Sequential Decision Models

Extended from Chapter 2 of
Sustainable Management of Natural Resources. Mathematical Models and Methods
by Luc DOYEN and Michel DE LARA

Michel De Lara
Cermics, École des Ponts ParisTech
Université Paris-Est
France

École des Ponts ParisTech

August 18, 2014

Outline of the presentation

1. Stylized examples of sequential decision models
2. Clothing a sequential decision problem in formal garb
3. A glimpse at some more complex models
4. General remarks on numerical issues

Outline of the presentation

Outline of the presentation

1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

Exploitation of an exhaustible resource

The dynamic of the exhaustible resource is

$$\underbrace{S(t+1)}_{\text{future stock}} = \underbrace{S(t)}_{\text{stock}} - \underbrace{h(t)}_{\text{extraction}} \quad , \quad t = t_0, t_0+1, \ldots, T-1$$

Chuquicamata (copper, Chile)

where

- $S(t)$ stock of resource at the beginning of period $[t, t+1[$
- $h(t)$ extraction during $[t, t+1[$

Different requirements are formulated as static constraints

- Physical constraints

$$0 \leq h(t) \leq S(t)$$

- Stronger conservation constraint

$$S^{\flat} \leq S(t)$$

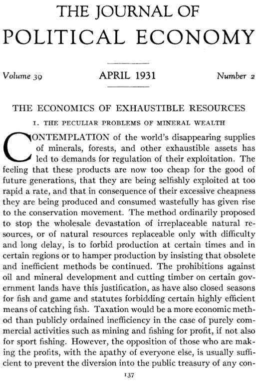
where $S^{\flat} > 0$ stands for some minimal resource standard

- Intergenerational equity: can we impose some guaranteed consumption level $h^{\flat} > 0$ along the generations t?

$$h^{\flat} \leq h(t)$$

In 1931, Harold Hotelling proposed to look for extraction paths that maximize discounted utility

Looking for an optimal decision path $h^\star(t_0), \ldots, h^\star(T-1)$ solving


THE JOURNAL OF
POLITICAL ECONOMY

Volume 39 APRIL 1931 *Number 2*

THE ECONOMICS OF EXHAUSTIBLE RESOURCES

I. THE PECULIAR PROBLEMS OF MINERAL WEALTH

CONTEMPLATION of the world's disappearing supplies of minerals, forests, and other exhaustible assets has led to demands for regulation of their exploitation. The feeling that these products are now too cheap for the good of future generations, that they are being selfishly exploited at too rapid a rate, and that in consequence of their excessive cheapness they are being produced and consumed wastefully has given rise to the conservation movement. The method ordinarily proposed to stop the wholesale devastation of irreplaceable natural resources, or of natural resources replaceable only with difficulty and long delay, is to forbid production at certain times and in certain regions or to hamper production by insisting that obsolete and inefficient methods be continued. The prohibitions against oil and mineral development and cutting timber on certain government lands have this justification, as have also closed seasons for fish and game and statutes forbidding certain highly efficient means of catching fish. Taxation would be a more economic method than publicly ordained inefficiency in the case of purely commercial activities such as mining and fishing for profit, if not also for sport fishing. However, the opposition of those who are making the profits, with the apathy of everyone else, is usually sufficient to prevent the diversion into the public treasury of any con-

137

$$\max_{h(t_0),\ldots,h(T-1)} \underbrace{\sum_{t=t_0}^{T-1} \Big(\frac{1}{1+r_e}\Big)^{t-t_0} \overbrace{\mathtt{L}\big(h(t)\big)}^{\text{utility}}}_{\text{discounted utility}}$$

- $\frac{1}{1+r_e}$ stands for a (social) discount factor
- $\mathtt{L}$ is a utility function of the consumption h, for instance,

$$\underbrace{\mathtt{L}(h)}_{\text{profit}} = \underbrace{p}_{\text{price}} h - \underbrace{\mathtt{Cost}(h, S)}_{\text{costs}}$$

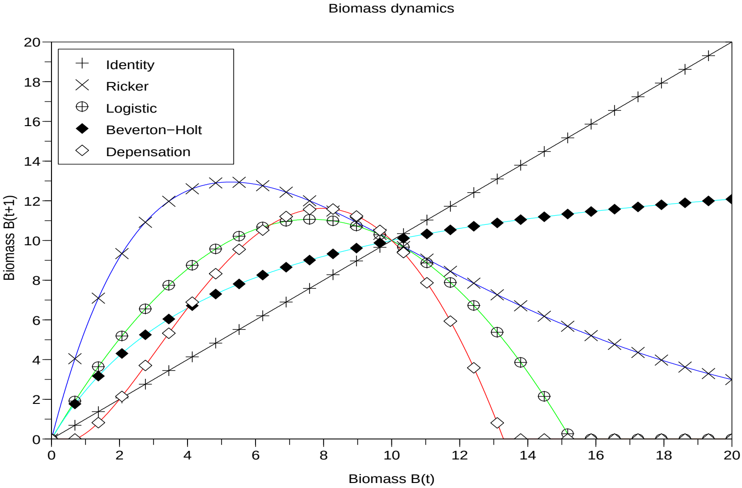
Outline of the presentation

First, we showcase biomass biological models

$$\underbrace{B(t+1)}_{\text{future biomass}} = \texttt{Biol}\big(\underbrace{B(t)}_{\text{biomass}}\big)$$

where

- $B(t)$ resource biomass (tonnes)
- biological dynamics
  $\texttt{Biol} : \mathbb{R}_+ \to \mathbb{R}_+$
  $(\texttt{Biol}(0) = 0)$



Comparaison of population dynamics `Biol` for $r = 1.9$, $K = 10$, $B^{\flat} = 2$

The linear model is the most simple

$$B(t+1) = RB(t) = \overbrace{(1+r)}^{\text{growth factor}} B(t)$$

- $\mathtt{Biol}(B) = RB$
- $r = R - 1$

  $= \text{natality rate - mortality rate}$

  is the per capita rate of growth

In 1835, Adolphe Quételet proposed to model resistance to geometric population growth as quadratic

Adolphe Quételet
Sur l'homme et le développement de ses facultés ou essai de physique sociale, 1835

> *La théorie de la population peut se réduire aux deux principes suivants, que je regarde comme devant servir désormais de principes fondamentaux à l'analyse du développement de la population et des causes qui l'influencent.*
>
> - *La population tend à croître selon une progression géométrique.*
> - *La résistance, ou la somme des obstacles à son développement, est, toutes choses égales d'ailleurs comme le carré de la vitesse avec laquelle la population tend à croître.*

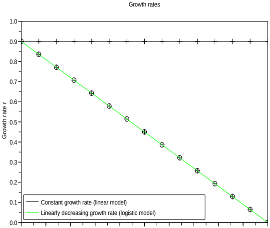
Pierre-François Verhulst proposed a linear density-dependent per capita rate of growth

- Pierre-François Verhulst claims

  j'ai tenté depuis longtemps de déterminer par l'analyse, la loi probable de la population, mais j'ai abandonné ce genre de recherches parce que les données de l'observation sont trop peu nombreuses pour que les formules puissent être vérifiées, de manière à ne laisser aucun doute sur leur exactitude

- and proposes to keep "l'hypothèse la plus simple que l'on puisse faire" the simplest hypothesis one can do, that is, a growth rate

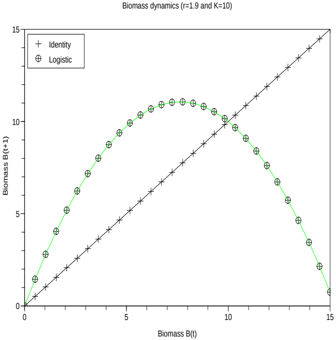
  linearly decreasing with the biomass



linear $\rightarrow$ logistic

$$r \rightarrow r\left(1 - \frac{B}{K}\right) \searrow \text{ with biomass } B$$

A linear density-dependent per capita rate of growth leads to the logistic model



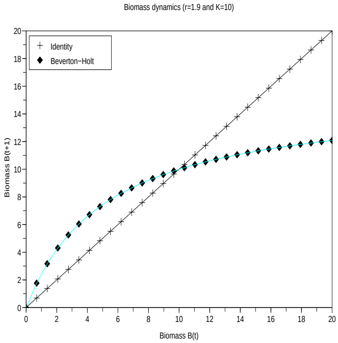
$$B(t+1) = B(t) + rB(t) \overbrace{\left(1 - \frac{B(t)}{K}\right)}^{\text{correction term}}$$

- $\texttt{Biol}(B) = B + r\left(1 - \frac{B}{K}\right) B$
- $r \geq 0$ is the per capita rate of growth (for small populations)
- K is the carrying capacity of the habitat: the lowest $K > 0$ which satisfies

$$\texttt{Biol}(K) = K$$

The Beverton-Holt model was introduced for fisheries

Beverton, R. J. H. and Holt, S. J.,
On the dynamics of exploited fish populations,
Fishery Investigations. Her Majesty's Stationery Office, London, 1957



$$B(t+1) = \frac{RB(t)}{1 + bB(t)}$$

$$\texttt{Biol}(B) = \frac{RB}{1 + bB}$$

where the carrying capacity is $K = \frac{R-1}{b}$

Now, we turn to harvesting models

Introduced for fisheries in 1954, the Schaefer model builds harvesting upon a biological model

M. B. Schaefer,
Some aspects of the dynamics of populations important to the management of commercial marine fisheries,
Bulletin of the Inter-American tropical tuna commission, 1954

$$\underbrace{B(t+1)}_{\text{future biomass}} = \texttt{Biol}\Big(\underbrace{B(t) - h(t)}_{\text{biomass}-\text{catches}} \Big) , \quad 0 \leq h(t) \leq B(t)$$

where $h(t)$ is the harvesting or catch at time t

1. catches occur at beginning of period $[t, t+1[$: $B(t) \to B(t) - h(t)$
2. then regeneration takes place during period $[t, t+1[$: $B(t) - h(t) \to \texttt{Biol}\big(B(t) - h(t)\big)$

Catches are related to harvesting effort

$$\underbrace{h}_{\text{catch}} = q \overbrace{E}^{\text{effort}} \underbrace{B}_{\text{biomass}}$$

- E is the harvesting effort: number of boats, equipment, etc.
- q is a catchability coefficient

More generally, $h = \texttt{Catch}(E, B)$

Cobb-Douglas production function

$$\texttt{Catch}(E, B) = qE^{\alpha}B^{\beta}$$

In Perú, 2nd world country for fish production,

fisheries depend of the ministry of production

Constraints may be inevitable (physical) or imposed (conservation)

- One cannot harvest more biomass than there is

$$0 \leq h(t) \leq B(t)$$

- Requiring a minimal resource biomass $B^{\flat} > 0$ (safety threshold) for all times is represented by a conservation constraint

$$B^{\flat} \leq B(t)$$

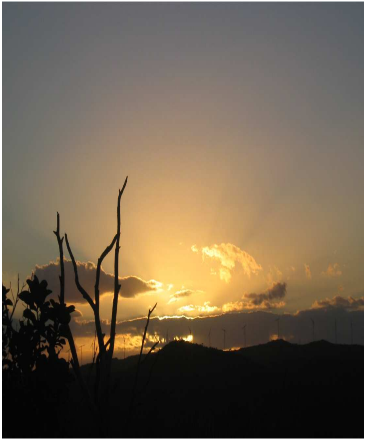
An optimal catch path can be looked after as solution of an intertemporal utility maximization

$$\max_{h(t_0),\ldots,h(T-1)} \Big(\overbrace{\sum_{t=t_0}^{T-1} \Big(\frac{1}{1+r_e}\Big)^{t-t_0} \mathtt{L}\big(h(t)\big) + \Big(\frac{1}{1+r_e}\Big)^{T-t_0} \underbrace{\mathtt{K}\big(B(T)\big)}_{\text{final utility}}}^{\text{discounted utility}} \Big)$$

- r_e is a discount rate
- $\frac{1}{1+r_e}$ is a discount factor
- $\mathtt{L}$ is a utility function
- Final term $\mathtt{K}\big(B(T)\big)$:
  existence or inheritance value of the biomass

Outline of the presentation

Let us scout a very stylized model of the climate-economy system



We lay out a dynamical model with

- two state variables

  environmental: atmospheric CO_2 concentration level $M(t)$

  economic: gross world product GWP $Q(t)$

- one decision variable, the emission abatement rate $a(t)$

A carbon cycle model "à la Nordhaus" is an example of *decision model*

- Time index t in years
- Economic production $Q(t)$ (GWP)

$$Q(t+1) = \overbrace{(1+g)}^{\text{economic growth}} Q(t)$$

- CO_2 concentration $M(t)$

$$M(t+1) = M(t) \underbrace{-\delta(M(t) - M_{-\infty})}_{\text{natural sinks}} + \alpha \overbrace{\texttt{Emiss}(Q(t))}^{\text{emissions}} \underbrace{(1-a(t))}_{\text{abatement}}$$

- Decision $a(t) \in [0, 1]$ is the abatement rate of CO_2 emissions

Data

- $M(t)$ $\mathrm{CO_2}$ atmospheric concentration, measured in ppm, parts per million (379 ppm in 2005)
- $M_{-\infty}$ pre-industrial atmospheric concentration (about 280 ppm)
- $\mathtt{Emiss}(Q(t))$ "business as usual" $\mathrm{CO_2}$ emissions (about 7.2 GtC per year between 2000 and 2005)
- $0 \leq a(t) \leq 1$ abatement rate reduction of $\mathrm{CO_2}$ emissions
- α conversion factor from emissions to concentration ($\alpha \approx 0.471$ ppm.GtC^{-1} sums up highly complex physical mechanisms)
- δ natural rate of removal of atmospheric $\mathrm{CO_2}$ to unspecified sinks ($\delta \approx 0.01$ year^{-1})

A concentration target is pursued to avoid danger

Limitation of concentrations of CO_2

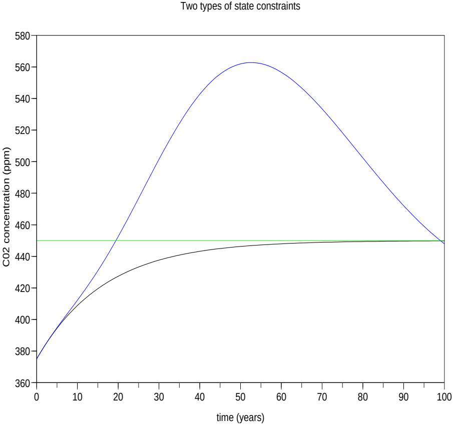
- below a tolerable threshold $M^\sharp$ (say 350 ppm, 450 ppm)
- at a specified date $T > 0$ (say year 2050 or 2100)

United Nations Framework Convention on Climate Change

"to achieve, (. . .), stabilization of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system"

$$\underbrace{M(T)}_{\text{concentration at horizon}} \leq \underbrace{M^\sharp}_{\text{threshold}}$$

Constraints capture different requirements



- The concentration has to remain below a tolerable level at the horizon T:

$$M(T) \leq M^{\sharp}$$

- More demanding: from the initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp}$$

$$t = t_0, \ldots, T$$

Constraints may be environmental, physical, economic

- The concentration has to remain below a tolerable level from initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp} , \quad t = t_0, \ldots, T$$

- Abatements are expressed as fractions

$$0 \leq a(t) \leq 1 , \quad t = t_0, \ldots, T-1$$

- As with "cap and trade", setting a ceiling on $\mathrm{CO_2}$ price amounts to cap abatement costs

$$\underbrace{\texttt{Cost}\big(a(t), Q(t)\big)}_{\text{costs}} \leq c^{\sharp} \, (100 \text{ euros / tonne } \mathrm{CO_2}) , \quad t = t_0, \ldots, T-1$$

Mixing dynamics, optimization and constraints yields a cost-effectiveness problem

- Minimize abatement costs

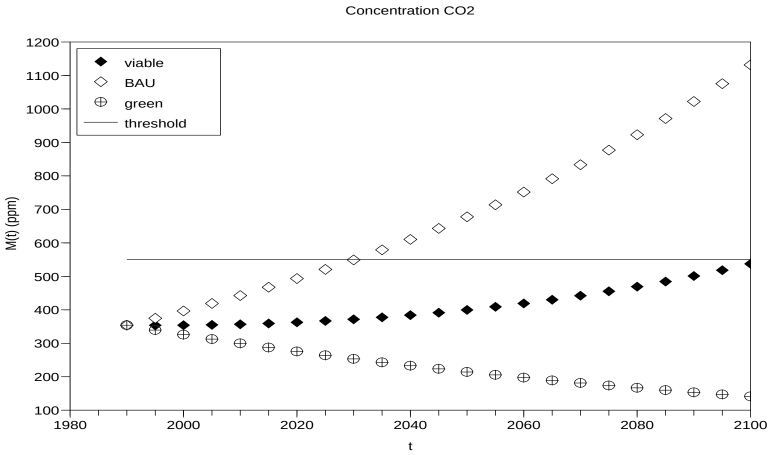
$$\min_{a(t_0),\dots,a(T-1)} \sum_{t=t_0}^{T-1} \left(\frac{1}{1+r_e}\right)^{t-t_0} \underbrace{\texttt{Cost}\big(a(t), Q(t)\big)}_{\text{abatement costs}}$$

- under the GWP-CO_2 dynamics

$$\begin{cases} M(t+1) & = M(t) - \delta(M(t) - M_{-\infty}) + \alpha \texttt{Emiss}\big(Q(t)\big)(1 - a(t)) \\ Q(t+1) & = (1+g)Q(t) \end{cases}$$

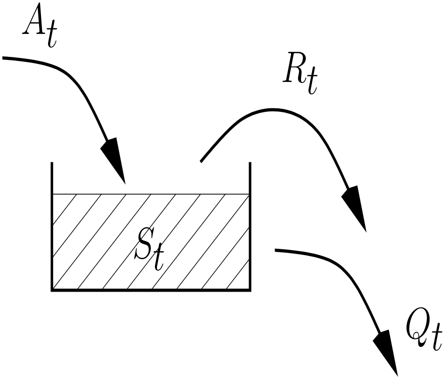
- and under target constraint

$$\underbrace{M(T) \leq M^{\sharp}}_{\text{CO2 concentration}}$$


Concentration CO2
viable
BAU
green
threshold
M(t) (ppm)
1200
1100
1000
900
800
700
600
500
400
300
200
100
1980
2000
2020
2040
2060
2080
2100
t

Outline of the presentation

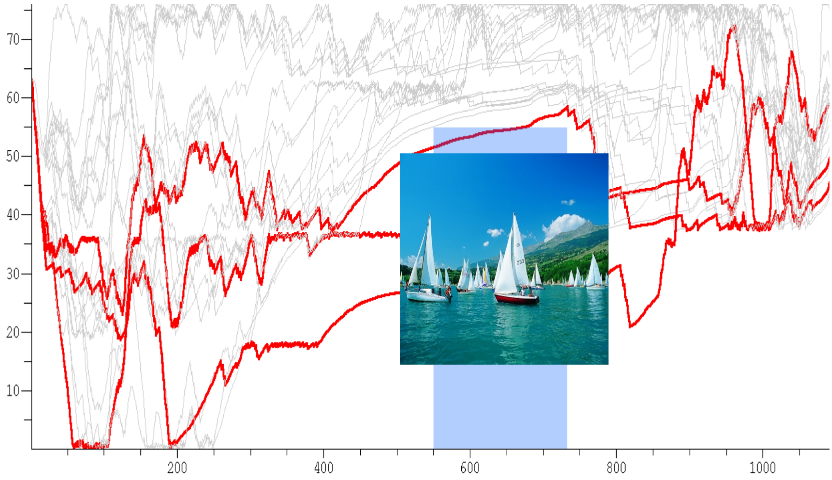
1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time


A_t
R_t
S_t
Q_t

Tourism issues impose constraints upon traditional economic management of a hydro-electric dam

- Maximizing the revenue from turbinated water
- under a tourism constraint of having enough water in July and August

The red stock trajectories fail to meet the tourism constraint in July and August



We consider a single dam nonlinear dynamical model in the decision-hazard setting

We can model the dynamics of the water volume in a dam by

$$\underbrace{S(t+1)}_{\text{future volume}} = \min\{S^\sharp, \underbrace{S(t)}_{\text{volume}} - \underbrace{q(t)}_{\text{turbined}} + \underbrace{a(t)}_{\text{inflow volume}}\}$$

- $S(t)$ volume (stock) of water at the beginning of period $[t, t+1[$
- $a(t)$ inflow water volume (rain, etc.) during $[t, t+1[$
- $q(t)$ turbined outflow volume during $[t, t+1[$
  - decided at the beginning of period $[t, t+1[$
  - chosen such that $0 \leq q(t) \leq \min\{S(t), q^\sharp\}$
  - supposed to depend on the stock $S(t)$ but not on the inflow water $a(t)$
- the setting is called decision-hazard:
  $a(t)$ is not available at the beginning of period $[t, t+1[$

In the risk-neutral economic approach, an optimal management maximizes the expected payoff

- Suppose that
  - at the horizon, the final volume $S(T)$ has a value $\mathtt{K}(S(T))$, the "final value of water"
  - turbined water $q(t)$ is sold at price $p(t)$, related to the price at which energy can be sold at time t
  - a probability $\mathbb{P}$ is given on the set $\Omega = \mathbb{R}^{T-t_0} \times \mathbb{R}^{T-t_0}$ of water inflows scenarios $\big(a(t_0), \ldots, a(T-1)\big)$ and prices scenarios $\big(p(t_0), \ldots, p(T-1)\big)$
- The traditional (risk-neutral) economic problem is to maximize the intertemporal payoff (without discounting if the horizon is short)

$$\max \mathbb{E}\left[\sum_{t=t_0}^{T-1}\left(\overbrace{p(t)}^{\text{price}}\ \overbrace{q(t)}^{\text{turbined}}\ \underbrace{-\epsilon q(t)^2}_{\text{turbined costs}}\right) + \overbrace{\mathtt{K}(S(T))}^{\text{final volume utility}}\right]$$

We now have a stochastic optimization problem, where the tourism constraint still needs to be dressed in formal clothes

- Traditional cost minimization/payoff maximization

$$\max \mathbb{E}\left[\sum_{t=t_0}^{T-1} \overbrace{p(t)q(t) - \epsilon q(t)^2}^{\text{turbined water payoff}} + \overbrace{\mathtt{K}(S(T))}^{\text{final volume utility}}\right]$$

- Tourism constraint

$$\text{volume} \;\; S(t) \geq S^{\flat} \; , \quad \forall t \in \{\, \text{July, August} \,\}$$

- In what sense should we consider this inequality which involves the random variables $S(t)$ for $t \in \{\, \text{July, August} \,\}$?

Robust / almost sure / probability constraint

- Robust constraints: for all the scenarios in a subset $\overline{\Omega} \subset \Omega$

$$S(t) \geq S^{\flat} , \quad \forall t \in \{ \text{July}, \text{August} \}$$

- Almost sure constraints

$$\text{Probability} \left\{ \begin{array}{l} \text{water inflow scenarios along which} \\ \text{the volumes } S(t) \text{ are above the} \\ \text{threshold } S^{\flat} \text{ for periods } t \text{ in summer} \end{array} \right\} = 1$$

- Probability constraints, with "confidence" level $p \in [0, 1]$

$$\text{Probability} \left\{ \begin{array}{l} \text{water inflow scenarios along which} \\ \text{the volumes } S(t) \text{ are above the} \\ \text{threshold } S^{\flat} \text{ for periods } t \text{ in summer} \end{array} \right\} \geq p$$

- and also by penalization, or in the mean, etc.

Our problem may be clothed as a stochastic optimization problem under a probability constraint

- The traditional economic problem is $\max \mathbb{E}\left[P(T)\right]$ where the payoff/utility criterion is

$$P(T) = \sum_{t=t_0}^{T-1} \overbrace{p(t)q(t) - \epsilon q(t)^2}^{\text{turbined water payoff}} + \overbrace{K(S(T))}^{\text{final volume utility}}$$

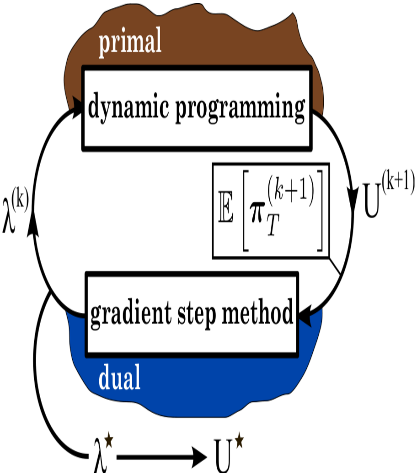
- and a failure tolerance is accepted

$$\text{Probability} \left\{ \begin{array}{l} \text{water inflow scenarios along which} \\ \text{the volumes } S(t) \geq S^{\flat} \\ \text{for periods } t \text{ in July and August} \end{array} \right\} \geq 90\%$$

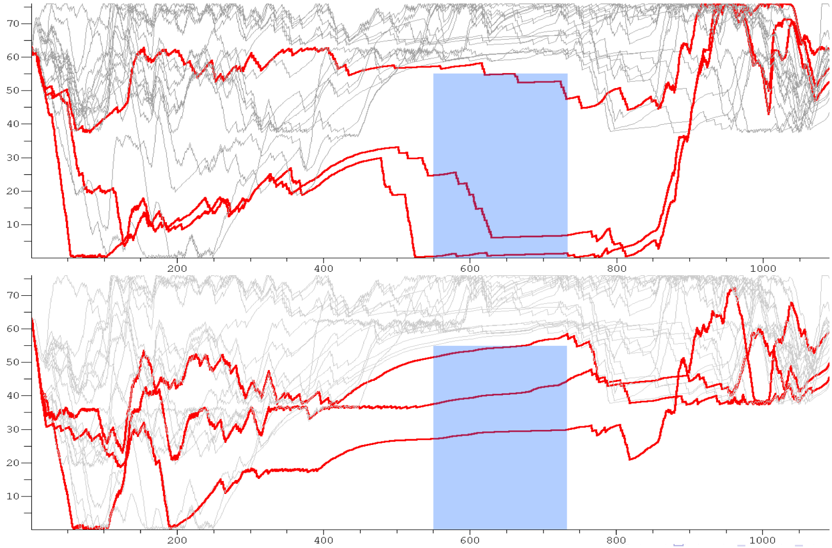
- Details concerning the theoretical and numerical resolution are available on demand ;-)

Details concerning the theoretical and numerical resolution are available on demand ;-)

- $\boldsymbol{\pi}_0 = 1$ and $\boldsymbol{\pi}_{t+1} = \begin{cases} \mathbf{1}_{\{x_{t+1} \geq x_{\text{ref}}\}} \times \boldsymbol{\pi}_t & \text{if } t \in \mathcal{T} \\ \boldsymbol{\pi}_t & \text{else} \end{cases}$
- $\mathbb{P}\big[x_\tau \geq x_{\text{ref}} \,, \quad \forall \tau \in \mathcal{T}\big]$ $= \mathbb{E}\big[\mathbf{1}_{\{x_\tau \geq x_{\text{ref}} \,, \ \forall \tau \in \mathcal{T}\}}\big]$ $= \mathbb{E}\big[\prod_{\tau \in \mathcal{T}} \mathbf{1}_{\{x_\tau \geq x_{\text{ref}}\}}\big]$ $= \mathbb{E}\big[\boldsymbol{\pi}_T\big]$



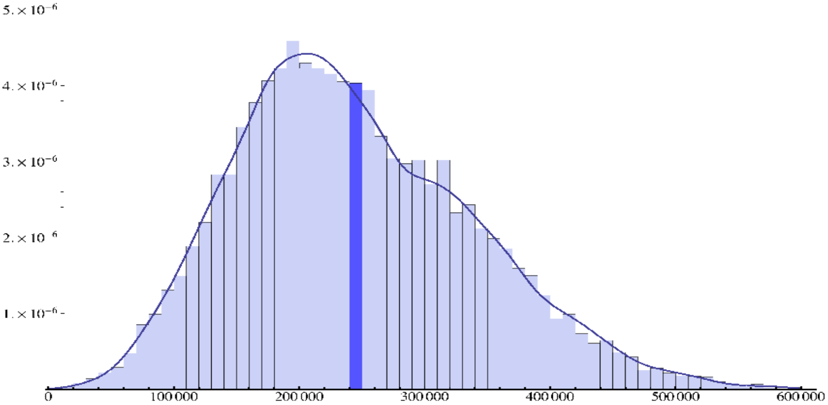
90% of the stock trajectories meet the tourism constraint in July and August



Our resolution approach brings a sensible improvement compared to standard procedures

OPTIMAL POLICIES	OPTIMIZATION		SIMULATION		
	Iterations	Time	Gain	Respect	Well behaviour
Standard	15	10 mn	**ref**	**0,9**	**no**
Convenient	**10**	**160 mn**	**-3.20%**	**0,9**	**yes**
Heuristic	10	160 mn	**-3.25%**	**0,9**	**yes**

However, though the expected payoff is optimal, the payoff effectively realized can be far from it



Summary

- Mineral resources, forestry, fisheries, climate and energy provide examples of sequential decision-making
- Decisions are made at discrete times: extraction, catches, abatment, turbined
- Objectives can be formulated as indicators not trespassing thresholds
- A particular objective can be distinguished to be optimized

Outline of the presentation

Outline of the presentation

1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

Examples of dynamical equations highlight the specific roles of state and control variables

In the above examples appear two types of variables

- state variables
- control variables

$$S(t+1) = S(t) - h(t)$$

$$B(t+1) = \texttt{Biol}\big(B(t) - h(t)\big)$$

$$\begin{cases} M(t+1) & = M(t) - \delta(M(t) - M_{-\infty}) + \alpha\texttt{Emiss}\big(Q(t)\big)(1 - a(t)) \\ Q(t+1) & = (1+g)Q(t) \end{cases}$$

$$S(t+1) = \min\{S^{\sharp}, S(t) - q(t) + a(t)\}$$

A control system connects input and output variables

Input variables

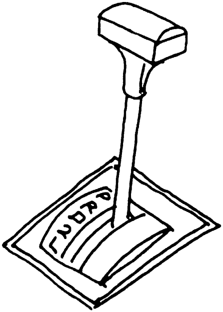
Control	wood logs
Uncertainty	wood humidity
	metal conductivity

Output variables

soup quality
water vapor
temperature (internal state)

Input control variables are in the hands of the decision-maker at successive time periods

Control variables $u(t)$ are those variables whose values the decision-maker can fix



- at successive time periods
  - annual catches
  - years, months: starting of energy units like nuclear plants
  - weeks, days, intra-day: starting of hydropower units
- within given bounds
  - fishing quotas
  - turbined capacity

Discrete-time nonlinear state-control systems are special input-output systems

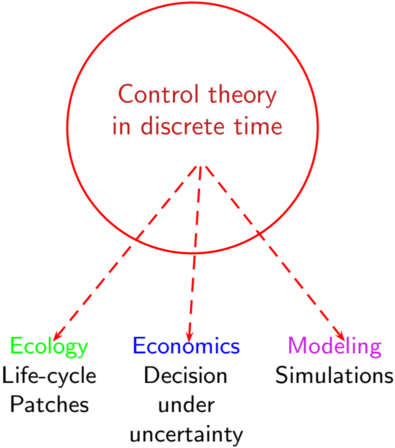
A specific output is distinguished, and is labeled state,
when the system may be written as

$$x(t+1) = \mathtt{Dyn}(t, x(t), u(t)), \quad t \in \mathbb{T} = \{t_0, t_0+1, \ldots, T-1\}$$

- the time $t \in \overline{\mathbb{T}} = \{t_0, t_0+1, \ldots, T-1, T\} \subset \mathbb{N}$ is discrete with initial time t_0 and horizon T ($T < +\infty$ or $T = +\infty$) (*the time period* $[t, t+1[$ *may be a year, a month, etc.*)
- the state variable $x(t)$ belongs to the finite dimensional *state space* $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$; (*stocks, biomasses, abundances, capital, etc.*)
- the control variable $u(t)$ is an element of the *control space* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ (*outflows, catches, harvesting effort, investment, etc.*)
- the dynamics $\mathtt{Dyn}$ maps $\mathbb{T} \times \mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$ (*storage, age-class model, population dynamics, economic model, etc.*)

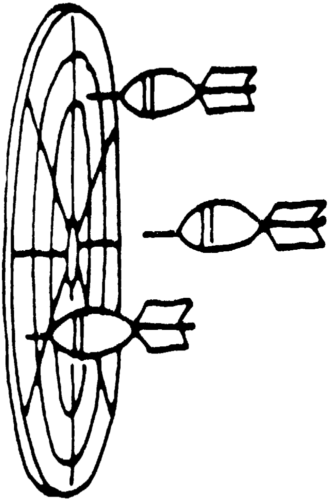
Outline of the presentation

We dress natural resources management issues in the formal clothes of control theory in discrete time



- Problems are framed as
  - find controls/decisions driving a dynamical system
  - to achieve various goals
- Three main ingredients are
  - controlled dynamics
  - constraints
  - criterion to optimize

We mathematically express the objectives pursued as control and state constraints



- For a state-control system, we cloth objectives as constraints
- and we distinguish
  control constraints (rather easy)
  state constraints (rather difficult)
- Viability theory deals with state constraints

Constraints may be explicit on the control variable

and are rather easily handled by reducing the decision set

Examples of control constraints

- Irreversibility constraints, physical bounds
  $0 \leq a(t) \leq 1 \ , \quad 0 \leq h(t) \leq B(t)$
- Tolerable costs $c\big(a(t), Q(t)\big) \leq c^{\sharp}$

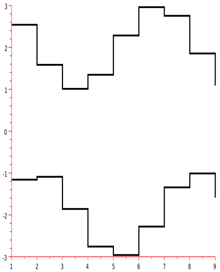
Control constraints / admissible decisions

$$\underbrace{u(t)}_{\text{control}} \in \underbrace{\mathbb{B}\big(t, x(t)\big)}_{\text{admissible set}} \ , \quad t = t_0, \ldots, T-1$$

Easy because control variables $u(t)$ are precisely those variables whose values the decision-maker can fix at any time within given bounds

Meeting constraints bearing on the state variable is delicate

due to the dynamics pipeline between controls and state



State constraints / admissible states

$$\underbrace{x(t)}_{\text{state}} \in \underbrace{\mathbb{A}(t)}_{\text{admissible set}} \quad , \quad t = t_0, \ldots, T$$

Examples ("tipping points")

- CO_2 concentration $M(t) \leq M^\sharp$
- biomass $B^\flat \leq B(t) \leq B^\sharp$

State constraints are mathematically difficult because of "inertia"

$$x(t) = \underbrace{\text{function}}_{\text{iterated dynamics}} \Big(\underbrace{u(t-1), \ldots, u(t_0)}_{\text{past controls}}, x(t_0) \Big)$$

Target and asymptotic state constraints are special cases

- Final state achieves some target

$$\underbrace{x(T)}_{\text{final state}} \in \underbrace{\mathbb{A}(T)}_{\text{target set}}$$

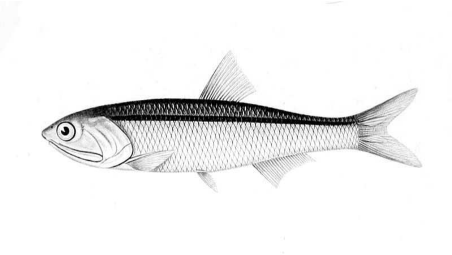
Example: CO_2 concentration

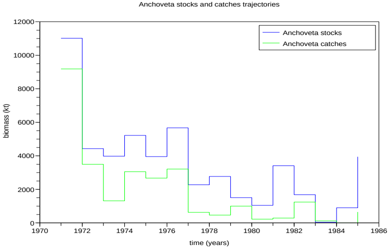
- State converges toward a target

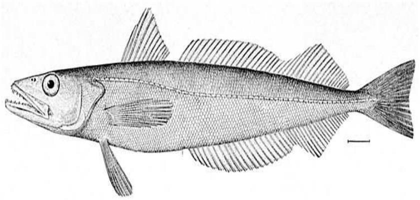
$$\underbrace{\lim_{t \to +\infty} x(t)}_{\text{asymptotic state}} \in \underbrace{\mathbb{A}(\infty)}_{\text{target set}}$$

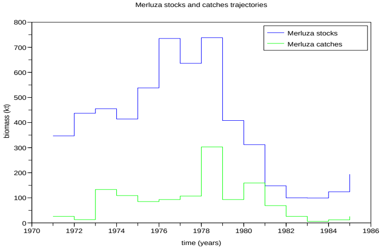
Example: convergence towards an endemic state in epidemiology

Anchoveta and merluza stock and catches trajectories, in Perú from 1971 to 1985









Trajectories are time sequences (of states and controls), also called paths

- Control trajectory

$$u(\cdot) := \underbrace{\big(u(t_0), u(t_0+1), \ldots, u(T-1)\big)}_{\text{control path}}$$

- State trajectory

$$x(\cdot) := \underbrace{\big(x(t_0), x(t_0+1), \ldots, x(T-1), x(T)\big)}_{\text{state path}}$$

- State-control trajectory

$$\big(x(\cdot), u(\cdot)\big) := \underbrace{\big(x(t_0), \ldots, x(T), u(t_0), \ldots, u(T-1)\big)}_{\text{state}-\text{control path}}$$

IMARPE data from 1971 to 1985 in thousands of tonnes (10^3 tons)

- anchoveta stocks [11019 4432 3982 5220 3954 5667 2272 2770 1506 1044 3407 1678 40 900 3944]
- merluza stocks [347 437 455 414 538 735 636 738 408 312 148 100 99 124 194]
- anchoveta captures [9184 3493 1313 3053 2673 3211 626 464 1000 223 288 1240 118 2 648]
- merluza captures [26 13 133 109 85 93 107 303 93 159 69 26 6 12 26]

A history is a whole path of states and controls, and the history set is the natural domain for an intertemporal optimization problem

- A state-control trajectory is called a history

$$\big(x(\cdot), u(\cdot)\big) := \underbrace{\big(x(t_0), \ldots, x(T), u(t_0), \ldots, u(T-1)\big)}_{\text{history}}$$

- The set of state and control trajectories is the so-called history set

$$\underbrace{\big(x(\cdot), u(\cdot)\big)}_{\text{history}} \in \underbrace{\mathbb{X}^{T+1-t_0} \times \mathbb{U}^{T-t_0}}_{\text{history set}}$$

Single dam histories

$$\big(S(\cdot), q(\cdot)\big) = \big(\overbrace{S(t_0), \ldots, S(T)}^{\text{stocks}}, \overbrace{q(t_0), \ldots, q(T-1)}^{\text{turbined}}\big)$$

State and control constraints reduce the set of admissible trajectories to account for feasibility issues

Admissible trajectories $\big(x(\cdot), u(\cdot)\big)$ in $\mathcal{T}^{\mathrm{ad}}(t_0, x_0)$ satisfy

- dynamics $x(t+1) = \mathtt{Dyn}\big(t, x(t), u(t)\big)$
- control constraints $u(t) \in \mathbb{B}\big(t, x(t)\big)$
- state constraints $x(t) \in \mathbb{A}(t)$

$$\mathcal{T}^{\mathrm{ad}}(t_0, x_0) := \left\{ \big(x(\cdot), u(\cdot)\big) \;\middle|\; \begin{array}{rll} x(t_0) = & x_0 \; , & \\ x(t+1) = & \mathtt{Dyn}\big(t, x(t), u(t)\big) \; , & t \in \mathbb{T} \\ u(t) \in & \mathbb{B}\big(t, x(t)\big) \; , & t \in \mathbb{T} \\ x(t) \in & \mathbb{A}(t) \; , & t \in \overline{\mathbb{T}} \end{array} \right\}$$

Admissible trajectories for a single dam dynamical model

$$\mathcal{T}^{\mathrm{ad}}(t_0, S_0) := \left\{ \big(S(\cdot), q(\cdot)\big) \;\middle|\; \begin{array}{rll} S(t_0) = & S_0 \; , & \\ S(t+1) = & \min\{S^\sharp, S(t) - q(t) + a(t)\} \; , & t \in \mathbb{T} \\ q(t) \in & [0, \min\{q^\sharp, S(t)\}] & t \in \mathbb{T} \end{array} \right\}$$

Outline of the presentation

1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

What is "optimization"?

Optimizing is obtaining the best compromise between needs and resources

Marcel Boiteux (président d'honneur d'Électricité de France)

- Needs: multiple targets
- Resources: multiple limits and multiple possible allocations
- Best compromise: value, trade-offs

An optimization problem can have multiple targets and limits
that can conflict with each other, requiring trade-offs

An intertemporal criterion assigns a value to each history

Intertemporal criterion

An (intertemporal) criterion or (intertemporal) objective function

$$\texttt{Crit}\Big(x(t_0), x(t_0+1), \ldots, x(T-1), x(T), u(t_0), u(t_0+1), \ldots, u(T-1)\Big)$$

is a function defined over the set of histories

$$\begin{aligned} \texttt{Crit}: \quad \mathbb{X}^{T+1-t_0} \times \mathbb{U}^{T-t_0} & \to \mathbb{R} \\ \big(x(\cdot), u(\cdot)\big) & \mapsto \texttt{Crit}\big(x(\cdot), u(\cdot)\big) \end{aligned}$$

Intertemporal payoff for a single dam

$$\texttt{Crit}\big(S(\cdot), q(\cdot)\big) = \sum_{t=t_0}^{T-1} \overbrace{\underbrace{p(t)}_{\text{price} \times} \underbrace{q(t)}_{\text{quantity}}}^{\text{turbined water profit}} + \overbrace{\texttt{Final}\big(S(T)\big)}^{\text{final stock utility}}$$

A criterion reflects the intertemporal preferences of the decision-maker (impatience, intergenerational equity, etc.)

- The additive and time-separable criterion

$$\texttt{Crit}\big(x(\cdot), u(\cdot)\big) = \sum_{t=t_0}^{T-1} \underbrace{\texttt{L}\big(t, x(t), u(t)\big)}_{\text{instantaneous gain}} + \underbrace{\texttt{Final}\big(T, x(T)\big)}_{\text{final gain}}$$

  is the most common and covers many well-known examples
  - Discounted present value (or net present value) $\sum_{t=t_0}^{T-1} \delta^{t-t_0}\texttt{L}\big(x(t), u(t)\big)$
  - Green Golden $\texttt{Final}\big(T, x(T)\big)$
  - Chichilnisky $\theta \sum_{t=t_0}^{T-1} \delta^{t-t_0}\texttt{L}\big(x(t), u(t)\big) + (1-\theta)\texttt{Final}\big(T, x(T)\big)$
- The Maximin or Rawls criterion

$$\texttt{Crit}\big(x(\cdot), u(\cdot)\big) = \min_{t=t_0,\dots,T-1} \texttt{L}\big(t, x(t), u(t)\big)$$

The most common additive and time-separable criterion allows for compensations between time periods

- The most usual criterion is additive and time-separable

$$\mathtt{Crit}(x(\cdot), u(\cdot)) = \sum_{t=t_0}^{T-1} \mathtt{L}(t, x(t), u(t)) + \mathtt{Final}(T, x(T))$$

- Additive criteria allow for possible compensations between time periods (like the sums of times spent on a graph)
- Environmental economists sanction the present value

$$\mathtt{Crit}(x(\cdot), u(\cdot)) = \overbrace{\sum_{t=t_0}^{+\infty} \left(\frac{1}{1+r_e}\right)^{t-t_0} \mathtt{L}(x(t), u(t))}^{\text{discounted utility}}$$

as "dictatorship of the present" (because of discounting)

Discounting erases the future

The French public discount rate

En France, le rapport *Révision du taux d'actualisation des investissements publics* (Commissariat général du Plan, groupe d'experts présidé par Daniel Lebègue, janvier 2005) a conduit à diviser par deux (de 8% à 4%) le taux d'actualisation à retenir pour évaluer la rentabilité des choix d'investissements publics

$$\frac{1}{1+r_e} = \frac{1}{1+0.04} \approx 0.96$$

The future in one hundred years is valued, seen from today, 2%

$$(\frac{1}{1+0.04})^{10} \approx 0.68 \;, \quad (\frac{1}{1+0.04})^{50} \approx 0.14 \;, \quad (\frac{1}{1+0.04})^{100} \approx 0.02$$

The Maximin focuses on minimal utility over time

- Equity: a focus on the poorest generation / utility level of the least advantaged generation
- The maximin form in the finite horizon case

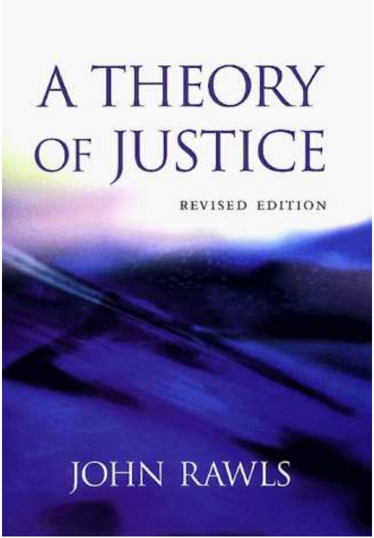
$$\texttt{Crit}\big(x(\cdot), u(\cdot)\big) = \underbrace{\min_{t=t_0,\ldots,T-1} \overbrace{\texttt{L}\big(t, x(t), u(t)\big)}^{\text{generation utility}}}_{\text{worse generation utility}}$$

- In the infinite horizon case

$$\texttt{Crit}\big(x(\cdot), u(\cdot)\big) = \min_{t=t_0,\ldots,+\infty} \texttt{L}\big(t, x(t), u(t)\big)$$

- There can be no compensations between time periods
- John Rawls, *A Theory of Justice*, 1971

John Bordley Rawls (1921–2002)



- John Bordley Rawls was an American philosopher and a leading figure in moral and political philosophy, famous for having written *A Theory of Justice* (1971)
- Two of John Rawls's younger brothers died as children – from illnesses they contracted from him
- Rawls believed he developed his life-long stutter as a result of guilt over his brothers' deaths

The Green Golden criterion is a "dictatorship of the future"

- In the finite horizon case

$$\mathtt{Crit}\big(x(\cdot), u(\cdot)\big) = \mathtt{Final}\big(T, \underbrace{x(T)}_{\text{state}}\big)$$

- In the infinite horizon case

$$\mathtt{Crit}\big(x(\cdot), u(\cdot)\big) = \liminf_{T \to +\infty} \mathtt{Final}\big(T, x(T)\big)$$

- The Green Golden criterion values only the final state and none of the controls (no consumption)

The Chichilnisky criterion is in-between

- The Chichilnisky form with ponderation parameter $\theta \in [0,1]$

$$\texttt{Crit}\big(x(\cdot),u(\cdot)\big) = \underbrace{\theta \sum_{t=t_0}^{T-1} \Big(\frac{1}{1+r_e}\Big)^{t-t_0} \texttt{L}\big(x(t),u(t)\big)}_{\text{dictatorship of the present}} + (1-\theta) \underbrace{\texttt{Final}\big(T,x(T)\big)}_{\text{dictatorship of the future}}$$

- Sustainability: to reconcile $\begin{cases} \text{present} \\ \text{future} \end{cases}$
- In the infinite horizon case

$$\texttt{Crit}\big(x(\cdot),u(\cdot)\big) = \theta \sum_{t=t_0}^{+\infty} \texttt{L}\big(t,x(t),u(t)\big) + (1-\theta) \liminf_{T\to+\infty} \texttt{Final}\big(T,x(T)\big)$$

Summary

- Discrete-time nonlinear state-control systems
  are special input-output dynamical systems
  - control = input
  - state = specific output satisfying a dynamical equation
- Trajectories are time sequences (of states and controls), also called paths
- State and control constraints reduce the set of admissible trajectories to account for feasibility issues
- A history is a whole path of states and controls, and an intertemporal criterion assigns a value to each history
- A criterion reflects the intertemporal preferences of the decision-maker (impatience, intergenerational equity, etc.)
- An optimal trajectory maximizes the criterion over all admissible trajectories

Outline of the presentation

Outline of the presentation

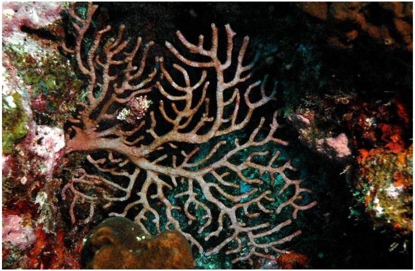
1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

The New Caledonia lagoon hosts rich trophic webs

- Abore reef reserve (15 000 ha) New Caledonia
- large coral reef ecosystem: 374 species
- differing in mobility, taxonomy (41 families) and feeding habits
- 7 clusters, each cluster forming a trophic group

Species differ in mobility, taxonomy and feeding habits






Diet composition varies with groups of species

	Piscivores (Pi)	Macro carnivores (MC)	Micro carnivores (mC)	Coral feeders (Co)	Herbivores (He)	Microalgae Detritivores (mAD)	Zooplankton feeders (Zoo)
Group for the model	X_1	X_2	X_4	X_4	X_3	X_3	X_4
Species richness	46	112	50	26	10	73	54
Diet composition (%)							
- Nekton	77	10	2	0	0	0.1	1
- Macroinvertebrates	21	82	20	2	0	2	1
- Microinvertebrates	0.3	6	67	11	3	5	6
- Zooplankton	1	0.4	3	2	0	3	79
- Other plankton	0	0	0	0	0	0	0.3
- Macroalgae	0	0.3	1	0	66	3	0.3
- Microalgae	0	1	5	7	28	80	11
- Coral	0	0.3	2	77	0	1	0.3
- Detritus	0	0.3	1	1	4	6	0.2
Maxi. adult size (cm)	77	38	17	16	39	24	13

A Lotka-Volterra model is the simplest trophic model

- N_i abundance of species i (number of individuals or density)

$$N_i(t+1) = N_i(t) \overbrace{\Big(R_i + \sum_{j=1}^{n} S_{ij} N_j(t)\Big)}^{\text{density-dependent growth factor}}$$

- Species i consumes species j when $S_{ij} > 0$
- Species i is the prey of species j if $S_{ij} < 0$
- The strength of direct intra-specific interactions is given by $S_{ii} < 0$

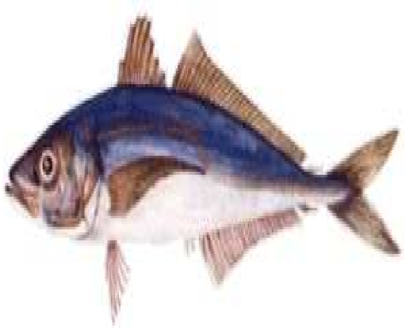
Example of interaction coefficients

$$S = \begin{pmatrix} -0.093 & 0.013 & 0.013 & 0.013 \\ -0.106 & -0.012 & 0.002 & 0.002 \\ -0.076 & -0.01 & 0. & 0. \\ -0.53 & -0.069 & 0. & 0. \end{pmatrix}$$

Outline of the presentation

1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

Populations can be described by abundances at ages



Jack Mackrel abundances (Chilean data) are measured in thousand of individuals

13651022	thousand of age < 1 (recruits)
7495888	thousand of age $\in [1, 2[$
6804151	
4191318	
4582943	
2500338	
1139182	
523261	
269328	
166390	
95606	thousand of age ≥ 11

We now line up the ingredients of a harvested population age-class dynamical model

- Time $t \in \mathbb{N}$ measured in years
- Abundances at age $N = (N_a)_{a=1,\dots,A} \in \mathbb{X} = \mathbb{R}_+^A$
- $a \in \{1, \dots, A\}$ age class index
  - $A = 3$ for anchovy
  - $A = 8$ for hake
  - $A = 40$ for bacalao
- Control variable $\lambda \in \mathbb{U} = \mathbb{R}_+$ is fishing effort

One year older every year. . .

Except for the recruits ($a = 1$) and the last age class ($a = A$),

$$N_a(t+1) = e^{-\overbrace{(\underbrace{M_{a-1}}_{\text{natural}} + \underbrace{\lambda(t) F_{a-1}}_{\text{fishing}})}^{\text{mortality}}} N_{a-1}(t), \quad a = 2, \ldots, A-1$$

where

- M_a stands for
  the natural mortality-at-age a
- F_a is the harvesting mortality rate
  of individuals of age a,
  also called exploitation pattern-at-age a,
  related to the mesh size for instance
- the control variable $\lambda(t)$ is the fishing effort,
  or the exploitation pattern multiplier

The last age-class may comprise a plus-group

- N_A is the abundance of individuals of age above $A-1$ (and not equal, like for other classes)
- To account for this specificity, one considers the dynamics

$$\begin{aligned} N_A(t+1) &= N_{A-1}(t)\exp\big(-(M_{A-1}+\lambda(t)F_{A-1})\big) \\ &+ \underbrace{\pi}_{0 \text{ or } 1} N_A(t)\exp\big(-(M_A+\lambda(t)F_A)\big) \end{aligned}$$

- The parameter $\pi \in \{0,1\}$ is related to the existence of a so-called plus-group
  - if we neglect the survivors older than age A, then $\pi = 0$ (an example is anchovy)
  - if we consider the survivors older than age A, then $\pi = 1$, and the last age class is a plus group (an example is hake)

The stock-recruitment function mathematically turns spawning stock biomass into future recruits abundance

- The spawning stock biomass is

$$\mathrm{SSB}(N) = \sum_{a=1}^{A} \underbrace{\gamma_a}_{\text{proportion}} \overbrace{\mu_a}^{mass} \underbrace{N_a}_{\text{abundance}}$$

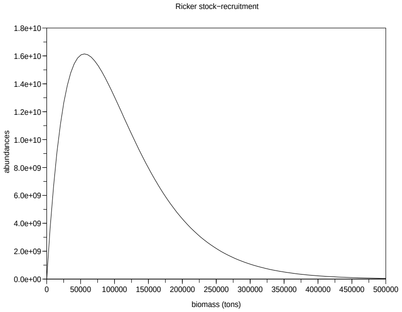
  - γ_a proportion of matures-at-age a
  - μ_a weight-at-age a

- The stock-recruitment relationship $\mathtt{S/R}$ turns biomass into abundance

$$\underbrace{N_1(t+1)}_{\text{future recruits}} = \mathtt{S/R}\Big(\underbrace{\mathrm{SSB}\big(N(t)\big)}_{\text{spawning biomass}} \Big)$$

Here are traditional examples of stock-recruitment functions

Recruitment involves complex biological and environmental processes that fluctuate in time, and are difficult to integrate into a population model



- constant: $\mathtt{S/R}(B) = R$
- linear: $\mathtt{S/R}(B) = rB$
- Beverton-Holt: $\mathtt{S/R}(B) = \frac{B}{\alpha+\beta B}$
- Ricker: $\mathtt{S/R}(B) = \alpha B \mathrm{e}^{-\beta B}$

And here are the state vector and the control

- The state vector $N(t)$ is forged with abundances at age

$$N(t) = \begin{pmatrix} N_1(t) \\ N_2(t) \\ \vdots \\ N_{A-1}(t) \\ N_A(t) \end{pmatrix} \in \mathbb{R}_+^A$$

- The scalar control $\lambda(t)$ is the fishing effort multiplier

A harvested population age-class model is an *A*—dimensional controlled dynamical system

$$
\begin{aligned}
N_1(t+1) &= \mathtt{S/R}\Big(\overbrace{\mathtt{SSB}(N(t))}^{\text{spawning biomass}} \Big) \quad \text{recruitment} \\
N_2(t+1) &= e^{-(M_1+\lambda(t)F_1)} N_1(t) \\
N_a(t+1) &= e^{-\overbrace{(\underbrace{M_{a-1}}_{\text{natural}} + \underbrace{\lambda(t)F_{a-1}}_{\text{fishing}})}^{\text{mortality}}} N_{a-1}(t), \quad a = 2, \ldots, A-1 \\
N_{A-1}(t+1) &= e^{-(M_{A-2}+\lambda(t)F_{A-2})} N_{A-2}(t) \\
N_A(t+1) &= e^{-(M_{A-1}+\lambda(t)F_{A-1})} N_{A-1}(t) + \underbrace{\pi e^{-(M_A+\lambda(t)F_A)}}_{\text{plus group}} N_A(t)
\end{aligned}
$$

Catches and production formulas depend on effort and abundances

- Catches (Baranov catch equation)

$$\mathtt{Catch}_a(\lambda, N) = \frac{\lambda F_a}{\lambda F_a + M_a}\Big(1 - \exp\big(-(M_a + \lambda F_a)\big)\Big) N_a$$

- Production (summing catches over mean weights per age)

$$\mathtt{Yield}(\lambda, N) = \sum_{a=1}^{A} \mu_a \, \mathtt{Catch}_a(\lambda, N)$$

- Thus quotas on catches are related to the fishing effort multiplier $\lambda(t)$

Here are different possible objectives

- *International Council for the Exploration of the Sea* (ICES) *precautionary approach* on spawning biomass requires that

$$\mathtt{SSB}\big(N(t)\big) \geq B_{\text{lim}}$$

- Guaranteed production is achieved by

$$\mathtt{Yield}\big(\lambda(t), N(t)\big) \geq Y^{\flat}$$

- Economists usually aim at optimizing the discounted rent

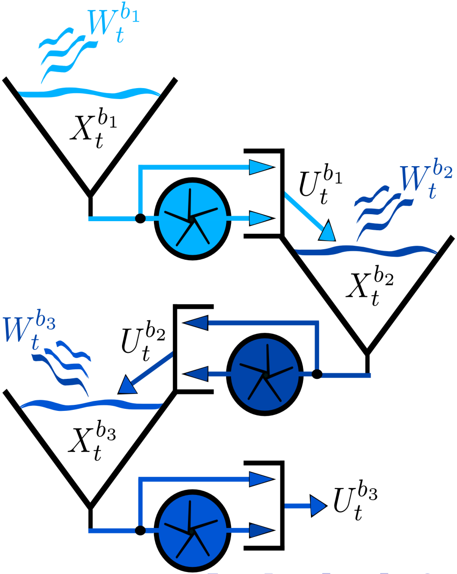
$$\max_{\lambda(\cdot)} \sum_{t=t_0}^{+\infty} \Big(\frac{1}{1+r_e}\Big)^{t-t_0} \Big(p\mathtt{Yield}\big(\lambda(t), N(t)\big) - c\lambda(t)\Big)$$

where p is a unit price, whereas c is a unit cost

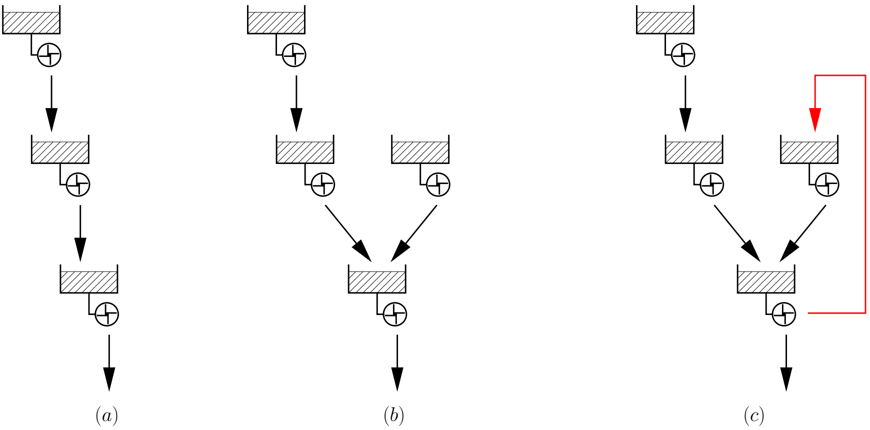
Outline of the presentation

1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

Complexity increases with interconnected dams

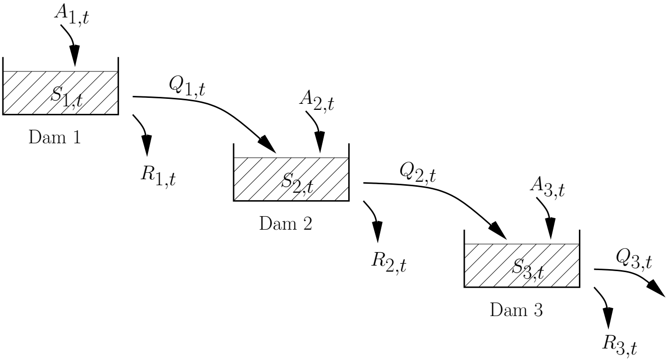


Typology of hydro-valleys



dams in cascade | converging valleys | pumping

Sketch of a cascade model with dams $i = 1, \ldots, N$



$a_i(t)$: inflow into dam i at time t (rain, run off water)
$S_i(t)$: volume in dam i at time t (water volume)
$q_i(t)$: turbined from dam i at time t (valued at price $p_i(t)$)
$r_i(t)$: spilled volume from dam i at time t (irrigation. . .)

Outline of the presentation

Outline of the presentation

The ICES precautionary approach is an example of policy

The precautionary approach (PA) may be sketched as follows

- The condition $\mathtt{SSB}(N) \geq B_{\text{lim}}$ is checked
- If valid, the following usual advice is given:

$$\overbrace{\lambda_{UA}(N)}^{\text{effort}} = \max\{\lambda \in \mathbb{R}_+ \mid \quad \lambda \leq \lambda_{\text{lim}} \text{ and } \underbrace{\mathtt{SSB}\big(\mathtt{Dyn}(N, \lambda)\big)}_{\text{future spawning biomass}} \geq B_{\text{lim}}\}$$

Open loop solutions are control paths

$$x(t+1) = \mathtt{Dyn}\big(t, x(t), u(t)\big)$$

- Stationary (open-loop): stationary sequences

$$u : \underbrace{t \in \mathbb{T}}_{\text{time}} \mapsto \underbrace{u(t) \equiv u_{\mathrm{E}} \in \mathbb{U}}_{\text{control}}$$

Example: maximum sustainable yield

- Open-loop: time-dependent sequences (planning, scheduling)

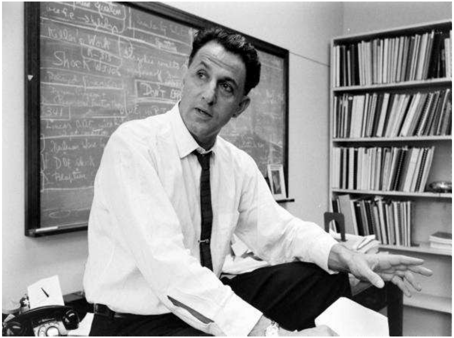
$$u : \underbrace{t \in \mathbb{T}}_{\text{time}} \mapsto \underbrace{u(t) \in \mathbb{U}}_{\text{control}}$$

Example: Pontryagin approach to optimal control

Solutions are no longer control paths, but are policies

From planning 🕐 to contingent planning 🕐× 👁

Again the intriguing thought: A solution is not merely a set of functions of time, or a set of numbers, but a rule telling the decisionmaker what to do; a policy. (Richard Bellman)

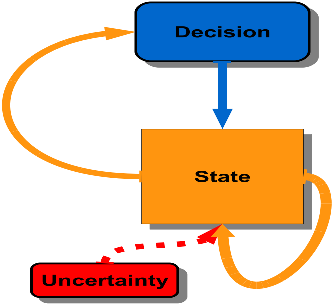


Richard Ernest Bellman (August 26, 1920 – March 19, 1984) was an applied mathematician, celebrated for his invention of dynamic programming in 1953, and important contributions in other fields of mathematics

Wikipedia

The concept of policy as a contingent planning

However, the thought was finally forced upon me that the desired solution in a control process was a policy:
'Do thus-and-thus if you find yourself in this portion of state space with this amount of time left.'



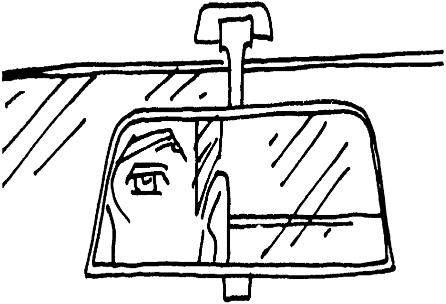
Richard Bellman autobiography, Eye of the Hurricane

- Closed-loop: state feedback (decision rule)

$$\texttt{Pol} : \underbrace{(t,x) \in \mathbb{T} \times \mathbb{X}}_{\text{(time, state)}} \mapsto u = \underbrace{\texttt{Pol}(t,x) \in \mathbb{U}}_{\text{control}}$$

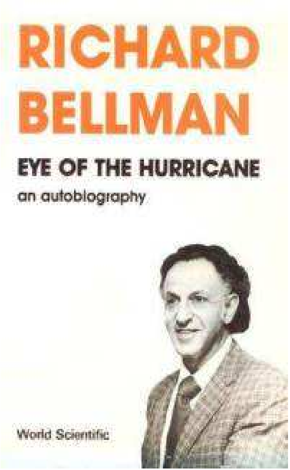
- Going from planning to contingent planning, we have considerably enlarged the set of solutions
  - an open-loop solution is an element of $\mathbb{U}^{\mathbb{T}}$
  - whereas now it is an element of $\mathbb{U}^{\mathbb{T}\times\mathbb{X}}$

"The blind cat does not catch mice"



- A decision rule depends on online information
- State feedback decision rules are natural solutions given by dynamic programming methods
- Adaptive decision rules
  - Appropriate for managing uncertain systems
  - More robust

How clouded the crystal ball looks beforehand

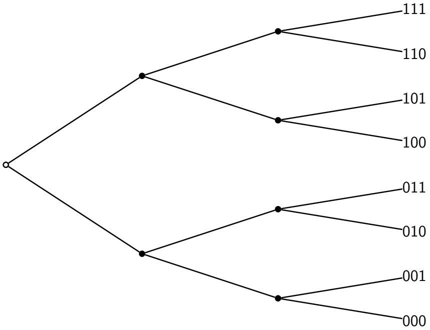


What is worth noting about the foregoing development is that I should have seen the application of dynamic programming to control theory several years before. I should have, but I didn't. It is very well to start a lecture by saying, 'Clearly, a control process can be regarded as a multistage decision process in which. . . ,' but it is a bit misleading.
Scientific developments can always be made logical and rational with sufficient hindsight. It is amazing, however, how clouded the crystal ball looks beforehand. We all wear such intellectual blinders and make such inexplicable blunders that it is amazing that any progress is made at all.

Outline of the presentation

1. Stylized examples of sequential decision models
   - Exploitation of an exhaustible resource
   - Management of a renewable resource
   - Mitigation policies for carbon dioxyde emissions
   - Single dam management under tourism constraint
2. Clothing a sequential decision problem in formal garb
   - State-control dynamical systems
   - State-control constraints and feasability/viability
   - Criterion and optimality
3. A glimpse at some more complex models
   - A trophic web example
   - A single species age-classified model of fishing
   - Interconnected dam models
4. General remarks on numerical issues
   - Open and closed loop solutions to sequential decision problems
   - Computational explosion with time

A simple decision tree illustrates the exponential growth



- A binary decision $u \in \{0, 1\}$ on horizon $T = 52$ weeks
  $\Longrightarrow 2^{52}$ possible paths $u(\cdot)$
- On a computer
  - RAM: 8 GBytes = $8(1\ 024)^3 = 2^{33}$ bytes
  - a double-precision real: 8 bytes = 2^3 bytes

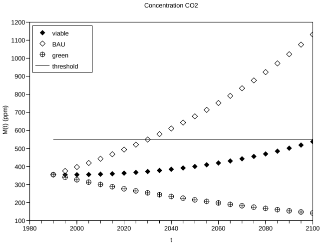
  $\Longrightarrow 2^{30}$ double-precision reals
  $<< 2^{52}$ possible controls paths

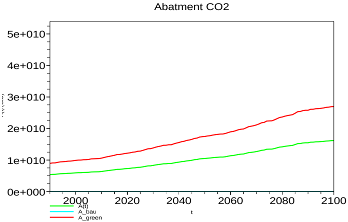
The rice and chessboard tale

It is said that the inventor of the game of chess was invited by his sovereign who was so pleased that he asked him to choose his price. The king was suprised, enven offended, by the inventor's answer: for the first square of the chessboard, he would receive one grain of rice; two for the second, four for the third, etc., doubling the amount from one square to the other. The total number is around 1,000 times the global production of rice in 2010.

Simulations can help

- Test specific policies by computing outputs for a given initial state x_0
- Example: projections of mitigation policies

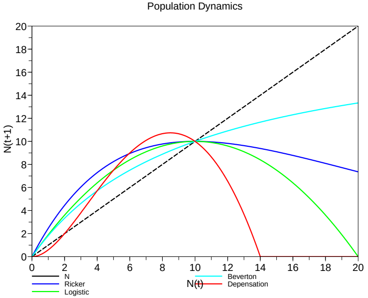


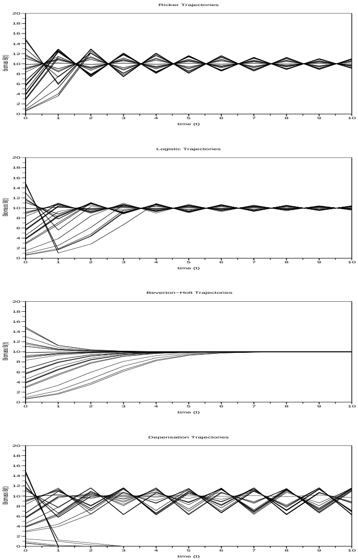


- Problem: the outputs, like $\texttt{L}\big(x(t), u(t)\big)$, etc. depend on
  - the parameters of the model
  - the functional form of the model

Sensitivity to functional forms is a delicate issue

Example: different population dynamics with $R = 2$, $K = 10$





Summary and conclusion

- We have unwrapped control theory with an eye
  to formalize sustainability issues in mathematical suits
- We have laid out management examples with objective framed as constraints or criterion
- The firt step towards sustainability consists in equating it with permanency
- Hence, we explore equilibrium and stability in the next chapter