Hidden Convexity in the ℓ_0 Pseudonorm

Algorithms in Generalized Convexity and Application to Sparse Optimization

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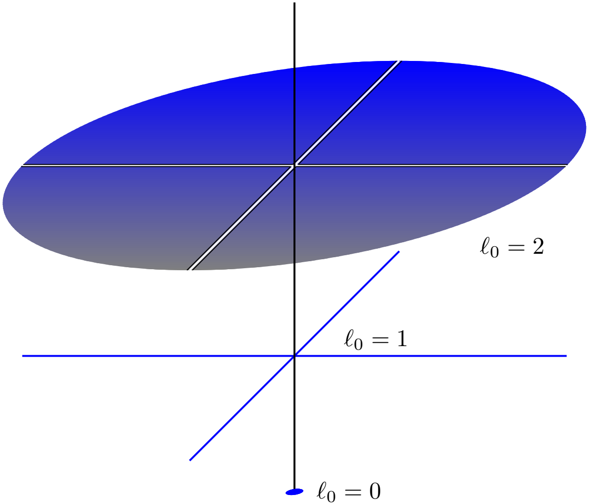
Here are the level sets
of the (highly nonconvex) ℓ_0 pseudonorm in $\mathbb{R}^2$



The ℓ_0 pseudonorm is ~~not~~ a ~~norm~~

Let $n \in \mathbb{N}^*$ be a fixed natural number

- For any vector $x \in \mathbb{R}^n$, we define its ℓ_0 pseudonorm(x) by

$$\ell_0(x) = \text{number of nonzero components of } x = \sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq 0\}}$$

- The function ℓ_0 pseudonorm : $\mathbb{R}^n \to [\![0, n]\!] = \{0, 1, \ldots, n\}$ satisfies 3 out of 4 axioms of a norm
  - we have $\ell_0(x) \geq 0$ ✓
  - we have $\Big(\ell_0(x) = 0 \iff x = 0 \Big)$ ✓
  - we have $\ell_0(x + x') \leq \ell_0(x) + \ell_0(x')$ ✓
  - But... instead of 1-homogeneity, it is 0-homogeneity that holds true

$$\ell_0(\rho x) = \ell_0(x) \; , \;\; \forall \rho \neq 0$$

WHY STUDY A FUNCTION
THAT IS
ALMOST SURELY CONSTANT?

The ℓ_0 pseudonorm is used in typical sparse optimization problems

- Spark of a matrix A

$$\mathrm{spark}(A) = \min \left\{ \ell_0(x) \,\middle|\, Ax = 0 \;,\;\; x \neq 0 \right\}$$

- Compressed sensing: recovery of a sparse signal $x \in \mathbb{R}^n$ from a measurement $b = Ax$

$$\min_{\substack{x \in \mathbb{R}^n \\ Ax = b}} \ell_0(x)$$

- Least squares sparse regression (best subset selection):

$$\text{for } k \in [\![1, n]\!] \qquad \min_{\substack{x \in \mathbb{R}^n \\ \ell_0(x) \leq k}} \|Ax - b\|^2$$

"explaining" the output b by at most k components of x

Fenchel conjugacy ($\star$) *versus* E-Capra conjugacy ($\cent$) for the ℓ_0 pseudonorm

- Fenchel conjugacy ($\star$)

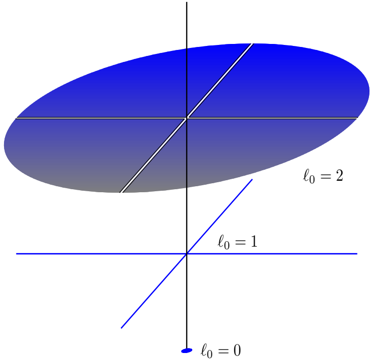
$$\ell_0^{\star\star'} = 0$$

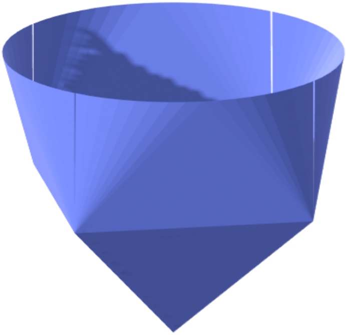
- E-Capra conjugacy ($\cent$)

$$\boxed{\ell_0^{\cent\cent'} = \ell_0}$$

[Chancelier and De Lara, 2021]

The ℓ_0 pseudonorm coincides, on the unit sphere, with the proper convex lower semicontinuous ℓ_0-cup function $\mathcal{L}_0 = \ell_0^{¢\star\prime}$





Towards algorithms?

- As motivation, we consider the sparse optimization problem, where C is a nonempty closed convex subset of $\mathbb{R}^n$,

$$\min_{x \in C} \ell_0(x) = \min_{x \in \mathbb{R}^n} \Big\{ \underbrace{\ell_0(x)}_{\substack{\text{E-Capra convex} \\ \ell_0^{\cent\cent'} = \ell_0}} + \overbrace{\underbrace{\iota_C(x)}_{\substack{\text{proper convex lsc} \\ \iota_C^{\star\star'} = \iota_C}}}^{\substack{\text{indicator} \\ \text{function}}} \Big\}$$

  where $\cent$ is the so-called E-Capra coupling
- Can we design algorithms using the above property that the pseudonorm ℓ_0 is E-Capra convex?

END OF THE TEASER

Talk outline

Outline of the presentation

Crash course on generalized convexity [5 min]

Capra conjugacies [20 min]

Towards Capra-algorithms in sparse optimization? [15 min]

Conclusion [1 min]

Additional material

Motivation: Legendre transform and Fenchel conjugacy in convex analysis

Definition

Two vector spaces $\mathcal{X}$ and $\mathcal{Y}$, paired by a bilinear form $\langle\ ,\ \rangle$, give rise to the classic Fenchel conjugacy

$$f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{\star} \in \overline{\mathbb{R}}^{\mathcal{Y}}$$

given by the Legendre transform

$$f^{\star}(y) = \sup_{x \in \mathcal{X}} \Big(\langle x , y \rangle + \big(-f(x) \big) \Big) , \ \ \forall y \in \mathcal{Y}$$

Coupling functions

Coupling function between sets

- Let be given two sets $\mathcal{X}$ ("primal") and $\mathcal{Y}$ ("dual")
  not necessarily paired vector spaces (nodes and arcs, etc.)
- We consider a coupling function

$$c : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}$$

We also use the notation $\mathcal{X} \overset{c}{\leftrightarrow} \mathcal{Y}$ for a coupling [Moreau, 1966-1967, 1970]

In duality in convex analysis, one uses the bilinear coupling

$$c(x, y) = \langle x \, , \, y \rangle$$

and, on a Hilbert space, the scalar product

$$c(x, y) = \langle x \mid y \rangle$$

Euclidean Constant Along Primal RAys (Capra) coupling

- On the Euclidean space $\mathbb{R}^n$, the Euclidean-Capra coupling (E-Capra) $\mathbb{R}^n \overset{¢}{\longleftrightarrow} \mathbb{R}^n$ is given by

$$\forall y \in \mathbb{R}^n \; , \quad \begin{cases} ¢(x,y) & = \dfrac{\langle x \mid y \rangle}{\|x\|_2} = \dfrac{\langle x \mid y \rangle}{\sqrt{\langle x \mid x \rangle}} \; , \quad \forall x \in \mathbb{R}^n \backslash \{0\} \\ ¢(0,y) & = 0 \end{cases}$$

- The coupling E-Capra has the property of being Constant Along Primal RAys (Capra)

Fenchel-Moreau conjugacies

Fenchel-Moreau conjugate of a function

$$f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^c \in \overline{\mathbb{R}}^{\mathcal{Y}}$$

Definition

The c-Fenchel-Moreau conjugate $f^c : \mathcal{Y} \to \overline{\mathbb{R}}$
of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is defined by

$$f^c(y) = \sup_{x \in \mathcal{X}} \Big(c(x,y) \stackrel{\cdot}{+} \big(-f(x)\big) \Big) , \quad \forall y \in \mathcal{Y}$$

We use the Moreau lower and upper additions on $\overline{\mathbb{R}}$
that extend the usual addition with

$$(+\infty) \stackrel{\cdot}{+} (-\infty) = (-\infty) \stackrel{\cdot}{+} (+\infty) = -\infty$$
$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$$

E-Capra-conjugate of the ℓ_0 pseudonorm

$$\ell_0^{\cent}(y) = \sup_{x\in\mathbb{R}^n} \Big\{ \cent(x,y) \dot{+} \big(-\ell_0(x)\big) \Big\}$$

$$= \sup \Big\{ 0, \sup_{x\neq 0} \Big\{ \frac{\langle x \mid y \rangle}{\|x\|_2} - \ell_0(x) \Big\} \Big\}$$

$$= \sup \Big\{ 0, \sup_{s\in S_2} \big\{ \langle s \mid y \rangle - \ell_0(s) \big\} \Big\}$$

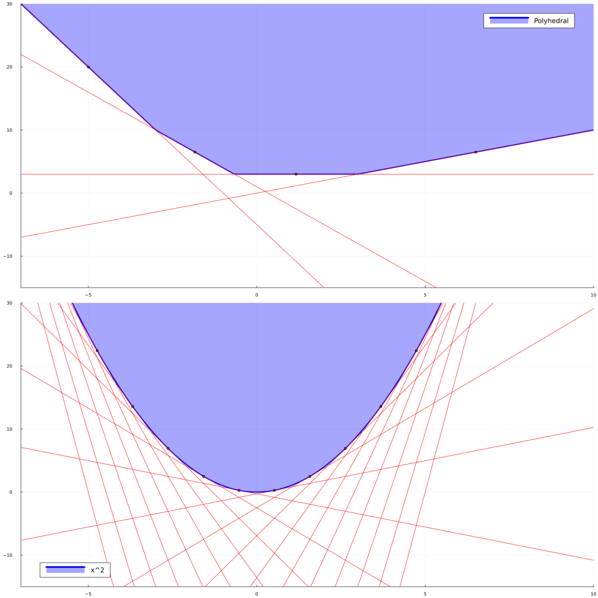
where $S_2 \subset \mathbb{R}^n$ is the Euclidean unit sphere

$$= \sup \Big\{ 0, \sup_{j\in[\![1,d]\!]} \Big\{ \underbrace{\sup_{\substack{s\in S_2 \\ \ell_0(s)=j}} \langle s \mid y \rangle}_{\substack{\text{top-}(2,j)\text{ norm} \\ \|y\|^{\top}_{2,j} = \sqrt{\sum_{l=1}^{k} |y_{\nu(l)}|^2}}} - j \Big\} \Big\}$$

$$= \sup_{j\in[\![1,d]\!]} \Big[\|y\|^{\top}_{2,j} - j \Big]_{+}$$

Biconjugates and duality

Motivation: duality in convex analysis



Reverse coupling and Fenchel-Moreau biconjugate

With the coupling c, we associate the reverse coupling c'

$$c' : \mathcal{Y} \times \mathcal{X} \to \overline{\mathbb{R}} \,, \;\; c'(y,x) = c(x,y) \,, \;\; \forall (y,x) \in \mathcal{Y} \times \mathcal{X}$$

$$f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}}$$

$$g \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto g^{c'} \in \overline{\mathbb{R}}^{\mathcal{X}}$$

Reverse coupling and Fenchel-Moreau biconjugate

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$$f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^c \in \overline{\mathbb{R}}^{\mathcal{Y}}$$

$$g \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto g^{c'} \in \overline{\mathbb{R}}^{\mathcal{X}}$$

$$g^{c'}(x) = \sup_{y \in \mathcal{Y}} \Big(c(x,y) \dot{+} \big(-g(y)\big) \Big) \; , \; \forall x \in \mathcal{X}$$

$$f^{cc'}(x) = \big(f^c\big)^{c'}(x) = \sup_{y \in \mathcal{Y}} \Big(c(x,y) \dot{+} \big(-f^c(y)\big) \Big) \; , \; \forall x \in \mathcal{X}$$

In generalized convexity,
one defines so-called c-convex functions

$$f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^c \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto f^{cc'} \in \overline{\mathbb{R}}^{\mathcal{X}}$$

For any function $f : \mathcal{X} \to \overline{\mathbb{R}}$, one has that

$$\boxed{f^{cc'} \leq f}$$

Definition

The function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is said to be c-convex if

$$f^{cc'} = f$$

c-convex functions have dual representations as suprema of elementary functions (abstract convexity)

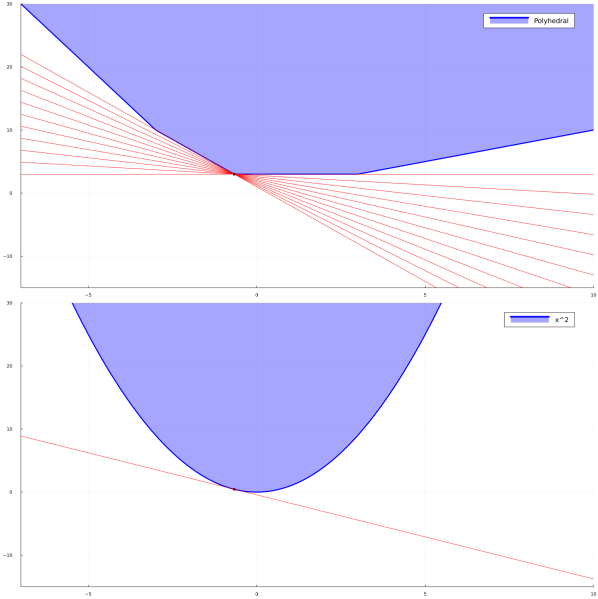
If the function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is c-convex, we have that

$$f(x) = \sup_{y \in \mathcal{Y}} \underbrace{\Big(c(x,y) \dot{+} \big(-f^{c}(y) \big) \Big)}_{\text{elementary function of } x} , \quad \forall x \in \mathcal{X}$$

Example: $\star$-convex functions
$=$ closed convex functions
$=$ proper convex lsc or $\equiv -\infty$ or $\equiv +\infty$
$=$ suprema of affine functions

Subdifferential

Motivation: subgradients in convex analysis



(Upper) subdifferential $\partial^c f : \mathcal{X} \rightrightarrows \mathcal{Y}$ of a conjugacy

For any function $f : \mathcal{X} \to \overline{\mathbb{R}}$ and $x \in \mathcal{X}$, $y \in \mathcal{Y}$

Definition

Upper subdifferential (following [Martinez-Legaz and Singer, 1995])

$$y \in \partial^c f(x) \iff f(x) = c(x,y) \dot{+} \left(-f^c(y)\right)$$

The upper subdifferential $\partial^c f$ has the property that

$$\partial^c f(x) \neq \emptyset \implies f(x) = \max_{y \in \partial^c f(x)} \Big(c(x,y) \dot{+} \left(-f^c(y)\right) \Big)$$

$$\implies \underbrace{f(x) = f^{cc'}(x)}_{\text{the function } f \text{ is } c\text{-convex at } x}$$

Wrap-up on generalized/abstract convexity

- Generalized convexity
  - coupling function between two sets
    $c : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}$
  - conjugacy and biconjugacy
    $f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto f^{cc'} \in \overline{\mathbb{R}}^{\mathcal{X}}$
  - generalized convex functions
    $f = f^{cc'}$
  - subdifferential
    $\partial^{c} f(x) \subset \mathcal{Y}$
- Abstract convexity
  - set of elementary functions
  - abstract convex envelope:
    supremum of lower elementary functions
  - abstract convex function:
    equal to its abstract convex envelope
  - subdifferential:
    tight lower elementary functions

Outline of the presentation

Outline of the presentation

We introduce the coupling E-Capra between $\mathbb{R}^n$ and itself

Definition

The Euclidean-Capra coupling (E-Capra) $\mathbb{R}^n \overset{¢}{\longleftrightarrow} \mathbb{R}^n$ is given by

$$\forall y \in \mathbb{R}^n , \quad \begin{cases} ¢(x,y) & = \dfrac{\langle x \mid y \rangle}{\|x\|_2} = \dfrac{\langle x \mid y \rangle}{\sqrt{\langle x \mid x \rangle}} , \quad \forall x \in \mathbb{R}^n \backslash \{0\} \\ ¢(0,y) & = 0 = \dfrac{0}{0} \end{cases}$$

The coupling E-Capra has the property of being
Constant Along Primal RAys (Capra)

E-Capra = Fenchel coupling after primal normalization

- We define the primal radial projection ϱ as

$$\varrho : \mathbb{R}^n \to S_2 \cup \{0\} \; , \;\; \varrho(x) = \begin{cases} \dfrac{x}{\|x\|_2} & \text{if } \; x \neq 0 \\ \\ \dfrac{0}{0} = 0 & \text{if } \; x = 0 \end{cases}$$

- so that the coupling E-Capra

$$¢(x, y) = \langle \varrho(x) \mid y \rangle \; , \;\; \forall x \in \mathbb{R}^n \; , \;\; \forall y \in \mathbb{R}^n$$

appears as the Fenchel coupling after primal normalization (and the coupling E-Capra is one-sided linear)

The E-Capra conjugacy shares properties with the Fenchel conjugacy

Proposition

- For any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,
  the $¢$-Fenchel-Moreau conjugate is given by

$$f^{¢} = \big(\inf \left[f \mid \varrho \right]\big)^{\star} \qquad \text{where}$$

$$\inf \left[f \mid \varrho \right](x) = \begin{cases} \inf_{\rho > 0} f(\rho x) & \text{if } x \in S_2 \cup \{0\} \\ +\infty & \text{if } x \notin S_2 \cup \{0\} \end{cases}$$

- For any function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$,
  the $¢'$-Fenchel-Moreau conjugate is given by

$$g^{¢'} = g^{\star'} \circ \varrho$$

The E-Capra-convex functions are 0-homogeneous and coincide, on the unit sphere, with a closed convex function

Proposition

¢-convexity of the function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$

$$\iff h = h^{¢¢'}$$

$$\iff h = \underbrace{\left(h^{¢}\right)^{\star'}}_{\text{convex lsc function}} \circ \varrho$$

$\iff$ hidden convexity in the function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$

there exists a closed convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$

such that $h = f \circ \varrho$, that is, $h(x) = f\left(\frac{x}{\|x\|_2}\right)$

The ℓ_0 pseudonorm is E-Capra-convex

Notation

- The Euclidean top-$(2,k)$ norm is also known as the $(2,k)$-*symmetric gauge norm*, or *Ky Fan vector norm*

$$\|y\|^{\top}_{2,k} = \sqrt{\sum_{l=1}^{k} |y_{\nu(l)}|^2} \ , \quad |y_{\nu(1)}| \geq |y_{\nu(2)}| \geq \cdots \geq |y_{\nu(n)}|$$

- We denote the level sets of the ℓ_0 pseudonorm by

$$\ell_0^{\leq k} = \left\{ x \in \mathbb{R}^n \,\middle|\, \ell_0(x) \leq k \right\} \ , \quad \forall k \in [\![0, n]\!]$$

  and its elements are call k-sparse vectors
- For any subset $W \subset \mathbb{R}^n$, its indicator function ι_W is

$$\iota_W(w) = \begin{cases} 0 & \text{if } w \in W \\ +\infty & \text{if } w \notin W \end{cases}$$

The ℓ_0 pseudonorm and the E-Capra-coupling

Theorem

The ℓ_0 pseudonorm,
the indicator functions $\iota_{\ell_0^{\leq k}}$ of its level sets
and the Euclidean top-$(2,k)$ norms $\|\cdot\|_{2,k}^{\top}$ are related by

$$\iota_{\ell_0^{\leq k}}^{\cent} = \|\cdot\|_{2,k}^{\top} \ , \quad k \in [\![0, n]\!]$$

$$\ell_0^{\cent} = \sup_{j \in [\![0,n]\!]} \Big[\|\cdot\|_{2,j}^{\top} - j \Big]$$

$$\ell_0^{\cent\cent'} = \ell_0$$

The ℓ_0 pseudonorm displays hidden convexity

The ℓ_0 pseudonorm displays a convex factorization property

Theorem

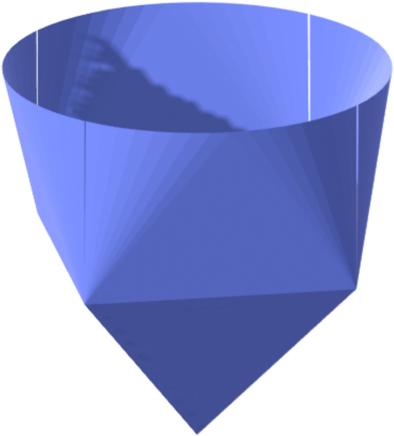
As the ℓ_0 pseudonorm is E-Capra-convex, we get that

$$\ell_0 = \ell_0^{¢¢'} = \ell_0^{¢\star'} \circ \varrho = \underbrace{\left(\ell_0^{¢}\right)^{\star'}}_{\text{convex lsc function } \mathcal{L}_0} \circ \overbrace{\varrho}^{\text{radial projection}}$$

As a consequence, the ℓ_0 pseudonorm coincides,
on the Euclidean unit sphere S_2,
with a proper convex lsc function,
the Euclidean ℓ_0-cup function $\mathcal{L}_0 = \ell_0^{¢\star'}$

$$\ell_0(x) = \mathcal{L}_0(x) \ , \ \ \forall x \in S_2$$

Graph of the Euclidean ℓ_0-cup function $\mathcal{L}_0 = \ell_0^{¢\star\prime}$



Best proper convex lsc lower approximation of the ℓ_0 pseudonorm on the Euclidean unit ball

Theorem

The Euclidean ℓ_0-cup function $\mathcal{L}_0 = \ell_0^{¢\star'}$ is
the best convex lsc lower approximation of the ℓ_0 pseudonorm
on the Euclidean unit ball B_2

$$\text{best convex lsc function} \quad \mathcal{L}_0(x) \leq \ell_0(x) \;, \;\; \forall x \in B_2$$

and, as seen above, coincides with the ℓ_0 pseudonorm

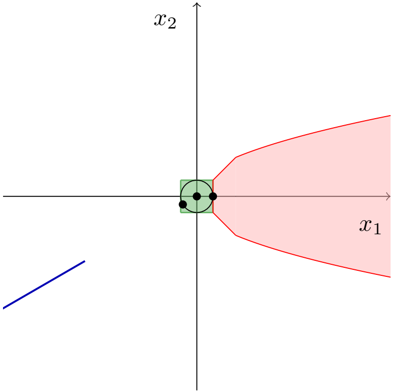
on the Euclidean unit sphere S_2

$$\ell_0(x) = \mathcal{L}_0(x) \;, \;\; \forall x \in S_2$$

E-Capra subdifferential of the ℓ_0 pseudonorm
(with Adrien Le Franc)

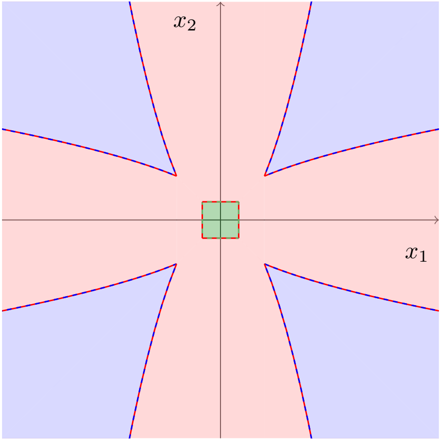
Capra-subdifferential of the ℓ_0 pseudonorm on $\mathbb{R}^2$

Illustration at three points (black dots)



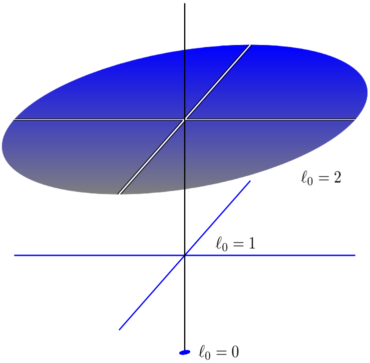
$$\partial_{¢}\ell_0(0,0)\ ,\ \ \partial_{¢}\ell_0(1,0)\ ,\ \ \partial_{¢}\ell_0(-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2})$$

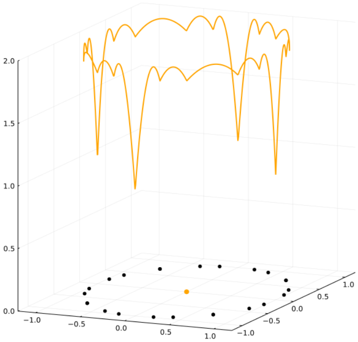
Capra-subdifferential of the ℓ_0 pseudonorm on $\mathbb{R}^2$



$$\partial_{¢}\ell_0(0) \bigcup \left\{ \bigcup_{\ell_0(x)=1} \partial_{¢}\ell_0(x) \right\} \bigcup \left\{ \bigcup_{\ell_0(x)=2} \partial_{¢}\ell_0(x) \right\}$$

Lower approximation of the ℓ_0 pseudonorm by a finite number of elementary E-Capra-functions





Variational formulas

We recall the Euclidean $(2,k)$-support norms $\|\cdot\|_{2,k}^{\top\star}$

- The dual norm of the top-$(2,k)$ norm $\|\cdot\|_{2,k}^{\top}$

$$\|\cdot\|_{2,k}^{\top\star} = \left(\|\cdot\|_{2,k}^{\top}\right)_{\star}$$

  is called the (Euclidean) $(2,k)$-support norm [Argyriou, Foygel, and Srebro, 2012]

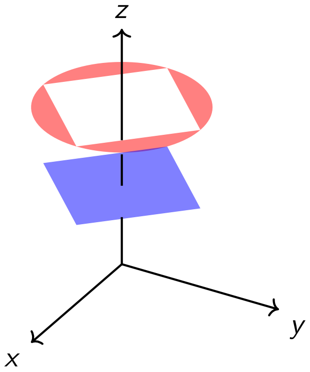
- We have the following inclusions between unit balls

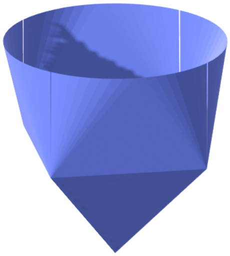
$$B_{(1)}^{\top\star} \subset \cdots \subset B_{(\ell-1)}^{\top\star} \subset B_{(\ell)}^{\top\star} \subset \cdots \subset B_{(n)}^{\top\star} = B$$

Proposition

The proper convex lsc function $\mathcal{L}_0$ is the convex envelope of the following piecewise constant function

$$L_0(x) = \begin{cases} 0 & \text{if } x = 0, \\ \ell & \text{if } x \in B^{\top\star}_{(\ell)} \setminus B^{\top\star}_{(\ell-1)}\ , \quad \ell \in [\![1, n]\!] \\ +\infty & \text{if } x \notin B^{\top\star}_{(n)} = B \end{cases}$$





Variational formulas for the ℓ_0 pseudonorm

Proposition

$$\ell_0(x) = \frac{1}{\|x\|_2} \min_{\substack{x^{(1)}\in\mathbb{R}^n,\ldots,x^{(n)}\in\mathbb{R}^n \\ \sum_{\ell=1}^n \left\|x^{(\ell)}\right\|_{2,\ell}^{\top\star} \leq \|x\|_2 \\ \sum_{\ell=1}^n x^{(\ell)} = x}} \sum_{\ell=1}^n \ell \left\|x^{(\ell)}\right\|_{2,\ell}^{\top\star}, \ \forall x \in \mathbb{R}^n$$

$$\ell_0(x) = \sup_{y\in\mathbb{R}^n} \inf_{\ell\in[\![1,n]\!]} \left(\frac{\langle x \mid y \rangle}{\|x\|_2} - \left[\|y\|_{2,\ell}^{\top} - \ell \right]_+ \right), \ \forall x \in \mathbb{R}^n \setminus \{0\}$$

Outline of the presentation

Work has gone on along two paths

	Norm Euclidean	Norm orthant-strictly monotonic	Norm any	1-homogeneous nonnegative function
ℓ_0 pseudonorm	¢-convex ($\ell_0^{¢¢'} = \ell_0$) hidden convexity variational formula [Chancelier and De Lara, 2021] subdifferential [Le Franc et al., 2022]	difference of norms [Chancelier and De Lara, 2022b]		
$\varphi \circ \ell_0$ $\varphi : \mathbb{N} \to \overline{\mathbb{R}}$ nondecreasing		¢-convex ($(\varphi \circ \ell_0)^{¢¢'} = \varphi \circ \ell_0$) hidden convexity variational formula subdifferential [Chancelier and De Lara, 2022c]		
$\varphi \circ \ell_0$ $\varphi : \mathbb{N} \to \overline{\mathbb{R}}$ any			$(\varphi \circ \ell_0)^{¢¢'}$ variational inequality subdifferential [Chancelier and De Lara, 2022a]	
$F \circ$ support $F : 2^{[\![1,d]\!]} \to \overline{\mathbb{R}}$ any			$(F \circ \mathrm{support})^{¢¢'}$ variational inequality subdifferential [preprint]	
0-homogeneous function				best lower approximation [preprint]

We introduce the coupling Capra

- Let be given $\mathcal{X}$ and $\mathcal{Y}$, two vector spaces paired by a bilinear form $\langle \cdot , \cdot \rangle$
- Suppose that $\mathcal{X}$ is equipped with a (source) norm $\|\cdot\|$

Definition

[Chancelier and De Lara, 2022a]

The coupling Capra $\mathcal{X} \overset{¢}{\longleftrightarrow} \mathcal{Y}$ is given by

$$\forall y \in \mathcal{Y} \, , \quad \begin{cases} ¢(x,y) & = \dfrac{\langle x , y \rangle}{\|x\|} \, , \quad \forall x \in \mathcal{X} \backslash \{0\} \\ ¢(0,y) & = 0 \end{cases}$$

In what follows, $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$
with norm $\|\cdot\|$ having unit ball B and unit sphere S

Orthant-monotonic and orthant-strictly monotonic norms

Orthant-monotonic norms

For any $x \in \mathbb{R}^n$, we denote by $|x|$
the vector of $\mathbb{R}^n$ with components $|x_i|$, $i \in [\![1, n]\!]$

Definition

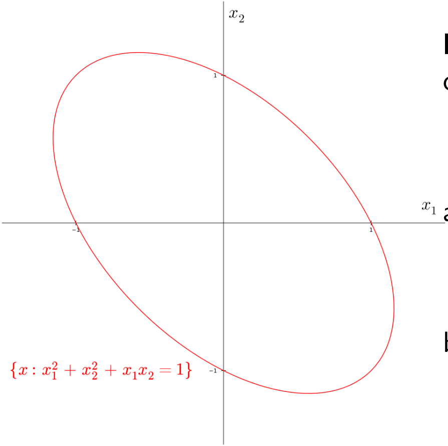
A norm $\|\cdot\|$ on the space $\mathbb{R}^n$ is called orthant-monotonic [Gries, 1967]
if, for all x, x' in $\mathbb{R}^n$, we have

$$|x| \leq |x'| \text{ and } x \circ x' \geq 0 \implies \|x\| \leq \|x'\|$$

where $x \circ x' = (x_1 x'_1, \ldots, x_n x'_n)$
is the Hadamard (entrywise) product

$$\left.\begin{array}{c} |x_1| \leq |x'_1| \,, \quad \ldots \,, \quad |x_n| \leq |x'_n| \\ \text{and} \\ x_1 x'_1 \geq 0 \,, \quad \ldots \,, \quad x_n x'_n \geq 0 \end{array}\right\} \implies \|x\| \leq \|x'\|$$

Example of unit sphere of a non orthant-monotonic norm



In the bottom right orthant, consider

$$|(0, -1)| \leq |(0.5, -1)|$$

and

$$(0, -1) \circ (0.5, -1) \geq (0, 0)$$

but

$$1 = \|(0, -1)\| > \|(0.5, -1)\|$$

Orthant-strictly monotonic norms

[Chancelier and De Lara, 2022b]

Definition

A norm $\|\cdot\|$ on the space $\mathbb{R}^n$ is called orthant-strictly monotonic if, for all x, x' in $\mathbb{R}^n$, we have

$$|x| < |x'| \text{ and } x \circ x' \geq 0 \implies \|x\| < \|x'\|$$

where $|x| < |x'|$ means that
there exists $j \in [\![1, n]\!]$ such that $|x_j| < |x'_j|$

Intuition: $\epsilon \neq 0 \implies \|(0, *, 0, *, *, 0)\| < \|(0, *, \epsilon, *, *, 0)\|$

Examples of orthant-strictly monotonic norms

$$\|x\|_\infty = \sup_{i \in [\![1,n]\!]} |x_i| \text{ and } \|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} \text{ for } p \in [1, \infty[$$

with unit ball B_p and unit sphere S_p

- All the ℓ_p-norms $\|\cdot\|_p$ on the space $\mathbb{R}^n$, for $p \in [1, \infty]$, are monotonic, hence orthant-monotonic

$$\ell_1, \ell_2, \ell_\infty$$

- All the ℓ_p-norms $\|\cdot\|_p$ on the space $\mathbb{R}^n$, for $p \in [1, \infty[$, are orthant-strictly monotonic

$$\ell_1, \ell_2, \cancel{\ell_\infty}$$

$$|\epsilon| < 1 \implies \|(1,0)\|_\infty = 1 = \|(1,\epsilon)\|_\infty$$

Orthant-strictly monotonic norms and Capra-convexity

Capra-subdifferentiability properties of the ℓ_0 pseudonorm

[Chancelier and De Lara, 2022c]

Proposition

If both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_\star$ are orthant-strictly monotonic, we have that

$$\partial_{¢}\ell_0(x) \neq \emptyset \ , \ \ \forall x \in \mathbb{R}^n \ ,$$

that is, the pseudonorm ℓ_0 is Capra-subdifferentiable on $\mathbb{R}^n$ and, as a consequence,

$$\ell_0^{¢¢'} = \ell_0$$

Best convex lower approximation of the ℓ_0 pseudonorm on the ℓ_p-unit balls, $p \in [1, \infty]$

Theorem

The best convex lsc lower approximation $\mathcal{L}_0$ of ℓ_0

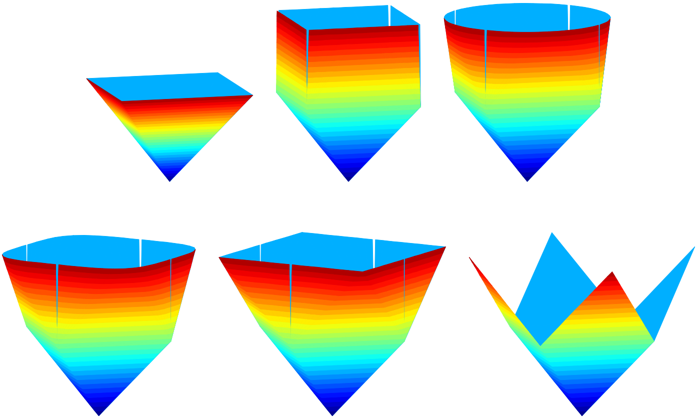
$$\text{best convex lsc function} \quad \mathcal{L}_0(x) \leq \ell_0(x) \ , \ \ \forall x \in B_p$$

on the unit ball B_p is $\ell_0^{¢\star\prime}$, and coincides with the ℓ_0 pseudonorm

$$\ell_0(x) = \mathcal{L}_0(x) \ , \ \ \forall x \in S_p$$

on the unit sphere S_p

Tightest closed convex function below the ℓ_0 pseudonorm on the ℓ_p-unit balls on $\mathbb{R}^2$ for $p \in \{1, 1.1, 2, 4, 300, \infty\}$



Capra-subdifferential of the ℓ_0 pseudonorm

Capra-subdifferential of the ℓ_0 pseudonorm

- $\{\|\cdot\|^{\mathcal{R}}_{(j)}\}_{j\in[\![1,n]\!]}$ and $\{\|\cdot\|^{\mathcal{R}}_{(j),\star}\}_{j\in[\![1,n]\!]}$,
associated coordinate-k and dual coordinate-k norms
- $\{B^{\mathcal{R}}_{(j)}\}_{j\in[\![1,n]\!]}$ and $\{B^{\mathcal{R}}_{(j),\star}\}_{j\in[\![1,n]\!]}$, corresponding unit balls

Proposition

[Chancelier and De Lara, 2022a]
The Capra-subdifferential of the ℓ_0 pseudonorm is given by

$$\text{if } x = 0, \qquad \partial_{\text{¢}}\ell_0(0) = \bigcap_{j\in[\![1,n]\!]} jB^{\mathcal{R}}_{(j),\star}$$

$$\text{if } x \neq 0 \text{ and } \ell_0(x) = \ell, \qquad \partial_{\text{¢}}\ell_0(x) = N\Big(B^{\mathcal{R}}_{(\ell)}, \frac{x}{\|x\|^{\mathcal{R}}_{(\ell)}}\Big) \cap Y_\ell$$

$$\text{where} \qquad Y_\ell = \Big\{ y \in \mathcal{Y} \,\Big|\, \ell \in \operatorname*{arg\,max}_{j\in[\![0,n]\!]} \big(\|y\|^{\mathcal{R}}_{(j),\star} - j\big) \Big\}\,, \;\; \forall \ell \in [\![0,n]\!]$$

Exposed faces and normal cones

For any nonempty closed convex subset $C \subset \mathcal{X}$,
where $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$,

- the exposed face $F_\perp(C, y)$ of C by any dual vector $y \in \mathcal{Y}$ is

$$F_\perp(C, y) = \arg\max_{x \in C} \langle x \mid y \rangle$$

- the normal cone $N(C, x)$ of C at any primal vector $x \in C$ is defined by the conjugacy relation

$$x \in C \text{ and } y \in N(C, x) \iff x \in F_\perp(C, y)$$

The family of all normal cones is the normal fan $\mathcal{N}(C)$

Coordinate-k norms and their dual norms

Courtesy of Basile and Lionel Pournin

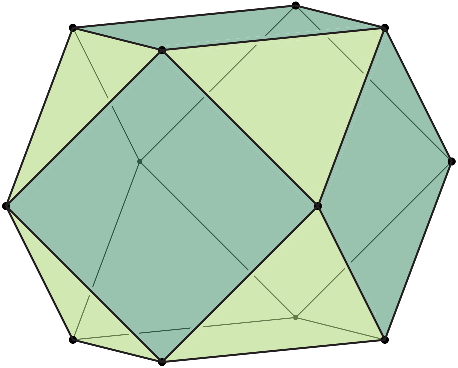


Figure: Unit ball $\overline{\mathrm{co}}\left(\ell_0^{\leq 2} \cap S_1\right)$ when $n = 3$

Extreme points of the coordinate-k norm unit ball are k-sparse

For any source norm $\|\cdot\|$ on $\mathbb{R}^n$, and for $k \in [\![1, d]\!]$,

- the coordinate-k norm $\|\cdot\|^{\mathcal{R}}_{(k)}$ has unit ball

$$B^{\mathcal{R}}_{(k)} = \underbrace{\overline{\mathrm{co}}\left(\ell_0^{\leq k} \cap S\right)}_{\text{closed convex hull}} = \underbrace{\mathrm{co}\left(\ell_0^{\leq k} \cap S\right)}_{\text{convex hull}}$$

- hence the extreme points of $B^{\mathcal{R}}_{(k)}$ belong to $\ell_0^{\leq k} \cap S \subset \ell_0^{\leq k}$, hence are k-sparse vectors

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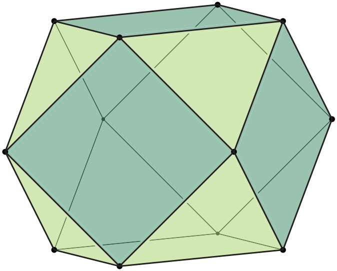
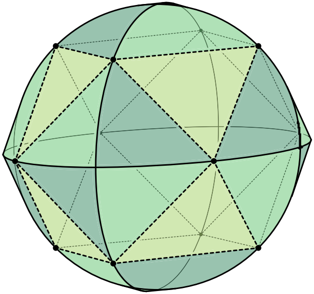
This is how we define

- a sequence $\left\{\|\cdot\|^{\mathcal{R}}_{(k)}\right\}_{k \in [\![1,n]\!]}$ of coordinate-k norms
- a sequence $\left\{\|\cdot\|^{\mathcal{R}}_{(k),\star}\right\}_{k \in [\![1,n]\!]}$ of dual coordinate-k norms

Courtesy of Basile and Lionel Pournin



(a) Unit ball $\overline{\mathrm{co}}\left(\ell_0^{\leq 2} \cap S_1\right)$ when $n = 3$



(b) Unit ball $\overline{\mathrm{co}}\left(\ell_0^{\leq 2} \cap S_2\right)$ when $n = 3$

Coordinate and dual coordinate norms induced by the ℓ_p-norms $\|\cdot\|_p$

For $y \in \mathbb{R}^n$, ν is a permutation of $[\![1, n]\!]$ such that $|y_{\nu(1)}| \geq |y_{\nu(2)}| \geq \cdots \geq |y_{\nu(n)}|$

source norm $\|\cdot\|$	$\|\cdot\|^{\mathcal{R}}_{(k),\star}$	$\|\cdot\|^{\mathcal{R}}_{(k)}$
$\|\cdot\|_p$	top-(q,k) norm $\|y\|^{\top}_{q,k}$ $= \left(\sum_{l=1}^{k} |y_{\nu(l)}|^q\right)^{\frac{1}{q}}$	(p,k)-support norm $\|x\|^{\top\star}_{p,k}$ no analytic expression
$\|\cdot\|_1$	top-(∞,k) norm ℓ_∞-norm $\|y\|^{\top}_{\infty,k} = \|y\|_\infty$	$(1,k)$-support norm ℓ_1-norm $\|x\|^{\top\star}_{1,k} = \|x\|_1$
$\|\cdot\|_2$	top-$(2,k)$ norm $\|y\|^{\top}_{2,k} = \sqrt{\sum_{l=1}^{k} |y_{\nu(l)}|^2}$ $\|y\|^{\top}_{2,1} = \|y\|_\infty$	$(2,k)$-support norm $\|x\|^{\top\star}_{2,k}$ no analytic expression $\|x\|^{\top\star}_{2,1} = \|x\|_1$
$\|\cdot\|_\infty$	top-$(1,k)$ norm $\|y\|^{\top}_{1,k} = \sum_{l=1}^{k} |y_{\nu(l)}|$ $\|y\|^{\top}_{1,1} = \|y\|_\infty$	(∞,k)-support norm $\|x\|^{\top\star}_{\infty,k} = \max\{\frac{\|x\|_1}{k}, \|x\|_\infty\}$ $\|x\|^{\top\star}_{1,1} = \|x\|_1$

Why do top-k and k-support norms pop up?

Generalized top and support norms

We reformulate sparsity in terms of coordinate subspaces

- For any $K \subset [\![1, n]\!]$, we introduce the (coordinate) subspace

$$\mathcal{R}_K = \left\{ y \in \mathbb{R}^n \,\middle|\, y_j = 0 \,, \ \forall j \notin K \right\} \subset \mathbb{R}^n$$

- The connection with the level sets of the ℓ_0 pseudonorm is

$$\ell_0^{\leq k} = \bigcup_{|K| \leq k} \mathcal{R}_K \,, \ \forall k \in [\![0, n]\!]$$

- We denote by $\pi_K : \mathbb{R}^n \to \mathcal{R}_K$ the orthogonal projection
- For any vector $y \in \mathbb{R}^n$, $\pi_K(y) \in \mathbb{R}^n$ is the vector whose components coincide with those of y, except for those outside of K that vanish

$$y = (*, *, *, *, *, *) \to \pi_{\{2,4,5\}}(y) = (0, *, 0, *, *, 0)$$

We define generalized top-k and k-support dual norms

Definition

For any source norm $\|\cdot\|$ on $\mathbb{R}^n$, for any $k \in [\![1, n]\!]$, we call

- generalized top-k dual norm the norm

$$\|y\|^{\top}_{\star,(k)} = \underbrace{\sup_{|K| \leq k} \| \overbrace{\pi_K(y)}^{\substack{k\text{-sparse}\\ \text{projection}\\ \text{on } \mathcal{R}_K}} \|_\star}_{\text{exploring all } k\text{-sparse projections}} , \quad \forall y \in \mathbb{R}^n$$

- generalized k-support dual norm the dual norm

$$\|\cdot\|^{\top\star}_{\star,(k)} = \left(\|\cdot\|^{\top}_{\star,(k)}\right)_\star$$

Coordinate norms and dual norms *versus* generalized top-k and k-support dual norms

Proposition

If the source norm $\|\cdot\|$ is orthant monotonic,
for all $k \in [\![1, n]\!]$,

k-coordinate norm		k-support dual norm
$\|\cdot\|^{\mathcal{R}}_{(k)}$	$=$	$\|\cdot\|^{\top\star}_{\star,(k)}$

dual k-coordinate norm		top-k dual norm
$\|\cdot\|^{\mathcal{R}}_{(k),\star}$	$=$	$\|\cdot\|^{\top}_{\star,(k)}$

so that, if S is the unit sphere of the source norm $\|\cdot\|$,

$$B^{\mathcal{R}}_{(k)} = \mathrm{co}\left(\ell_0^{\leq k} \cap S\right) = B^{\top\star}_{\star,(k)}$$

Where do we stand?

- We have Capra couplings $¢$ for which the pseudonorm ℓ_0
  - has nonempty Capra-subdifferential

$$\partial_{¢}\ell_0 \neq \emptyset$$

  - hence is Capra-convex (equal to its Capra-biconjugate)

$$\ell_0^{¢¢'} = \ell_0$$

- This looks promising to study sparse optimization problems

But...

Outline of the presentation

Archetypal sparse optimization problems

- For $X \subset \mathbb{R}^n$ a nonempty set,

$$\min_{x \in X} \ell_0(x)$$

is an optimization problem for which any point in X is a local minimizer! Jean-Baptiste Hiriart-Urruty and Hai Le. A variational approach of the rank function. *TOP: An Official Journal of the Spanish Society of Statistics and Operations Research*, 21 (2):207–240, 2013.

- For $k \in [\![1, n]\!]$ and a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$\min_{\ell_0(x) \leq k} f(x)$$

- For $\gamma > 0$ and a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$\min_{x \in \mathbb{R}^n} \left(f(x) + \gamma \ell_0(x) \right)$$

Outline of the presentation

Good news :-)
the Fermat rule holds true for the Capra coupling

$$x^* \in \arg\min f \iff 0 \in \partial_{\text{¢}} f(x^*)$$

Good news :-)
the Fermat rule holds true for the Capra coupling

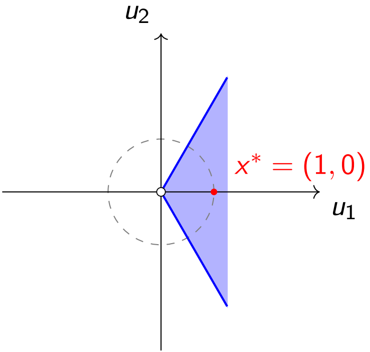
$$x^* \in \arg\min f \iff 0 \in \partial_{¢} f(x^*)$$

As an application, we get that

$$x^* \in \arg\min_{x \in X} \ell_0(x) \iff 0 \in \partial_{¢}(\ell_0 + \iota_X)(x^*)$$

But. . .

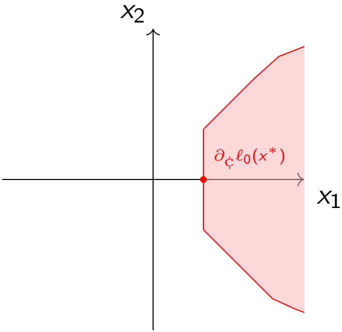
Bad news :-(
when zero is in the subdifferential of the sum...

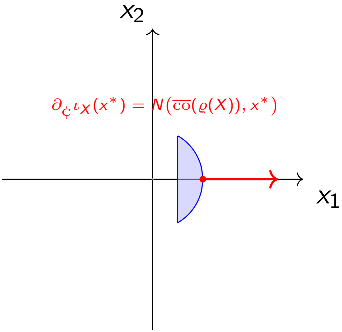


$$x^* \in \arg\min_{X} \ell_0 \iff 0 \in \underbrace{\partial_{\text{Ç}}(\ell_0 + \iota_X)(x^*)}_{\text{subdifferential of the sum}}$$

… but zero is not in the sum of the subdifferentials

$$\underbrace{\partial_{\text{ç}}\ell_0(x^*) + \partial_{\text{ç}}\iota_X(x^*)}_{0\notin} \subsetneq \underbrace{\partial_{\text{ç}}\big(\ell_0 + \iota_X\big)(x^*)}_{0\in}$$





Who is to blame? Capra or ℓ_0?
(with Seta Rakotomandimby)

Primal-dual pair in the Capra-subdifferential of an absolute function

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an absolute function and $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ be an absolute norm, meaning that

$$f(x) = f(|x|) \; , \;\; \forall x \in \mathbb{R}^n$$
$$\|x\| = \||x|\| \; , \;\; \forall x \in \mathbb{R}^n$$

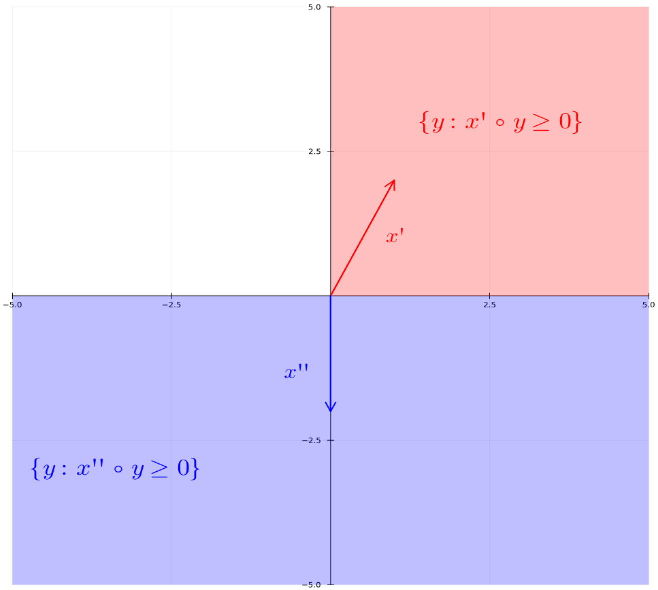
Then, we have that

$$y \in \partial_{\text{¢}} f(x) \implies x \circ y \geq 0$$

where $x \circ y = (x_1 y_1, \ldots, x_n y_n)$

NB: this property also holds true with the classic Rockafellar-Moreau subdifferential in convex analysis

Illustration of $x \circ y \geq 0$

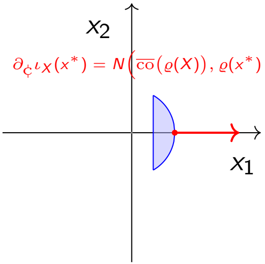


Capra-subdifferential of an indicator function

Proposition

Let $X \subset \mathbb{R}^n$ be a nonempty set. Then, for any $x \in \mathbb{R}^n$

$$\partial_{¢}\iota_X(x) = \begin{cases} \overbrace{N\Big(\overline{\mathrm{co}}\big(\varrho(X)\big), \varrho(x)\Big)}^{\text{normal cone}} & \text{if } x \in X \\ \emptyset & \text{if } x \notin X \end{cases}$$



- The Capra-subdifferential of ι_X at x^* is the normal cone of the convex subset $\overline{\mathrm{co}}\big(\varrho(X)\big) \subset B$ at $\varrho(x^*) \in S$, hence points outward
- The Rockafellar-Moreau subdifferential of ι_X at x^* is the normal cone of X at x^*

$0 \in \partial_{\text{¢}} f(x) + \partial_{\text{¢}} \iota_X(x)$ is much too strong a condition

Under the previous assumptions, we get that

$$0 \in \partial_{\text{¢}} f(x) + \partial_{\text{¢}} \iota_X(x) \implies 0 = \underbrace{\overbrace{y'}^{\partial_{\text{¢}} f(x)}}_{x \circ y' \geq 0} + \overbrace{y''}^{\partial_{\text{¢}} \iota_X(x)}$$

$$\implies \underbrace{y'' \in N\Big(\overline{\mathrm{co}}\big(\varrho(X)\big), \varrho(x)\Big)}_{y'' \text{ is outward}} \text{ and } \underbrace{x \circ y'' \leq 0}_{y'' \text{ is inward}}$$

- In general, this will give $y'' = 0$, that is, $0 \in \partial_{\text{¢}} f(x)$
- Thus, necessarily, $x \in X$ must be a global minimum of f over all $\mathbb{R}^n$, which is much too strong...

Where do we stand?

- We had good hope to handle sparse optimization problems with the Capra coupling that makes the pseudonorm ℓ_0 Capra convex
- But, in a simple sparse optimization problem, it is not true that the subdifferential of the sum is equal to the sum of the subdifferentials
- And not having practical qualification conditions is an obstacle to many numerical methods

Outline of the presentation

Crash course on generalized convexity [5 min]

Capra conjugacies [20 min]
- Euclidean Capra conjugacy
- Capra conjugacies

Towards Capra-algorithms in sparse optimization? [15 min]
- Good and bad news about the Fermat rule (with Adrien Le Franc and Seta Rakotomandimby)
- Capra-cuts method (with Seta Rakotomandimby)
- The geometry of sparsity-inducing unit balls (with Antoine Deza and Lionel Pournin)

Conclusion [1 min]

Additional material

Minimization problems from compressed sensing

- Goal: recovery of a sparse signal $x \in \mathbb{R}^n$ from a measurement $b \in \mathbb{R}^m \setminus \{0\}$, where $m < n$
- Measurements are modeled by $A \in \mathbb{R}^{m \times n}$ such that

$$Ax = b$$

- Minimization approach for the recovery

$$\min_{\substack{x \in \mathbb{R}^n \\ Ax = b}} \ell_0(x)$$

Using a Capra-polyhedral approximation for ℓ_0

- For a suitable (infinite) subset $Y \subset \bigcup_{x'} \partial_{\cent} \ell_0(x')$ of Capra-subgradients of ℓ_0, we have that

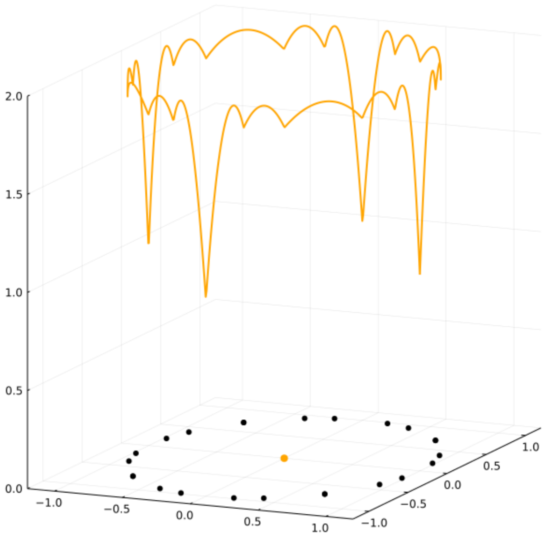
$$\ell_0(x) = \sup_{y \in Y} \langle \varrho(x), y \rangle - \ell_0^{\cent}(y) \; , \;\; \forall x \in \mathbb{R}^n$$

- Idea: using a Capra-"polyhedral" approximation f of ℓ_0 in the minimization problem

$$f(x) = \max_{y \in \tilde{Y}} \langle \varrho(x), y \rangle - \ell_0^{\cent}(y)$$

where $\tilde{Y} \subset Y$ and $\tilde{Y}$ finite $\rightsquigarrow$ cutting plane-like method

Illustration of a Capra-polyhedral approximation for ℓ_0



Abstract cutting plane method

[Rubinov, 2000, §9.2.3]

Definition

Let $\mathcal{W}$ be a set, $H \subset \overline{\mathbb{R}}^{\mathcal{W}}$ be a set of elementary functions, and $f : \mathcal{W} \to \overline{\mathbb{R}}$ be a H-convex function

1. Set $k := 0$. Choose an arbitrary initial point $w_0 \in \mathcal{W}$
2. Find an abstract subgradient $h_k \in \partial^H f(w_k)$
   Let $f_{-1} = -\infty$ and set

$$f_k = \max\{f_{k-1}, \underbrace{h_k}_{\substack{\text{new cut} \\ \text{in } \partial^H f(w_k)}} \}$$

3. Find an optimal solution $\widehat{w} \in \arg\min_{w \in \mathcal{W}} f_k(w)$
4. Set $k := k + 1$, $w_k = \widehat{w}$
   Repeat from Step 2 until a stop condition is satisfied

Still problems with ℓ_0

- The pseudonorm ℓ_0 is abstract Capra-convex
- ... but ℓ_0 is not continuous and its abstract Capra-subgradients

$$\Big\{x \mapsto \langle \varrho(x), y\rangle - \ell_0^{\cent}(y)\Big\}_{y \in \cup_{x'} \partial_{\cent} \ell_0(x')}$$

  are not uniformly continuous
- So the pseudonorm ℓ_0 does not satisfy any assumptions of established theoretical convergence results [Pallaschke and Rolewicz, 1997, Theorem 9.1.1]
- Also, numerically, we observe no convergence for simple examples in dimension $n = 3$

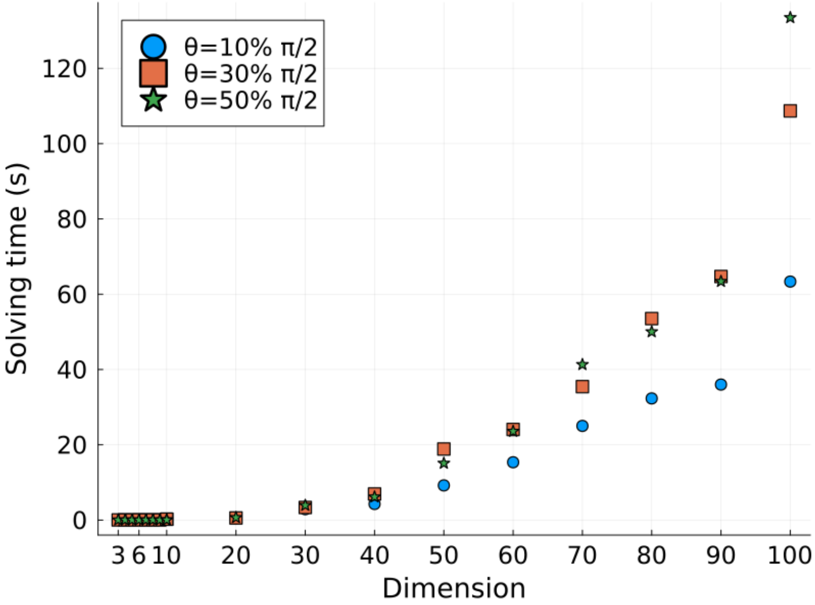
However for ℓ_1/ℓ_2 !

- ℓ_1/ℓ_2 is a surrogate function for ℓ_0 in compressed sensing
- ℓ_1/ℓ_2 is Capra-convex
  (and an absolute function so Fermat rule is no help)
- and ℓ_1/ℓ_2 is continuous
  and the following Capra-abstract subgradients

$$\left\{x \mapsto \langle \varrho(x), y \rangle - \ell_0^{¢}(y)\right\}_{y \in \{-1,0,1\}^n}$$

  are uniformly continuous
- Most assumptions of theoretical convergence results
  [Pallaschke and Rolewicz, 1997, Theorem 9.1.1] are satisfied

Solving time for the ratio of two norms



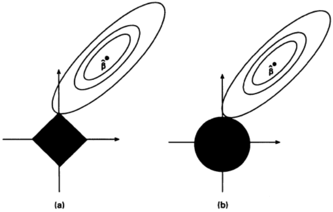
Work needs to be done for theoretical guarantees

- Convergence results
  [Pallaschke and Rolewicz, 1997, Theorem 9.1.1]
  [Rubinov, 2000, Proposition 9.2]
- But the assumptions do not fit our case:
  need to be adapted

Outline of the presentation

The intuition behind lasso

$$\min_{x \in \mathbb{R}^n} \left(f(x) + \gamma \|x\|_1\right)$$



$$\min_{x \in \mathbb{R}^n} \left(f(x) + \gamma \|x\|_2\right)$$

Comments of [Tibshirani, 1996, Figure 2]

> *"The lasso solution is the first place that the contours touch the square, and this will sometimes occur at a corner, corresponding to a zero coefficient. The picture for ridge regression is shown in Fig. 2(b): there are no corners for the contours to hit and hence zero solutions will rarely result."*

Geometric (alignment) expression of optimality condition

- We consider an optimal solution x^* of

$$\min_{x \in \mathbb{R}^n} \left(f(x) + \gamma \|x\| \right)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function, $\gamma > 0$ and $\|\cdot\|$ is a norm with unit ball B

- By the Fermat rule, when $x^* \neq 0$,

$$0 \in \nabla f(x^*) + \gamma \partial \|\cdot\|(x^*) \iff \frac{x^*}{\|x^*\|} \in \underbrace{F_\perp(B, -\nabla f(x^*))}_{\substack{\text{face of the unit ball } B \\ \text{exposed by } -\nabla f(x^*)}}$$

- The norm $\|\cdot\|$ may be qualified as sparsity-inducing if information about the support of x^* and the exposed faces of the unit ball B can be recovered from one another [Fan, Jeong, Sun, and Friedlander, 2020]

Design of sparsity inducing norms/balls
for k-sparse vectors with given k

Courtesy of Basile and Lionel Pournin

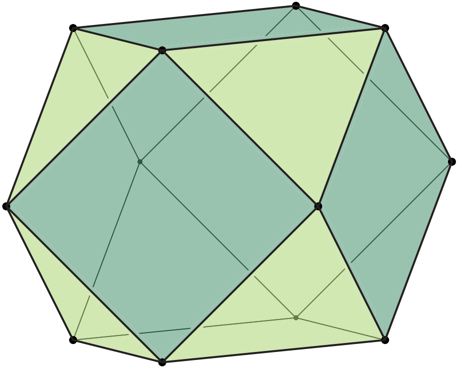


Figure: Unit ball $\overline{\mathrm{co}}\left(\ell_0^{\leq 2} \cap S_1\right)$ when $n = 3$

How to design a sparsity inducing unit ball?

For $k \in [\![1, d]\!]$

- consider the k-sparse vectors in $\ell_0^{\leq k}$
- as they do not form a compact set,
  intersect $\ell_0^{\leq k}$ with a unit sphere S (or a unit ball B)
- form the convex hull and obtain a new

$$\text{unit ball} \quad B_{(k)}^{\mathcal{R}} = \mathrm{co}\big(\ell_0^{\leq k} \cap S\big)$$

whose extreme points belong to $\ell_0^{\leq k} \cap S \subset \ell_0^{\leq k}$,
hence are k-sparse vectors

Does this procedure induces sparsity? If yes, in what sense?

Support identification of a k-sparse vector in the exposed face of a generalized k-support dual norm (1/2)

Theorem

Let $k \in [\![1, n]\!]$. If the source norm $\|\cdot\|$ is orthant-monotonic, then

$$B^{\mathcal{R}}_{(k)} = \mathrm{co}\big(\ell_0^{\leq k} \cap S\big) = B^{\top\star}_{\star,(k)}$$

and, for any nonzero dual vector $y \in \mathcal{Y} \setminus \{0\}$,
the two following statements are equivalent

(i) $x \in \ell_0^{\leq k} \cap F_{\perp}(B^{\top\star}_{\star,(k)}, y)$

(ii) there exists $K^* \in \arg\max_{|K| \leq k} \|\pi_K(y)\|_\star$
such that $x \in \pi_{K^*}\big(B \cap F_{\perp}(B, \pi_{K^*}(y))\big) \subset \mathcal{R}_{K^*}$

As a consequence, we get that

$$\mathrm{supp}(x) \subset K^*$$

Support identification of a k-sparse vector in the exposed face of a generalized k-support dual norm (2/2)

$$\text{Consider a vector } x \in \overbrace{\ell_0^{\leq k}}^{k\text{-sparse}} \cap \overbrace{F_{\perp}(B_{\star,(k)}^{\top\star}, y)}^{\text{exposed face}}$$

1. From $x \in \ell_0^{\leq k}$, we only know that

$$\text{there exists } K \subset [\![1, n]\!] \text{ with } |K| \leq k \text{ such that}$$

$$\mathrm{supp}(x) \subset K$$

2. From $x \in F_{\perp}(B_{\star,(k)}^{\top\star}, y)$, we add information and obtain that

$$\text{there exists } K^* \in \arg\max_{|K| \leq k} \|\pi_K(y)\|_{\star} \text{ such that}$$

$$\mathrm{supp}(x) \subset K^*$$

Support identification

Corollary

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function, $\gamma > 0$ and $\|\cdot\|$ be a norm

Then, an optimal solution x^* of

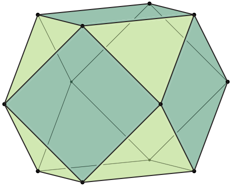
$$\min_{x \in \mathbb{R}^n} \left(f(x) + \gamma \|x\|_{\star,(k)}^{\top\star} \right)$$

has support

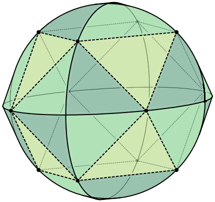
$$\mathrm{supp}(x^*) \subset \bigcup_{\substack{K^* \in \arg\max_{|K| \leq k} \\ \|\pi_K(-\nabla f(x^*))\|_\star}} K^*$$

Especially interesting when the $\arg\max_{|K|\leq k}$ is unique, because then the optimal solution x^* is k-sparse

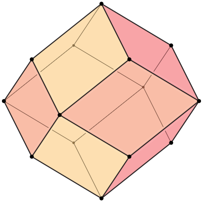
Geometry of sparsity inducing balls



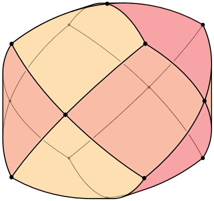
(a) Unit ball $B^{\top\star}_{\infty,2}$ when $n = 3$



(b) Unit ball $B^{\top\star}_{2,2}$ when $n = 3$



(c) Unit ball $B^{\top}_{1,2}$ when $n = 3$



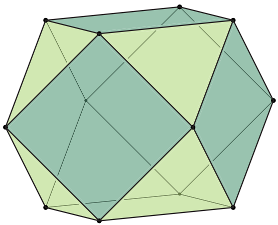
(d) Unit ball $B^{\top}_{2,2}$ when $n = 3$

Figure: Four top (6c and 6d) and support (7a and 7b) unit balls, either obtained from the ℓ_1 source norm (7a and 6c) or from the ℓ_2 source norm (7b and 6d)

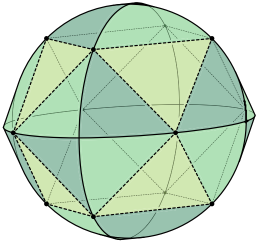
Additional geometric properties

Proposition

For any $k \in [\![1, n]\!]$, all the proper faces of $B_{2,k}^{\top\star}$ are hypersimplices, and the normal fan of $B_{2,k}^{\top\star}$ refines the normal fan of $B_{\infty,k}^{\top\star}$



(a) Unit ball $B_{\infty,2}^{\top\star}$ when $n = 3$



(b) Unit ball $B_{2,2}^{\top\star}$ when $n = 3$

Figure: Two support norm unit balls, either obtained from the ℓ_1 source norm (7a) or from the ℓ_2 source norm (7b)

Outline of the presentation

Crash course on generalized convexity [5 min]

Capra conjugacies [20 min]

Towards Capra-algorithms in sparse optimization? [15 min]

Conclusion [1 min]

Additional material

- So-called generalized convexity and Fenchel-Moreau conjugacy are extensions of duality beyond convex analysis
- The Capra-coupling ¢ and induced Capra-conjugacy seem promising to handle sparsity in optimization as the pseudonorm ℓ_0 satisfies

$$\partial_{¢}\ell_0 \neq \emptyset \text{ hence } \ell_0^{¢¢'} = \ell_0$$

  but we have problems handling sums like $\ell_0 + \iota_X$:-(
- So, our working program is now to study
  - the ℓ_0-cup function $\mathcal{L}_0 = \ell_0^{¢\star'}$
  - the geometry of unit balls of norms related to the Capra-coupling ¢ and to the pseudonorm ℓ_0
  - lower bound convex programs

Andreas Argyriou, Rina Foygel, and Nathan Srebro. Sparse prediction with the k-support norm. In *Proceedings of the 25th International Conference on Neural Information Processing Systems - Volume 1*, NIPS'12, pages 1457–1465, USA, 2012. Curran Associates Inc.

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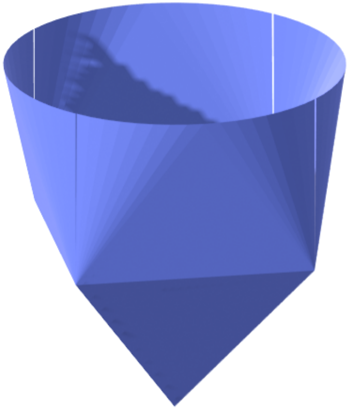
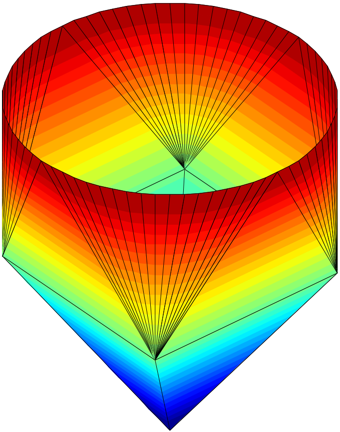
Thank you :-)





Outline of the presentation

The ℓ_0 pseudonorm is (almost) a convex-composite function

- [Chancelier and De Lara, 2021]

$$\boxed{\ell_0(x) = \underbrace{\mathcal{L}_0}_{\text{proper convex lsc}}\Big(\frac{x}{\|x\|}\Big)} \,, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

- As a consequence,
  if $C \subset \mathbb{R}^n$ is a closed convex set with $0 \notin C$,

$$\min_{x \in C} \ell_0(x) = \min_{x \in \mathbb{R}^n} \Big\{ \mathcal{L}_0\big(\frac{x}{\|x\|}\big) + \iota_C(x) \Big\}$$

  or if $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper convex lsc function,

$$\min_{x \in \mathbb{R}^n, \ell_0(x) \leq k} f(x) = \min_{x \in \mathbb{R}^n} \Big\{ f(x) + \underbrace{\iota_{B^{\top\star}_{(k)}}}_{\substack{(2,k)\text{-support norm} \\ \text{unit ball}}}\big(\frac{x}{\|x\|}\big) \Big\}$$

Graded sequence of norms

We define graded sequence of norms

A graded sequence of norms detects the number of nonzero components of a vector in $\mathbb{R}^n$

when the sequence becomes stationary

Definition

We say that a sequence $\{\|\cdot\|_k\}_{k \in [\![1,n]\!]}$ of norms is (increasingly) graded with respect to the ℓ_0 pseudonorm if, for any $y \in \mathbb{R}^n$ and $l \in [\![1,n]\!]$, we have

$$\ell_0(y) = \ell \iff \|y\|_1 \leq \cdots \leq \|y\|_{\ell-1} < \|y\|_\ell = \cdots = \|y\|_n$$

or, equivalently, $k \in [\![1,n]\!] \mapsto \|y\|_k$ is nondecreasing and

$$\ell_0(y) \leq \ell \iff \|y\|_\ell = \|y\|_n$$

Graded sequences are suitable for so-called "difference of convex" (DC) optimization methods to tackle sparse $\ell_0(y) \leq l$ constraints

Orthant-strictly monotonic dual norms produce graded sequences of norms

Proposition

If the dual norm $\|\cdot\|_\star$ of the source norm $\|\cdot\|$ is orthant-strictly monotonic, then the sequence

$$\underbrace{\left\{\|\cdot\|^{\top}_{\star,(k)}\right\}_{k\in[\![1,n]\!]}}_{\text{generalized top-}k\text{ dual norm}} = \underbrace{\left\{\|\cdot\|^{\mathcal{R}}_{(k),\star}\right\}_{k\in[\![1,n]\!]}}_{\text{dual-}k\text{ coordinate norm}}$$

is graded with respect to the ℓ_0 pseudonorm

Thus, we can produce families of graded sequences of norms suitable for "difference of convex" (DC) optimization methods to tackle sparse constraints

Fenchel *versus* Capra conjugacies for ℓ_0

[Chancelier and De Lara, 2022a], [Chancelier and De Lara, 2022c]
If both the source norm and its dual are orthant-strictly monotonic

Fenchel conjugacy	Capra conjugacy
$\iota^{\star}_{\ell_0^{\leq k}} = \iota_{\{0\}}, \; k \neq 0$	$\iota^{\cent}_{\ell_0^{\leq k}} = \|\cdot\|^{\mathcal{R}}_{(k),\star} = \|\cdot\|^{\top}_{\star,(k)}$
$\ell_0^{\star} = \iota_{\{0\}}$	$\ell_0^{\cent} = \sup_{\ell \in [\![0,n]\!]} \Big[\|\cdot\|^{\mathcal{R}}_{(\ell),\star} - \ell \Big]$ $= \sup_{\ell \in [\![0,n]\!]} \Big[\|\cdot\|^{\top\star}_{\star,(\ell)} - \ell \Big]$
$\iota^{\star\star'}_{\ell_0^{\leq k}} = 0$	$\iota^{\cent\cent'}_{\ell_0^{\leq k}} = \iota_{\ell_0^{\leq k}}$
$\ell_0^{\star\star'} = 0$	$\ell_0^{\cent\cent'} = \ell_0$

Lower bounds for the pseudonorm ℓ_0

Best ratio of norms [Chancelier and De Lara, 2022a]

- For any $\varphi : [\![0, d]\!] \to [0, +\infty[$, such that $\varphi(j) > \varphi(0) = 0$ for all $j \in [\![1, d]\!]$, there exists a norm $\|\cdot\|^{\mathcal{R}}_{(\varphi)}$ such that

$$\frac{\|x\|^{\mathcal{R}}_{(\varphi)}}{\|x\|} \leq \varphi\big(\ell_0(x)\big) \;, \;\; \forall x \in \mathbb{R}^n \backslash \{0\}$$

where $\|\cdot\|^{\mathcal{R}}_{(\varphi)}$ is characterized by its dual norm

$$\|y\|^{\mathcal{R}}_{(\varphi),\star} = \sup_{j \in [\![1,d]\!]} \frac{\|y\|^{\mathcal{R}}_{(j),\star}}{\varphi(j)} \;, \;\; \forall y \in \mathbb{R}^n$$

- For $\|\cdot\| = \|\cdot\|_p$ with $p > 1$, and $\varphi_\alpha(j) = j^{1/\alpha}$ for $\alpha > 0$,

$$\left(\frac{\big(\|x\|_p\big)^{\mathcal{R}}_{(\varphi_\alpha)}}{\|x\|_p}\right)^{\alpha} \leq \ell_0(x) \;, \;\; \forall x \in \mathbb{R}^n \backslash \{0\}$$

$$\left(\frac{\|x\|_1}{\|x\|_p}\right)^{p} \leq \ell_0(x) \;, \;\; \forall x \in \mathbb{R}^n \backslash \{0\}$$

Lower bound convex programs for exact sparse optimization

Concave dual problem for exact sparse optimization

From
$$\sup_{y \in \mathcal{Y}} \Big(\big(-f^{¢}(y)\big) \dotplus \big(-\iota_X^{-¢}(y)\big) \Big) \leq \inf_{x \in \mathcal{X}} \Big(f(x) \dotplus \iota_X(x) \Big)$$

we deduce that

$$\sup_{y \in \mathbb{R}^n} \Big(-\big(\inf \left[f \mid \varrho \right]\big)^{\star}(y) \dotplus \big(-\underbrace{\iota_{\ell_0^{\leq k}}^{-¢}(y)}_{\|y\|_{2,k}^{\top}}\big) \Big) \leq \inf_{\ell_0(x) \leq k} f(x)$$

Proposition

For any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we have the following lower bound

$$\sup_{y \in \mathbb{R}^n} \overbrace{\Big(-\big(\inf \left[f \mid \varrho \right]\big)^{\star}(y) - \|y\|_{2,k}^{\top} \Big)}^{\text{concave usc function}} \leq \inf_{\ell_0(x) \leq k} f(x)$$
$$= \inf_{\ell_0(x) \leq k} \inf \left[f \mid \varrho \right](x)$$

Convex primal problem for exact sparse optimization

Proposition

Under a mild technical assumption (“à la” Fenchel-Rockafellar), namely if $\big(\inf\big[f \mid \varrho\big]\big)^{\star}$ is a proper function, we have the following lower bound

$$\min_{\|x\|_{2,k}^{\top\star}\leq 1} \big(\inf\big[f \mid \varrho\big]\big)^{\star\star'}(x) \leq \inf_{\ell_0(x)\leq k} f(x) = \inf_{\ell_0(x)\leq k} \inf\big[f \mid \varrho\big](x)$$

The primal problem is the minimization of a closed convex function on the unit ball of the $(2,k)$-support norm $\|\cdot\|_{2,k}^{\top\star}$ (introduced in [Argyriou, Foygel, and Srebro, 2012])

Duality

Perturbation scheme

- Functions $k : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\varphi : [\![0,n]\!] \to \overline{\mathbb{R}}$ nondecreasing (ex: identity, $\iota_{[\![0,k]\!]}$) and original minimization problem

$$\inf_{w\in\mathbb{R}^n} \Big\{ k(w) \dotplus \varphi\big(\ell_0(w)\big) \Big\} = \inf_{w\in\mathbb{R}^n} \Big\{ k(w) \dotplus \big(\varphi \circ \ell_0\big)^{¢\star'}\big(\varrho(w)\big) \Big\}$$

because $\varphi \circ \ell_0 = \big(\varphi \circ \ell_0\big)^{¢¢'} = \big(\varphi \circ \ell_0\big)^{¢\star'} \circ \varrho$ [Chancelier and De Lara, 2022c]

- Rockafellian (perturbation scheme) $R : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$

$$R(w,x) = k(w) \dotplus \big(\varphi \circ \ell_0\big)^{¢\star'}\big(\varrho(w) + x\big) \; , \quad \forall (w,x) \in \mathbb{R}^n \times \mathbb{R}^n$$

- Value function

$$\varphi(x) = \inf_{w\in\mathbb{R}^n} \Big\{ k(w) \dotplus \big(\varphi \circ \ell_0\big)^{¢\star'}\big(\varrho(w) + x\big) \Big\} \; , \quad \forall x \in \mathbb{R}^n$$

Lagrangian and dual problem

- Fenchel coupling $\mathbb{R}^n \overset{\langle\cdot|\cdot\rangle}{\leftrightarrow} \mathbb{R}^n$, and Lagrangian $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given, for any $(w, y) \in \mathbb{R}^n \times \mathbb{R}^n$, by

$$\mathcal{L}(w,y) = \inf_{x\in\mathbb{R}^n} \Big\{ k(w) \dot{+} (\varphi \circ \ell_0)^{¢\star'}(\varrho(w) + x) - \langle x,\ y\rangle \Big\}$$
$$= k(w) \dot{+} \big(\langle \varrho(w),\ y\rangle - (\varphi \circ \ell_0)^{¢}(y)\big)$$

- Dual maximization problem

$$\varphi^{\star\star'}(0) = \sup_{y\in\mathbb{R}^n} \inf_{w\in\mathbb{R}^n} \mathcal{L}(w,y) = \sup_{y\in\mathbb{R}^n} \Big\{ \big(-k^{-¢}(y)\big) \underset{\cdot}{+} \big(-(\varphi \circ \ell_0)^{¢}(y)\big) \Big\}$$

- Original minimization problem (case "$\dot{+} = +$" when k proper)

$$\varphi(0) = \inf_{w\in\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} \mathcal{L}(w,y) = \inf_{w\in\mathbb{R}^n} \Big\{ k(w) \dot{+} \varphi(\ell_0(w)) \Big\}$$

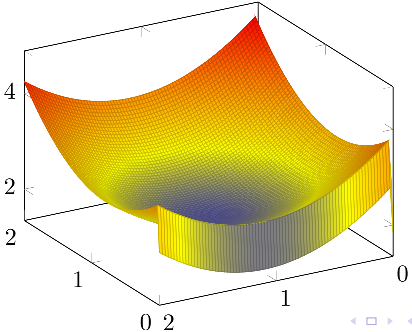
Numerics

A toy example

$$\min_{w \in \mathbb{R}^2} \overbrace{\left((w_1 - b_1)^2 + (w_2 - b_2)^2\right)}^{k(w)} + \ell_0(w)$$

$$\text{with} \quad b = (0.8, 1.1)$$

We have that $\{(0, b_2)\} = \{(0, 1.1)\} = \arg\min_{w \in \mathbb{R}^2} \left\{ k(w) + \ell_0(w) \right\}$



The toy example as a min-max problem

As $\ell_0(w) = \max_{y \in \mathbb{R}^2} \left\{ ¢(w, y) - \ell_0^{¢}(y) \right\}$, we obtain that

$$\min_{w \in \mathbb{R}^2} \left\{ k(w) + \ell_0(w) \right\} = \min_{w \in \mathbb{R}^2} \max_{y \in \mathbb{R}^2} \left\{ k(w) + ¢(w, y) - \ell_0^{¢}(y) \right\}$$

with

$$\ell_0^{¢}(y) = \sup_{k \in [\![1, n]\!]} \left[\|y\|_{2,k}^{\top} - k \right]_+$$

Generalized primal-dual proximal splitting

GPDPS Algorithm Christian Clason, Stanislav Mazurenko, and Tuomo Valkonen. Primal–dual proximal splitting and generalized conjugation in non-smooth non-convex optimization. *Applied Mathematics and Optimization*, 84(2):1239–1284, apr 2020.

Given a starting point (w_0, y_0) and step lengths $\tau_i, \omega_i, \sigma_i > 0$, iterate

$$
\begin{aligned}
w^{(i+1)} &:= \operatorname{prox}_{\tau_i k}\big(w^{(i)} - ¢_w(w^{(i)}, y^{(i)})\big)\\
\overline{w}^{(i+1)} &:= w^{(i+1)} + \omega_i(w^{(i+1)} - w^{(i)})\\
y^{(i+1)} &:= \operatorname{prox}_{\sigma_i \ell_0^{¢}}\big(y^{(i)} + \sigma_i ¢_y(\overline{w}^{(i+1)}, y^{(i)})\big)
\end{aligned}
$$

The prox of k is analytically computed (quadratic function), whereas the prox of $\ell_0^{¢}$ is numerically computed with the optimization algorithm `newuoa` by M.J.D. Powell

GPDPS convergence, varying the starting point