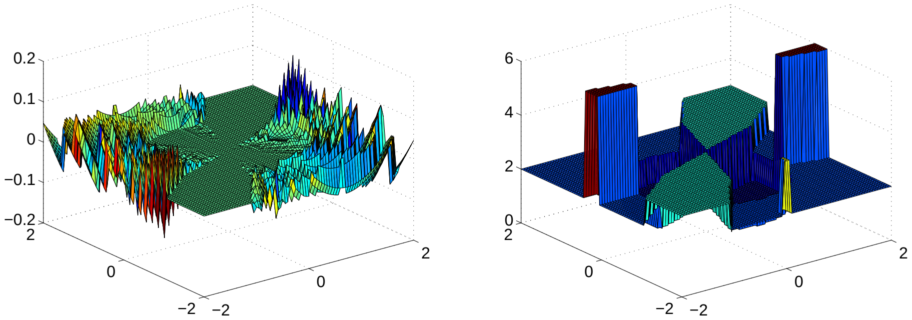


Figure: Backsubstitution error and optimal policy on the x_1,x_2 plane, $h = 0.1$ SG, McEneaney, Qu CDC 11

Error of the cod reduction method

Theorem (Coro of Qu 2012, improving Kluberg, McEneaney SICON 09)

Under technical conditions (ensuring in particular that the sol. of the switched LQ is finite), the approximation error of the maxplus cod reduction method is

$$\|v^{codr} - v\|_{\infty,[-1,1]^n} = O\big(\exp(\frac{\log k}{\epsilon}) \times polynomial(d)\big)$$

polynomial = RiccatiODE.

A certified coarse approximation can be obtained in a cod free way. (This bound does not take pruning into account, much better complexity in practice.)

however . . .

Curse of dimensionality is unavoidable

SG, McEneaney, Qu CDC 11: Cant approximate a $\mathscr{C}^2$ strictly convex function ψ by N affine max-plus finite elements in dimension d with an approximation error better than

$$\mathrm{cst} \times \frac{1}{N^{2/d}}$$

Corollary of techniques/results of Grüber on approximation of convex bodies.

$$\mathrm{cst} \sim \Big(\int_X (\det(\psi''_x))^{\frac{1}{2}} dx \Big)^{\frac{2}{d}} \qquad L_\infty \text{ case}$$

If the value function is flat in certain directions, the constant is zero (lower dimension).

A similar negative result holds for the L_1 norm.

Current works & TODO

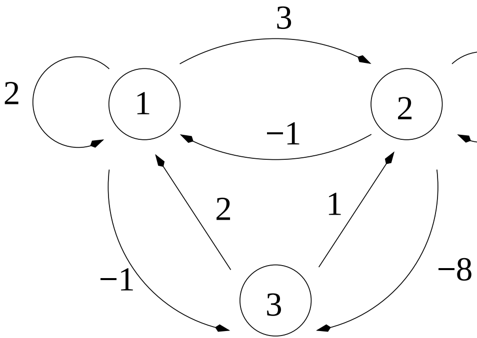
- equivalence between pruning and facility location for Bregman distances (McEneaney, SG, Qu).
- better complexity bounds ?
- relation between the template method of Manna et al in static analysis by abstract interpretation (Adjé, Goubault, SG), dimension = # lines of a program.
- higher dimensional examples (McEneaney, James, Sridharan - control of 2 q-bits = $SU(4)$, dim 15)
- extension to the stochastic case, comparison ($\neq$) with SDDP
- extension to general Hamiltonians (non cod free?)

Part II

Tropical methods for mean payoff zero-sum games ($=$ ergodic reward, average cost)

A repeated zero-sum stochastic game

Max and Min flip a coin to decide who makes the move.
Min always pays.



Solving the game

- $v_i^k :=$ value of the k-horizon game starting from node i.

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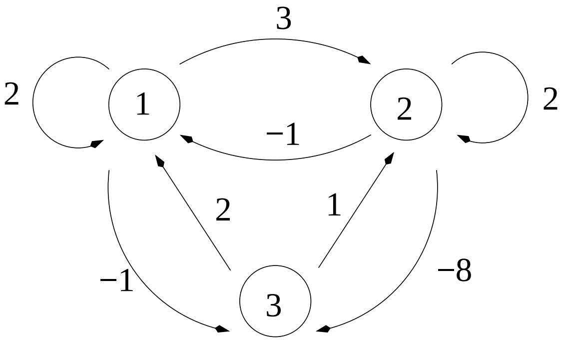
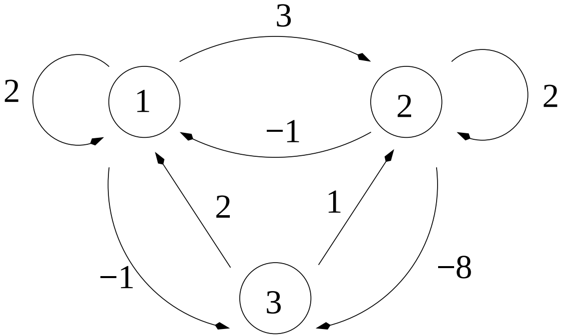
- value is defined as the mean reward of Max, assuming both players play optimally
- $v^k = (v_i^k) \in \mathbb{R}^n$
- $v^0 = 0$
- $v^{k+1} = T(v^k)$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is the Shapley operator


3
2
1
2
−1
2
2
1
−1
−8
3



$$v_i^{k+1} = \frac{1}{2}\Big(\max_{j:\, i\to j}(c_{ij} + v_j^k) + \min_{j:\, i\to j}(c_{ij} + v_j^k)\Big) \quad .$$

The mean payoff vector

$$\chi(T) := \lim_{k\to\infty} T^k(0)/k$$

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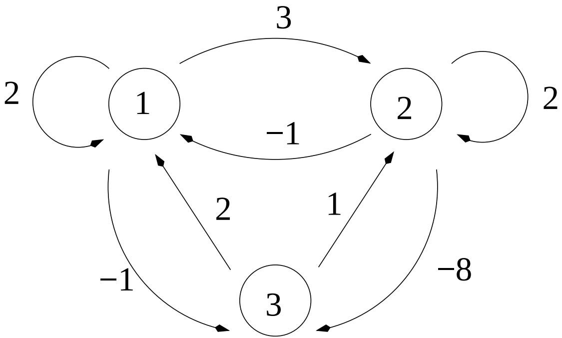
Think of x_i has a terminal bounty paid by Min to Max if the game ends in state i.



$$\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} \quad \begin{aligned} v_1 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 3+v_2, -1+v_3) + \min(2+v_1, \overline{3+v_2}, \overline{-1+v_3}) \\ v_2 &= \tfrac{1}{2}(\max(\overline{-1+v_1}, 2+v_2, -8+v_3) + \min(-1+v_1, 2+v_2, \overline{-8+v_3}) \\ v_3 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 1+v_2) + \min(2+v_1, \overline{1+v_2}) \end{aligned}$$

this game is fair

Optimality certificates

More generally, for $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$T(u) \geq u \implies \chi(T) \geq 0 \qquad \text{superfair}$$

$$T(u) \leq u \implies \chi(T) \leq 0 \qquad \text{subfair}$$

$$T(u) = \lambda + u \implies \chi(T) = (\lambda, \dots, \lambda) \ .$$

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Sufficient condition SG+Gunawardena, TAMS 2004: if $G(T)$ is strongly connected, then the additive eigenproblem $T(u) = \lambda + u$ with $\lambda \in \mathbb{R}$ is solvable

The latter game is a variant of the discrete Laplacian infinity (Oberman) or of Richman games or of stochastic Tug of war (Peres, Schramm, Sheffield, Scott, Wilson).

$$v_i = \frac{1}{2}(\max_{j:\, i\to j} v_j + \min_{j:\, i\to j} v_j) \enspace ,$$

v_i, $i \in$ boundary prescribed.

The Laplacian infinity:

$$\Delta_\infty v := \sum_{1\leq i,j\leq d} \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} = 0$$

More generally, zero-sum games = degenerate elliptic Hamilton-Jacobi PDE

Shapley operators, general games, state space $[n]$

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

- $[n] := \{1, \ldots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$

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$$T_i(x) = \max_{a\in A_i} \min_{b\in B_{i,a}} \Big(r_i^{ab} + \sum_{1\leq j\leq n} P_{ij}^{ab} x_j\Big), \qquad i \in [n]$$

- $[n] := \{1, \dots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$
- the game is deterministic if $P_{ij}^{ab} \in \{0,1\}$.

Shapley operators, general games, state space $[n]$

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T is order preserving and additively homogeneous:

$$x \leq y \implies T(x) \leq T(y)$$
$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$

Shapley operators, general games, state space $[n]$

Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov, Gunawardena, Sparrow; Singer, Rubinov),

$$T_i(x) = \sup_{y \in \mathbb{R}} \left(T_i(y) + \min_{1 \leq i \leq n} (x_i - y_i) \right)$$

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The operator approach

T is nonexpansive in the sup-norm:

$$\|T(x) - T(y)\|_\infty \leq \|x - y\|_\infty$$

The nonexpansiveness axiom was noted very early in dynamic programming (e.g., Blackwell), also in PDE (e.g., Crandall, Tartar).

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Games $\subset$ nonexpansive mappings in Banach spaces

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Games $\subset$ nonexpansive mappings in Banach spaces

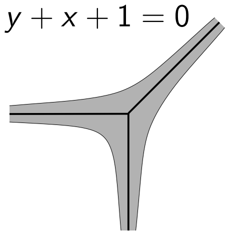
$$\chi(T) := \lim_{k\to\infty} T^k(x)/k \qquad \text{mean payoff vector ?}$$

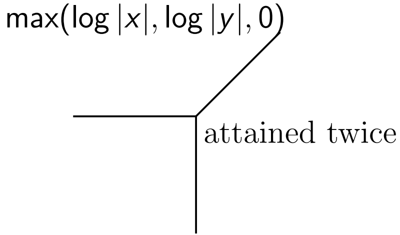
Mean ergodic theorems for non-expansive maps, in the 70' - 80' Pazy, Reich, Kohlberg, Neyman, ...

The tropical point of view arises with log glasses

Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \dots, \log|z_n|) \mid (z_1, \dots, z_n) \in V\} \ .$$





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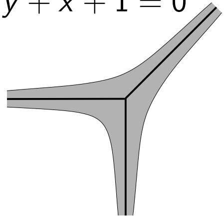
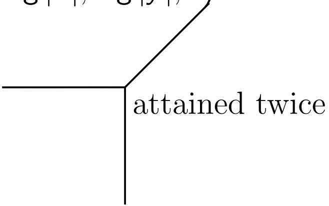
$y + x + 1 = 0$



$\max(\log|x|, \log|y|, 0)$



$$|y| \leq |x| + 1,\ |x| \leq |y| + 1,\ 1 \leq |x| + |y|$$

Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

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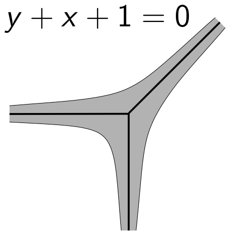
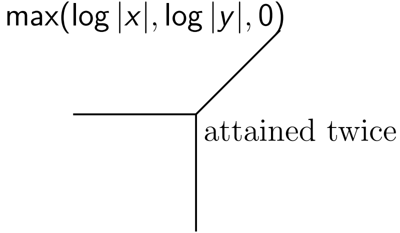




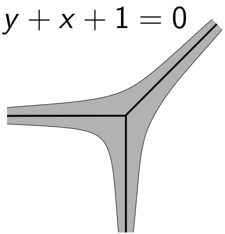
$X := \log|x|$, $Y := \log|y|$

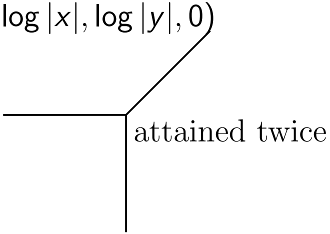
$Y \leq \log(e^X + 1)$, $X \leq \log(e^Y + 1)$, $0 \leq \log(e^X + e^Y)$

Viro's log-glasses, related to Maslov's dequantization

$$a +_h b := h \log(e^{a/h} + e^{b/h}), \qquad h \to 0^+$$

With h-log glasses, the amoeba of the line retracts to the tropical line as $h \to 0^+$





$$\max(a, b) \leq a +_h b \leq h \log 2 + \max(a, b)$$

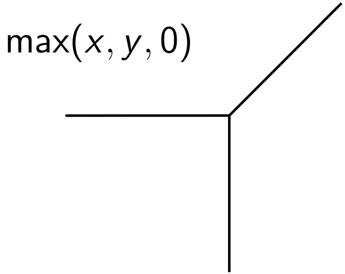
From tropical convexity to games

Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\text{“}ax + by + c\text{”}$$

is attained at least twice.



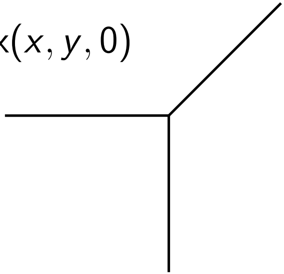
Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

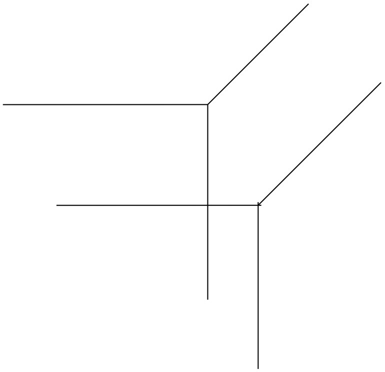
$$\max(a + x, b + y, c)$$

is attained at least twice.

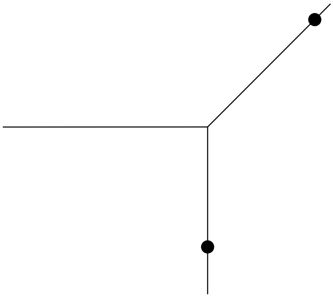
$\max(x, y, 0)$



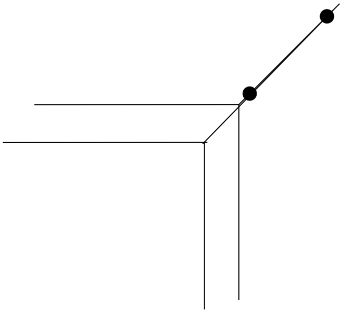
Two generic tropical lines meet at a unique point



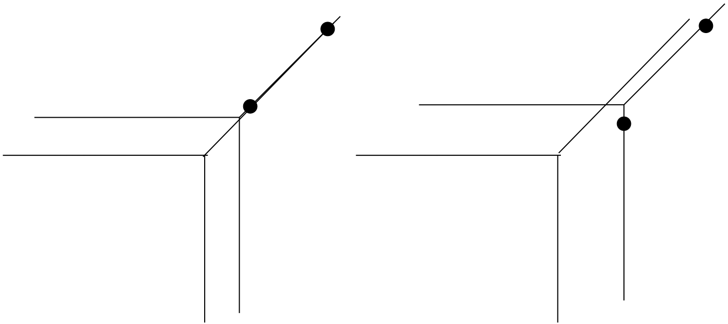
By two generic points passes a unique tropical line



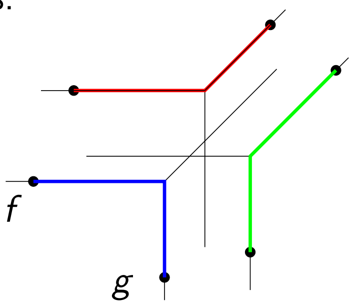
non generic case



non generic case resolved by perturbation



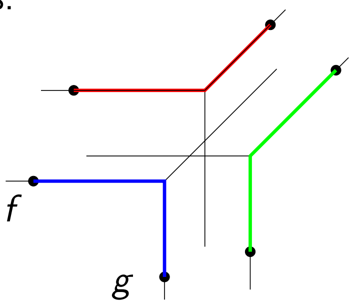
Tropical segments:



$$[f,g] := \{\text{“}\lambda f + \mu g\text{”} \mid \lambda,\mu \in \mathbb{R}\cup\{-\infty\},\ \text{“}\lambda+\mu = 1\text{”}\}.$$

(The condition “$\lambda,\mu \geq 0$” is automatic.)

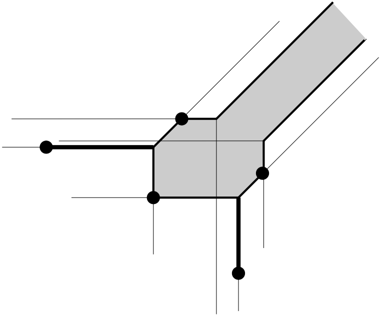
Tropical segments:



$[f,g] := \{\sup(\lambda+f,\mu+g) \mid \lambda,\mu \in \mathbb{R}\cup\{-\infty\},\ \max(\lambda,\mu)=0\}$.

(The condition $\lambda,\mu \geq -\infty$ is automatic.)

Tropical convex set: $f, g \in C \implies [f, g] \in C$



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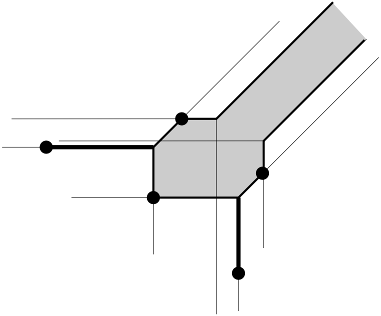
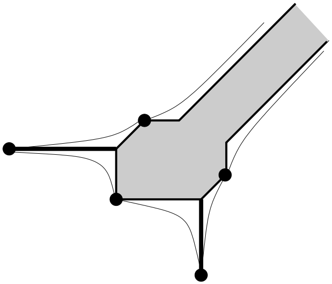


Tropical convex cone: ommit "$\lambda + \mu = 1$", i.e., replace $[f, g]$ by $\{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}\}$

Tropical convex sets are deformations of classical convex sets



Briec and Horvath 04

$$[a, b] := \{\lambda a +_p \mu b,\ \lambda, \mu \geq 0, \lambda +_p \mu = 1\}$$

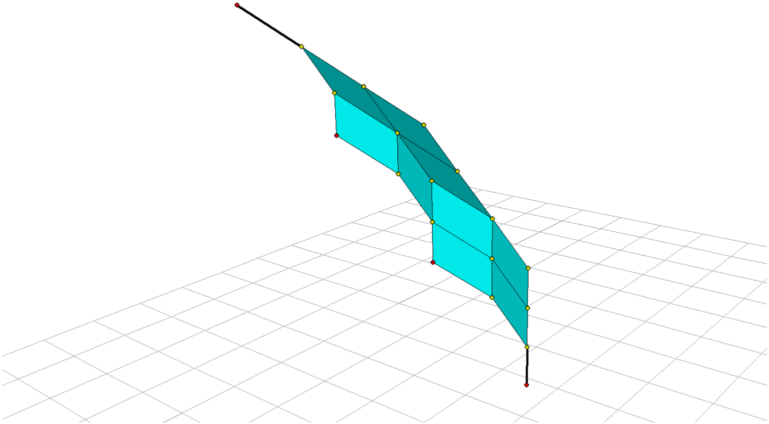
$$a +_p b = (a^p + b^p)^{1/p}$$

Homogeneization

A convex set C in $\mathbb{R}^n_{\max}$ is a cross section of a convex cone $\hat{C}$ in $\mathbb{R}^{n+1}_{\max}$,

$$\hat{C} := \{(\lambda + u, \lambda) \mid u \in C, \lambda \in \mathbb{R}_{\max}\}$$

A tropical polytope with four vertices



Structure of the polyhedral complex: Develin, Sturmfels

Back to Shapley operators

$$T_i(x) = \max_{a \in A} \min_{b \in B} \big(r_i^{ab} + \sum_{j \in [n]} P_{ij}^{ab} x_j \big), \qquad i \in [n] =: \Omega$$

T is order preserving and additively homogeneous:

$$x \leq y \implies T(x) \leq T(y)$$
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Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic)
Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \left(T_i(y) + \min_{1 \le i \le n} (x_i - y_i) \right)$$

From games to tropical convexity

If T is order preserving and additively homogeneous, then the set of subsolutions

$$C = \{u \mid T(u) \geq u\}$$

(showing that the game is superfair) is a tropical (max-plus) convex cone.

Supersolutions constitute a min-plus convex cone.

Infinite dimensional tropical convex sets

Similarly, if S^t is the semigroup of the Isaacs equation

$$v_t - H(x, Dv, D^2v) = 0, \qquad H(x, p, \cdot) \text{ order preserving}$$

S^t is order preserving and additively homogeneous

$$\begin{aligned} C &= \{u \mid S^t u \geq u, \ \forall t \geq 0\} \\ &= \{u \mid -H(x, Du, D^2u) \leq 0\} \end{aligned}$$

is a tropical convex cone.

Discounted case / terminating games

T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

If T is order preserving and additively subhomogeneous, then

$$C = \{u \mid T(u) \geq u\}$$

is a tropical (max-plus) convex set.

Tropical sesquilinear form and Hilbert's metric

$$\begin{aligned} x/v &:= \max\{\lambda \mid \text{“}\lambda v\text{”} \leq x\} \\ &= \min_i (x_i - v_i) \quad \text{if } x, v \in \mathbb{R}^n \ . \end{aligned}$$

$$\delta(x,y) = \text{“}(x/y)(y/x)\text{”} = \min_i (x_i - y_i) + \min_j (y_j - x_j)$$

$d = -\delta$ is the (additive) Hilbert's projective metric

$$d(x,y) = \|x - y\|_H, \ \ \|z\|_H := \max_{1 \leq i \leq d} z_i - \min_{1 \leq i \leq d} z_i \ .$$

Shapley operators are nonexpansive in this metric.

Projection on a tropical cone

If the tropical convex cone $C \subset \mathbb{R}^n_{\max}$ generated by U is stable stable by arbitrary sups (closed in Scott topology -non-Haussdorf-):

$$\begin{aligned} P_C(x) &= \max\{v \in C \mid v \leq x\} \\ &= \max_{u \in U} (x/u) + u \ . \end{aligned}$$

Similar to $\quad P_C(x) = \sum_{u \in U} \langle x, u \rangle u$

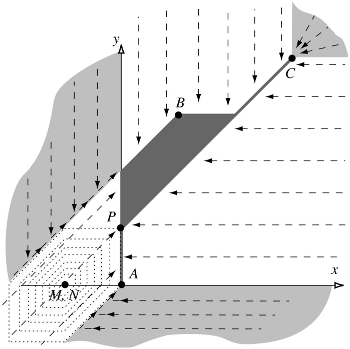
$$C = \mathrm{Col}(A), \quad [P_C(x)]_i = \max_{k \in [p]} \min_{j \in [n]} (A_{ik} - A_{jk} + x_j), \quad i \in [n]$$

Cuninghame-Green; Gondran, Minoux; Cohen, SG, Quadrat; Ardila; Joswig,

Best approximation in Hilbert's projective metric

Prop.(Cohen, SG, Quadrat, in Bensoussan Festschrift 01)

$$d(x, P_{\mathcal{V}}(x)) = \min_{y\in\mathcal{V}} d(x, y) \quad .$$



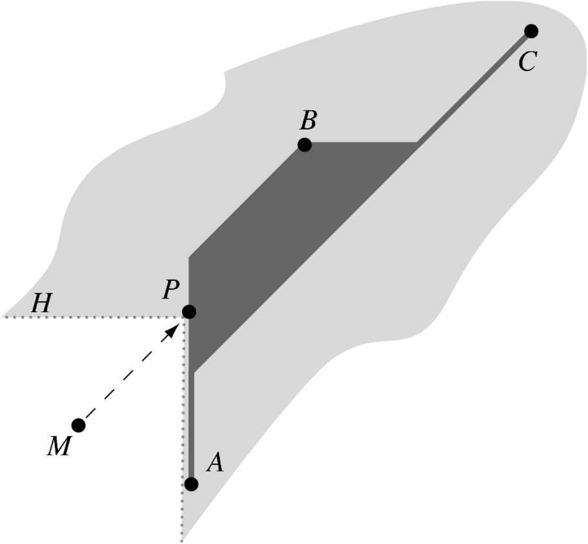
y
B
C
P
A
x
M, N

Separation

Goes back to Zimmermann 77, Hilbert metric construction in Cohen, SG, Quadrat in Ben01, LAA04.
C closed linear cone of $\mathbb{R}^d_{\max}$, or complete semimodule
If $y \notin C$, then, the tropical half-space

$$\mathcal{H} := \{v \mid y/v \leq P_C(y)/v\}$$

contains C and not y.
Compare with the optimality condition for the projection on a convex cone C: $\langle y - P_C(y), v\rangle \leq 0, \forall v \in C$


C
B
P
H
M
A

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leq bx\text{”}\}$$

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

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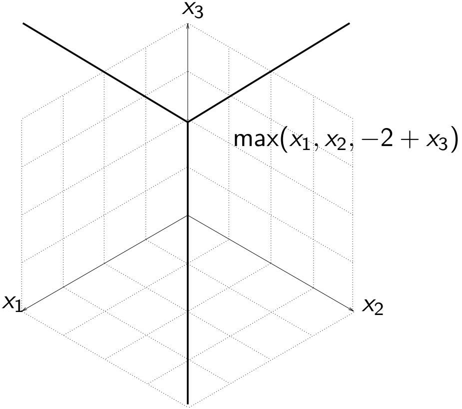

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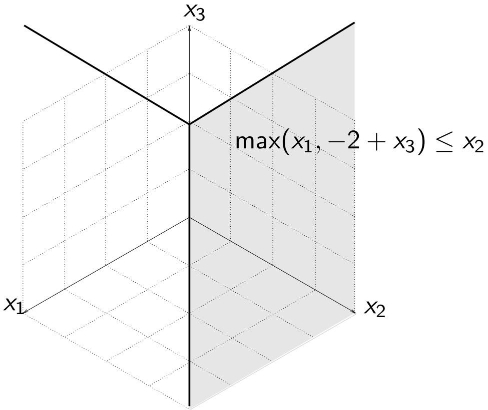

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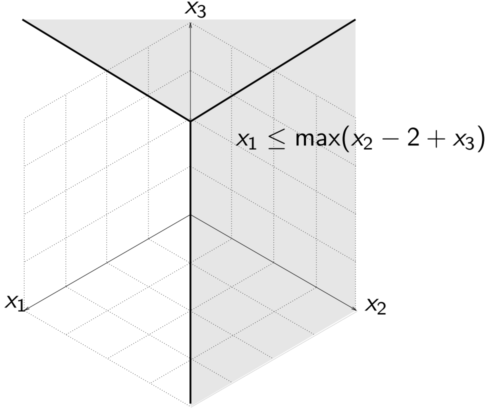

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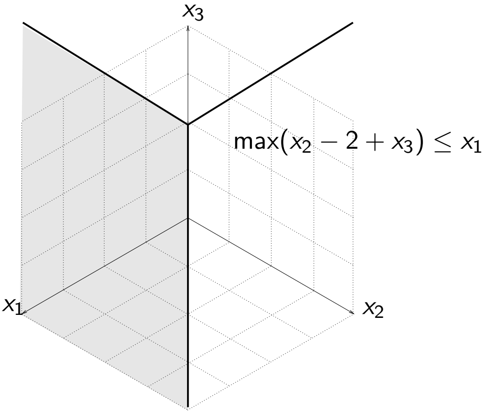


Corollary (Zimmermann; Samborski, Shpiz; Cohen, SG, Quadrat, Singer; Develin, Sturmfels; Joswig...)

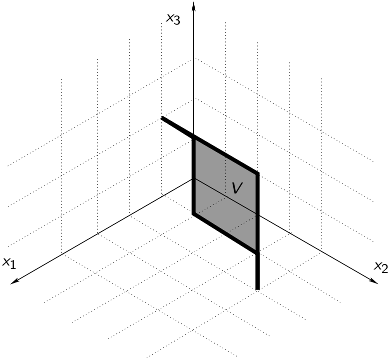
A tropical convex cone closed (in the Euclidean topology) is the intersection of tropical half-spaces.

$\mathbb{R}_{\max}$ is equipped with the topology of the metric $(x, y) \mapsto \max_i |e^{x_i} - e^{y_i}|$ inherited from the Euclidean topology by log-glasses.

The apex $-P_C(y)$ of the algebraic separating half-space $\mathcal{H}$ above may have some $+\infty$ coordinates, and therefore may not be closed in the Euclidean topology (always Scott closed). The proof needs a perturbation argument, this is where the assumption that C is closed (and not only stable by arbitrary sups = Scott closed) is needed.

Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



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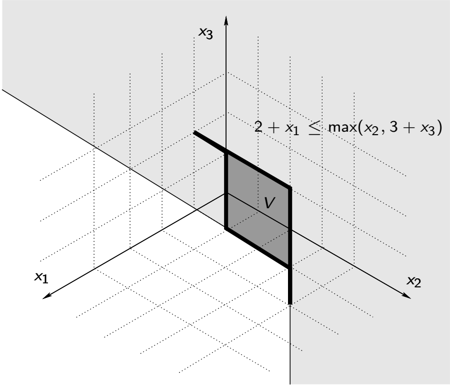

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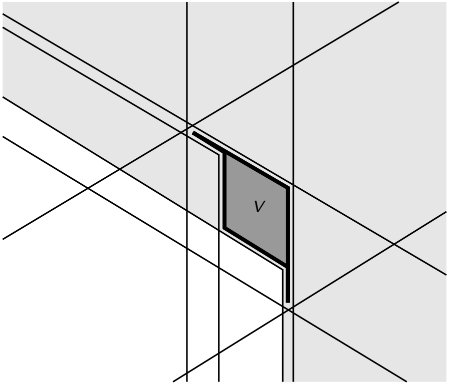


More generally, all the results of classical convexity have tropical analogues, but sometimes more degenerate. . .

- generation by extreme points Helbig; SG, Katz 07; Butkovič, Sergeev, Schneider 07; Choquet Akian, SG, Walsh 09, Poncet 11 infinite dim.
- projection / best-approximation : Cohen, SG, Quadrat 01,04; Singer
- Hahn-Banach analytic Litvinov, Maslov, Shpiz 00; Cohen, SG, Quadrat 04; geometric Zimmermann 77, Cohen, SG, Quadrat 01,05; Develin, Sturmfels 04, Joswig 05
- cyclic projections Butkovic, Cuninghame-Green TCS03; SG, Sergeev 06
- Radon, Helly, Carathéodory, Colorful Carathéodory, Tverberg: SG, Meunier DCG09

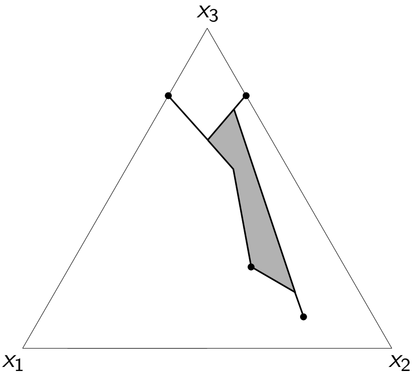
Correspondence between tropical convexity and zero-sum games

Theorem (Akian, SG, Guterman, IJAC 2012)

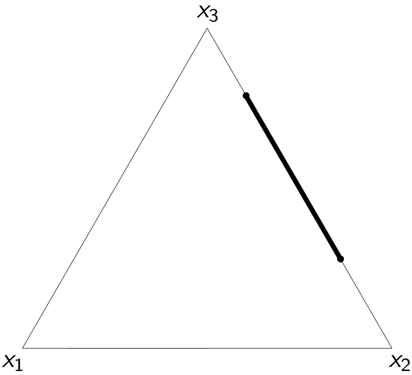
TFAE:

- *C closed tropical convex cone*
- *$C = \{u \in (\mathbb{R} \cup \{-\infty\})^n \mid u \leq T(u)\}$ for some Shapley operator T*

and MAX has at least one winning state ($\exists i, \chi_i(T) \geq 0$) if and only if $C \neq \{(-\infty, \ldots, -\infty)\}$. Moreover, tropical polyhedra correspond to deterministic games with finite action spaces. Then, state i is winning iff $u_i \neq -\infty$ for some $u \in C$.



states 1,2,3 winning



states 2,3 winning

The Shapley operator T is defined on $\mathbb{R}^n$, but extends continuously to $(\mathbb{R} \cup \{-\infty\})^n$,

$$T(x) = \inf\{ T(z) \mid z \geq x, \quad z \in \mathbb{R}^n\} \ .$$

Setting $x_i = -\infty$ prohibits MAX from reaching state i.

Proof. (Part 1, Equivalence). A closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \text{“}A_i x \leq B_i x\text{”}$$

Proof. (Part 1, Equivalence). A closed tropical convex cone can be written as

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where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \qquad a_{ij}, b_{ik} \in \mathbb{R} \cup \{-\infty\}$$

$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \le k \le n} b_{ik} + x_k \quad .$$

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$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k , \qquad a_{ij}, b_{ik} \in \mathbb{R} \cup \{-\infty\}$$

$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \leq k \leq n} b_{ik} + x_k \quad .$$

$$x \leq T(x) \iff \max_{1 \leq j \leq n} a_{ij} + x_j \leq \max_{1 \leq k \leq n} b_{ik} + x_k , \ \forall i \in I \quad .$$

$$H_i : \max_{1\le j\le n} a_{ij} + x_j \le \max_{1\le k\le n} b_{ik} + x_k$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\le k\le n} b_{ik} + x_k \ .$$

Interpretation of the game

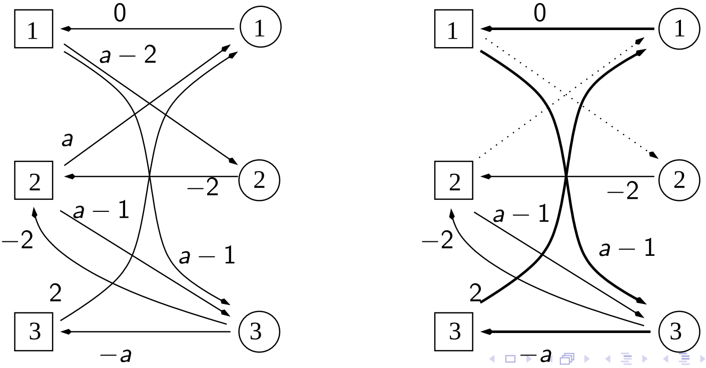
- State of MIN: variable x_j, $j \in \{1, \dots, n\}$
- State of MAX: half-space H_i, $i \in I$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

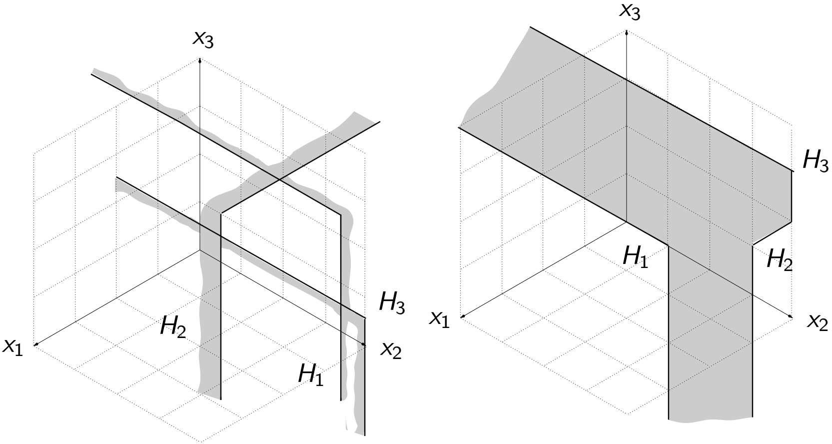
$$x_1 \leq a + \max(x_2 - 2, x_3 - 1) \qquad (H_1)$$

$$-2 + x_2 \leq a + \max(x_1, x_3 - 1) \qquad (H_2)$$

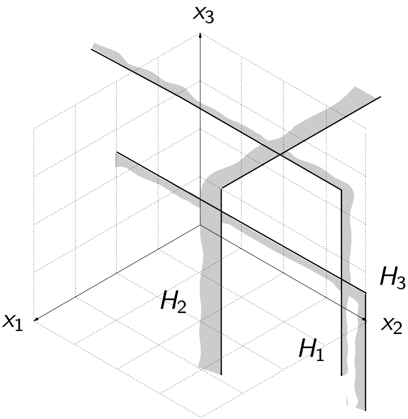
$$\max(x_2 - 2, x_3 - a) \leq x_1 + 2 \qquad (H_3)$$

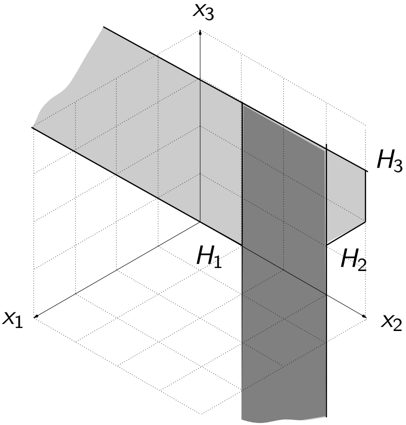
value $\chi(T)_j = (2a+1)/2$, $\forall j$.





$a = -3/2$, victorious strategy of Min: certificate of emptyness involving $\leq n$ inequalities (Helly)





$a = 1$, victorious strategy of Max: tropical polytrope $\neq \emptyset$ included in the convex set

Proof in the polyhedral case (cont.)

Relies on Kohlberg's theorem 1980.

A nonexpansive piecewise affine map $T:\mathbb{R}^n\to\mathbb{R}^n$ admits an invariant half-line

$$\exists v\in\mathbb{R}^n,\ \vec{\eta}\in\mathbb{R}^n,\quad T(v+t\vec{\eta})=v+(t+1)\vec{\eta}\ .$$

$$\chi(T)=\vec{\eta}$$

The vector u such that $T(u)\geq u$ is obtained from v,η (hint: $u_i=-\infty$ if $\vec{\eta}_i<0$).

Non polyhedral case

The mean payoff starting from some state i

$$\chi_i(T) = \lim_{k\to\infty} [T^k(x)]_i/k, \quad , x \in \mathbb{R}^n$$

may not exist.

However,

$$\overline{\chi}(T) = \lim_{k\to\infty} \max_{i\in[n]} [T^k(x)]_i/k$$

does exist,

$$\exists i \in [n], \qquad \chi_i(T) = \overline{\chi}(T)$$

and we can certify it

$$\exists u \in (\mathbb{R} \cup \{-\infty\})^n, u_i \neq -\infty, \qquad T(u) = \chi_i(T) + u \ .$$

Existence of coordinates for which the mean payoff $\chi_i(T) = \overline{\chi}(T)$ is a Denjoy-Wolff type theorem, SG, Gunawardena (TAMS 2004), extended by SG, Vigeral (Math. Proc. Cambridge Phil. Soc., 2011).

Existence of the certificate u follows from a non-linear Collatz-Wielandt theorem of Nussbaum (LAA, 1986).

$$\exists u \in (\mathbb{R} \cup \{-\infty\})^n, \qquad T(u) \geq \overline{\chi}(T) + u \enspace .$$

Application: policy iteration for zero-sum games with mean payoff

Write the Shapley operator T as

$$T = \inf_{\sigma} T^{\sigma}$$

where σ is a strategy of MIN, and T^{σ} is the Shapley operator of a one-player stochastic game. Assume finite action spaces, perfect information (finite number of strategies, T, T^{σ} polyhedral).

Idea: compute an invariant half-line of T^σ

$$T^\sigma(v^\sigma + t\vec{\eta}^\sigma) = v^\sigma + (t+1)\vec{\eta}^\sigma \quad , \forall t \geq 0.$$

This is a one player problem (Howard, Denardo, Fox, multichain policy iteration).

New strategy π is obtained by selection:

$$T(v^\sigma + t\vec{\eta}^\sigma) = T^\pi(v^\sigma + t\vec{\eta}^\sigma) \quad , \text{ for large } t$$

Hoffman and Karp 66: $\vec{\eta}^\pi < \vec{\eta}^\sigma$, so terminates, if every stochastic matrix is irreducible. Otherwise, may cycle ! (cyclic does occur for Richman games).

Solution to cycling: using the tropical spectral theorem

Cochet-Terrasson, SG, Gunawardena DSS 99, CRAS 98 (deterministic games), Cochet-Terrasson, SG CRAS 06, Akian, Cochet-Terrasson, **Detournay**, SG (stochastic games, PIGAMES library)

One player, stochastic case, wlog $\chi(H) = 0$:

$$H_i(x) = \max_{b \in B_i}(r_i^b + \sum_j P_{ij}^b x_j)$$

Theorem (Akian, SG; NLA TMA 03)

Any fixed point v of H is uniquely determined by its values v_i, $i \in C$, where C is the set of critical nodes (aka projected stochastic Aubry set).

C is the set of points which are visited infinitely often, a.s., when following an optimal strategy for the mean payoff problem.

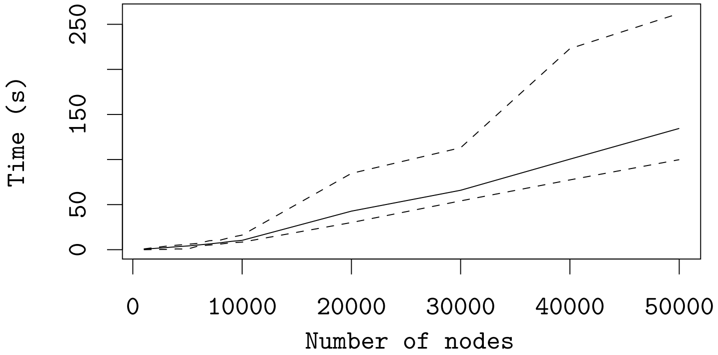
Theorem (Cochet-Terrasson, SG, Gunawardena DSS 99, deterministic case, Akian, Cochet-Terrasson, Detournay, SG 12 stochastic case)

If the previous strategy is σ, improved strategy π, select the new bias vector v^π in such a way that

$$v_i^\pi = v_i^\sigma, \qquad \forall i \in C(\pi) \ .$$

Then, the same policy is never selected twice.

PIGAMES library (Detournay)



Richman games. random graphs. 10 actions per state.
CPU time (s) on a single core Xeon 2.93Ghz.
PIGAMES (C library) soon public on inria gforge, see Detournay PhD thesis for documentation.

random Richman 10^6 states, 10^7 actions, typically 12 iterations, 5 hours on a single core

- Condon 92, Zwick and Paterson, TCS 96: mean payoff games are in NP $\cap$ co-NP;
- Ye: PI, one player, (strongly) polynomial time if the discount factor is fixed.
- Friedmann, LICS 2009 PI can be exponential time for mean payoff problems
- Edmonds conjectured that NP $\cap$ co-NP $=$ P , if one believes this, there should be a polynomial time algorithm for mean payoff games
- for stochastic mean payoff games, even pseudo polynomial time is not known Boros, Elbassioni, Gurvich, Makino.

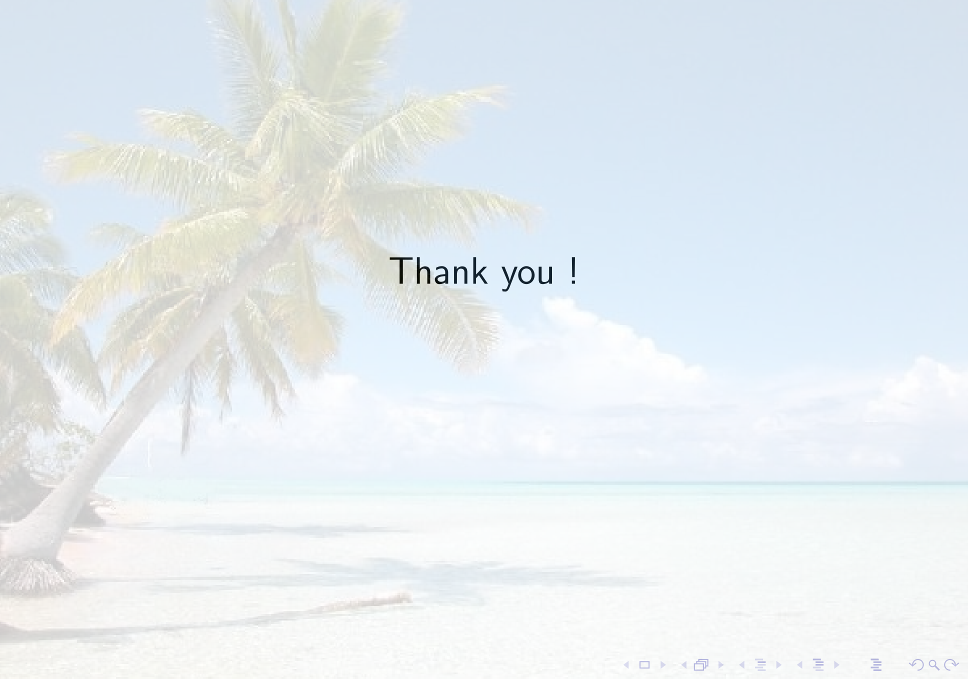
Concluding remarks

Connections between

- Zero-sum games
- metric geometry
- tropical algebra

motivated by

- algorithmic issues (complexity of mean payoff games is unsettled).



Thank you !