Rank-Based Norms, Capra-Conjugacies and the Rank Function

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Some examples about matrix rank

We denote by rk the rank function,
defined on $\mathcal{M}_{m,n}$, the space of real matrices with m rows and n columns,
which gives the number of independent columns of a matrix and satisfies

$$0 \leq \mathrm{rk}(M) \leq \min(m, n) \quad \text{and} \quad \mathrm{rk}(M) = 0 \iff M = 0$$

Rank of some matrices

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{rank } 0} \qquad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{rank } 1} \qquad \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}}_{\text{rank } 1} \qquad \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\text{rank } 2}$$

The rank function is a discontinuous function

$$\mathrm{rk} \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 1, \forall \epsilon \neq 0 \quad \text{whereas} \quad \mathrm{rk} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Background on norms

Definition

A function $|||\cdot||| : E \to \mathbb{R}_+$ is a norm on a vector space E if it satisfies

1. $|||x||| = 0 \iff x = 0, \forall x \in E$ (point-separating)
2. $|||\lambda x||| = |\lambda| |||x|||, \forall x \in E, \forall \lambda \in \mathbb{R}$ (homogeneity)
3. $|||x + y||| \leq |||x||| + |||y|||, \forall x, y \in E$ (triangle inequality)

Some matrix norms

$$\left|\left|\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|\right|\right| = \sqrt{a^2 + b^2 + c^2 + d^2} \,, \qquad \left|\left|\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|\right|\right| = |a| + 2|b| + 3|c| + 4|d|$$

How do we define a dual norm

We would like to write

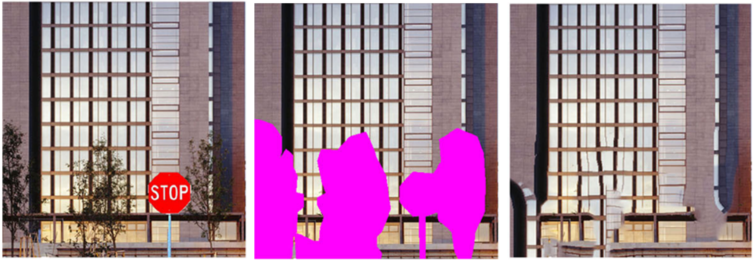
$$\underbrace{\mathrm{Tr}(MN^{\mathrm{T}})}_{\text{scalar product}} \leq \underbrace{|||M|||}_{\text{source norm}} \times \underbrace{|||N|||_\star}_{\text{dual norm}}$$

and a solution is to define

$$|||N|||_\star \equiv \sup_{|||M||| \leq 1} \mathrm{Tr}(MN^{\mathrm{T}}) \,, \quad \forall N \in \mathcal{M}_{m,n}$$

An optimization problem using the rank function

Image completion



(a) initial image M^* (b) area to modify Ω (c) final result

Figure: A sparse image completion

A way to solve it

$$\min_{M \text{ s.t. } M_{i,j}=M^*_{i,j}, \forall (i,j)\notin\Omega} \text{rk}(M)$$

Main result: variational lower bound of the rank function

- We denote by $\mathcal{M}_{m,n}$ the space of real matrices with m rows and n columns, and $d = \min(m, n)$

Theorem

Let $|||\cdot|||$ be a (source) norm on the space $\mathcal{M}_{m,n}$ of matrices
We have the following variational lower bound of the rank function

$$\mathrm{rk}(M) \geq \frac{1}{|||M|||} \min_{\substack{M^{(1)}\in\mathcal{M}_{m,n},\ldots,M^{(d)}\in\mathcal{M}_{m,n} \\ \sum_{r=1}^{d} |||M^{(r)}|||_{(r)}^{\mathrm{rk}} \leq |||M||| \\ \sum_{r=1}^{d} M^{(r)} = M}} \sum_{r=1}^{d} r \underbrace{|||M^{(r)}|||_{(r)}^{\mathrm{rk}}}_{\textit{rank based norms}}$$

for all matrix $M \in \mathcal{M}_{m,n} \setminus \{0\}$

Outline of the presentation

Let $|||\cdot|||$ be a norm on the space $\mathcal{M}_{m,n}$ of matrices, that we call *source (matrix) norm*

Proposition

The following expression

$$|||N|||^{\mathrm{rk}}_{(r),\star} = \sup_{\mathrm{rk}(M)\leq r, |||M|||\leq 1} \mathrm{Tr}(MN^{\mathrm{T}}) \; , \; \forall N \in \mathcal{M}_{m,n} \; , \; \forall r \in [\![1, d]\!]$$

define a nondecreasing sequence $\left\{|||\cdot|||^{\mathrm{rk}}_{(r),\star}\right\}_{r\in[\![1,d]\!]}$ *of norms on* $\mathcal{M}_{m,n}$

Rank Based Norms

Generalized r-rank matrix norms

- The matrix norms in the nondecreasing sequence $\left\{ |||\cdot|||^{\mathrm{rk}}_{(r),\star} \right\}_{r \in [\![1,d]\!]}$ are called generalized dual r-rank matrix norms and satisfy

$$|||\cdot|||^{\mathrm{rk}}_{(1),\star} \leq \cdots \leq |||\cdot|||^{\mathrm{rk}}_{(d),\star} = |||\cdot|||_{\star}$$

- By taking their dual norms $|||\cdot|||^{\mathrm{rk}}_{(r)} = \left(|||\cdot|||^{\mathrm{rk}}_{(r),\star} \right)_{\star}$, we obtain a nonincreasing sequence $\left\{ |||\cdot|||^{\mathrm{rk}}_{(r)} \right\}_{r \in [\![1,d]\!]}$ of norms on $\mathcal{M}_{m,n}$ called generalized r-rank matrix norms which satisfy

$$|||\cdot|||^{\mathrm{rk}}_{(1)} \geq \cdots \geq |||\cdot|||^{\mathrm{rk}}_{(d)} = |||\cdot|||$$

Outline of the presentation

Preliminaries for unitarily invariant source norms

A very common class of norms consists of the unitarily invariant norms, such as the Frobenius norm

$$\|M\| = \sqrt{\mathrm{Tr}(M^{\mathrm{T}}M)}$$

Definition

A square matrix $M \in \mathcal{M}_{n,n}$ is orthogonal if, by definition, $M^{\mathrm{T}}M = MM^{\mathrm{T}} = I_n$
We denote by $\mathcal{O}_n$ the set of orthogonal matrices of size n

$$\mathcal{O}_n = \left\{ M \in \mathcal{M}_{n,n} \,\middle|\, M^{\mathrm{T}}M = MM^{\mathrm{T}} = I_m \right\}$$

Some examples

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{O}_2, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}_3, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}_4$$

The Singular Value Decomposition (SVD)

Theorem

For any matrix $M \in \mathcal{M}_{m,n}$, there exists two orthogonal matrices $U \in \mathcal{O}_m, V \in \mathcal{O}_n$ and a diagonal matrix $\operatorname{diag}(s_1, \ldots, s_d)$ with $s_1 \geq \ldots \geq s_d \geq 0$ such that

$$M = U \operatorname{diag}(s_1, \ldots, s_d) V^{\mathrm{T}}$$

The $s_1, \ldots, s_d$ are called the singular values of the matrix M

An example

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}}_{\operatorname{diag}(s_1,\ldots,s_d)} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}}_{V^{\mathrm{T}}}$$

Main result for unitarily invariant source norm

- We recall that the vector $s(M)$ of singular values of a matrix M consists of the square roots of the eigenvalues of the square positive matrix $M^{\mathrm{T}}M$ arranged in nonincreasing order

$$s_1(M) \geq s_2(M) \geq \cdots \geq s_d(M) \geq 0$$

- In the case of an unitarily invariant source norm, the generalized r-rank matrix norms $|||\cdot|||_{(r)}^{\mathrm{rk}}$ are given by

$$\underbrace{|||\cdot|||_{(r)}^{\mathrm{rk}}}_{\text{matrix norm}} = \underbrace{\|\cdot\|_{(r)}^{\mathcal{R}}}_{\text{vector norm}} \circ \overbrace{s}^{}_{\text{vector of singular values}}$$

that is,

$$|||M|||_{(r)}^{\mathrm{rk}} = \|(s_1(M), \ldots, s_d(M))\|_{(r)}^{\mathcal{R}}, \ \forall M \in \mathcal{M}_{m,n}$$

where $\|\cdot\|_{(r)}^{\mathcal{R}}$ denotes the coordinate vector norms on $\mathbb{R}^d$, recently introduced in relation to the l_0 function

Unitarily invariant norm

We recall that a unitarily invariant norm on $\mathcal{M}_{m,n}$ is a matrix norm such that

$$|||UMV||| = |||M||| \ , \ \ \forall M \in \mathcal{M}_{m,n}, \forall U \in \mathcal{O}_m, \forall V \in \mathcal{O}_n$$

We recall that a *symmetric absolute norm* is a vector norm $\|\cdot\|$ on $\mathbb{R}^d$ which satisfies the following properties

- $\|\cdot\|$ is *absolute* in the sense that $\| \, |x| \, \| = \|x\|$, for any $x \in \mathbb{R}^d$, where $|x| = (|x_1|, \ldots, |x_d|)$
- $\|\cdot\|$ is *symmetric* (or *permutation invariant*), that is, $\left\|(x_{\nu(1)}, \ldots, x_{\nu(d)})\right\| = \|(x_1, \ldots, x_d)\|$, for any $x \in \mathbb{R}^d$ and for any permutation ν of the indices in $[\![1, d]\!]$

Proposition (Von Neumann)

A norm $|||\cdot|||$ on the space $\mathcal{M}_{m,n}$ of matrices is unitarily invariant if and only if there exists a symmetric absolute norm $\|\cdot\|$ on $\mathbb{R}^d$ such that

$$|||\cdot||| = \|\cdot\| \circ s \quad \text{that is,} \quad |||M||| = \left\| \left(s_1(M), \ldots, s_d(M) \right) \right\| \ , \forall M \in \mathcal{M}_{m,n}$$

In that case, one has the following relation between dual norms

$$|||\cdot|||_\star = \|\cdot\|_\star \circ s$$

We connect rank-based matrix norms and coordinate vector norms

Back to the main proposition

Proposition

When the source norm $|||\cdot|||$ on $\mathcal{M}_{m,n}$ is unitarily invariant, with associated symmetric absolute norm $\|\cdot\|$ on $\mathbb{R}^d$, then both
the generalized dual r-rank matrix norms $\left\{ |||\cdot|||^{\mathrm{rk}}_{(r),\star} \right\}_{r \in [\![1,d]\!]}$
and the generalized r-rank matrix norms $\left\{ |||\cdot|||^{\mathrm{rk}}_{(r)} \right\}_{r \in [\![1,d]\!]}$
are unitarily invariant, with

$$|||\cdot|||^{\mathrm{rk}}_{(r)} = \|\cdot\|^{\mathcal{R}}_{(r)} \circ s \ , \ \ \forall r \in [\![1, d]\!]$$

$$|||\cdot|||^{\mathrm{rk}}_{(r),\star} = \|\cdot\|^{\mathcal{R}}_{(r),\star} \circ s \ , \ \ \forall r \in [\![1, d]\!]$$

Outline of the presentation

Examples of unitarily invariant source norms

- For any $p \in [1, \infty[$, we define the ℓ_p norm on $\mathbb{R}^d$ by

$$\|x\|_{\ell_p} = \left(\sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}$$

  as well as $\|x\|_{\ell_\infty} = \max_{i \in [\![1,d]\!]} |x_i|$, for any vector $x \in \mathbb{R}^d$
- The Schatten p-norm on the space $\mathcal{M}_{m,n}$ as the unitarily invariant norm

$$|\!|\!| M |\!|\!|_{s_p} = \|s(M)\|_{\ell_p} \ , \ \ \forall M \in \mathcal{M}_{m,n} \ , \ \ \forall p \in [1, \infty]$$

Examples of unitarily invariant source norms

- The Schatten 1-norm is the nuclear norm

$$|||M|||_{s_1} = \|s(M)\|_{\ell_1} = \sum_{i=1}^{d} s_i(M) \ , \ \ \forall M \in \mathcal{M}_{m,n}$$

- The Schatten 2-norm is the Frobenius norm

$$\sqrt{\mathrm{Tr}(MM^{\mathrm{T}})} = |||M|||_{s_2} = \|s(M)\|_{\ell_2} = \sqrt{\sum_{i=1}^{d} s_i(M)^2} \ , \ \ \forall M \in \mathcal{M}_{m,n}$$

- The Schatten ∞-norm is the spectral norm

$$|||M|||_{s_\infty} = \|s(M)\|_{\ell_\infty} = s_1(M) \ , \ \ \forall M \in \mathcal{M}_{m,n}$$

- the Ky Fan k-norms on the space $\mathcal{M}_{m,n}$ are

$$\|s(M)\|_{1,k}^{\mathrm{tn}} = \sum_{i=1}^{k} s_i(M) \ , \ \ \forall M \in \mathcal{M}_{m,n} \ , \ \ \forall k \in [\![1, d]\!]$$

Case of unitarily invariant source matrix norms

source norm $\|\|\|\cdot\|\|\|$	$\|\|\|\cdot\|\|\|^{\mathrm{rk}}_{(r)}$, $r \in [\![1, d]\!]$	$\|\|\|\cdot\|\|\|^{\mathrm{rk}}_{(r),\star}$, $r \in [\![1, d]\!]$
Schatten 1-norm = nuclear norm	Schatten 1-norm for all $r \in [\![1, d]\!]$	Schatten ∞-norm for all $r \in [\![1, d]\!]$
Schatten 2-norm = Frobenius norm	$\\|s(M)\\|^{\mathrm{sn}}_{2,r}$	$\\|s(N)\\|^{\mathrm{tn}}_{2,r} = \sqrt{\sum_{i=1}^{r} s_i(N)^2}$
Schatten p-norm	$\\|s(M)\\|^{\mathrm{sn}}_{p,r}$	$\\|s(N)\\|^{\mathrm{tn}}_{q,r} = \left(\sum_{i=1}^{r} s_i(N)^q\right)^{\frac{1}{q}}$
Schatten ∞-norm = spectral norm	$\\|s(M)\\|^{\mathrm{sn}}_{\infty,r} = s_1(M)$ Schatten ∞-norm	$\\|s(N)\\|^{\mathrm{tn}}_{1,r} = \sum_{i=1}^{r} s_i(N)$
Ky Fan k-norm $k \in [\![1, d]\!]$	Schatten 1-norm for all $r \in [\![1, k]\!]$	Schatten ∞-norm for all $r \in [\![1, k]\!]$

Table: Generalized r-rank matrix norms $\left\{\|\|\|\cdot\|\|\|^{\mathrm{rk}}_{(r)}\right\}_{r\in[\![1,d]\!]}$ and generalized dual r-rank matrix norms $\left\{\|\|\|\cdot\|\|\|^{\mathrm{rk}}_{(r),\star}\right\}_{r\in[\![1,d]\!]}$ associated with by Schatten and Ky Fan source norms $\|\|\|\cdot\|\|\|$ (for $p \in [1, \infty]$, and where $1/p + 1/q = 1$)

Where do we stand and where are we going to

- We have introduced a source norm
- We have defined rank-based norms
- We now show how these rank-based norms appear in variational formulations for the rank function
- For this purpose, we will now present the so-called Capra-couplings and conjugacies

Outline of the presentation

Classical Fenchel coupling

- We use the Moreau lower addition for infinite values, that is

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = -\infty$$

- The classical Fenchel coupling between $\mathcal{M}_{m,n}$ and $\mathcal{M}_{m,n}$ is defined as

$$\langle M,\ N \rangle = \mathrm{Tr}(MN^{\mathrm{T}}) \ , \ \ \forall M, N \in \mathcal{M}_{m,n}$$

- For any function $F : \mathcal{M}_{m,n} \to \overline{\mathbb{R}}$, the associated Fenchel conjugates are defined by

$$F^{\star}(N) = \sup_{M \in \mathcal{M}_{m,n}} \Big(\langle M,\ N \rangle \dot{+} \big(-F(M) \big) \Big)$$

$$F^{\star\star'}(M) = \sup_{N \in \mathcal{M}_{m,n}} \Big(\langle M,\ N \rangle \dot{+} \big(-F^{\star}(N) \big) \Big)$$

- The Fenchel conjugacy is useful for convex functions, which is not the case for the rank function though, as its Fenchel biconjugate is null
- As the rank function satisfies $\mathrm{rk}(M) = \mathrm{rk}(\lambda M) \ , \ \ \forall \lambda \neq 0$ ($0-$ homogenous), the following $\mathrm{CAPRA}-$ *couplings* are more adapted

Outline of the presentation

CAPRA-couplings

Definition

Let $|||\cdot|||$ be a source matrix norm on $\mathcal{M}_{m,n}$
The CAPRA-coupling ¢, between $\mathcal{M}_{m,n}$ and $\mathcal{M}_{m,n}$, associated with $|||\cdot|||$, is defined by

$$\forall M, N \in \mathcal{M}_{m,n} \; , \;\; ¢(M, N) = \begin{cases} \frac{\mathrm{Tr}(MN^{\mathrm{T}})}{|||M|||} & \text{if } M \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The CAPRA-coupling as its name indicates, is Constant Along Primal RAys :

$$¢(\lambda M, N) = ¢(M, N) \; , \forall \lambda \in \mathbb{R}^*$$

CAPRA-conjugates for matrices

- Here the conjugates define a function with the same domain and codomain than the source function $\overline{\mathbb{R}}^{\mathcal{M}_{m,n}} \overset{conjugacy}{\longrightarrow} \overline{\mathbb{R}}^{\mathcal{M}_{m,n}}$
- For any function $F : \mathcal{M}_{m,n} \to \overline{\mathbb{R}}$, the *¢-Fenchel-Moreau conjugate*, or CAPRA-*conjugate*, is the function $F^{¢} : \mathcal{M}_{m,n} \to \overline{\mathbb{R}}$ defined by

$$F^{¢}(N) = \sup_{M \in \mathcal{M}_{m,n}} \Big(¢(M,N) \dotplus \big(-F(M)\big) \Big) \; , \; \forall N \in \mathcal{M}_{m,n}$$

  and the *¢-Fenchel-Moreau biconjugate*, or CAPRA-*biconjugate*, is the function $F^{¢¢'} : \mathcal{M}_{m,n} \to \overline{\mathbb{R}}$ defined by

$$F^{¢¢'}(M) = \sup_{N \in \mathcal{M}_{m,n}} \Big(¢(M,N) \dotplus \big(-F^{¢}(N)\big) \Big) \; , \; \forall M \in \mathcal{M}_{m,n}$$

- For any function F, the following inequality holds true

$$F \geq F^{¢¢'}$$

  which will bring us to the variational formula for the rank function,
  as we will be able to express $\mathrm{rk}^{¢¢'}(M)$ in term of the rank based norms

Outline of the presentation

Formulas for conjugate and biconjugate of the rank function

Proposition

Let $|||\cdot|||$ be a source matrix norm on $\mathcal{M}_{m,n}$, and $¢$ be the associated Capra*-coupling*
For any function $\varphi : [\![0,d]\!] \to \mathbb{R}$, we have that (with the convention that $|||\cdot|||^{\mathrm{rk}}_{(0),\star} = 0$)

$$(\varphi \circ \mathrm{rk})^{¢}(N) = \sup_{i \in [\![0,d]\!]} \Big\{ \underbrace{|||N|||^{\mathrm{rk}}_{(i),\star}}_{\text{rank matrix norm}} - \varphi(i) \Big\} , \quad \forall N \in \mathcal{M}_{m,n}$$

and, for any function $\varphi : [\![0,d]\!] \to \mathbb{R}_+$ (that is, with nonnegative finite values) and such that $\varphi(0) = 0$, we have that

$$(\varphi \circ \mathrm{rk})^{¢¢'}(M) = \frac{1}{|||M|||} \min_{\substack{M^{(1)} \in \mathcal{M}_{m,n}, \dots, M^{(d)} \in \mathcal{M}_{m,n} \\ \sum_{r=1}^{d} |||M^{(r)}|||^{\mathrm{rk}}_{(r)} \leq |||M||| \\ \sum_{r=1}^{d} M^{(r)} = M}} \sum_{r=1}^{d} \varphi(r) \underbrace{|||M^{(r)}|||^{\mathrm{rk}}_{(r)}}_{\text{rank matrix norm}} ,$$

$\forall M \in \mathcal{M}_{m,n} \setminus \{0\}$

Theorem

Let $\|\!|\cdot|\!\|$ be a source norm on the space $\mathcal{M}_{m,n}$ of matrices, with associated sequence $\left\{ \|\!|\cdot|\!\|^{\mathrm{rk}}_{(r)} \right\}_{r \in [\![1,d]\!]}$ of generalized r-rank matrix norms

Then, we have the following variational lower bound of the rank function

$$\mathrm{rk}(M) \geq \frac{1}{\|\!|M|\!\|} \min_{\substack{M^{(1)} \in \mathcal{M}_{m,n}, \dots, M^{(d)} \in \mathcal{M}_{m,n} \\ \sum_{r=1}^{d} \|\!|M^{(r)}|\!\|^{\mathrm{rk}}_{(r)} \leq \|\!|M|\!\| \\ \sum_{r=1}^{d} M^{(r)} = M}} \sum_{r=1}^{d} r \|\!|M^{(r)}|\!\|^{\mathrm{rk}}_{(r)} \,, \quad \forall M \in \mathcal{M}_{m,n} \setminus \{0\}$$

Corollary

Moreover, if the source norm $|||\cdot|||$ is the Frobenius norm given by

$$|||M||| = \sum_{i=1}^{\mathrm{rk}(M)} s_i^2(M) = \sum_{i \leq m, j \leq n} M_{i,j}^2 \ , \ \ \forall M \in \mathcal{M}_{m,n}$$

the previous inequality is an equality, that is,

$$\mathrm{rk}(M) = \frac{1}{|||M|||} \min_{\substack{M^{(1)} \in \mathcal{M}_{m,n}, \ldots, M^{(d)} \in \mathcal{M}_{m,n} \\ \sum_{r=1}^{d} |||M^{(r)}|||_{(r)}^{\mathrm{rk}} \leq |||M||| \\ \sum_{r=1}^{d} M^{(r)} = M}} \sum_{r=1}^{d} r |||M^{(r)}|||_{(r)}^{\mathrm{rk}} \ , \ \ \forall M \in \mathcal{M}_{m,n} \setminus \{0\}$$

Outline of the presentation

Conclusion

- We have introduced a source norm
- We have defined rank-based matrix norms
- We have defined CAPRA-couplings and conjugates on the space of matrices
- We have obtained variational formulations for the rank function that involve rank-based matrix norms

Back to the image completion problem

The problem was to solve

$$\min_{M \text{ s.t. } M_{i,j}=M^*_{i,j}, \forall (i,j)\notin\Omega} \mathrm{rk}(M)$$

With our result, the problem can actually be written as

$$\min_{\substack{M\in\mathcal{M}_{m,n}, M^{(1)}\in\mathcal{M}_{m,n},\ldots,M^{(d)}\in\mathcal{M}_{m,n} \\ \sum_{r=1}^d |||M^{(r)}|||_{(r)}^{\mathrm{rk}} \leq |||M||| \\ \sum_{r=1}^d M^{(r)}=M \\ M_{i,j}=M^*_{i,j}, \forall (i,j)\notin\Omega}} \frac{1}{|||M|||} \sum_{r=1}^{d} r |||M^{(r)}|||_{(r)}^{\mathrm{rk}}$$

Pros and cons

- A smoother formulation
- An optimization problem with many more variables

Our organization, encountered problems and what we have done

How we have worked

1. Background work
   - handling of the subject (reading papers, books. . .)
   - review of scholar notions of analysis
     (norms, Hölder and Cauchy-Schwarz inequality, convexity. . .)
   - review of linear algebra
     (orthogonal matrices, singular values, classical properties of the rank function. . .)
2. Active research period and main contributions
   - we have elaborated several proofs of the same result
   - that the rank based norms are norms,
   - the formula for unitarily invariant source norm,
   - the case of equality, etc.
3. Timing
   - Soon writing of a formal paper in parallel with the research
   - followed by and intensive only writing phase

Our organization, encountered problems and what we have done

Problems we faced

- Time constraint: we had to give the abstract the 21 May, the paper the 28 May
- Hard to deal with the case of general norms:
  most of our results tackle the case of unitarily invariant norms
- Difficulties with LaTeX layout, as we were beginners in LaTeX before this project

Work done

- Paper submitted to NeurIPS meeting (Neural Information Processing Systems)
- We have planned to submit to others journals or conferences in optimization in case of a reject from NeurIPS, which is possible as the paper is really theoretical and specialized
- The paper is available on HAL and arXiv websites