Witsenhausen Intrinsic Model
Games in Product Form

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Charles Darwin and the peacock's tail

In a letter to botanist Asa Gray — dated 3 April 1860, one year after the publication of *The Origin of Species* — Charles Darwin writes

> *The sight of a feather in a peacock's tail, whenever I gaze at it, makes me sick!*

Indeed, this embarrasing cumbersome tail is a handicap for survival (like escaping predators)

Informational asymetry in mating/ signaling games

- In 1871, Charles Darwin published
  The Descent of Man, and Selection in Relation to Sex
  and proposed that the peacock's tail had evolved because females preferred to mate with males with more elaborate ones
- In 1975, biologist Amotz Zahavi published
  Mate Selection-A Selection for a Handicap

  > *These handicaps are of use to the selecting sex since they test the quality of the mate. [...] The understanding that a handicap, which tests for quality, can evolve as a consequence of its advantage to the individual, may provide an explanation for many puzzling evolutionary problems.*

- In 2013, mathematicians Pierre Bernhard and Frédéric Hamelin published
  Simple signaling games of sexual selection (Grafen's revisited)

Information in game theory

Game theory is concerned with strategic interactions:
my best choice depends on the other players

Strategic interactions originate from two sources

- Payoffs and beliefs
  - My payoff depends on the other players actions
  - I have beliefs about Nature (like other players types)
- Information
  - Information — who knows what and when — plays a crucial role in competitive contexts
  - Concealing, cheating, lying, deceiving are effective strategies

Game = game form + preferences

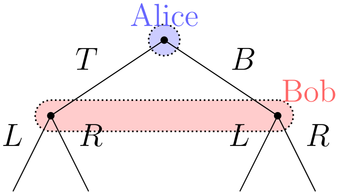
A (mathematical) game is made of two parts

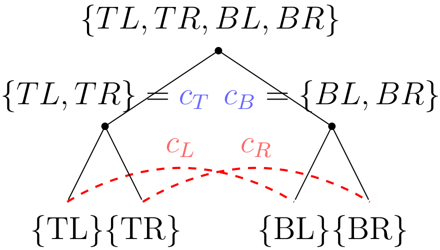
1. A game form, *game frame*, *ruleset*, or *outcome function*
   is the set of rules that govern a game and determine its outcome based on each player's strategies

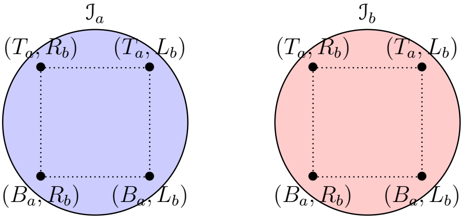
   $$\text{strategy space} \mapsto \text{outcome space}$$

2. A family of preferences, one per player, on the outcome space

Three game forms (for two players Alice and Bob): Kuhn, Alós-Ferrer and Ritzberger, Witsenhausen







Roadmap

1. Introduce the Witsenhausen intrinsic model (W-model), and illustrate its potential to handle informational interactions
2. Extend the W-model to games in product form (W-games)

Outline of the presentation

Witsenhausen intrinsic model (W-model) [15 min]

Games in product form (W-game) [10 min]

Conclusion

Outline of the presentation

Witsenhausen intrinsic model (W-model) [15 min]

Games in product form (W-game) [10 min]

Conclusion

Algebras, σ-algebras/fields, partition fields

Let $\mathcal{Z}$ be a set

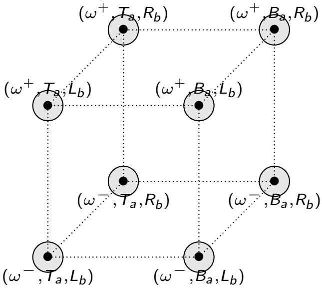
- An algebra (or field) on $\mathcal{Z}$ is a nonempty collection $\mathfrak{Z}$ of subsets of $\mathcal{Z}$ (identified with a subset $\mathfrak{Z} \subset 2^{\mathcal{Z}}$) which is stable under complementation and finite union (hence, under finite intersection)
- A σ-algebra (or σ-field) on $\mathcal{Z}$ is a nonempty collection $\mathfrak{Z}$ of subsets of $\mathcal{Z}$ (identified with a subset $\mathfrak{Z} \subset 2^{\mathcal{Z}}$) which is stable under complementation and countable union (hence, under countable intersection)
- A partition field (or π-field) on $\mathcal{Z}$ is a nonempty collection $\mathfrak{Z}$ of subsets of $\mathcal{Z}$ (identified with a subset $\mathfrak{Z} \subset 2^{\mathcal{Z}}$) which is stable under complementation and unlimited union (hence, under unlimited intersection)

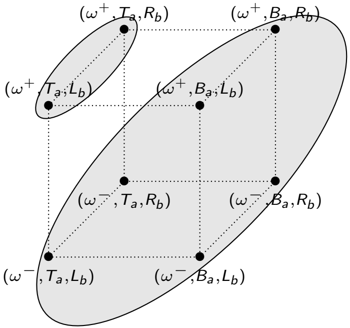
The couple $(\mathcal{Z}, \mathfrak{Z})$ is called a measurable space

Examples of σ-fields and partition fields

Let $\mathcal{Z}$ be a set

- $\mathfrak{Z} = \{\emptyset, \mathcal{Z}\}$ is the trivial σ-field (or trivial π-field)
- $\mathfrak{Z} = 2^{\mathcal{Z}}$ is the complete σ-field (or complete π-field)
- The atoms of a partition field are
  the minimal elements for the inclusion $\subset$ relation,
  and they form a partition of $\mathcal{Z}$ into undistinguishable elements





Operations on σ-fields

Let $\mathcal{Z}$ be a set and $\{\mathfrak{Z}_i\}_{i \in I}$ be a family of σ-fields

- $\bigwedge_{i \in I} \mathfrak{Z}_i$ is the largest σ-field included in all the $\mathfrak{Z}_i$, for $i \in I$ (it coincides with $\bigcap_{i \in I} \mathfrak{Z}_i$)
- $\bigvee_{i \in I} \mathfrak{Z}_i$ is the smallest σ-field that contains all the $\mathfrak{Z}_i$, for $i \in I$

Let $\{(\mathcal{Z}_i, \mathfrak{Z}_i)\}_{i \in I}$ be a family of measurable spaces

- $\bigotimes_{i \in I} \mathfrak{Z}_i$ is a (product) σ-field on the (product) set $\prod_{i \in I} \mathcal{Z}_i$ ($\bigotimes_{i \in I} \mathfrak{Z}_i$ is the smallest σ-field that contains all the cylinders)

Outline of the presentation

Agents, actions, Nature, configuration space

We distinguish an individual from an agent

- An individual who makes a first, followed by a second action, is represented by two agents (two decision makers)
- An individual who makes a sequence of actions — one for each period $t = 0, 1, 2, \ldots, T - 1$ — is represented by T agents, labelled $t = 0, 1, 2, \ldots, T - 1$
- N individuals — each i of whom makes a sequence of actions, one for each period $t = 0, 1, 2, \ldots, T_i - 1$ — is represented by $\prod_{i=1}^{N} T_i$ agents, labelled by

$$(i, t) \in \bigcup_{j=1}^{N} \{j\} \times \{0, 1, 2, \ldots, T_j - 1\}$$

Agents, actions and action spaces

- Let A be a (finite or infinite) set,
  whose elements are called agents (or decision-makers)
- With each agent $a \in A$ is associated a measurable space

$$(\mathcal{U}_a, \mathfrak{U}_a)$$

  where
  - the set $\mathcal{U}_a$ is the set of actions for agent a,
    where he makes one action $u_a \in \mathcal{U}_a$
  - the set $\mathfrak{U}_a \subset 2^{\mathcal{U}_a}$ is a σ-field (σ-algebra)

Examples

- $A = \{0, 1, 2, \ldots, T-1\}$ (T sequential actions),
  $(\mathcal{U}_a, \mathfrak{U}_a) = (\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$
- Energy producer's action: (peak, off-peak) prices (€)

$$u^{\mathrm{L}} = (\overline{u}^{\mathrm{L}}, \underline{u}^{\mathrm{L}}) \in \mathcal{U}^{\mathrm{L}} = \left\{(x, y) \in \mathbb{R}^2 \mid x \geq y\right\} \subset \mathbb{R}^2$$

Nature space

With Nature is associated a measurable space

$$(\Omega, \mathfrak{G})$$

where

- the set Ω is the set of states of Nature (uncertainties, scenarios, etc.) $\omega \in \Omega$
- the set $\mathfrak{G} \subset 2^{\Omega}$ is a σ-field (σ-algebra) (at this stage of the presentation, we do not need to equip $(\Omega, \mathfrak{G})$ with a probability distribution, as we only focus on information)

Examples

- Exogenous Nature: electricity demand (kWh)

$$\omega^{\mathsf{e}} \in \Omega^{\mathsf{e}} = \mathbb{R}_+$$

- Energy producer type: unitary production cost (€/kWh)

$$\omega^{\mathtt{L}} \in \Omega^{\mathtt{L}} = \mathbb{R}_+$$

The configuration space is a product space

Configuration space

The configuration space is the product space

$$\mathcal{H} = \Omega \times \mathcal{U}_A = \Omega \times \prod_{a \in A} \mathcal{U}_a$$

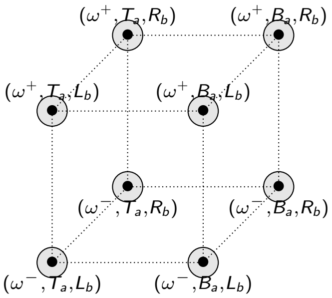
equipped with the product σ-field, called configuration field

$$\mathfrak{H} = \mathfrak{G} \otimes \mathfrak{U}_A = \mathfrak{G} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$$

so that $(\mathcal{H}, \mathfrak{H})$ is a measurable space

Example of configuration space

$(\mathcal{H}, \mathfrak{H})$



- product configuration space

$$\mathcal{H} = \Omega \times \prod_{a \in A} \mathcal{U}_a$$

- product configuration field

$$\mathfrak{H} = \mathfrak{G} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$$

Remark: a finite σ-field is represented by the partition of its atoms (minimal elements for inclusion)
Here, $\mathfrak{H} = 2^{\mathcal{H}}$ is represented by the partition of singletons

Example of configuration space in demand response

- Nature

$$\Omega = \underbrace{\mathbb{R}_+}_{\text{electricity demand}} \times \underbrace{\mathbb{R}_+}_{\text{unitary production cost}} \times \underbrace{\mathbb{R}_+}_{\text{unwillingness to shift}} = \mathbb{R}^3_+$$

The configuration space is the product space $\mathcal{H} = \Omega \times \mathcal{U}^{\mathtt{L}} \times \mathcal{U}^{\mathtt{F}}$

$$\mathcal{H} = \underbrace{\mathbb{R}^3_+}_{\text{Nature}} \times \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x \geq y\}}_{\text{(peak, off-peak) prices}} \times \underbrace{\{(\alpha,\beta) \in \mathbb{R}^2_+ \mid \alpha + \beta = 1\}}_{\text{consumption shift}}$$

Information fields

Information fields express dependencies

Information field of an agent

The information field of agent $a \in A$ is a σ-field

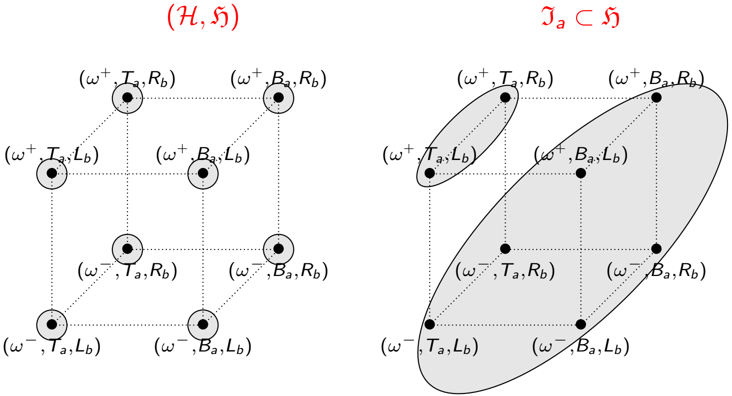
$$\mathfrak{I}_a \subset \mathfrak{H} = \mathfrak{G} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$$

which is a subfield of the product configuration field

- The subfield $\mathfrak{I}_a$ of the configuration field $\mathfrak{H}$ represents the information available to agent a when the agent chooses an action
- Therefore, the information of agent a may depend
  - on the states of Nature
  - and on other agents' actions

In the finite case, information fields are represented by the partition of its atoms

The information field of agent $a \in A$ is a subfield $\mathfrak{I}_a \subset \mathfrak{H} = \mathfrak{G} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$ which can, in the finite case, be represented by the partition of its atoms



Example of energy producer information field in demand response

The energy producer information field $\mathfrak{I}^{\mathrm{L}}$ is a subfield of the σ-field associated with the configuration space $\mathfrak{H} = \mathfrak{G}^{\mathrm{e}} \otimes \mathfrak{G}^{\mathrm{L}} \otimes \mathfrak{G}^{\mathrm{F}} \otimes \mathfrak{U}^{\mathrm{L}} \otimes \mathfrak{U}^{\mathrm{F}}$

$$\underbrace{\mathfrak{I}^{\mathrm{L}}}_{\substack{\text{leader's} \\ \text{information} \\ \text{field}}} = \underbrace{\{\emptyset, \Omega^{\mathrm{e}}\}}_{\substack{\text{cannot see} \\ \text{consumer's} \\ \text{demand}}} \otimes \underbrace{\mathfrak{G}^{\mathrm{L}}}_{\substack{\text{knows his} \\ \text{production} \\ \text{cost}}} \otimes \underbrace{\{\emptyset, \Omega^{\mathrm{F}}\}}_{\substack{\text{cannot see} \\ \text{consumer's} \\ \text{unwillingness} \\ \text{to shift}}} \otimes \underbrace{\{\emptyset, \mathcal{U}^{\mathrm{L}}\}}_{\substack{\text{absence of} \\ \text{self-information}}} \otimes \underbrace{\{\emptyset, \mathcal{U}^{\mathrm{F}}\}}_{\substack{\text{cannot see} \\ \text{consumer's} \\ \text{action}}}$$

Definition of the W-model (2 basic objects, 1 axiom)

W-model

A W-model $\left(A, (\Omega, \mathfrak{G}), (\mathcal{U}_a, \mathfrak{U}_a)_{a \in A}, (\mathfrak{I}_a)_{a \in A}\right)$
consists of 2 basic objects

(W-BO1a) the sample space $(\Omega, \mathfrak{G})$
equipped with a σ-field

(W-BO1b) the collection $(\mathcal{U}_a, \mathfrak{U}_a)_{a \in A}$
of agents' actions equipped with σ-fields

(W-BO2) the collection $(\mathfrak{I}_a)_{a \in A}$
of agents' information subfields of $\mathfrak{H} = \mathfrak{G} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$

and 1 axiom imposed on them

(W-Axiom1) for all agent $a \in A$, absence of self-information holds

$$\mathfrak{I}_a \subset \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \bigotimes_{b \in A \setminus \{a\}} \mathfrak{U}_b$$

Outline of the presentation

Alice and Bob

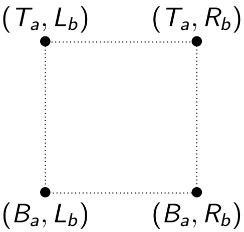
"Alice and Bob" configuration space

Alice and Bob are playing simultaneously

Example

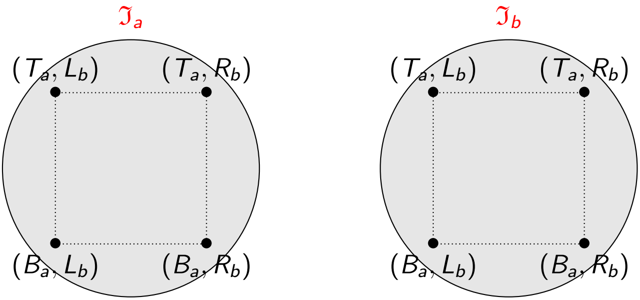
- no Nature
- two agents a (Alice) and b (Bob)
- two possible actions each $\mathcal{U}_a = \{T_a, B_a\}$, $\mathcal{U}_b = \{R_b, L_b\}$
- product configuration space (4 elements)

$$\mathcal{H} = \{T_a, B_a\} \times \{R_b, L_b\}$$



"Alice and Bob" information partitions

Alice and Bob are playing simultaneously

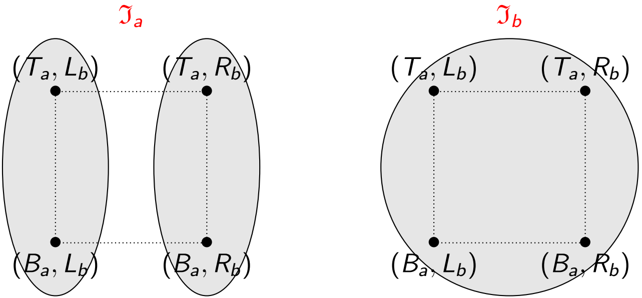


- $\mathfrak{I}_a = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$ (trivial σ-field)
  Alice knows nothing
- $\mathfrak{I}_b = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$ (trivial σ-field)
  Bob knows nothing

Alice knows Bob's action

"Alice and Bob" information partitions

Alice knows Bob's action



- $\mathfrak{I}_b = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$ (trivial σ-field)
  Bob knows nothing
- $\mathfrak{I}_a = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b\}, \{L_b\}, \{R_b, L_b\}\}$
  (cylindrical σ-field by absence of self-information)
  Alice knows what Bob does
  (as she can distinguish between Bob's actions $\{R_b\}$ and $\{L_b\}$)

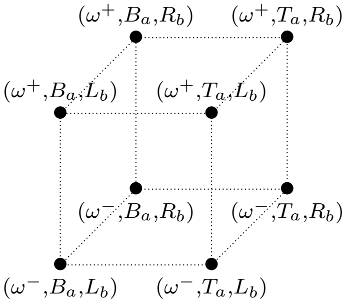
Alice, Bob and a coin tossing

"Alice, Bob and a coin tossing" configuration space

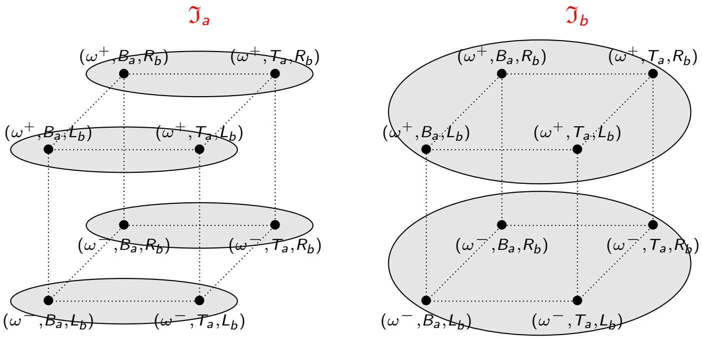
Example

- two states of Nature $\Omega = \{\omega^+, \omega^-\}$ (heads/tails)
- two agents a and b
- two possible actions each: $\mathcal{U}_a = \{T_a, B_a\}$, $\mathcal{U}_b = \{R_b, L_b\}$
- product configuration space (8 elements)

$$\mathcal{H} = \{\omega^+, \omega^-\} \times \{T_a, B_a\} \times \{R_b, L_b\}$$



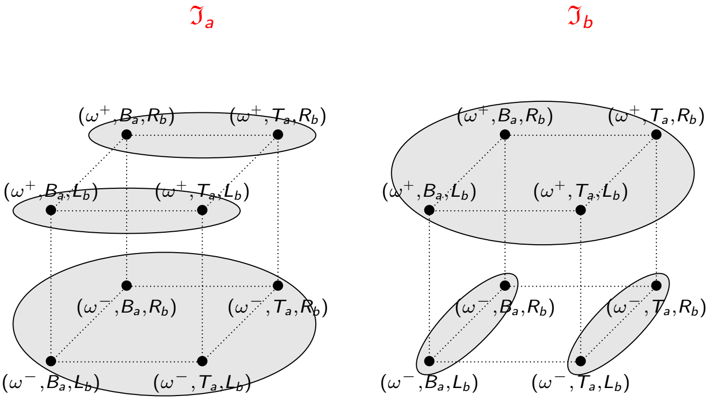
"Alice, Bob and a coin tossing" information partitions



$$\mathfrak{I}_b = \overbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}^{\text{Bob knows Nature's move}} \otimes \overbrace{\{\emptyset, \{T_a, B_a\}\}}^{\text{Bob does not know what Alice does}} \otimes \{\emptyset, \mathcal{U}_b\}$$

$$\mathfrak{I}_a = \underbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}_{\text{Alice knows Nature's move}} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \underbrace{\{\emptyset, \{R_b\}, \{L_b\}, \{R_b, L_b\}\}}_{\text{Alice knows what Bob does}}$$

"Alice, Bob and a coin tossing" information partitions



Outline of the presentation

Witsenhausen intrinsic model (W-model) [15 min]
- Agents, actions, Nature, configuration space, information fields
- Examples (basic)
- Examples (more advanced)
- Strategies, playability and solution map

Games in product form (W-game) [10 min]
- Game in product form (W-game)
- Normal form of a W-game
- Mixed and behavioral strategies

Conclusion

Stochastic control

Stochastic control

- Infinite (nonatomic) agents $A = [0, +\infty[$
- Decision of agent t taken in a set $\mathcal{U}_t$
- Filtration $\{\mathfrak{G}_t\}_{t \geq 0}$ of the sample space $(\Omega, \mathfrak{G})$

$$s \leq t \implies \mathfrak{G}_s \subset \mathfrak{G}_t \subset \mathfrak{G}$$

- Information of (nonanticipative) agent t is either modeled as

$$\mathfrak{I}_t \subset \underbrace{\mathfrak{G}_t}_{\text{partial observation of nature}} \otimes \underbrace{\bigotimes_{s \geq 0} \{\emptyset, \mathcal{U}_s\}}_{\text{no observation of actions}}$$

or as

$$\mathfrak{I}_t \subset \mathfrak{G}_t \otimes \underbrace{\bigotimes_{r < t} \mathfrak{U}_r}_{\text{memory of past actions}} \otimes \underbrace{\bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s\}}_{\text{no observation of future actions}}$$

Mean-field/dynamic game

Mean-field/dynamic game: data for information structures

- Infinite (mean-field) number of players $p \in P$
  (finite for dynamical games)
- Time either discrete, $t \in \mathbb{N}$, or continuous, $t \in [0, +\infty[$
- Agents are couples $a = (p, t)$
  making decisions in measurable sets $(\mathcal{U}_t^p, \mathfrak{U}_t^p)$
- Filtration $\{\mathfrak{G}_t\}_{t \geq 0}$ of the sample space $(\Omega, \mathfrak{G})$

$$s \leq t \implies \mathfrak{G}_s \subset \mathfrak{G}_t \subset \mathfrak{G}$$

Dynamic game: information structures

Information $\mathfrak{I}_{(p,t)}$ of (nonanticipative) agent (p,t)

$$\mathfrak{I}_{(p,t)} \subset \overbrace{\mathfrak{G}_t}^{\text{partial observation of nature}} \otimes \overbrace{\bigotimes_{p \in P} \bigotimes_{s \geq 0} \{\emptyset, \mathcal{U}_s^p\}}^{\text{no observation of actions}}$$

$$\mathfrak{I}_{(p,t)} \subset \mathfrak{G}_t \otimes \left(\underbrace{\bigotimes_{r<t} \mathfrak{U}_r^p}_{\text{memory of one's past actions}} \otimes \underbrace{\bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s^p\}}_{\text{no observation of one's future actions}} \right) \otimes \underbrace{\bigotimes_{q \in P \setminus \{p\}} \bigotimes_{s \geq 0} \{\emptyset, \mathcal{U}_s^q\}}_{\text{no observation of other players actions}}$$

$$\mathfrak{I}_{(p,t)} \subset \mathfrak{G}_t \otimes \bigotimes_{q \in P} \left(\underbrace{\bigotimes_{r<t} \mathfrak{U}_r^q}_{\text{memory of any player's past actions}} \otimes \underbrace{\bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s^q\}}_{\text{no observation of any player's future actions}} \right)$$

Mean-field game: information structures

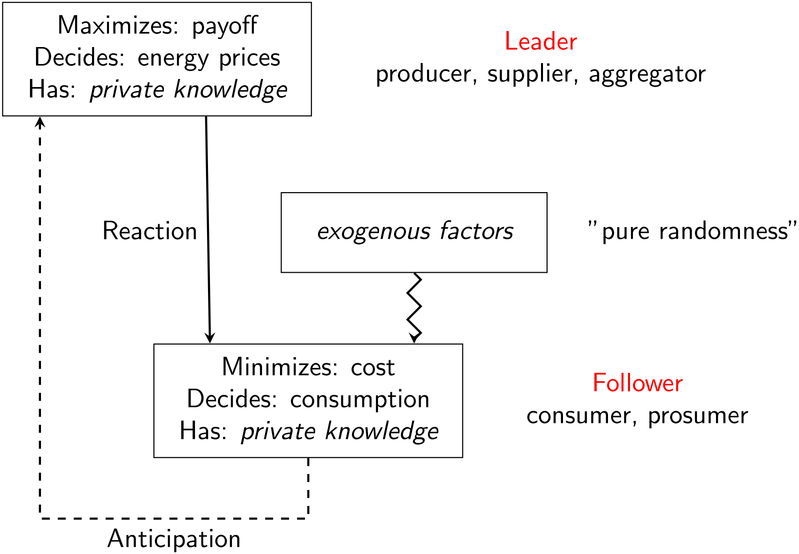
Information $\mathfrak{I}_{(p,t)}$ of (nonanticipative) agent (p, t)

$$\mathfrak{I}_{(p,t)} = \underbrace{\mathfrak{M}_t}_{\text{mean-field information}} \otimes \underbrace{\bigotimes_{q \in P} \bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s^q\}}_{\text{no observation of any player's future actions}}$$

$$\underbrace{\mathfrak{M}_t}_{\text{symmetric w.r.t. all players}} \subset \mathfrak{G}_t \otimes \underbrace{\bigotimes_{q \in P} \bigotimes_{r < t} \mathfrak{U}_r^q}_{\text{memory of all player's past actions}}$$

Principal-agent models
or
Leader-follower models

Leader-follower model in energy demand response



Principal-agent or leader-follower models with two decision-makers

A branch of Economics studies so-called principal-agent models, which can easily be expressed with Witsenhausen intrinsic model

To avoid confusion (between agent and agent...), we will shift to the vocable leader-follower models

- The model exhibits two decision-makers
  - the leader $\mathtt{L}$ makes actions $u_{\mathtt{L}}$ in $(\mathcal{U}_{\mathtt{L}}, \mathfrak{U}_{\mathtt{L}})$
  - the follower $\mathtt{F}$ makes actions $u_{\mathtt{F}}$ in $(\mathcal{U}_{\mathtt{F}}, \mathfrak{U}_{\mathtt{F}})$
- and Nature, corresponding to private information (or type) of the leader $\mathtt{L}$ or of the follower $\mathtt{F}$
  - Nature selects ω in $(\Omega, \mathfrak{G})$

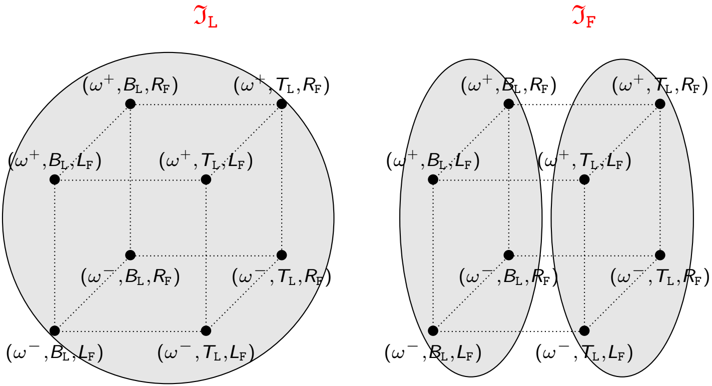
Classical information patterns in game theory

Now, we will make the information structure more specific

- Stackelberg leadership model
- Moral hazard (hidden action)
- Adverse selection (hidden type)
- Signaling

Stackelberg leadership model

Example of a (binary) Stackelberg leadership W-model



Stackelberg leadership model

- The follower $\mathtt{F}$ may partly observe the action of the leader $\mathtt{L}$

$$\mathfrak{I}_{\mathtt{F}} \overbrace{\subset}^{\substack{\text{partly}\\ \text{observes}}} \underbrace{\mathfrak{G}}_{\text{Nature}} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \underbrace{\mathfrak{U}_{\mathtt{L}}}_{\substack{\text{action of}\\ \text{the leader } \mathtt{L}}}$$

- whereas the leader $\mathtt{L}$ observes at most the state of Nature

$$\mathfrak{I}_{\mathtt{L}} \overbrace{\subset}^{\substack{\text{observes}\\ \text{at most}}} \underbrace{\mathfrak{G}}_{\text{Nature}} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

- As a consequence, the system is sequential
  - with the leader $\mathtt{L}$ as first decision-maker
  - and the follower $\mathtt{F}$ as second decision-maker

Moral hazard (hidden action)

Moral hazard (hidden action)

- An insurance company (the leader $\mathtt{L}$)
  - cannot observe the efforts of the insured (the follower $\mathtt{F}$) to avoid risky behavior,
  - whereas it faces the hazard that the insured person behaves "immorally"
    (like playing with matches at home)
- Moral hazard (hidden action) occurs when the actions of the follower $\mathtt{F}$ are hidden to the leader $\mathtt{L}$

$$\mathfrak{I}_{\mathtt{L}} \overbrace{\subset}^{\substack{\text{observes}\\ \text{at most}}} \underbrace{\mathfrak{G}}_{\text{Nature}} \otimes \underbrace{\{\emptyset, \mathcal{U}_{\mathtt{F}}\}}_{\substack{\text{hidden}\\ \text{action}}} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

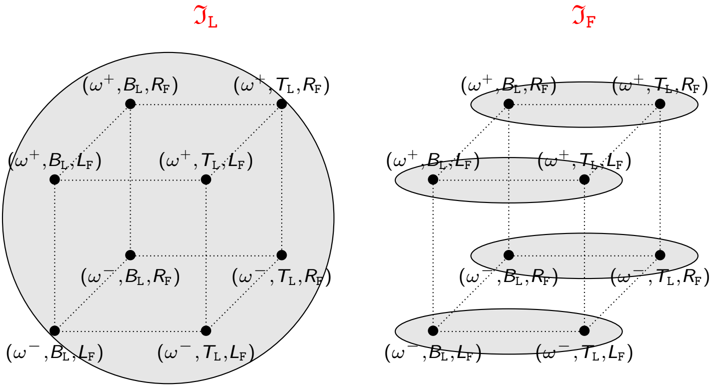
- In case of moral hazard, the system is sequential with the leader as first decision-maker

Adverse selection

Adverse selection

- In the absence of observable information on potential customers (the follower F),
- an insurance company (the leader L) offers a unique price for a contract,
- hence screens and selects the “bad” ones

Example of a (binary) adverse selection W-model



Adverse selection

Adverse selection occurs when

- the follower F knows the state of nature (her/his own type, or private information) and observes the leader L action (contract)

$$\underbrace{\mathfrak{G}}_{\text{Nature}} \otimes \{\emptyset, \mathcal{U}_{\mathrm{F}}\} \otimes \underbrace{\mathfrak{U}_{\mathrm{L}}}_{\substack{\text{action of}\\ \text{the leader L}}} \overbrace{\subset}^{\text{is observed}} \mathfrak{I}_{\mathrm{F}}$$

- but the leader L does not know the state of nature, that is, the agent F type

$$\mathfrak{I}_{\mathrm{L}} \overbrace{=}^{\text{knows}} \underbrace{\{\emptyset, \Omega\} \otimes \{\emptyset, \mathcal{U}_{\mathrm{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathrm{L}}\}}_{\text{nothing}}$$

In case of adverse selection, the system is sequential

Signaling

Signaling

- In biology, a peacock signals its "good genes" (genotype) by its lavish tail (phenotype)
- In economics, a worker signals her/his working ability (productivity) by her/his educational level (diplomas)

Signaling

There is room for signaling

- when the leader L knows the state of nature (her/his own type)

$$\mathfrak{I}_{\mathtt{L}} \overbrace{=}^{\text{knows}} \underbrace{\mathfrak{G}}_{\text{Nature}} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

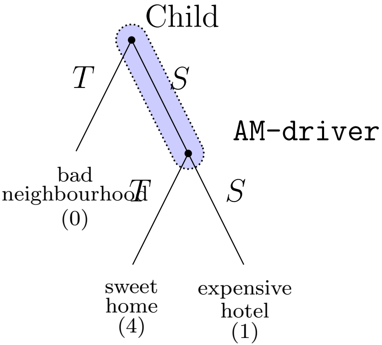
- whereas the follower F does not know the state of nature,
  that is, the leader L type,
  but the follower F observes the leader L action

$$\mathfrak{I}_{\mathtt{F}} \overbrace{=}^{\text{observes}} \{\emptyset, \Omega\} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \underbrace{\mathfrak{U}_{\mathtt{L}}}_{\substack{\text{action of} \\ \text{the leader L}}}$$

Thus, the leader L may reveal her/his type
by her/his action which is observable by the follower F

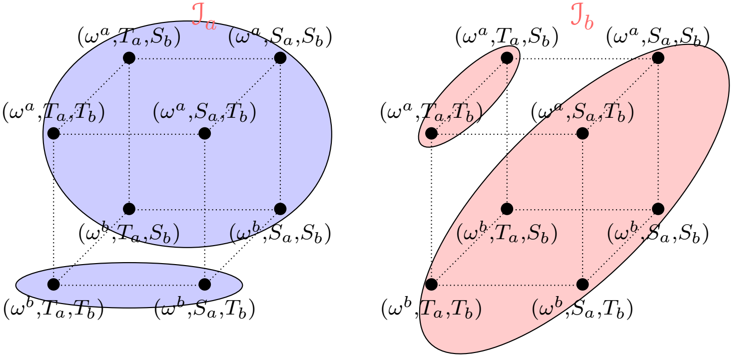
Absent-minded driver

Absent-minded driver



- S=Stay, T=Turn
- "paradox" that raised a problem in game theory
- the player looses public time, as plays "SS" "ST" cross the information set twice
- cannot be modelled *per se* in tree models (violates "no-AM" axiom)

A W-model for the absent-minded driver



$$\mathfrak{I}_a = \{\emptyset, \overbrace{\underbrace{\{\omega_a\} \times \mathcal{U}_a \times \mathcal{U}_b}_{\text{agent } a \text{ is whether the first one to act}} \cup \underbrace{\{\omega_b\} \times \{S_b\} \times \mathcal{U}_a}_{\text{or he acts second after agent } b \text{ has chosen } S}}^{\text{agent } a \text{ makes a move}}, \overbrace{\underbrace{\{\omega_b\} \times \{T_b\} \times \mathcal{U}_a}_{\text{agent } b \text{ chose } T \text{ and finished the game}}}^{\text{agent } a \text{ doesn't make a move}}, \mathcal{H}\}$$

$$\mathfrak{I}_b = \{\emptyset, \{\omega_b\} \times \mathcal{U}_a \times \mathcal{U}_b \cup \{\omega_a\} \times \{S_a\} \times \mathcal{U}_b, \{\omega_a\} \times \{T_a\} \times \mathcal{U}_b, \mathcal{H}\}$$

What land have we covered?
What comes next?

- The stage is in place; so are the actors
  - agents
  - Nature
  - information
- How can actors play?
  - strategies
  - playability

Outline of the presentation

Witsenhausen intrinsic model (W-model) [15 min]

- Agents, actions, Nature, configuration space, information fields
- Examples (basic)
- Examples (more advanced)
- Strategies, playability and solution map

Games in product form (W-game) [10 min]

- Game in product form (W-game)
- Normal form of a W-game
- Mixed and behavioral strategies

Conclusion

Strategies

Information is the fuel of W-strategies

W-strategy of an agent

A (pure) W-strategy of agent a is a mapping

$$\lambda_a : (\mathcal{H}, \mathfrak{H}) \to (\mathcal{U}_a, \mathfrak{U}_a)$$

which is measurable w.r.t. the information field $\mathfrak{I}_a$, that is,

$$\lambda_a^{-1}(\mathfrak{U}_a) \subset \mathfrak{I}_a$$

This condition expresses the property that
a W-strategy for agent a
may only depend upon the information $\mathfrak{I}_a$ available to the agent

Leader's information field and strategies

The leader information field $\mathfrak{I}^{\mathrm{L}}$ is a subfield of the σ-field associated with the configuration space $\mathfrak{H} = \mathfrak{G}^{\mathrm{e}} \otimes \mathfrak{G}^{\mathrm{L}} \otimes \mathfrak{G}^{\mathrm{F}} \otimes \mathfrak{U}^{\mathrm{L}} \otimes \mathfrak{U}^{\mathrm{F}}$

$$\underbrace{\mathfrak{I}^{\mathrm{L}}}_{\text{leader's information field}} = \underbrace{\{\emptyset, \Omega^{\mathrm{e}}\}}_{\text{cannot see consumer's demand}} \otimes \underbrace{\mathfrak{G}^{\mathrm{L}}}_{\text{knows his production cost}} \otimes \underbrace{\{\emptyset, \Omega^{\mathrm{F}}\}}_{\text{cannot see consumer's unwillingness to shift}} \otimes \underbrace{\{\emptyset, \mathcal{U}^{\mathrm{L}}\}}_{\text{absence of self-information}} \otimes \underbrace{\{\emptyset, \mathcal{U}^{\mathrm{F}}\}}_{\text{cannot see consumer's action}}$$

A leader's strategy is a mapping $\lambda^{\mathrm{L}} : (\mathcal{H}, \mathfrak{H}) \to (\mathcal{U}^{\mathrm{L}}, \mathfrak{U}^{\mathrm{L}})$ measurable with respect to his information field $\mathfrak{I}^{\mathrm{L}}$: $(\lambda^{\mathrm{L}})^{-1}(\mathfrak{U}^{\mathrm{L}}) \subset \mathfrak{I}^{\mathrm{L}}$

$$\underbrace{u^{\mathrm{L}}}_{\text{electricity prices}} = \underbrace{\lambda^{\mathrm{L}}}_{\text{leader's strategy}} \Big(\not{\omega}^{\mathrm{e}}, \underbrace{\omega^{\mathrm{L}}}_{\text{production costs}}, \not{\omega}^{\mathrm{F}}, \not{u}^{\mathrm{L}}, \not{u}^{\mathrm{F}} \Big)$$

Set of W-strategies

Set of W-strategies of an agent

We denote the set of (pure) W-strategies of agent a by

$$\Lambda_a = \left\{ \lambda_a : (\mathcal{H}, \mathfrak{H}) \to (\mathcal{U}_a, \mathfrak{U}_a) \,\middle|\, \lambda_a^{-1}(\mathfrak{U}_a) \subset \mathfrak{I}_a \right\}$$

and the set of W-strategies of all agents is

$$\Lambda = \Lambda_A = \prod_{a \in A} \Lambda_a$$

Examples of W-strategies

Consider a W-model with two agents a and b,
and suppose that σ-fields $\mathfrak{U}_a$, $\mathfrak{U}_b$ and $\mathfrak{G}$ contain the singletons

- Absence of self-information

$$\mathfrak{I}_a \subset \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b \subset \mathfrak{G} \otimes \mathfrak{U}_a \otimes \{\emptyset, \mathcal{U}_b\}$$

  Then, W-strategies λ_a and λ_b have the form

$$\lambda_a(\omega, \cancel{u_a}, u_b) = \widetilde{\lambda_a}(\omega, u_b) \ , \ \ \lambda_b(\omega, u_a, \cancel{u_b}) = \widetilde{\lambda_b}(\omega, u_a)$$

- Sequential W-model

$$\mathfrak{I}_a = \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b = \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

  Then, W-strategies λ_a and λ_b have the form

$$\lambda_a(\omega, u_b, \cancel{u_a}) = \widetilde{\lambda}_a(\omega, u_b) \ , \ \ \lambda_b(\omega, \cancel{u_b}, \cancel{u_a}) = \widetilde{\lambda}_b(\omega)$$

Playability

Playability

- In the Witsenhausen's intrinsic model,
  agents make actions in an order which is not fixed in advance
- Briefly speaking, playability ("solvability" in Witsenhausen's terms)
  is the property that, for each state of Nature,
  the agents' actions are uniquely determined by their W-strategies

Playability problem

The playability (solvability) problem consists in finding

- for any collection $\lambda = \{\lambda_a\}_{a \in A} \in \Lambda_A$ of W-strategies
- for any state of Nature $\omega \in \Omega$

actions $u \in \mathcal{U}_A$ satisfying
the implicit ("closed loop") equation

$$u = \lambda(\omega, u)$$

or, equivalently, the family of "closed loop" equations

$$\boxed{u_a = \lambda_a(\omega, \{u_b\}_{b \in A}) \ , \ \ \forall a \in A}$$

$$u = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \dots & x_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix} u \ , \quad u = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ 0 & x_{22} & x_{23} & \dots & x_{2n} \\ 0 & 0 & x_{33} & \dots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{dn} \end{bmatrix} u$$

Playability property

Playability property

A W-model displays the playability property when the "closed loop" equation $u = \lambda(\omega, u)$ has a unique solution for any collection $\lambda = \{\lambda_a\}_{a \in A} \in \Lambda_A$ of W-strategies and for any state of Nature $\omega \in \Omega$, that is,

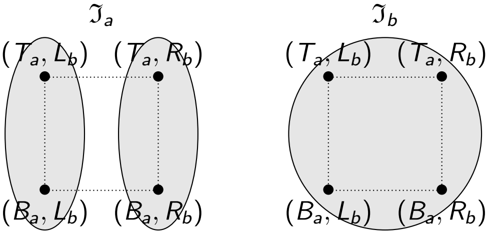
$$\forall \lambda = (\lambda_a)_{a \in A} \in \Lambda_A \ , \ \ \forall \omega \in \Omega \ , \ \ \exists ! u \in \mathcal{U}_A \ , \ \ u = \lambda(\omega, u)$$

or, equivalently, when

$$\forall \lambda = (\lambda_a)_{a \in A} \in \Lambda_A \ , \ \ \forall \omega \in \Omega \ , \ \ \exists ! u \in \mathcal{U}_A \ , \\ u_a = \lambda_a(\omega, \{u_b\}_{b \in A}) \ , \ \ \forall a \in A$$

Playability is a property of the information structure

Sequentiality



Sequential W-model

$$\mathfrak{I}_a = \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b = \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

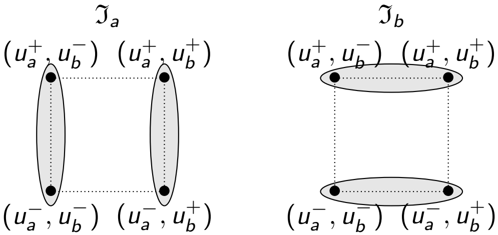
The closed-loop equations

$$u_a = \lambda_a(\omega, u_b, \cancel{u_a}) = \widetilde{\lambda}_a(\omega, u_b) \ , \ \ u_b = \lambda_b(\omega, \cancel{u_b}, \cancel{u_a}) = \widetilde{\lambda}_b(\omega)$$

always displays a unique solution (u_a, u_b),
whatever $\omega \in \Omega$ and W-strategies λ_a and λ_b

Playability is a property of the information structure

Deadlock



W-model with deadlock

$$\mathfrak{I}_a = \{\emptyset, \Omega\} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \;, \;\; \mathfrak{I}_b = \{\emptyset, \Omega\} \otimes \mathfrak{U}_a \otimes \{\emptyset, \mathcal{U}_b\}$$

The closed-loop equations

$$u_a = \lambda_a(\cancel{u_a}, u_b) = \tilde{\lambda}_a(u_b) \;, \;\; u_b = \lambda_b(u_a, \cancel{u_b}) = \tilde{\lambda}_b(u_a)$$

may display zero solutions, one solution or multiple solutions, depending on the W-strategies λ_a and λ_b

Playability makes it possible to define a solution map from states of Nature towards configurations

Suppose that the playability property holds true

Solution map

We define the solution map

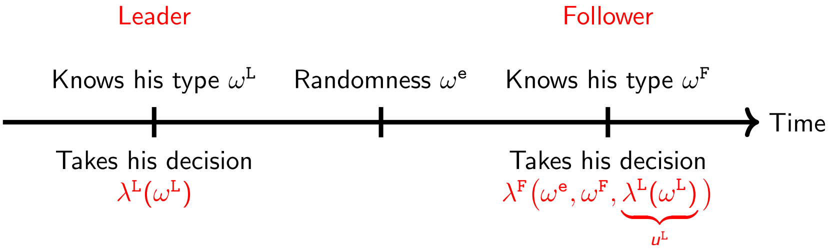
$$S_\lambda : \Omega \to \mathcal{H} = \Omega \times \mathcal{U}_A = \Omega \times \prod_{a \in A} \mathcal{U}_a$$

that maps states of Nature towards configurations, by

$$(\omega, u) = S_\lambda(\omega) \iff u = \lambda(\omega, u) \ , \ \ \forall (\omega, u) \in \Omega \times \mathcal{U}_A$$

We include the state of Nature ω in the image of $S_\lambda(\omega)$, so that we map the set Ω towards the configuration space $\mathcal{H}$, making it possible to interpret $S_\lambda(\omega)$ as a configuration driven by the W-strategy λ (in classical control theory, a state trajectory is produced by a policy)

A sequential (hence playable) information structure



When playability holds true, the solution map
is the mapping $S_{\lambda^{\mathrm{L}},\lambda^{\mathrm{F}}} : \Omega \to \mathcal{H}$ which gives for every state of Nature
the unique outcome

$$S_{\lambda^{\mathrm{L}},\lambda^{\mathrm{F}}}(\omega^{\mathrm{e}}, \omega^{\mathrm{L}}, \omega^{\mathrm{F}}) = \Big(\omega^{\mathrm{e}}, \omega^{\mathrm{L}}, \omega^{\mathrm{F}}, \underbrace{\lambda^{\mathrm{L}}(\omega^{\mathrm{L}})}_{u^{\mathrm{L}}}, \underbrace{\lambda^{\mathrm{F}}\big(\omega^{\mathrm{e}}, \omega^{\mathrm{F}}, \lambda^{\mathrm{L}}(\omega^{\mathrm{L}})\big)}_{u^{\mathrm{F}}}\Big)$$

In the sequential case, the solution map is given by iterated composition

- In the sequential case

$$\mathfrak{I}_b = \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\} \ , \ \ \mathfrak{I}_a = \mathfrak{G} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b$$

- W-strategies λ_b and λ_a have the form

$$\lambda_b(\omega, \not{u_b}, \not{u_a}) = \widetilde{\lambda}_b(\omega) \ , \ \ \lambda_a(\omega, \not{u_a}, u_b) = \widetilde{\lambda}_a(\omega, u_b)$$

- so that the solution map is

$$S_\lambda(\omega) = \Big(\omega, \widetilde{\lambda}_a\Big(\omega, \widetilde{\lambda}_b(\omega)\Big), \widetilde{\lambda}_b(\omega)\Big)$$

- because the system of equations $u = \lambda(\omega, u)$ here writes

$$u_b = \lambda_b(\omega, \not{u_a}, \not{u_b}) = \widetilde{\lambda}_b(\omega) \ , \ \ u_a = \lambda_a(\omega, \not{u_a}, u_b) = \widetilde{\lambda}_a(\omega, u_b)$$

With playability, hence with a solution map, one obtains a game form

Game form

A playable W-model induces a game form
by means of the outcome mapping

$$S : \Lambda \to \mathcal{H}^{\Omega}$$
$$\lambda \mapsto S_{\lambda}$$

from strategies towards the outcome space $\mathcal{H}^{\Omega}$ (random variables)

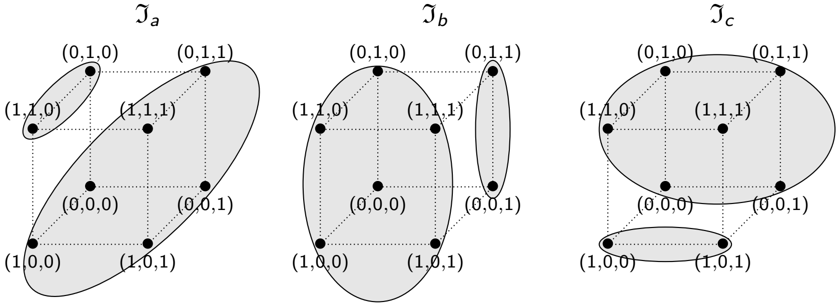
If the W-model is not playable, we get a set-valued mapping (correspondence)

A game that can be played but that cannot start: the clapping hand game

- [Three players:] Alice, Bob and Carol
  are sitting around a circular table, with their eyes closed
- [Two decisions:] Each of them has to decide either to extend her/his left hand to the left or to extend her/his right hand to the right
- [Information:] when two hands touch, the remaining player is informed (say, a clap is directly conveyed to her/his ears);
  when two hands do not touch, the remaining player is not informed
- [Strategies:] for each player, a strategy is a mapping
  $\{\text{clap}, \text{no clap}\} \to \{\text{left}, \text{right}\}$
- [Playability:] for each triplet of strategies — one for each of Alice, Bob and Carol — there is a unique outcome of extended hands:
  the game is playable
- [No tree:] however, the game cannot start,
  hence this playable game cannot be written on a tree

Playable noncausal example [Witsenhausen, 1971]

- No Nature, $A = \{a, b, c\}$, $\mathcal{U}_a = \mathcal{U}_b = \mathcal{U}_c = \{0, 1\}$
- Set of configurations $\mathcal{H} = \{0, 1\}^3$, and information fields
  $\mathfrak{I}_a = \sigma(u_b(1 - u_c))$, $\mathfrak{I}_b = \sigma(u_c(1 - u_a))$, $\mathfrak{I}_c = \sigma(u_a(1 - u_b))$
- The "game" can be played but... cannot be started (no first agent)



What land have we covered?
What comes next?

- The stage is in place; so are the actors
  - agents
  - Nature
  - information
- Actors know how they can play
  - W-strategies
  - playability
- In a noncooperative context,
  we will now define players as "team leaders of agents"
  - playing mixed strategies
  - endowed with preferences
    (in particular, with objectives and beliefs)

What comes next?

- Players and preferences
- Game in product form (W-game)
- Normal form of a W-game
- Mixed and behavioral strategies

Outline of the presentation

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Witsenhausen intrinsic model (W-model) [15 min]

- Agents, actions, Nature, configuration space, information fields
- Examples (basic)
- Examples (more advanced)
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Games in product form (W-game) [10 min]

- Game in product form (W-game)
- Normal form of a W-game
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Conclusion

Players

A player holds a team of executive agents

- The set of players is denoted by P (finite or infinite set)
- Every player $p \in P$ has
  a team of executive agents

$$A^p \subset A$$

  where $(A^p)_{p \in P}$ forms a partition of the set A of agents

$$A = \underbrace{\bigcup_{p \in P} A^p}_{\text{partition}}$$

- A player is a team leader

Preferences

Preference relations on random variables

- As a playable W-model induces a game form by means of the outcome mapping

$$
\begin{aligned}
S \colon \Lambda &\to \mathcal{H}^{\Omega} \\
\lambda &\mapsto S_{\lambda}
\end{aligned}
$$

from strategies towards the outcome space $\mathcal{H}^{\Omega}$ (random variables)

- the preferences of each player $p \in P$ are represented by a preference relation $\preceq^{p}$ on (a subset of) the outcome space $\mathcal{H}^{\Omega}$ (random variables)

Game in product form

Game in product form

[Heymann, De Lara, and Chancelier, 2022]

Game in product form / W-game

A game in product form or W-game is
a W-model, endowed with an additional layer made of

- with a partition of the set of agents,
  whose atoms are the players
- where each player $p \in P$ is endowed with
  a preference relation $\preceq^p$ on (a subset of) the outcome space $\mathcal{H}^{\Omega}$
  (random variables)

Outline of the presentation

Preferences deduced from
objective functions and beliefs

Players can be endowed with objective functions and beliefs

Every player $p \in P$ has

- a team of executive agents

$$A^p \subset A$$

where $(A^p)_{p \in P}$ forms a partition of the set A of agents

- a criterion/objective function/payoff (to be maximized)

$$j^p : \mathcal{H} \to [-\infty, +\infty[$$

a $\mathfrak{H}$-measurable function over the configuration space $\mathcal{H}$ (where the value $-\infty$ stands for possible constraints on configurations)

- a belief

$$\beta^p : \mathfrak{G} \to [0, 1]$$

a probability distribution over the states of Nature $(\Omega, \mathfrak{G})$

Preference relation on random variables

Preference relation on random variables

Let $\mathbf{X}, \mathbf{Y}$ be two elements in (a subset of) the outcome space $\mathcal{H}^{\Omega}$ (random variables)

$$\mathbf{X} \preceq^{p} \mathbf{Y} \iff \mathbb{E}_{\beta^p}\left[j^p \circ \mathbf{X}\right] \leq \mathbb{E}_{\beta^p}\left[j^p \circ \mathbf{Y}\right]$$

when the integrals (expectations) make sense

Strategy profiles

Strategy profiles

- A strategy for player p is an element of

$$\Lambda^p = \prod_{a \in A_p} \Lambda_a$$

- The set of strategies for all players is

$$\prod_{p \in P} \Lambda^p = \prod_{p \in P} \prod_{a \in A_p} \Lambda_a = \prod_{a \in A} \Lambda_a = \Lambda_A$$

- A strategy profile is

$$\lambda = (\lambda^p)_{p \in P} \in \prod_{p \in P} \Lambda^p$$

- When we focus on player p, we write

$$\lambda = (\lambda^{-p}, \lambda^p) \in \Lambda^p \times \underbrace{\prod_{p' \neq p} \Lambda_{A_{p'}}}_{\Lambda^{-p}}$$

How player p evaluates a strategy profile λ

- Measurable solution map attached to $\lambda \in \Lambda_A$ is

$$S_\lambda : \Omega \to \mathcal{H}$$

- Measurable criterion (costs or payoffs) is

$$j^p : \mathcal{H} \to \overline{\mathbb{R}}$$

- The composition of criteria with the solution map provides a random variable

$$j^p \circ S_\lambda : \Omega \to \overline{\mathbb{R}}$$

- The random variable can be integrated w.r.t. the belief β^p, yielding

$$\mathbb{E}_{\beta^p}\left[j^p \circ S_\lambda \right] \in \overline{\mathbb{R}}$$

where $\mathbb{E}_{\beta^p}$ denotes the mathematical expectation w.r.t. the probability β^p on $(\Omega, \mathfrak{G})$

W-game in normal form

We can now turn a W-game into a W-game in normal form

Every player $p \in P$ has

- a strategy set

$$\Lambda^p = \prod_{a \in A_p} \Lambda_a$$

- a normal form objective function
  from strategies profiles to the (extended) real numbers

$$\lambda \in \prod_{p \in P} \Lambda^p \mapsto \mathbb{E}_{\beta^p}\left[j^p \circ S_\lambda\right] = \mathbb{E}_{\beta^p \circ S_\lambda^{-1}}\left[j^p\right]$$

Nash equilibrium

Normal form objective functions, Nash equilibrium

Data of player $p \in P$

$$d^p = (j^p, \beta^p)$$

Normal form objective function for player $p \in P$

$$J^p(\lambda^p, \lambda^{-p}; d^p) = \mathbb{E}_{\beta^p}[j^p \circ S_{\lambda^p, \lambda^{-p}}]$$

Set of best responses for player $p \in P$

$$\Lambda^p_{\mathcal{N}}(\underline{\lambda}^{-p}; d^p) = \arg\min_{\lambda^p \in \Lambda^p} J^p(\lambda^p, \underline{\lambda}^{-p}; d^p) \subset \Lambda^p$$

A strategy profile $\underline{\lambda} = (\underline{\lambda}^p)_{p \in P} \in \Lambda^P$ is said to be a Nash equilibrium if

$$\underline{\lambda}^p \in \Lambda^p_{\mathcal{N}}(\underline{\lambda}^{-p}; d^p) \ , \ \ \forall p \in P$$

Outline of the presentation

Pure W-strategies profiles

- A pure W-strategy for player p is an element of

$$\Lambda_{A^p} = \prod_{a \in A^p} \Lambda_a$$

- The set of pure W-strategies for all players is

$$\prod_{p \in P} \Lambda_{A^p} = \prod_{p \in P} \prod_{a \in A^p} \Lambda_a = \prod_{a \in A} \Lambda_a = \Lambda_A$$

- A W-strategy profile is

$$\lambda = (\lambda^p)_{p \in P} \in \prod_{p \in P} \Lambda_{A^p}$$

- When we focus on player p, we write

$$\lambda = (\lambda^p, \lambda^{-p}) \in \Lambda_{A^p} \times \underbrace{\prod_{p' \neq p} \Lambda_{A^{p'}}}_{\Lambda^{-p}}$$

Mixed and behavioral strategies “à la Aumann”

For any player $p \in P$ and agent $a \in A^p$, we denote by

- $\big(\mathcal{W}_a, \mathfrak{W}_a\big)$ a copy of the Borel space $\big([0,1], \mathfrak{B}([0,1])\big)$
- ℓ_a a copy of the Lebesgue measure on $\big(\mathcal{W}_a, \mathfrak{W}_a\big) = \big([0,1], \mathfrak{B}([0,1])\big)$

and we define a probability space (random generator) $(\mathcal{W}^p, \mathfrak{W}^p, \ell^p)$ attached to player p by

$$\mathcal{W}^p = \prod_{a \in A^p} \mathcal{W}_a \ , \ \ \mathfrak{W}^p = \bigotimes_{a \in A^p} \mathfrak{W}_a \ , \ \ \ell^p = \bigotimes_{a \in A^p} \ell_a$$

and we also set

$$\mathcal{W} = \prod_{p \in P} \mathcal{W}^p \ , \ \ \mathfrak{W} = \bigotimes_{p \in P} \mathfrak{W}^p \ , \ \ \ell = \bigotimes_{p \in P} \ell^p$$

Mixed, behavioral and pure strategies “à la Aumann”: definition

For the player $p \in P$,

- an A-mixed strategy is a family $m^p = \{m_a\}_{a \in A^p}$ of measurable mappings

$$m_a : \Big(\prod_{b \in A^p} \mathcal{W}_b \times \mathcal{H}, \bigotimes_{b \in A^p} \mathfrak{W}_b \otimes \mathfrak{I}_a \Big) \to (\mathcal{U}_a, \mathfrak{U}_a) \ , \ \ \forall a \in A^p$$

- an A-behavioral strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that

$$m_a^{-1}(\mathfrak{U}_a) \subset \Big(\mathfrak{W}_a \otimes \bigotimes_{b \in A^p \setminus \{a\}} \{\emptyset, \mathcal{W}_b\} \otimes \mathfrak{I}_a \Big) \ , \ \ \forall a \in A^p$$

- an A-pure strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that

$$m_a^{-1}(\mathfrak{U}_a) \subset \bigotimes_{b \in A^p} \{\emptyset, \mathcal{W}_b\} \otimes \mathfrak{I}_a \ , \ \ \forall a \in A^p$$

Mixed, behavioral and pure strategies "à la Aumann": interpretation

For the player $p \in P$,

- an A-mixed strategy is a family $m^p = \{m_a\}_{a \in A^p}$ such that, for any configuration $h \in \mathcal{H}$,

$$m_a(\cdot, h) : \Big(\prod_{b \in A^p} \mathcal{W}_b, \bigotimes_{b \in A^p} \mathfrak{W}_b, \bigotimes_{a \in A^p} \ell_a \Big) \to (\mathcal{U}_a, \mathfrak{U}_a) \ , \quad \forall a \in A^p$$

  is a random variable
- an A-behavioral strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that, for any configuration $h \in \mathcal{H}$, the random variables $\{m_a(\cdot, h)\}_{a \in A^p}$ are independent
- an A-pure strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that, for any configuration $h \in \mathcal{H}$, the random variables $\{m_a(\cdot, h)\}_{a \in A^p}$ are constant

Outline of the presentation

Witsenhausen intrinsic model (W-model) [15 min]

Games in product form (W-game) [10 min]

Conclusion

Potential of W-models and W-games

W-models and W-games cover

- deterministic games (with finite or measurable action sets)
- deterministic dynamic games (countable time span)
- Bayesian games
- stochastic dynamic games (countable time span)
- games in Kuhn extensive form (countable time span)

For games with continuous time span,
the W-model has to be adapted (configuration-orderings)

Conclusion

- a rich language
- a lot of open questions, and a lot of things not yet properly defined
- we are looking for feedback

Thank you :-)

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Example: Don Juan wants to get married

Don Juan wants to get married

- Player Don Juan p is considering giving a phone call
  to his ex-lovers q, r (players),
  asking them if they want to marry him
- Don Juan selects one of his ex-lovers in the set $\{q, r\}$ and phones her
- If the answer to the first phone call is "yes",
  Don Juan marries the first called ex-lover
  (and decides not to give a second phone call)
- If the answer to the first phone call is "no",
  Don Juan makes a second phone call to the remaining ex-lover
- In that case, the remaining ex-lover answers "yes" or "no"

Agents, decisions, players

- Four agents partitioned in three players

$$A = \{ \overbrace{p_1, p_2}^{\text{Don Juan } p} , \overbrace{q}^{\text{ex-lover } q} , \overbrace{r}^{\text{ex-lover } r} \}$$

because player Don Juan p makes decisions
at possibly two occasions, hence has two executive agents p_1, p_2

- No Nature, but finite decisions sets

$$\mathcal{U}_{p_1} = \{q, r\} \; , \; \mathcal{U}_{p_2} = \{q, r, \partial\} \; , \; \mathcal{U}_q = \{Y, N\} \; , \; \mathcal{U}_r = \{Y, N\}$$

  - Agent p_1 selects an ex-lover in the set $\mathcal{U}_{p_1} = \{q, r\}$ and phones her
  - Agent p_2 either stops (decision ∂) or selects an ex-lover in $\{q, r\}$
  - Agents q, r either say "yes" or "no", hence select a decision in the set $\{Y, N\}$

- The finite decisions sets $\mathcal{U}_{p_1}, \mathcal{U}_{p_2}, \mathcal{U}_q, \mathcal{U}_r$ are equipped with the complete finite σ-fields $\mathfrak{U}_{p_1} = 2^{\mathcal{U}_{p_1}}, \mathfrak{U}_{p_2} = 2^{\mathcal{U}_{p_2}}, \mathfrak{U}_q = 2^{\mathcal{U}_q}, \mathfrak{U}_r = 2^{\mathcal{U}_r}$

Information structure: Don Juan

$$\mathcal{H} = \mathcal{U}_{p_1} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r$$

- When agent Don Juan p_1 makes the first phone call, he knows nothing, represented by his trivial information field

$$\mathfrak{I}_{p_1} = \{\emptyset, \mathcal{U}_{p_1}\} \otimes \{\emptyset, \mathcal{U}_{p_2}\} \otimes \{\emptyset, \mathcal{U}_q\} \otimes \{\emptyset, \mathcal{U}_r\}$$

- The agent Don Juan p_2 remembers who Don Juan p_1 called first, and knows the answer, which is represented by his information field

$$\mathfrak{I}_{p_2} = \{\emptyset, \mathcal{U}_{p_1} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r,$$

$$\underbrace{\{q\}}_{\text{remembering}} \times \overbrace{\{\emptyset, \mathcal{U}_{p_2}\}}^{\text{absence of self-information}} \times \underbrace{\mathfrak{U}_q}_{\text{knowing the answer}} \times \{\emptyset, \mathcal{U}_r\},$$

$$\{r\} \times \{\emptyset, \mathcal{U}_{p_2}\} \times \{\emptyset, \mathcal{U}_q\} \times \mathfrak{U}_r\}$$

Information structure: ex-lovers

- If ex-lover q receives a phone call from Don Juan, she does not know if she was called first or second, hence she cannot distinguish the elements in the set

$$\{\underbrace{(q,q),(q,r),(q,\partial)}_{\text{called first}}, \quad \underbrace{(r,q)}_{\text{called second}}\}$$

so that her information field is

$$\mathfrak{I}_q = \{\emptyset, \underbrace{\{(q,q),(q,r),(q,\partial),(r,q)\}}_{\text{called}}, \underbrace{\{(r,r),(r,\partial)\}}_{\text{not called}}, \mathcal{U}_{p_1} \times \mathcal{U}_{p_2}\}$$
$$\otimes \{\emptyset, \mathcal{U}_q\} \otimes \{\emptyset, \mathcal{U}_r\}$$

- Conversely, ex-lover r is equipped with the σ-field

$$\mathfrak{I}_r = \{\emptyset, \{(r,r),(r,q),(r,\partial),(q,r)\}, \{(q,q),(q,\partial)\}, \mathcal{U}_{p_1} \times \mathcal{U}_{p_2}\} \otimes \{\emptyset, \mathcal{U}_q\} \otimes \{\emptyset, \mathcal{U}_r\}$$

A causal but nonsequential system

If Don Juan p_1 calls ex-lover q first, the agents play in the following order

$$p_1 \to q \to p_2 \to r$$

and conversely

- Configuration space

$$\mathcal{H} = \mathcal{U}_{p_1} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r$$

- Configuration space partition

$$\mathcal{H}_q = \{q\} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r \ , \ \ \mathcal{H}_r = \{r\} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r$$

- A non constant history-ordering mapping is

$$\varphi : \mathcal{H} \to \{(p_1, q, p_2, r), (p_1, r, p_2, q)\}$$

  such that

$$\varphi_{|\mathcal{H}_q} \equiv (p_1, q, p_2, r) \ , \ \ \varphi_{|\mathcal{H}_r} \equiv (p_1, r, p_2, q)$$