Fenchel-Moreau Conjugates of Inf-Transforms and Application to the Stochastic Bellman Equation

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Examples of inf-transforms in optimization

- Perturbation of constraints $\mathcal{Y} \rightsquigarrow \mathcal{Y}(x)$ gives

$$\inf_{y \in \mathcal{Y}} g(y) \rightsquigarrow \inf_{y \in \mathcal{Y}(x)} g(y)$$

and value function

$$f(x) = \inf_{y} \Big(\underbrace{\delta_{\mathcal{Y}(x)}(y)}_{\mathcal{K}(x,y)} \dot{+} g(y) \Big)$$

- Two-stage linear stochastic programming

$$f_s(x) = \inf_{y} \Big(\langle c_s \,, x \rangle \dot{+} \langle p_s \,, y \rangle \dot{+} \delta_{\{y \geq 0 \,,\ A_s x + b_s + y \geq 0\}} \Big)$$

Examples of inf-transforms in optimization (continued)

- Product from the left by a (linear) operator L

$$(Lg)(x) = \inf_y \left(\underbrace{\delta_{Ly=x}}_{\mathcal{K}(x,y)} \dot{+} g(y) \right)$$

- Moreau-Yosida approximation of g

$$f(x) = \inf_y \left(\underbrace{\frac{1}{\alpha}\|x-y\|^2}_{\mathcal{K}(x,y)} \dot{+} g(y) \right)$$

- Inf-convolution of g_1 and g_2

$$f(x) = \inf_y \left(\underbrace{g_1(x-y)}_{\mathcal{K}(x,y)} \dot{+} g_2(y) \right)$$

Examples of inf-transforms in optimization (continued)

- Lasso problem

$$f(x) = \inf_y \left(\frac{1}{2}\|x - Ay\|_2^2 \dot{+} \underbrace{\lambda\|y\|_1}_{\text{sparsity, regularization}} \right)$$

- Supervised learning and sparsity

$$f(x) = \inf_y \left(\underbrace{l(x, Ay)}_{\text{loss function}} \dot{+} \underbrace{\lambda\|y\|_0}_{\text{l0 pseudo-norm}} \right)$$

- Bregman "distance"

$$f(x) = \inf_y \left(\underbrace{H(x) - H(y) - \langle \nabla H(x), x - y \rangle}_{\text{Bregman "distance" } \mathcal{K}(x,y)} \dot{+} g(y) \right)$$

Examples of inf-transforms in optimization (continued)

- Upper envelope representations

$$V(\tau, \xi) = \inf_{\xi'} \Big(E(\tau, \xi, \xi') + g(\xi') \Big)$$

and Hamilton-Jacobi equation

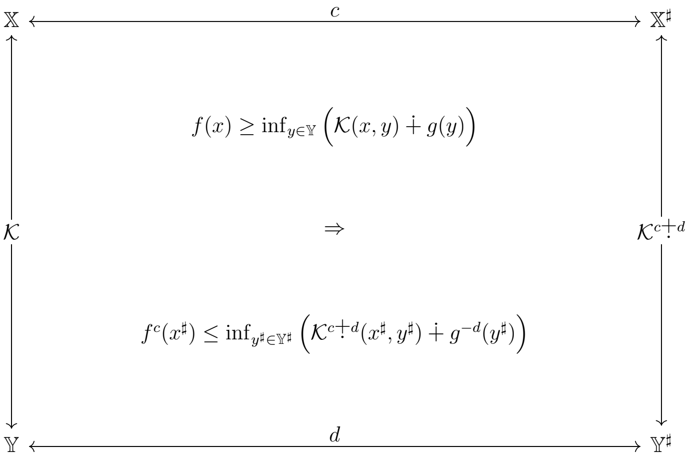
Question: what about their Fenchel conjugate

$$f^{\star}(x^{\sharp}) = \sup_{x \in \mathbb{X}} \Big(\Big\langle x \, , x^{\sharp} \Big\rangle \dot{+} \big(-f(x) \big) \Big) \ ?$$

(hence what about dual problems?)

Main result

Two couplings c and d, and an inf-operation with kernel $\mathcal{K}$



Outline of the presentation

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Basic spaces

- We introduce a first couple of spaces in bilinear duality

$$\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}} \text{ and } \mathbb{X}^{\sharp} = \mathbb{R}^{n_{\mathbb{X}}}$$

- and a second couple of spaces in bilinear duality

$$\mathbb{Y} = \mathbb{L}^{p}\big((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^{n_{\mathbb{X}}}\big) \text{ and } \mathbb{Y}^{\sharp} = \mathbb{L}^{q}\big((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^{n_{\mathbb{X}}}\big)$$

  p and q-integrable random variables with values in $\mathbb{R}^{n_{\mathbb{X}}}$, where

  - $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space
  - $1 \leq p < +\infty$ and q are such that $1/p + 1/q = 1$
- Random variables, elements of $\mathbb{Y} = \mathbb{L}^{p}\big((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^{n_{\mathbb{X}}}\big)$ will be denoted by bold letters like $\mathbf{X}$ and elements of $\mathbb{Y}^{\sharp} = \mathbb{L}^{q}\big((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^{n_{\mathbb{X}}}\big)$ by $\mathbf{X}^{\sharp}$
- All Fenchel conjugates will be denoted by

$$g^{\star} , \quad g^{(-\star)}$$

Ingredients for a stochastic optimal control problem

- Let time $t = 0, 1, \dots, T$ be discrete, with $T \in \mathbb{N}^*$
- Consider a stochastic optimal control problem with
  - state space $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$
  - control space $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$
  - white noise process $\{\mathbf{W}_t\}_{t=1,\dots,T}$
    taking values in uncertainty space $\mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$
    and defined over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- For each time $t = 0, 1, \dots, T-1$, we have
  - dynamics $F_t : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{X}$
  - instantaneous costs $L_t : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to [0, +\infty]$
  - final cost $K : \mathbb{X} \to [0, +\infty]$

We introduce the Bellman functions

- We define Bellman functions by,
  for all $x \in \mathbb{X}$ and $t = T-1, \ldots, 0$,

$$V_T(x) = K(x)$$

$$V_t(x) = \inf_{\mathbf{X},\mathbf{U}} \mathbb{E}\Big[\sum_{s=t}^{T-1} L_s(\mathbf{X}_s, \mathbf{U}_s, \mathbf{W}_{s+1}) \dot{+} K(\mathbf{X}_T) \Big]$$

  where $\mathbf{X}_t = x \in \mathbb{X}$, $\mathbf{X}_{s+1} = F_s(\mathbf{X}_s, \mathbf{U}_s, \mathbf{W}_{s+1})$ and $\sigma(\mathbf{U}_s) \subset \sigma(\mathbf{X}_s)$, for $s = t, \ldots, T-1$

- If the Bellman functions are measurable, they satisfy the backward Bellman inequation, for $t = T-1, \ldots, 0$

$$\overbrace{V_t(x)}^{\text{Bellman function}} \geq \inf_{u \in \mathbb{U}} \mathbb{E}\Big[\underbrace{L_t(x, u, \mathbf{W}_{t+1})}_{\text{instantaneous cost}} \dot{+} V_{t+1}\big(\underbrace{F_t(x, u, \mathbf{W}_{t+1})}_{\text{dynamics}} \big) \Big]$$

Fenchel conjugates of the Bellman functions

Theorem

The Bellman functions satisfy the backward inequalities

$$V_t(x) \geq \inf_{\mathbf{X}} \left(\inf_{u \in \mathbb{U}} \Big(\big(-\mathcal{H}(x,u,\cdot) \big)^{(-\star)}(\mathbf{X}) \Big) \dotplus \mathbb{E}\big[V_{t+1}(\mathbf{X}) \big] \right)$$

for $t = T-1, \ldots, 0$, where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(x,u,\mathbf{X}^\sharp) = \mathbb{E}\Big[L_t(x,u,\mathbf{W}_{t+1}) \dotplus \Big\langle F_t(x,u,\mathbf{W}_{t+1}) \, , \mathbf{X}^\sharp \Big\rangle \Big]$$

Moreover, letting $\{V_t^\star\}_{t=0,1,\ldots,T}$ be the Fenchel conjugates of the Bellman functions, we have, for all $x^\sharp \in \mathbb{X}^\sharp$ and $t = T-1, \ldots, 0$,

$$V_t^\star(x^\sharp) \leq \inf_{\mathbf{X}^\sharp} \left(\sup_{u \in \mathbb{U}} \Big(\mathcal{H}(\cdot,u,\mathbf{X}^\sharp)^\star(x^\sharp) \Big) \dotplus \mathbb{E}\big[V_{t+1}^\star(\mathbf{X}^\sharp) \big] \right)$$

Outline of the presentation

Obtaining upper and lower estimates in approximations of Bellman functions (I)

- Suppose that the Bellman functions $\{V_t\}_{t=0,1,\ldots,T}$ satisfy the Bellman equation and are convex l.s.c. and proper, that is,

$$V_t = V_t^{\star\star} \ , \ \ \forall t = 0, 1, \ldots, T$$

- This is the case in Stochastic Dual Dynamic Programming (SDDP), when
  - the dynamics F_t are jointly linear in state and control
  - the instantaneous costs L_t are jointly convex in state and control
  - the final cost K is convex
  - together with technical assumptions

Obtaining upper and lower estimates in approximations of Bellman functions (II)

- The Fenchel conjugates $\{V_t^\star\}_{t=0,1,\ldots,T}$ of the Bellman functions are convex l.s.c. and proper, by construction
- Suppose that they satisfy a "Bellman like" equation

$$V_t^\star(x^\sharp) = \inf_{\mathbf{X}^\sharp} \left(\sup_{u \in \mathbb{U}} \Big(\mathcal{H}(\cdot, u, \mathbf{X}^\sharp)^\star(x^\sharp) \Big) \dotplus \mathbb{E}\big[V_{t+1}^\star(\mathbf{X}^\sharp) \big] \right)$$

for $t = T-1, \ldots, 0$

Obtaining upper and lower estimates in approximations of Bellman functions (III)

- With the Bellman operators deduced from the Bellman equation and "Bellman like" equation, one can produce (by an adequate algorithm like the SDDP algorithm) lower bound functions

$$\forall k \in \mathbb{N} \qquad \begin{cases} \underline{V}_{t,(k)} \leq \underline{V}_{t,(k+1)} \leq V_t \\ \underline{\widetilde{V}}_{t,(k)} \leq \underline{\widetilde{V}}_{t,(k+1)} \leq V_t^\star \end{cases}$$

  that are piecewise affine
- Since the Bellman functions $\{V_t\}_{t=0,1,\dots,T}$ are convex l.s.c. and proper, we deduce that

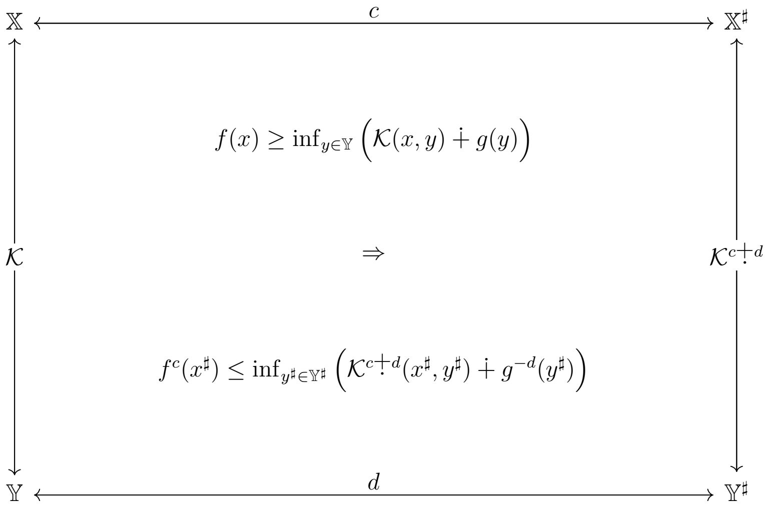
$$\underline{V}_{t,(k)} \leq \underline{V}_{t,(k+1)} \leq V_t = V_t^{\star\star} \leq \underline{\widetilde{V}}_{t,(k+1)}^\star \leq \underline{\widetilde{V}}_{t,(k)}^\star$$

- Thus, we can control the evolution of the SDDP algorithm

Outline of the presentation

Main result

Two couplings c and d, and an inf-operation with kernel $\mathcal{K}$



The Fenchel conjugacy

Definition

Two vector spaces $\mathbb{X}$ and $\mathbb{X}^\sharp$, paired by a bilinear product $\langle\ ,\ \rangle$, (in the sense of convex analysis),
give rise to the classic Fenchel conjugacy

$$f^\star(x^\sharp) = \sup_{x \in \mathbb{X}} \Big(\big\langle x \,, x^\sharp \big\rangle \underset{\cdot}{+} \big(- f(x) \big) \Big) \,, \quad \forall x^\sharp \in \mathbb{X}^\sharp$$

for any function $f : \mathbb{X} \to \overline{\mathbb{R}}$

Fenchel conjugate	Fourier transform
$\sup \to +$	
$+ \to \times$	
$\sup_{x \in \mathbb{X}} \Big(\langle x \,, x^\sharp \rangle \underset{\cdot}{+} \big(- f(x) \big) \Big)$	$\int_{x \in \mathbb{X}} \exp(\langle x \,, x^\sharp \rangle) f(x) dx$

Outline of the presentation

Moreau lower and upper additions

- The Moreau lower addition extends the usual addition with

$$(+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty$$

- The Moreau upper addition extends the usual addition with

$$(+\infty) \stackrel{\cdot}{+} (-\infty) = (-\infty) \stackrel{\cdot}{+} (+\infty) = +\infty$$

Background on couplings and Fenchel-Moreau conjugacies

- Let be given two sets $\mathbb{X}$ ("primal") and $\mathbb{X}^\sharp$ ("dual")
- Consider a coupling function $c : \mathbb{X} \times \mathbb{X}^\sharp \to \overline{\mathbb{R}} = [-\infty, +\infty]$
- We also use the notation $\mathbb{X} \overset{c}{\leftrightarrow} \mathbb{X}^\sharp$ for a coupling

Definition

The c-Fenchel-Moreau conjugate of a function $f : \mathbb{X} \to \overline{\mathbb{R}}$,
with respect to the coupling c,
is the function $f^c : \mathbb{X}^\sharp \to \overline{\mathbb{R}}$ defined by

$$f^c(x^\sharp) = \sup_{x \in \mathbb{X}} \Big(c(x, x^\sharp) \dotplus \big(-f(x) \big) \Big) , \quad \forall x^\sharp \in \mathbb{X}^\sharp$$

Fenchel-Moreau conjugate $(\max, +)$	Kernel transform $(+, \times)$
$\sup_{x \in \mathbb{X}} \Big(c(x, x^\sharp) \dotplus \big(-f(x) \big) \Big)$	$\int_{x \in \mathbb{X}} c(x, x^\sharp) f(x) dx$

Background on couplings and Fenchel-Moreau conjugacies

With the coupling c, we associate the reverse coupling c'

$$c' : \mathbb{X}^\sharp \times \mathbb{X} \to \overline{\mathbb{R}} \; , \;\; c'(x^\sharp, x) = c(x, x^\sharp) \; , \;\; \forall (x^\sharp, x) \in \mathbb{X}^\sharp \times \mathbb{X}$$

- The c'-Fenchel-Moreau conjugate of a function $g : \mathbb{X}^\sharp \to \overline{\mathbb{R}}$, with respect to the coupling c', is the function $g^{c'} : \mathbb{X} \to \overline{\mathbb{R}}$

$$g^{c'}(x) = \sup_{x^\sharp \in \mathbb{X}^\sharp} \Big(c(x, x^\sharp) \dotplus \big(- g(x^\sharp) \big) \Big) \; , \;\; \forall x \in \mathbb{X}$$

- The c-Fenchel-Moreau biconjugate $f^{cc} : \mathbb{X} \to \overline{\mathbb{R}}$ of a function $f : \mathbb{X} \to \overline{\mathbb{R}}$ is given by

$$f^{cc}(x) = \sup_{x^\sharp \in \mathbb{X}^\sharp} \Big(c(x, x^\sharp) \dotplus \big(- f^{c}(x^\sharp) \big) \Big) \; , \;\; \forall x \in \mathbb{X}$$

The $(-c)$-Fenchel-Moreau conjugate of $g : \mathbb{X} \to \overline{\mathbb{R}}$ is given by

$$g^{-c}(x^\sharp) = \sup_{x \in \mathbb{X}} \Big(\big(- c(x, x^\sharp) \big) \dotplus \big(- g(x) \big) \Big) \; , \;\; \forall x^\sharp \in \mathbb{X}^\sharp$$

Fenchel inequality with a general coupling

- Conjugacies are special cases of dualites, that make it possible to obtain dual problems

$$\sup_{x^\sharp \in \mathbb{X}^\sharp} \Big(\big(-f^c(x^\sharp) \big) \dot{+} \big(-g^{-c}(x^\sharp) \big) \Big) \leq \inf_{x \in \mathbb{X}} \Big(f(x) \dot{+} g(x) \Big)$$

- In particular, optimization under constraints $x \in X$ gives

$$\sup_{x^\sharp \in \mathbb{X}^\sharp} \Big(\big(-f^c(x^\sharp) \big) \dot{+} \big(-\delta_X^{-c}(x^\sharp) \big) \Big) \leq \inf_{x \in \mathbb{X}} \Big(f(x) \dot{+} \delta_X(x) \Big)$$

where $\delta_X(x) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X \end{cases}$

- Hence, the issue is to find a coupling c that gives nice expressions for f^c and δ_X^{-c}

Outline of the presentation

Couplings for discrete convexity

"Primal"	"Dual"	Coupling
$\mathbb{R}^n$	$\mathbb{R}^n$	$\star(x, x^\sharp) = \langle x \, , x^\sharp \rangle$
$\mathbb{Z}^n$	$\mathbb{Z}^n$	$\bullet(z, p) = \langle z \, , p \rangle$
$\mathbb{Z}^n$	$\mathbb{R}^n$	$(\star)(z, x^\sharp) = \langle z \, , x^\sharp \rangle$
$\mathbb{Z}^n$	$\mathbb{R}^n \times \mathbb{R}^n$	$c\big(z, (x, x^\sharp)\big) = \langle z \, , x^\sharp \rangle \dot{+} \big(-\delta_{\mathcal{N}(x)}(z) \big)$

where integer neighbours $\mathcal{N}(x)$ of $x \in \mathbb{R}^n$ are such that

$$\{z\} \subset \mathcal{N}(z) \subset \mathbb{Z}^n \; , \quad \forall z \in \mathbb{Z}^n$$

Conjugacies for discrete convexity

- For any function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$, we define

$$f^{\bullet}(p) = \sup_{z \in \mathbb{Z}^n} \Big(\langle z \,, p \rangle \dot{+} \big(-f(z) \big) \Big) \qquad \forall p \in \mathbb{Z}^n$$

$$f^{(\star)}(x^{\sharp}) = \sup_{z \in \mathbb{Z}^n} \Big(\Big\langle z \,, x^{\sharp} \Big\rangle \dot{+} \big(-f(z) \big) \Big) \qquad \forall x^{\sharp} \in \mathbb{R}^n$$

$$f^{c}(x, x^{\sharp}) = \sup_{z \in \mathcal{N}(x)} \Big(\Big\langle z \,, x^{\sharp} \Big\rangle \dot{+} \big(-f(z) \big) \Big) \qquad \forall (x, x^{\sharp}) \in \mathbb{R}^n \times \mathbb{R}^n$$

- and we have the following relations

$$f^{\bullet} = f^{(\star)} \qquad \text{on } \mathbb{Z}^n$$

$$f^{(\star)} = \big(f \dot{+} \delta_{\mathbb{Z}^n} \big)^{\star} \qquad \text{on } \mathbb{R}^n$$

$$\sup_{x \in \mathbb{R}^n} f^{c}(x, \cdot) = f^{(\star)} \qquad \text{on } \mathbb{R}^n$$

Biconjugacies for discrete convexity

- For any function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$, we define

$$f^{\bullet\bullet}(z) = \sup_{p \in \mathbb{Z}^n} \Big(\langle z \,, p \rangle \dotplus \big(-f^{\bullet}(p) \big) \Big) \qquad \forall z \in \mathbb{Z}^n$$

$$f^{(\star)(\star)}(z) = \sup_{x^\sharp \in \mathbb{R}^n} \Big(\Big\langle z \,, x^\sharp \Big\rangle \dotplus \big(-f^{(\star)}(x^\sharp) \big) \Big) \qquad \forall z \in \mathbb{Z}^n$$

$$f^{cc}(z) = \sup_{x \in \mathcal{N}^{-1}(z)} \sup_{x^\sharp \in \mathbb{R}^n} \Big(\Big\langle z \,, x^\sharp \Big\rangle \dotplus \big(-f^{c}(x, x^\sharp) \big) \Big) \qquad \forall z \in \mathbb{Z}^n$$

- and we have the following relations

$$f^{\bullet\bullet} = \Big(\big(f \dotplus \delta_{\mathbb{Z}^n} \big)^{\star} \dotplus \delta_{\mathbb{Z}^n} \Big)^{\star'} \qquad \text{on } \mathbb{Z}^n$$

$$f^{(\star)(\star)} = f^{(\star)\star'} = \big(f \dotplus \delta_{\mathbb{Z}^n} \big)^{\star\star} \geq f^{\bullet\bullet} \qquad \text{on } \mathbb{Z}^n$$

$$f^{cc} \geq f^{\bullet\bullet} \qquad \text{on } \mathbb{Z}^n$$

Convex extensible functions

- For any function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$, we define the convex closure $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$\overline{f}(x) = \sup_{x^\sharp \in \mathbb{R}^n, \alpha \in \mathbb{R}} \left\{ \left\langle x \,, x^\sharp \right\rangle + \alpha \;\middle|\; \left\langle z \,, x^\sharp \right\rangle + \alpha \leq f(z) \,, \;\; \forall z \in \mathbb{Z}^n \right\}$$

- Convex closure and Fenchel biconjugate are related by

$$\overline{f}(x) = \left(f \dotplus \delta_{\mathbb{Z}^n}\right)^{\star\star}(x) = f^{(\star)\star'}(x) \,, \;\; \forall x \in \mathbb{R}^n$$

Definition

We say that the function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is convex extensible if

$$f(z) = \overline{f}(z) \,, \;\; \forall z \in \mathbb{Z}^n$$

We introduce a suitable coupling $(\star)$ for which convex extensible functions = $(\star)$-convex functions

- Integer space $\mathbb{Z}^n$ coupled with real space $\mathbb{R}^n$ by the bilinear coupling $(\star) = \langle \cdot \,, \cdot \rangle$
- The conjugate $f^{(\star)} : \mathbb{R}^n \to \overline{\mathbb{R}}$ of $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is given by

$$f^{(\star)}(x^\sharp) = \left(f \dot{+} \delta_{\mathbb{Z}^n}\right)^\star(x^\sharp) = \sup_{z \in \mathbb{Z}^n} \left(\left\langle z \,, x^\sharp \right\rangle \dot{+} \left(-f(z)\right) \right) \,, \quad \forall x^\sharp \in \mathbb{R}^n$$

- The biconjugate $f^{(\star)(\star)} : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is given by

$$f^{(\star)(\star)}(z) = \sup_{x^\sharp \in \mathbb{R}^n} \left(\left\langle z \,, x^\sharp \right\rangle \dot{+} \left(- f^{(\star)}(x^\sharp) \right) \right) \,, \quad \forall z \in \mathbb{Z}^n$$

Proposition

$$f \textit{ is convex extensible} \iff f = f^{(\star)(\star)}$$

Local convex extension

- For any function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$, we define the local convex extension $\widetilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$\widetilde{f}(x) = \sup_{x^\sharp \in \mathbb{R}^n, \alpha \in \mathbb{R}} \left\{ \left\langle x , x^\sharp \right\rangle + \alpha \;\middle|\; \left\langle z , x^\sharp \right\rangle + \alpha \leq f(z) \; , \; \forall z \in \mathcal{N}(x) \right\}$$

where integer neighbours $\mathcal{N}(x)$ of $x \in \mathbb{R}^n$ are such that

$$\{z\} \subset \mathcal{N}(z) \subset \mathbb{Z}^n \; , \; \forall z \in \mathbb{Z}^n$$

- The local convex extension is larger than the convex extension:

$$\widetilde{f}(x) \geq \overline{f}(x) \; , \; \forall x \in \mathbb{R}^n$$

- When integer neighbours are few, the local convex extension coincides with the original function on the integers:

$$\{z\} = \mathcal{N}(z) \; , \; \forall z \in \mathbb{Z}^n \Rightarrow \widetilde{f}(z) = f(z) \; , \; \forall z \in \mathbb{Z}^n$$

We introduce a suitable coupling

- Integer space $\mathbb{Z}^n$ coupled with real space $\mathbb{R}^n \times \mathbb{R}^n$ by localization of the bilinear coupling $\langle \cdot \, , \cdot \rangle$ w.r.t. neighbours $\mathcal{N}(x) \subset \mathbb{Z}^n$:

$$c\big(z, (x, x^\sharp)\big) = \Big\langle z \, , x^\sharp \Big\rangle \dotplus \big(-\delta_{\mathcal{N}(x)}(z)\big) = \Big\langle z \, , x^\sharp \Big\rangle \dotplus \big(-\delta_{\mathcal{N}^{-1}(z)}(x)\big)$$

- The c-conjugate $f^c : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ of $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is

$$f^c(x, x^\sharp) = \sup_{z \in \mathcal{N}(x)} \Big(\Big\langle z \, , x^\sharp \Big\rangle \dotplus \big(-f(z)\big) \Big) \, , \ \ \forall (x, x^\sharp) \in \mathbb{R}^n \times \mathbb{R}^n$$

- The c-biconjugate $f^{cc} : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is

$$f^{cc}(z) = \sup_{x \in \mathcal{N}^{-1}(z)} \sup_{x^\sharp \in \mathbb{R}^n} \Big(\Big\langle z \, , x^\sharp \Big\rangle \dotplus \big(-f^c(x, x^\sharp)\big) \Big) \, , \ \ \forall z \in \mathbb{Z}^n$$

Integrally convex functions

- We have that

$$f^c(x, x^\sharp) \leq f^{(\star)}(x^\sharp)$$

- The local convex extension satisfies

$$\widetilde{f}(x) = \sup_{x^\sharp \in \mathbb{R}^n} \Big(\Big\langle x \, , x^\sharp \Big\rangle \dot{+} \big(- f^c(x, x^\sharp) \big) \Big) \geq f^{(\star)\star'}(x) = \overline{f}(x)$$

- The c-biconjugate $f^{cc} : \mathbb{Z}^n \to \overline{\mathbb{R}}$ satisfies

$$f^{cc}(z) = \sup_{x \in \mathcal{N}^{-1}(z)} \widetilde{f}(x) \, , \;\; \forall z \in \mathbb{Z}^n$$

Definition

We say that the function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is integrally convex if

$$\widetilde{f}(x) = \overline{f}(x) = f^{(\star)\star'}(x) \, , \;\; \forall x \in \mathbb{R}^n$$

When integer neighbours are few, all functions are c-convex functions!

- From $\{z\} = \mathcal{N}(z)\ ,\ \ \forall z \in \mathbb{Z}^n$ and

$$\widetilde{f}(z) \leq \sup_{x \in \mathcal{N}^{-1}(z)} \widetilde{f}(x) = f^{cc}(z) \leq f(z) = \widetilde{f}(z)\ ,\ \ \forall z \in \mathbb{Z}^n$$

  we deduce that

$$\widetilde{f}(z) = \sup_{x \in \mathcal{N}^{-1}(z)} \widetilde{f}(x) = f^{cc}(z) = f(z)\ ,\ \ \forall z \in \mathbb{Z}^n$$

- Therefore, if the function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is integrally convex:

$$\overline{f}(z) = \widetilde{f}(z) = \sup_{x \in \mathcal{N}^{-1}(z)} \widetilde{f}(x) = f(z)\ ,\ \ \forall z \in \mathbb{R}^n$$

Outline of the presentation

Optimal transport

The optimal transport problem is

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\,d\pi(x,y)$$

where

- the sets $\mathcal{X}$ and $\mathcal{Y}$ are two Polish spaces
- we denote by $\mathcal{P}(\mathcal{X}\times\mathcal{Y})$, $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$ the corresponding probability spaces (rectangle and marginals)
- the set $\Pi(\mu,\nu)$ is made of probabilities $\pi \in \mathcal{P}(\mathcal{X}\times\mathcal{Y})$ on the rectangle, whose marginals are $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$
- the measurable cost function $c : \mathcal{X}\times\mathcal{Y} \to [0,+\infty]$, where $c(x,y)$ represents the cost to move from $x \in \mathcal{X}$ towards $y \in \mathcal{Y}$

We introduce a suitable coupling between probabilities and functions

- We denote by $C_b^0(\mathcal{X})$ and $C_b^0(\mathcal{Y})$ the spaces of continuous bounded functions
- We introduce the bilinear coupling

$$\mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y}) \stackrel{\beta}{\longleftrightarrow} C_b^0(\mathcal{X}) \times C_b^0(\mathcal{Y})$$

$$\beta\big((\mu, \nu); (\psi, \phi)\big) = \int_{\mathcal{Y}} \phi(y) d\nu(y) - \int_{\mathcal{X}} \psi(x) d\mu(x)$$

Conjugacy properties in optimal transport

$$C(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)d\pi(x,y)$$

$$D(\psi,\phi) = \sup_{x\in\mathcal{X},y\in\mathcal{Y}} \left[\phi(y) - \psi(x) - c(x,y)\right] = \sup_{y\in\mathcal{Y}} \Big(\phi(y) + \psi^{-c}(y)\Big)$$

We have the following conjugacy equalities and inequalities

$$\begin{aligned} C^{\beta}(\psi,\phi) &= D(\psi,\phi) = D^{\beta'\beta}(\psi,\phi) \\ C(\mu,\nu) &\geq C^{\beta\beta'}(\mu,\nu) = D^{\beta'}(\mu,\nu) \\ &= \sup_{\psi,\phi} \Big(\int_{\mathcal{Y}} \phi(y)d\nu(y) - \int_{\mathcal{X}} \psi(x)d\mu(x) - D(\psi,\phi) \Big) \\ &\geq \sup_{\phi-\psi\leq c} \Big(\int_{\mathcal{Y}} \phi(y)d\nu(y) - \int_{\mathcal{X}} \psi(x)d\mu(x) \Big) \end{aligned}$$

Outline of the presentation

We introduce a sum coupling

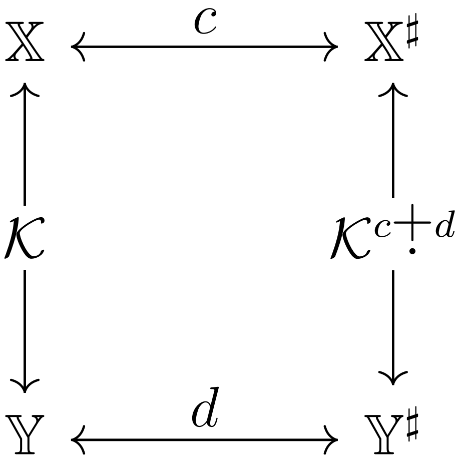
- Let be given two "primal" sets $\mathbb{X}$, $\mathbb{Y}$
- and two "dual" sets $\mathbb{X}^\sharp$, $\mathbb{Y}^\sharp$,
- together with two coupling functions

$$c : \mathbb{X} \times \mathbb{X}^\sharp \to \overline{\mathbb{R}} \,, \quad d : \mathbb{Y} \times \mathbb{Y}^\sharp \to \overline{\mathbb{R}}$$

We define the sum coupling $c \dotplus d$ — coupling the "primal" product set $\mathbb{X} \times \mathbb{Y}$ with the "dual" product set $\mathbb{X}^\sharp \times \mathbb{Y}^\sharp$ — by

$$\begin{aligned} c \dotplus d : (\mathbb{X} \times \mathbb{Y}) \times (\mathbb{X}^\sharp \times \mathbb{Y}^\sharp) &\to \overline{\mathbb{R}} \,, \\ \big((x,y),(x^\sharp,y^\sharp)\big) &\mapsto c(x,x^\sharp) \dotplus d(y,y^\sharp) \end{aligned}$$

A kernel $\mathcal{K}$, two couplings c and d and a new kernel $\mathcal{K}^{c \dagger d}$



We introduce the conjugate of a kernel bivariate function w.r.t. a sum coupling

With any kernel bivariate function

$$\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}} \ ,$$

defined on the "primal" product set $\mathbb{X} \times \mathbb{Y}$,
we associate the conjugate, with respect to the coupling $c \dotplus d$,
defined on the "dual" product set $\mathbb{X}^\sharp \times \mathbb{Y}^\sharp$, by

$$\mathcal{K}^{c \dotplus d}(x^\sharp, y^\sharp) = \sup_{x \in \mathbb{X}, y \in \mathbb{Y}} \Big(c(x, x^\sharp) \dotplus d(y, y^\sharp) \dotplus \big(-\mathcal{K}(x, y) \big) \Big)$$

$$\forall (x^\sharp, y^\sharp) \in \mathbb{X}^\sharp \times \mathbb{Y}^\sharp$$

Main result: Fenchel-Moreau conjugation inequalities with three couplings

Theorem

For any bivariate function $\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$
and univariate functions
$f : \mathbb{X} \to \overline{\mathbb{R}}$ *and* $g : \mathbb{Y} \to \overline{\mathbb{R}}$,
all defined on the "primal" sets, we have that

$$f(x) \geq \inf_{y \in \mathbb{Y}} \Big(\mathcal{K}(x,y) \dot{+} g(y) \Big) \ , \ \ \forall x \in \mathbb{X} \Rightarrow$$

$$f^{c}(x^{\sharp}) \leq \inf_{y^{\sharp} \in \mathbb{Y}^{\sharp}} \Big(\mathcal{K}^{c \dot{+} d}(x^{\sharp}, y^{\sharp}) \dot{+} g^{-d}(y^{\sharp}) \Big) \ , \ \ \forall x^{\sharp} \in \mathbb{X}^{\sharp}$$

Main result

Two couplings c and d, and an inf-operation with kernel $\mathcal{K}$

$$f(x) \geq \inf_{y \in \mathbb{Y}} \Big(\mathcal{K}(x,y) \dotplus g(y) \Big) \Rightarrow$$

$$\underbrace{f^{c}(x^\sharp)}_{\substack{c-\text{Fenchel-}\\\text{Moreau conjugate}\\\text{of } f}} \leq \inf_{y^\sharp \in \mathbb{Y}^\sharp} \Big(\underbrace{\mathcal{K}^{c \dotplus d}(x^\sharp, y^\sharp)}_{\substack{c \dotplus d-\text{Fenchel-}\\\text{Moreau conjugate}\\\text{of } \mathcal{K}}} \Big) \dotplus \underbrace{g^{-d}(y^\sharp)}_{\substack{(-d)-\text{Fenchel-}\\\text{Moreau conjugate}\\\text{of } g}}$$

Main result

Two couplings c and d, and an inf-operation with kernel $\mathcal{K}$

$$f(x) \geq \inf_{y \in \mathbb{Y}} \Big(\mathcal{K}(x,y) \dotplus g(y) \Big) \Rightarrow$$

$$\underbrace{f^{c}(x^{\sharp})}_{c-\text{Fenchel-Moreau conjugate of } f} \leq \inf_{y^{\sharp} \in \mathbb{Y}^{\sharp}} \Big(\underbrace{\mathcal{K}^{c \dotplus d}(x^{\sharp}, y^{\sharp})}_{c \dotplus d-\text{Fenchel-Moreau conjugate of } \mathcal{K}} \Big) \dotplus \underbrace{g^{-d}(y^{\sharp})}_{(-d)-\text{Fenchel-Moreau conjugate of } g}$$

- The left hand side assumption is a primal inequality, which is rather weak (upper bound for an infimum)
- whereas the right hand side conclusion is a dual inequality, which is rather strong (lower bound for an infimum)

Main result (second formulation)

Three couplings c, d and $\mathcal{K}$

$$f \dotplus \underbrace{g^{-\mathcal{K}}}_{\substack{(-\mathcal{K})-\text{Fenchel-}\\ \text{Moreau conjugate}\\ \text{of } g}} \geq 0 \quad \Rightarrow$$

$$\underbrace{f^{c}}_{\substack{c-\text{Fenchel-}\\ \text{Moreau conjugate}\\ \text{of } f}} \dotplus \underbrace{(g^{-d})^{-\mathcal{K}^{c} \dotplus d}}_{\substack{(-\mathcal{K}^{c} \dotplus d)-\text{Fenchel-}\\ \text{Moreau conjugate}\\ \text{of } g^{-d}}} \leq 0$$

Outline of the presentation

Outline of the presentation

The duality equality case

The duality equality case is the property that

$$f(x) = \inf_{y \in \mathbb{Y}} \Big(\mathcal{K}(x,y) \dot{+} g(y) \Big) \ , \ \forall x \in \mathbb{X} \Rightarrow$$

$$f^c(x^\sharp) = \inf_{y^\sharp \in \mathbb{Y}^\sharp} \Big(\mathcal{K}^{c \dot{+} d}(x^\sharp, y^\sharp) \dot{+} g^{-d}(y^\sharp) \Big) \ , \ \forall x^\sharp \in \mathbb{X}^\sharp$$

Sufficient conditions for the duality equality case

Corollary

Consider any bivariate function $\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ and univariate functions $f : \mathbb{X} \to \overline{\mathbb{R}}$ and $g : \mathbb{Y} \to \overline{\mathbb{R}}$, all defined on the "primal" sets.
The equality case holds when

1. $g^{(-d)(-d)} = g$
2. *the following function has a saddle point (or no duality gap)*

$$\big((x, y), y^\sharp\big) \in (\mathbb{X} \times \mathbb{Y}) \times \mathbb{Y}^\sharp \mapsto \Big(c(x, x^\sharp) \dot{+} \big(-\mathcal{K}(x, y)\big) \dot{+} d(y, y^\sharp)\Big) \dot{+} g^{-d}(y^\sharp)$$

3. *the two coupling functions $c : \mathbb{X} \times \mathbb{X}^\sharp \to \mathbb{R}$ and $d : \mathbb{Y} \times \mathbb{Y}^\sharp \to \mathbb{R}$, and the kernel $\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ all take finite values*

Sufficient conditions for the duality equality case

Corollary

Consider any bivariate function $\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ and univariate functions $f : \mathbb{X} \to \overline{\mathbb{R}}$ and $g : \mathbb{Y} \to \overline{\mathbb{R}}$, all defined on the "primal" sets. We define

$$\mathcal{K}_{x^\sharp}(y) = -\Big(\mathcal{K}(\cdot, y)^c(x^\sharp)\Big) = \inf_{x \in \mathbb{X}} \Big(\big(-c(x, x^\sharp)\big) \dot{+} \mathcal{K}(x, y)\Big)$$

The equality case holds true when

$$\sup_{y \in \mathbb{Y}} \Big(\big(-\mathcal{K}_{x^\sharp}(y)\big) \underset{\cdot}{+} \big(-g(y)\big)\Big) = \inf_{y^\sharp \in \mathbb{Y}^\sharp} \Big(\mathcal{K}^d_{x^\sharp}(y^\sharp) \dot{+} g^{-d}(y^\sharp)\Big)$$

Sufficient conditions for the duality equality case

Corollary

Consider any bivariate function $\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$
and univariate functions $f : \mathbb{X} \to \overline{\mathbb{R}}$ *and* $g : \mathbb{Y} \to]-\infty, +\infty]$,
all defined on the "primal" sets.
The equality case holds true when

1. *the coupling* $d : \mathbb{Y} \times \mathbb{Y}^{\sharp} \to \mathbb{R}$ *is the duality bilinear form* $\langle \, , \rangle$ *between* $\mathbb{Y}$ *and its algebraic dual* $\mathbb{Y}^{\sharp}$
2. *the function* g *is a proper convex function (the function* g *never takes the value* $-\infty$ *and is not identically equal to* $+\infty$*),*
3. *for any* $x^{\sharp} \in \mathbb{X}^{\sharp}$*, the function* $\mathcal{K}_{x^{\sharp}}$ *is a proper convex function*
4. *for any* $x^{\sharp} \in \mathbb{X}^{\sharp}$*, the function* g *is continuous at some point where* $\mathcal{K}_{x^{\sharp}}$ *is finite*

Outline of the presentation

Perturbation + Fenchel-Moreau duality

- To design a dual problem to the original problem

$$\inf_{y \in \mathbb{Y}} \big(h(y) \dot{+} g(y) \big)$$

- take a bivariate function $\mathcal{K} : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$, where $\mathbb{X}$ is a perturbation set, such that $\mathcal{K}(0, y) = h(y)$
- then define $f(x) = \inf_{y \in \mathbb{Y}} \Big(\mathcal{K}(x, y) \dot{+} g(y) \Big)$, $\forall x \in \mathbb{X}$
- then take two couplings c and d, and obtain

$$f^c(x^\sharp) \leq \inf_{y^\sharp \in \mathbb{Y}^\sharp} \Big(\mathcal{K}^{c \dot{+} d}(x^\sharp, y^\sharp) \dot{+} g^{-d}(y^\sharp) \Big) \ , \ \ \forall x^\sharp \in \mathbb{X}^\sharp$$

- and finally obtain the dual problem

$$\inf_{y \in \mathbb{Y}} \big(h(y) \dot{+} g(y) \big) = f(0) \geq$$

$$f^{cc}(0) \geq \underbrace{\sup_{y^\sharp \in \mathbb{Y}^\sharp} \Big(\mathcal{K}^{c \dot{+} d}(\cdot, y^\sharp)^c(0) \dot{+} \big(-g^{-d}(y^\sharp) \big) \Big)}_{\text{dual problem}}$$

Outline of the presentation

Fenchel conjugate of inf-convolution

The classic inf-convolution

$$(g_1 \square g_2)(x) = \inf_{y_1 + y_2 = x} \Big(g_1(y_1) \dot{+} g_2(y_2) \Big)$$

satisfies

$$(g_1 \square g_2)^\star = g_1^\star \underset{\cdot}{+} g_2^\star$$

Definition of generalized inf-convolution

Definition

- Let be given three sets $\mathbb{X}$, $\mathbb{Y}_1$ and $\mathbb{Y}_2$
- For any trivariate convoluting function

$$\mathcal{I} : \mathbb{Y}_1 \times \mathbb{X} \times \mathbb{Y}_2 \to \overline{\mathbb{R}} \,,$$

we define the $\mathcal{I}$-inf-convolution
of two functions $g_1 : \mathbb{Y}_1 \to \overline{\mathbb{R}}$ and $g_2 : \mathbb{Y}_2 \to \overline{\mathbb{R}}$ by

$$(g_1 \overset{\mathcal{I}}{\square} g_2)(x) = \inf_{y_1 \in \mathbb{Y}_1, y_2 \in \mathbb{Y}_2} \Big(g_1(y_1) \dotplus \underbrace{\mathcal{I}(y_1, x, y_2)}_{\text{convoluting function}} \dotplus g_2(y_2) \Big)$$

Definition of generalized inf-convolution

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$$(g_1 \overset{\mathcal{I}}{\square} g_2)(x) = \inf_{y_1 \in \mathbb{Y}_1, y_2 \in \mathbb{Y}_2} \Big(g_1(y_1) \dotplus \underbrace{\mathcal{I}(y_1, x, y_2)}_{\text{convoluting function}} \dotplus g_2(y_2) \Big)$$

The classic inf-convolution corresponds to
$\mathcal{I}(y_1, x, y_2) = \delta_x(y_1 + y_2)$:
$$(g_1 \square g_2)(x) = \inf_{y_1 + y_2 = x} \Big(g_1(y_1) \dotplus g_2(y_2) \Big)$$

Fenchel-Moreau conjugate of generalized inf-convolution

Proposition

Let be given three "primal" sets $\mathbb{X}$, $\mathbb{Y}_1$, $\mathbb{Y}_2$ *and three "dual" sets* $\mathbb{X}^\sharp$, $\mathbb{Y}_1^\sharp$, $\mathbb{Y}_2^\sharp$, *together with three coupling functions*

$$\mathbb{X} \overset{c}{\leftrightarrow} \mathbb{X}^\sharp \; , \;\; \mathbb{Y}_1 \overset{d_1}{\leftrightarrow} \mathbb{Y}_1^\sharp \; , \;\; \mathbb{Y}_2 \overset{d_2}{\leftrightarrow} \mathbb{Y}_2^\sharp$$

For any univariate functions $f : \mathbb{X} \to \overline{\mathbb{R}}$,
$g_1 : \mathbb{Y}_1 \to \overline{\mathbb{R}}$ *and* $g_2 : \mathbb{Y}_2 \to \overline{\mathbb{R}}$,
all defined on the "primal" sets, we have that

$$f(x) \geq \big(g_1 \overset{\mathcal{I}}{\square} g_2\big)(x) \; , \;\; \forall x \in \mathbb{X} \Rightarrow$$

$$f^c(x^\sharp) \leq \big(g_1^{(-d_1)} \overset{\mathcal{I}^\sharp}{\square} g_2^{(-d_2)}\big)(x^\sharp) \; , \;\; \forall x^\sharp \in \mathbb{X}^\sharp \; ,$$

where the convoluting function $\mathcal{I}^\sharp$ *on the "dual" sets is given by*

$$\mathcal{I}^\sharp = \underline{\mathcal{I}}^{c \dotplus d_1 \dotplus d_2}$$

The $\mathcal{I}$-inf-convolution is minus the Fenchel-Moreau conjugate of a sum

Proposition

The *$\mathcal{I}$-inf-convolution* *is given by*

$$g_1 \overset{\mathcal{I}}{\square} g_2 = -(g_1 \dotplus g_2)^{\mathcal{I}}$$

Fenchel-Moreau conjugate of generalized inf-convolution

Proposition

If there exist two coupling functions

$$\Gamma_1 : \mathbb{X}^\sharp \times \mathbb{Y}_1 \to \overline{\mathbb{R}} \,, \;\; \Gamma_2 : \mathbb{X}^\sharp \times \mathbb{Y}_2 \to \overline{\mathbb{R}} \,,$$

such that the partial c-Fenchel-Moreau conjugate of the convoluting function $\mathcal{I}$ splits as

$$\mathcal{I}(y_1, \cdot, y_2)^c(x^\sharp) = \Gamma_1(x^\sharp, y_1) \stackrel{\cdot}{+} \Gamma_2(x^\sharp, y_2) \,,$$

then the c-Fenchel-Moreau conjugate of the inf-convolution $g_1 \overset{\mathcal{I}}{\square} g_2$ is given by a sum as

$$\big(g_1 \overset{\mathcal{I}}{\square} g_2\big)^c = g_1^{\Gamma_1} \stackrel{\cdot}{+} g_2^{\Gamma_2}$$

Fenchel-Moreau conjugate of generalized inf-convolution

Proposition

If there exist two coupling functions

$$\Gamma_1 : \mathbb{X}^\sharp \times \mathbb{Y}_1 \to \overline{\mathbb{R}} \;, \;\; \Gamma_2 : \mathbb{X}^\sharp \times \mathbb{Y}_2 \to \overline{\mathbb{R}} \;,$$

such that the partial c-Fenchel-Moreau conjugate of the convoluting function $\mathcal{I}$ splits as

$$\mathcal{I}(y_1, \cdot, y_2)^c(x^\sharp) = \Gamma_1(x^\sharp, y_1) \dotplus \Gamma_2(x^\sharp, y_2) \;,$$

then the c-Fenchel-Moreau conjugate of the inf-convolution $g_1 \overset{\mathcal{I}}{\square} g_2$ is given by a sum as

$$\big(g_1 \overset{\mathcal{I}}{\square} g_2\big)^c = g_1^{\Gamma_1} \dotplus g_2^{\Gamma_2}$$

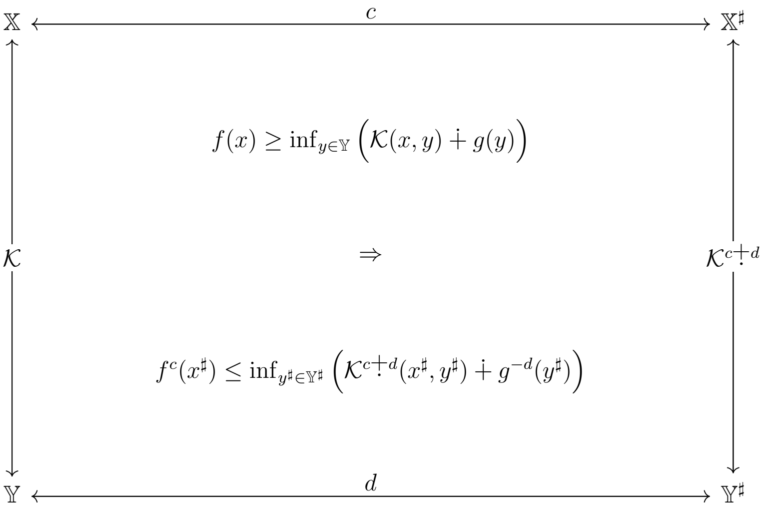
This generalizes $\big(g_1 \square g_2\big)^\star = g_1^\star \dotplus g_2^\star$

Outline of the presentation

Conclusion

- We have proven a new Fenchel-Moreau conjugation inequality with three couplings (and given sufficient conditions for the equality case)
- We have provided a general method to design dual problems by means of one kernel and two couplings
- We have introduced a generalized inf-convolution, and have provided formulas for Fenchel-Moreau conjugates
- We have shown that Fenchel conjugates of Bellman functions satisfy a "Bellman like" inequation, and we have sketched an application to the SDDP algorithm

Thank you:-)


$\mathbb{X}$
c
$\mathbb{X}^\sharp$
$f(x) \geq \inf_{y\in\mathbb{Y}} \Big(\mathcal{K}(x,y) \dot{+} g(y)\Big)$
$\mathcal{K}$
$\Rightarrow$
$\mathcal{K}^{c\dot{+}d}$
$f^c(x^\sharp) \leq \inf_{y^\sharp\in\mathbb{Y}^\sharp} \Big(\mathcal{K}^{c\dot{+}d}(x^\sharp, y^\sharp) \dot{+} g^{-d}(y^\sharp)\Big)$
$\mathbb{Y}$
d
$\mathbb{Y}^\sharp$