Equilibrium and Stability

Extended from Chapter 3 of
Sustainable Management of Natural Resources. Mathematical Models and Methods
by Luc DOYEN and Michel DE LARA

Michel De Lara
Cermics, École des Ponts ParisTech
Université Paris-Est
France

École des Ponts ParisTech

August 19, 2014

Equilibrium and stability analysis provides insight into factors driving world's fisheries non-sustainability

Towards sustainability in world fisheries

Pauly, D. V. Christensen, S. Guénette T.J. Pitcher, U.R. Sumaila, C.J. Walters, R. Watson and D. Zeller. Nature, 2002

insight review articles

Towards sustainability in world fisheries

Daniel Pauly, Villy Christensen, Sylvie Guénette, Tony J. Pitcher, U. Rashid Sumaila, Carl J. Walters, R. Watson & Dirk Zeller

Fisheries Centre, University of British Columbia, 2204 Main Mall, Vancouver, British Columbia, Canada V6T 1Z4 (e-mail: d.pauly@fisheries.ubc.ca)

Fisheries have rarely been 'sustainable'. Rather, fishing has induced serial depletions, long masked by improved technology, geographic expansion and exploitation of previously spurned species lower in the food web. With global catches declining since the late 1980s, continuation of present trends will lead to supply shortfall, for which aquaculture cannot be expected to compensate, and may well exacerbate. Reducing fishing capacity to appropriate levels will require strong reductions of subsidies. Zoning the oceans into unfished marine reserves and areas with limited levels of fishing effort would allow sustainable fisheries, based on resources embedded in functional, diverse ecosystems.

Fishing is the catching of aquatic wildlife, the equivalent of hunting bison, deer and rabbits on land. Thus, it is not surprising that industrial-scale fishing should generally not be sustainable: industrial-scale hunting, on land, would not be, either. What is surprising rather, is how entrenched the notion is that unspecified 'environmental change' caused, and continues to cause, the collapse of exploited fish populations. Examining the history of fishing and fisheries makes it abundantly clear that humans have had for thousands of years a major impact on target species and their supporting ecosystems[1]. Indeed, the archaeological literature contains many examples of ancient human fishing associated with gradual shifts, through time, to smaller sizes and the serial depletion of species that we now recognize as the symptoms of overfishing[1,2].

This literature supports the claim that, historically, fisheries have tended to be non-sustainable, although not unexpectedly there is a debate about the cause for this[3], and the exceptions[4]. The few uncontested historical examples of sustainable fisheries seem to occur where a superabundance of fish supported small human populations in challenging climates[5]. Sustainability occurred where fish populations were naturally protected by having a large part of their distribution outside of the range of fishing operations. Hence, many large old fecund females, which contribute overwhelmingly to the egg production that renews fish populations, remained untouched. How important such females can be is illustrated by the example of a single ripe female red snapper, Lutjanus campechanus, of 61 cm and 12.5 kg, which contains the same number of eggs (9,300,000) as 212 females of 42 cm and 1.1 kg each[6]. Where such natural protection was absent, that is, where the entire population was accessible to fishing gears, depletion ensued, even if the gear used seems inefficient in retrospect[7,8]. This was usually masked, however, by the availability of other species to target, leading to early instances of depletions observable in the changing size and species composition of fish remains, for example, in middens[9].

The fishing process became industrialized in the early nineteenth century when English fishers started operating steam trawlers, soon rendered more effective by power winches and, after the First World War, diesel engines[10]. The aftermath of the Second World War added another 'peace dividend' to the industrialization of fishing: freezer trawlers, radar and acoustic fish finders. The fleets of the Northern Hemisphere were ready to take on the world.

Fisheries science advanced over this time as well: the two world wars had shown that strongly exploited fish populations, such as those of the North Sea, would recover most, if not all, of their previous abundance when released from fishing[11]. This allowed the construction of models of single-species fish populations whose size is affected only by fishing pressure, expressed either as a fishing mortality rate (F, or catch/biomass ratio), or by a measure of fishing effort (f, for example, trawling hours per year) related to F through a catchability coefficient[12,13] (q): F = qf. Here, q represents the fraction of a population caught by one unit of effort, directly expressing the effectiveness of a gear. Thus, q should be monitored as closely as fishing effort itself, if the impact of fishing on a given stock, as expressed by F, is to be evaluated. Technology changes tend to increase q, leading to increases referred to as 'technology coefficient'[14], which quickly renders meaningless any attempts to limit fishing mortality by limiting only fishing effort.

The conclusion of these models, still in use even now (although in greatly modified forms; Box 1), is that adjusting fishing effort to some optimum level should generate 'maximum sustainable' yield, a notion that the fishing industry and the regulatory agencies eagerly adopted — if only in theory[15]. In practice, optimum effort levels were very rarely implemented (the Pacific halibut fishery is one exception[16]). Rather the fisheries expanded their reach, both offshore, by fishing deeper waters and remote seamounts[17], and by moving onto the then untapped resources of West Africa[18], southeast Asia[19], and other low-latitude and Southern Hemispheric regions[20].

Fisheries go global

In 1950, the newly founded Food and Agriculture Organization (FAO) of the United Nations began collection of global statistics. Fisheries in the early 1950s were at the onset of a period of extremely rapid growth, both in the Northern Hemisphere and along the coast of the countries of what is now known as the developing world. Everywhere that industrial-scale fishing (mainly trawling, but also purse seining

NATURE | VOL 418 | 8 AUGUST 2002 | www.nature.com/nature

- Open-access nature of many fisheries
- Common-pool fisheries that are managed non-cooperatively
- Sole-ownership fisheries with high discount rates and/or high price-to-cost ratios ones
- Payment of subsidies by governments to fishers, which generate 'profits' even when resources are overfished

Outline of the presentation

1. Equilibrium
2. Sustainable yield and related notions
3. Stability
4. Summary

Outline of the presentation

Equilibrium is the mathematical concept carrying the idea of sustainability as balance and stationarity

We consider autonomous dynamical systems, that is, without explicit dependence upon time t

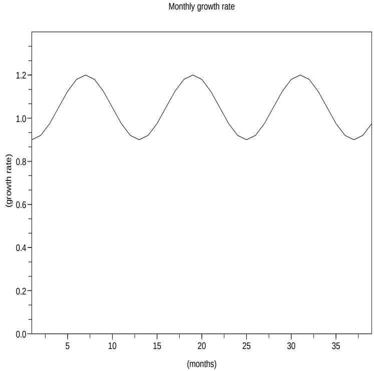
- Recall that
  - $x(t) \in \mathbb{X} = \mathbb{R}^n$ represents the state of the system
  - $u(t) \in \mathbb{U} = \mathbb{R}^p$ stands for the control
- In an autonomous dynamical system,
  the dynamics does not depend directly on time t

$$x(t+1) = \texttt{Dyn}\big(x(t), u(t)\big)$$

- Constraints are also time independent

$$\begin{cases} x(t) \in \mathbb{A} \\ u(t) \in \mathbb{B}(x(t)) \end{cases}$$

Example of non autonomous system



Monthly plant model

$$B(t+1) = R(t)B(t)$$

where the monthly growth rate is periodic, following the seasons

An equilibrium is a steady state of an autonomous dynamical system

Equilibrium under constraints

The state $x_{\mathrm{E}} \in \mathbb{X}$ is an equilibrium under constraints if there exists a control $u_{\mathrm{E}} \in \mathbb{U}$ satisfying

$$\begin{aligned} \mathtt{Dyn}(x_{\mathrm{E}}, u_{\mathrm{E}}) = x_{\mathrm{E}} & \quad \text{steady state} \\ x_{\mathrm{E}} \in \mathbb{A} & \quad \text{admissible state} \\ u_{\mathrm{E}} \in \mathbb{B}(x_{\mathrm{E}}) & \quad \text{admissible control} \end{aligned}$$

Outline of the presentation

Equilibria are trivial for an exhaustible resource

- Dynamics

$$\underbrace{S(t+1)}_{\text{future stock}} = \underbrace{S(t)}_{\text{stock}} - \underbrace{h(t)}_{\text{extraction}}$$

- The resource stock S_{E} and harvesting h_{E} are stationary whenever

$$S_{\mathrm{E}} = S_{\mathrm{E}} - h_{\mathrm{E}}$$

- The only equilibrium for any S_{E} is

$$h_{\mathrm{E}} = 0$$

Flooding in Cuzco Valley, Peru

Mitigation policies for carbon dioxyde emissions

- Assuming stationary emissions E_b

$$M(t+1) = M(t) \underbrace{-\delta(M(t) - M_{-\infty})}_{\text{natural sinks}} + \alpha \overbrace{E_b}^{\text{emissions}} \underbrace{(1 - a(t))}_{\text{abatement}}$$

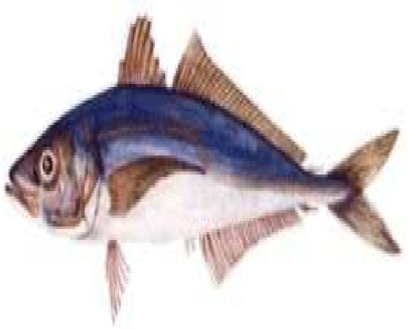
- Any equilibrium $(M_{\mathrm{E}}, a_{\mathrm{E}})$ satisfies

$$M_{\mathrm{E}} = M_{\mathrm{E}} - \delta(M_{\mathrm{E}} - M_{-\infty}) + \alpha E_b (1 - a_{\mathrm{E}}) \; ,$$

that is,

$$M_{\mathrm{E}} = M_{-\infty} + \frac{\alpha E_b (1 - a_{\mathrm{E}})}{\delta} \text{ with } 0 \leq a_{\mathrm{E}} \leq 1$$

Populations may be described by abundances at ages



Jack Mackrel abundances (Chilean data) in thousand of individuals

13651022	thousand of age < 1 (recruits)
7495888	thousand of age $\in [1, 2[$
6804151	
4191318	
4582943	
2500338	
1139182	
523261	
269328	
166390	
95606	thousand of age ≥ 11

Harvested population age-class dynamics

The spawning stock biomass is $\mathrm{SSB}(N) = \sum_{a=1}^{A} \underbrace{\gamma_a}_{\text{proportion}} \overbrace{\mu_a}^{\text{mass}} \underbrace{N_a}_{\text{abundance}}$

$$
\begin{aligned}
N_1(t+1) &= \mathtt{S/R}\Big(\overbrace{\mathtt{SSB}\big(N(t)\big)}^{\text{spawning biomass}} \Big) \quad \text{recruitment} \\
N_2(t+1) &= e^{-(M_1+\lambda(t)F_1)} N_1(t) \\
N_a(t+1) &= e^{-\overbrace{(\underbrace{M_{a-1}}_{\text{natural}} + \underbrace{\lambda(t)F_{a-1}}_{\text{fishing}})}^{\text{mortality}}} N_{a-1}(t) \quad a = 2, \ldots, A-1 \\
N_{A-1}(t+1) &= e^{-(M_{A-2}+\lambda(t)F_{A-2})} N_{A-2}(t) \\
N_A(t+1) &= e^{-(M_{A-1}+\lambda(t)F_{A-1})} N_{A-1}(t) + \underbrace{\pi e^{-(M_A+\lambda(t)F_A)}}_{\text{plus group}} N_A(t)
\end{aligned}
$$

An A age-classes equilibrium is solution of an algebraic system of A equations

$$\begin{pmatrix} \mathtt{S/R}\big(\mathtt{SSB}(N)\big) \\ N_1 \exp\big(-(M_1+\lambda F_1)\big) \\ N_2 \exp\big(-(M_2+\lambda F_2)\big) \\ \vdots \\ N_{A-2} \exp\big(-(M_{A-2}+\lambda F_{A-2})\big) \\ N_{A-1} \exp\big(-(M_{A-1}+\lambda F_{A-1})\big) \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_{A-1} \\ N_A \end{pmatrix}$$

$A - 1$ equations display a chained structure

The computation of an equilibrium $N_{\mathrm{E}}(\lambda)$, for $\lambda \geq 0$, gives

$$N_{a,\mathrm{E}}(\lambda) = s_a(\lambda) N_{1,\mathrm{E}}(\lambda)$$

for $a = 1, \ldots, A$, where

$$s_a(\lambda) = \exp\Big(-\big(M_1 + \cdots + M_{a-1} + \lambda(F_1 + \cdots + F_{a-1})\big)\Big)$$

is the proportion of equilibrium recruits which survive up to age a ($a = 2, \ldots, A$) while $s_1(\lambda) = 1$

Exercise: compute the equilibrium when the stock-recruitment is a Beverton-Holt function

- When the stock-recruitment is a Beverton-Holt function

$$\mathtt{S/R}(B) = \frac{B}{\alpha + \beta B}$$

- the equilibrium recruits are either $N_{1,\mathrm{E}}(\lambda) = 0$ or

$$N_{1,\mathrm{E}}(\lambda) = \frac{\mathtt{spr}(\lambda) - \alpha}{\beta \mathtt{spr}(\lambda)} \text{ when } \mathtt{spr}(\lambda) > \alpha$$

- where $\mathtt{spr}$ is the equilibrium spawners per recruits

$$\mathtt{spr}(\lambda) = \sum_{a=1}^{A} \gamma_a \mu_a s_a(\lambda)$$

- Exercice: case $A = 2$ and interpret the condition $\mathtt{spr}(\lambda)/\alpha > 1$

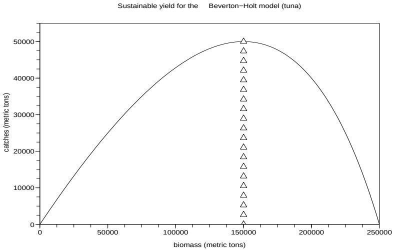
Summary

- An equilibrium is solution of an algebraic system of equations
- The number of equations is equal to the dimension of the state
- For control systems, equilibria are parameterized by the controls
- Constraints can restrict the number of admissible equilibria

Outline of the presentation

Outline of the presentation

1. Equilibrium
   - Definition of equilibrium (under const
   - Examples of equilibria
2. Sustainable yield and related notions
   - Sustainable yield for surplus model
   - Maximum sustainable equilibrium
   - Private property equilibrium
   - Common property equilibrium
   - Examples
3. Stability
   - Definition of stability of an equilibriu
   - Stability for dynamical linear systems
   - Linearized dynamics and stability
   - Examples
4. Summary



The sustainable yield for the Schaefer model is the harvest attached to a biomass equilibrium

- The Schaefer model builds upon a biological dynamics

$$B(t+1) = \texttt{Biol}\big(\underbrace{B(t)}_{\text{biomass}} - \underbrace{h(t)}_{\text{catches}} \big) \text{ with } 0 \leq h(t) \leq B(t)$$

- At equilibrium, we have that

$$B_{\mathrm{E}} = \texttt{Biol}(B_{\mathrm{E}} - h_{\mathrm{E}}) \text{ and } 0 \leq h_{\mathrm{E}} \leq B_{\mathrm{E}}$$

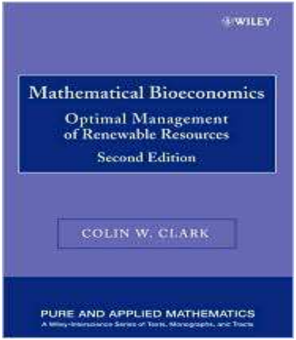
- The stationay harvest h_{E} solution to that equilibrium equation is the so-called sustainable yield

$$h_{\mathrm{E}} = \texttt{Sust}(B_{\mathrm{E}})$$

The sustainable yield is a surplus production that can be harvested in perpetuity without altering the stock level

The relation $\texttt{Biol}(B_E - h_E) = B_E$ may also be written as

$$\underbrace{h_E}_{\text{surplus}} = \underbrace{\texttt{Biol}(B_E - h_E)}_{\text{regeneration}} - \underbrace{(B_E - h_E)}_{\text{biomass after capture}} \geq 0$$



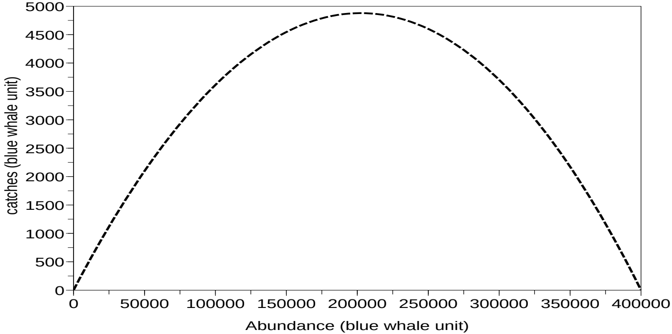
meaning that

> *a surplus production exists that can be harvested in perpetuity without altering the stock level*

(Clark, 1990)

Sustainable yield curve for the blue whale

$K = 400\ 000$ BWU
$R \approx 1.05$
Beverton-Holt



Exercise: compute the equilibrium when the biological dynamics is a Beverton-Holt function

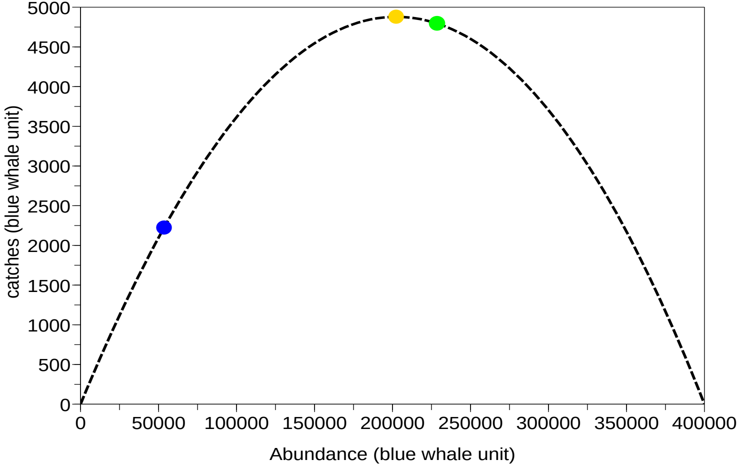
- Biological Beverton-Holt dynamics $\mathtt{Biol}(B) = \frac{RB}{1+bB}$
- Any equilibrium $(B_{\mathrm{E}}, h_{\mathrm{E}})$ satisfies

$$B_{\mathrm{E}} = \frac{R(B_{\mathrm{E}} - h_{\mathrm{E}})}{1 + b(B_{\mathrm{E}} - h_{\mathrm{E}})} \text{ with } 0 \leq h_{\mathrm{E}} \leq B_{\mathrm{E}}$$

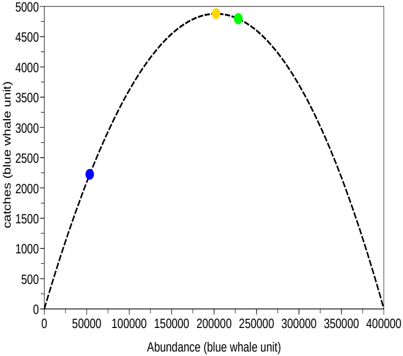
- Particular equilibria are
  - $(B_{\mathrm{E}}, h_{\mathrm{E}}) = (0, 0)$
  - $(B_{\mathrm{E}}, h_{\mathrm{E}}) = (K, 0)$ where $K = \frac{R-1}{b}$ is the carrying capacity
- The sustainable yield is

$$\mathtt{Sust}(B) = B - \frac{B}{R - bB} \text{ for } 0 \leq B \leq K$$

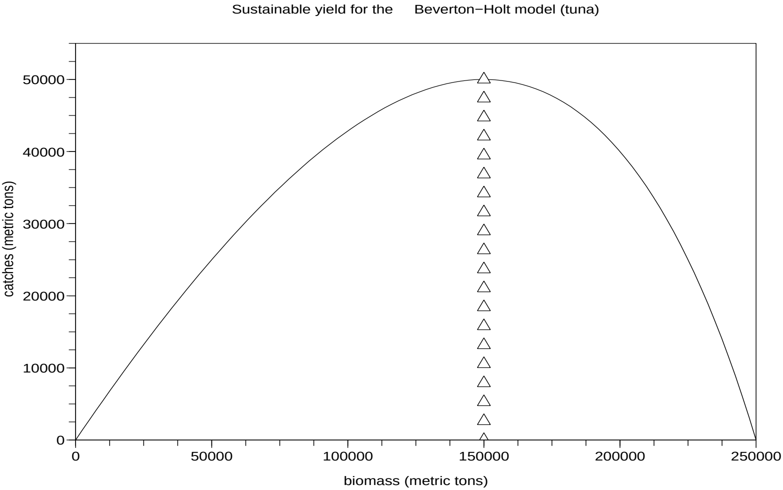
Three equilibria are worth being distinguished: MSE, PPE, CPE



Outline of the presentation



The sustainable yield has a hump-shape form as a function of biomass



The maximum sustainable yield is the maximum surplus production that can be harvested in perpetuity without altering the stock level

- The maximum sustainable equilibrium (MSE) is the biomass B_{MSE} solution of

$$\texttt{Sust}(B_{\text{MSE}}) = h_{\text{MSE}} = \max_{B \geq 0,\ h = \texttt{Sust}(B)} h = \max_{B \geq 0} \texttt{Sust}(B)$$

- The maximum catch h_{MSE} is called the maximum sustainable yield (MSY)

MSY is criticized for its simplistic assumptions

TRANSACTIONS of the AMERICAN FISHERIES SOCIETY

January 1977
VOLUME 106
NUMBER 1

An Epitaph for the Concept of Maximum Sustained Yield[1]

P. A. Larkin
Institute of Animal Resource Ecology, University of British Columbia
Vancouver, British Columbia V6T 1W5

About 30 years ago, when I was a graduate student, the idea of managing fisheries for maximum sustained yield was just beginning to really catch on. Of course, the ideas had already been around for quite a while. Baranov (1918) was the first to combine information on growth and abundance to develop a catch equation, and Russell (1931) and Graham (1935) brought the dynamic pool model to the forefront, but they were working from a base of natural history and fishery biology that had been growing for several decades.

By the late 1930s, in North America, the conservation movement was in full cry and fisheries, like other resources, were being illuminated in the glow of the Gospel of Efficiency (Hays 1969). In dozens of states and provinces, fish and game regulations were proliferated, commercial fisheries were increasingly documented, and there was a growing awareness of the necessary scientific base for management. Thompson and Bell (1934) came to the conclusion that too much fishing effort was at the heart of the halibut problem; Hile (1936) produced his classic on the cisco in Wisconsin; and the first steps were being taken to restore the Fraser River sockeye from the effects of overfishing and the Hell's Gate blockage.

The ten years following World War II were the golden age for the concept of maximum sustained yield. Ricker (1948) produced his famous "green book," the first version of his handbook (Ricker 1958); Fry (1947) developed the virtual population idea; and Schaefer (1954) proposed his method for estimating surplus production under nonequilibrium conditions. The literature crackled with new information and new ideas. The solidification of the concept of MSY, its application to fisheries here, there, and everywhere, was just under way. World fisheries catch was a mere 20 million tons, and there were signs in lots of places of irreligious practices such as harvesting more or less than should be harvested. In a mood of excitement about opportunities, coupled with determination to do it properly, the FAO emerged as a major actor in the international fisheries scene.

It was in consequence of this flowering of activity that the graduate students of those days had a missionary zeal about them, and as more than one wit has said, "They had a fine vocabulary of stained glass language." Briefly, the dogma was this: any species each year produces a harvestable surplus, and if you take that much, and no more, you can go on getting it forever and ever (Amen). You only need to have as much effort as is necessary to catch this magic amount, so to use more is wasteful of effort; to use less is wasteful of food. Basically, it was a puritanical philosophy in which the supreme powers were pretty harsh on people who enjoyed themselves rather than doing precisely the Right Thing. Armed with scientific knowledge about the number of fishermen and technological advances, the

[1] Keynote address to the American Fisheries Society Annual Meetings, Dearborn, Michigan, September 19–24, 1976.

1

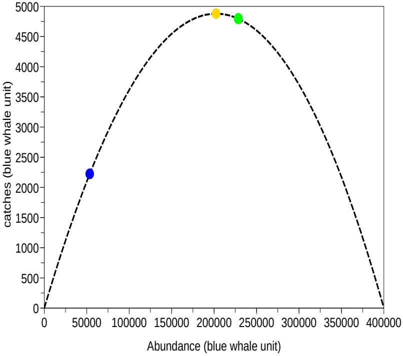
- MSY was developed in the early 1930s, and adopted in the 1950s by several international organizations and individual countries
- Larkin PA (1977) *An epitaph for the concept of maximum sustained yield* outlines deficiencies of MSY
  - monospecific dynamical model (no trophic relationships)
  - scalar dynamical model
    - no spatial variability
    - no age structure
    - no reproductive status
  - steady state approach
  - only benefits, no costs

Though being accused of having led to the collapse of many fisheries, the MSY remains the reference till now

- Accused of having led to the collapse of many fisheries
- Incorporated into the 1982
  United Nations Convention for the Law of the Sea
- Following the World Summit on Sustainable Development
  (Johannesburg, 2002),
  the signatory States undertook to restore and exploit their stocks at MSY
- About a quarter of world fisheries are overexploited
  in the sense that stocks are less than the MSE,
  the stock size that supports MSY

Outline of the presentation

1. Equilibrium
   - Definition of equilibrium (under const
   - Examples of equilibria
2. Sustainable yield and related notions
   - Sustainable yield for surplus model
   - Maximum sustainable equilibrium
   - Private property equilibrium
   - Common property equilibrium
   - Examples
3. Stability
   - Definition of stability of an equilibriur
   - Stability for dynamical linear systems
   - Linearized dynamics and stability
   - Examples
4. Summary



We introduce the rent or profit, which depends on price and costs

The economic rent or profit $\texttt{Rent}(h, B)$ is

$$\texttt{Rent}(h, B) = \underbrace{p}_{\text{price}} \overbrace{h}^{\text{harvest}} - \underbrace{\texttt{Cost}(h, B)}_{\text{harvest costs}}$$

where

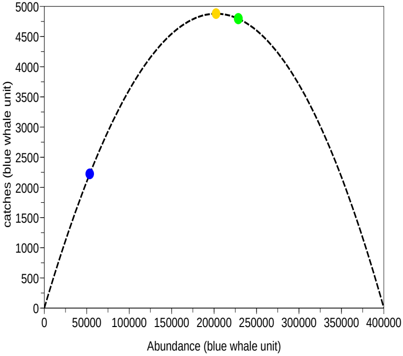
- p price per unit of harvested biomass ("price-taker")
- $\texttt{Cost}(h, B)$ harvesting costs

The private property equilibrium (PPE) maximizes the profit

- Among all sustainable equilibria on the curve $h = \mathtt{Sust}(B)$
- the private property equilibrium (PPE)
  is the one achieving the maximal rent

$$\mathtt{Rent}(h_{\mathrm{PPE}}, B_{\mathrm{PPE}}) = \max_{B \geq 0,\ h = \mathtt{Sust}(B)} \mathtt{Rent}(h, B)$$

Outline of the presentation



The common property equilibrium (CPE) makes the profit zero

- The assumption is that, when a resource is in open access, any positive rent is dissipated as newcomers exhaust it
- Among all sustainable equilibria on the curve $h = \mathtt{Sust}(B)$
- the common property equilibrium (CPE) is the one that makes the rent zero

$$\mathtt{Rent}(h_{\mathrm{CPE}}, B_{\mathrm{CPE}}) = 0 \quad \text{and} \quad h_{\mathrm{CPE}} = \mathtt{Sust}(B_{\mathrm{CPE}})$$

Let us consider the case where the cost is linear in the effort

- Catches are supposed to be linear in an effort variable

$$\underbrace{h}_{\text{catch}} = q \overbrace{E}^{\text{effort}} \underbrace{B}_{\text{biomass}}$$

- Assume harvest costs of the form

$$\texttt{Cost}(h, B) = \frac{ch}{qB} = c \underbrace{\frac{h}{qB}}_{\text{effort } E}$$

  - q catchability coefficient
  - c unit cost of effort
- $\texttt{Rent}(h, B) = ph - \texttt{Cost}(h, B) = \left(p - \frac{c}{qB}\right) h$

We derive the common property equilibrium (CPE) in the linear effort costs case

$$\texttt{Rent}(h, B) = \left(p - \frac{c}{qB} \right) h$$

$$\texttt{Rent}(h_{\mathrm{CPE}}, B_{\mathrm{CPE}}) = 0 \text{ and } h_{\mathrm{CPE}} = \texttt{Sust}(B_{\mathrm{CPE}})$$

$$\Downarrow$$

$$B_{\mathrm{CPE}} = \frac{c}{pq} \text{ and } h_{\mathrm{CPE}} = \texttt{Sust}(B_{\mathrm{CPE}})$$

leaving aside the trivial equilibrium $(B, h) = (0, 0)$

High price-to-cost ratio drives biomass to collapse

$$B_{\mathrm{CPE}} = \frac{c}{pq} = \frac{1/q}{\underbrace{p/c}_{\text{price to cost ratio}}} \propto \frac{1}{\text{price} \,/\, \text{cost}}$$

- The common property equilibrium B_{CPE} does not depend upon the biological dynamics `Biol`
- Conservation problem if high price-to-cost ratio: cost $c \searrow$ (sonar, satellite) or price $p \nearrow$ (scarcity) $\Rightarrow B_{\mathrm{CPE}} \searrow 0$

Outline of the presentation

Exercise: compute MSE and PPE when the biological dynamics is a Beverton-Holt function

- Consider the biological Beverton-Holt dynamics $\texttt{Biol}(B) = \frac{RB}{1+bB}$
- The sustainable yield is

$$\texttt{Sust}(B) = B - \frac{B}{R - bB} \text{ for } 0 \leq B \leq K = \frac{R-1}{b}$$

- The MSE achieves the maximum of $\texttt{Sust}$, that is,

$$\frac{d\,\texttt{Sust}(B)}{dB}_{|B=B_{\text{MSE}}} = 0 \iff 1 - \frac{R}{(R-bB)^2} = 0$$

- The MSE biomass is

$$B_{\text{MSE}} = \frac{R - \sqrt{R}}{b}$$

Here stands the Beverton-Holt PPE

- The sustainable yield is

$$\texttt{Sust}(B) = B - \frac{B}{R - bB} \text{ for } 0 \leq B \leq K = \frac{R-1}{b}$$

- The PPE achieves the maximum of the rent, that is,

$$\frac{d}{dB}_{|B=B_{\text{PPE}}} \Big((p - \frac{c}{qB}) \texttt{Sust}(B) \Big) = 0$$

- Show that the sign of the above derivative is positive at $B = B_{\text{MSE}}$
- The PPE is

$$B_{\text{PPE}} = \frac{R - \sqrt{R - \frac{b}{q} / \frac{p}{c}}}{b}$$

There is more biomass at equilibrium under private property equilibrium than under maximum sustainable equilibrium

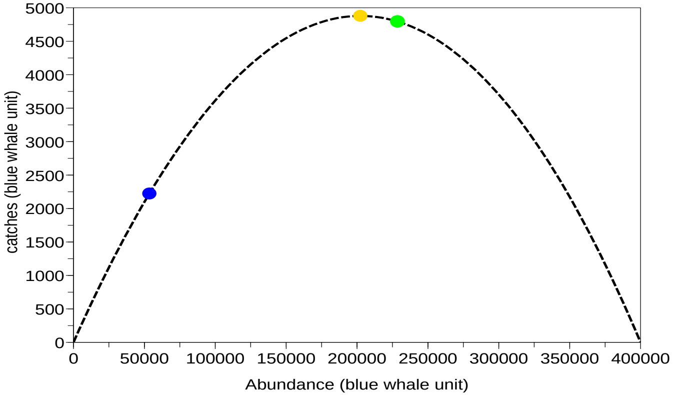
$$B_{\mathrm{PPE}} = \frac{R - \sqrt{R - \frac{b}{q} / \frac{p}{c}}}{b} > B_{\mathrm{MSE}} = \frac{R - \sqrt{R}}{b}$$

- $B_{\mathrm{PPE}} > B_{\mathrm{MSE}} \Longrightarrow$ more biomass at equilibrium
  - under private property equilibrium
  - than under maximum sustainable equilibrium
- To the first-order, we have that

$$B_{\mathrm{PPE}} - B_{\mathrm{MSE}} \propto 1 / \frac{p}{c} = \frac{c}{p} = \frac{\text{cost}}{\text{price}}$$

MSE, PPE, CPE for blue whale

K	$=$	400 000 BWU	$R \approx 1.05$
q	$=$	0.0 016 per whale catcher year	
c	$=$	600 000 \$ per whale catcher year	$p = 7\ 000$ \$ per BWU



The Aboré marine reserve dilapidation illustrates the "Tragedy of the Commons" / open access

- In August 1993, in New Caledonia the Aboré marine reserve has been re-opened to fishing
- The reserve had been closed during three years
- The benefits of these three years were dissipated in a few weeks
- Resources in open access suffer from over-exploitation

Open access dynamics displays CPE equilibrium

- We consider the open access dynamics

$$\begin{cases} B(t+1) = \texttt{Biol}\Big(B(t) - \overbrace{\texttt{Catch}\big(E(t), B(t)\big)}^{\text{catches}}\Big) \\ E(t+1) = E(t) + \alpha \underbrace{\texttt{Rent}(\texttt{Catch}\big(E(t), B(t)\big), E(t))}_{\text{rent}} \end{cases}$$

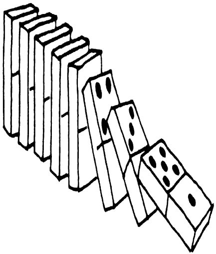
Peru's fishing overcapacity highlights the point that this model assumes a too flexible effort adaptation

- Gordon's bionomic equilibrium CPE is $B_{\text{CPE}} = \frac{c}{pq}$ when
  - catches $\texttt{Catch}(E, B) = qEB$,
  - rent $\texttt{Rent}(h, E) = ph - cE$

Summary

- The maximum sustainable yield (MSY) is the maximum surplus production that can be harvested in perpetuity without altering the stock level
- Though being accused of having led to the collapse of many fisheries, the MSY remains the reference till now
- The private property equilibrium (PPE) maximizes the profit
- The common property equilibrium (CPE) makes the profit zero
- Open access dynamics displays CPE equilibrium

Outline of the presentation



Outline of the presentation

Open loop stability of an equilibrium

The dynamics around an equilibrium x_{E}
with the fixed equilibrium decision u_{E} is

$$\begin{cases} x(t+1) = \mathtt{Dyn}(x(t), u_{\mathrm{E}}) \\ x(t_0) = x_0 \approx x_{\mathrm{E}} \end{cases}$$

Attractive equilibrium

Equilibrium x_{E} is attractive if there exists a neighborhood $\mathcal{N}(x_{\mathrm{E}})$ of x_{E} such that

$$\lim_{t \to +\infty} x(t) = x_{\mathrm{E}} \; , \quad \forall x_0 \in \mathcal{N}(x_{\mathrm{E}})$$

The property that an equilibrium is attractive
does not depend of the transitory phase $x(t_0), x(t_0+1), \ldots, x(T)$

The concept of asymptotic stability combines transitories and asymptotics properties

Asymptotically stable equilibrium

The equilibrium x_{E} is said to be asymptotically stable when any trajectory $x(t)$, $t = t_0, t_0 + 1, \ldots$, starting close enough to x_{E} ($x(t_0) \approx x_{\mathrm{E}}$)

- remains in the vicinity of x_{E} ($x(t) \approx x_{\mathrm{E}}$, $t = t_0, t_0 + 1, \ldots$): stability concerns transitories
- and converges towards x_{E} ($x(t) \to x_{\mathrm{E}}$): attractivity only focuses on asymptotics

Outline of the presentation

As a starter, we highlight stability of the zero equilibrium for scalar linear systems

- Scalar $x(t) \in \mathbb{R}$
- A dynamical linear scalar system has the form

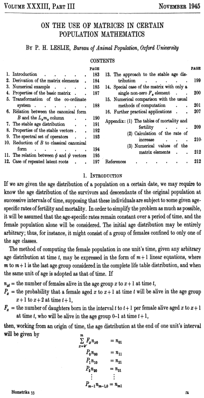
$$x(t+1) = Mx(t) \text{ where } M \in \mathbb{R}$$

- The solution has the expression $x(t) = M^{t-t_0} x(t_0)$
  - $|M| > 1$
  - $|M| < 1$
  - $|M| = 1$

Proposition

Equilibrium $x_E = 0$ is asymptotically stable if and only if $|M| < 1$

P. H. Leslie introduced mortality-natality matrix models in forestry


Volume XXXIII, Part III — November 1945

ON THE USE OF MATRICES IN CERTAIN POPULATION MATHEMATICS

By P. H. LESLIE, *Bureau of Animal Population, Oxford University*

CONTENTS

1. Introduction

If we are given the age distribution of a population on a certain date, we may require to know the age distribution of the survivors and descendants of the original population at successive intervals of time, supposing that these individuals are subject to some given age-specific rates of fertility and mortality. In order to simplify the problem as much as possible, it will be assumed that the age-specific rates remain constant over a period of time, and the female population alone will be considered. The initial age distribution may be entirely arbitrary; thus, for instance, it might consist of a group of females confined to only one of the age classes.

The method of computing the female population in one unit's time, given any arbitrary age distribution at time t, may be expressed in the form of $m+1$ linear equations, where m to $m+1$ is the last age group considered in the complete life table distribution, and when the same unit of age is adopted as that of time. If

n_{xt} = the number of females alive in the age group x to $x+1$ at time t,

P_x = the probability that a female aged x to $x+1$ at time t will be alive in the age group $x+1$ to $x+2$ at time $t+1$,

F_x = the number of daughters born in the interval t to $t+1$ per female alive aged x to $x+1$ at time t, who will be alive in the age group 0–1 at time $t+1$,

then, working from an origin of time, the age distribution at the end of one unit's interval will be given by

$$\begin{aligned} \sum_{x=0}^{m} F_x n_{x0} &= n_{01} \\ P_0 n_{00} &= n_{11} \\ P_1 n_{10} &= n_{21} \\ P_2 n_{20} &= n_{31} \\ \vdots \quad & \quad \vdots \\ P_{m-1} n_{m-1,0} &= n_{m1} \end{aligned}$$

Biometrika 33 — 12

$$N(t+1) = L\,N(t)$$

$$L = \begin{bmatrix} 1 - \overbrace{\mu_A}^{\text{mortality}} & 1-\mu_{A-1} & 0 & \cdots & 0 \\ 0 & 0 & 1-\mu_{A-2} & \ddots & 0 \\ & & & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & 1-\mu_1 \\ \underbrace{\gamma_A}_{\text{fertility}} & \gamma_{A-1} & \cdots & \cdots & \gamma_1 \end{bmatrix}$$

Leslie, P.H. (1945)
"The use of matrices in certain population mathematics"
Biometrika, 33(3), 183–212

The stability of the nul equilibrium of a linear system is related to the location of eigenvalues with respect to the unit disk

- Vector $x(t) \in \mathbb{R}^n$ and dynamical linear system

$$x(t+1) = \underbrace{M}_{\text{square matrix}} x(t)$$

- Eigenvalues $\lambda_i(M)$ of the square matrix M

Theorem

Equilibrium $x_{\mathrm{E}} = 0$ is asymptotically stable if and only if all eigenvalues have modulus strictly less than unity, that is,

$$|\underbrace{\lambda_i(M)}_{\text{eigenvalue}}| < 1 \; , \quad \forall i$$

Outline of the presentation

The dynamics of the increments is (almost) linear

$$x(t+1) = \mathtt{Dyn}(x(t), u_{\mathrm{E}})$$

$$x_{\mathrm{E}} = \mathtt{Dyn}(x_{\mathrm{E}}, u_{\mathrm{E}})$$

$$\Downarrow$$

$$\underbrace{x(t+1) - x_{\mathrm{E}}}_{\text{increment}} = \mathtt{Dyn}(x_{\mathrm{E}} + \underbrace{x(t) - x_{\mathrm{E}}}_{\text{increment}}, u_{\mathrm{E}}) - \mathtt{Dyn}(x_{\mathrm{E}}, u_{\mathrm{E}})$$

$$= \frac{\partial \mathtt{Dyn}}{\partial x}(x_{\mathrm{E}}, u_{\mathrm{E}}) \underbrace{\big(x(t) - x_{\mathrm{E}}\big)}_{\text{increment}} + \cdots$$

The Jacobian matrix provides the linear part of a nonlinear dynamics

$$\frac{\partial \mathtt{Dyn}}{\partial x}(x_{\mathrm{E}}, u_{\mathrm{E}}) = \begin{pmatrix} \frac{\partial \mathtt{Dyn}^1}{\partial x_1}(x_{\mathrm{E}}, u_{\mathrm{E}}) & \frac{\partial \mathtt{Dyn}^1}{\partial x_2}(x_{\mathrm{E}}, u_{\mathrm{E}}) & \cdots & \frac{\partial \mathtt{Dyn}^1}{\partial x_n}(x_{\mathrm{E}}, u_{\mathrm{E}}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathtt{Dyn}^n}{\partial x_1}(x_{\mathrm{E}}, u_{\mathrm{E}}) & \frac{\partial \mathtt{Dyn}^n}{\partial x_2}(x_{\mathrm{E}}, u_{\mathrm{E}}) & \cdots & \frac{\partial \mathtt{Dyn}^n}{\partial x_n}(x_{\mathrm{E}}, u_{\mathrm{E}}) \end{pmatrix}$$

where $\mathtt{Dyn}(x, u) = \begin{pmatrix} \mathtt{Dyn}^1(x_1, x_2, \ldots, x_n, u) \\ \vdots \\ \mathtt{Dyn}^n(x_1, x_2, \ldots, x_n, u) \end{pmatrix}$

Stability of an equilibrium can often be deduced from linearization of the dynamics at the equilibrium

Theorem

Assume that the dynamics $\mathtt{Dyn}$ *is a continuously differentiable mapping on a neighborhood of the equilibrium* $(x_{\mathrm{E}}, u_{\mathrm{E}})$. *Then,*

$$\forall i \ , \ |\underbrace{\lambda_i(\frac{\partial \mathtt{Dyn}}{\partial x}(x_{\mathrm{E}}, u_{\mathrm{E}}))}_{\text{eigenvalue}}| < 1 \implies x_{\mathrm{E}} \textit{ asymptotically stable}$$

for the nonlinear dynamical system

$$x(t+1) = \mathtt{Dyn}(x(t), u_{\mathrm{E}})$$

Outline of the presentation

Exercise: study stability when the biological dynamics is a Beverton-Holt function

- We consider the Schaefer model

$$B(t+1) = \frac{R(B(t) - h_{\mathrm{E}})}{1 + b(B(t) - h_{\mathrm{E}})} = \mathtt{Dyn}(B(t), h_{\mathrm{E}})$$

built upon the Beverton-Holt biological dynamics

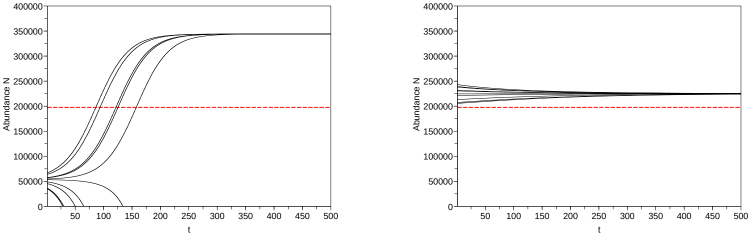
- Equilibrium: $h_{\mathrm{E}} = B_{\mathrm{E}} - \frac{B_{\mathrm{E}}}{R - bB_{\mathrm{E}}}$
- Exercise: compute $\frac{\partial \mathtt{Dyn}(B, h_{\mathrm{E}})}{\partial B}_{|B=B_{\mathrm{E}}} = \frac{\partial}{\partial B}_{|B=B_{\mathrm{E}}} [\frac{R(B-h_{\mathrm{E}})}{1+b(B-h_{\mathrm{E}})}] = (\frac{R-bB_{\mathrm{E}}}{\sqrt{R}})^2$
- Asymptotic stability if

$$B_{\mathrm{MSE}} < B_{\mathrm{E}} \leq K$$

  - MSE at frontier between stability and unstability
  - PPE is a stable equilibrium
  - CPE can be unstable

Blue whale: CPE versus PPE

K	$=$	400 000 BWU	$R \approx 1.05$
q	$=$	0.0 016 per whale catcher year	
c	$=$	600 000 \$ per whale catcher year	$p = 7\ 000$ \$ per BWU



CPE equilibrium: unstability   PPE equilibrium: stability

The red trajectory corresponds to the MSE

We show that, in a simple open access model, instability and extinction occur

$$\begin{cases} B(t+1) & = \mathtt{Biol}\Big(B(t) - qE(t)B(t)\Big) & = \mathtt{Dyn}^B\big(B(t), E(t)\big) \\ E(t+1) & = E(t) + \alpha\Big(pqE(t)B(t) - cE(t)\Big) & = \mathtt{Dyn}^E\big(B(t), E(t)\big) \end{cases}$$

The Jacobian matrix at equilibrium $(B_{\mathrm{E}}, E_{\mathrm{E}}) = (\frac{c}{pq}, E_{\mathrm{E}})$ is

$$\begin{pmatrix} (1-qE_{\mathrm{E}})\mathtt{Biol}'(B_{\mathrm{E}}(1-qE_{\mathrm{E}})) & -qB_{\mathrm{E}}\mathtt{Biol}'(B_{\mathrm{E}}(1-qE_{\mathrm{E}})) \\ \alpha pqE_{\mathrm{E}} & 1 \end{pmatrix}$$

In the linear biological dynamics case, the eigenvalues of the Jacobian matrix are outside the stability disk

- Assuming a linear biological dynamics $\texttt{Biol}(B) = RB$, the Jacobian matrix is

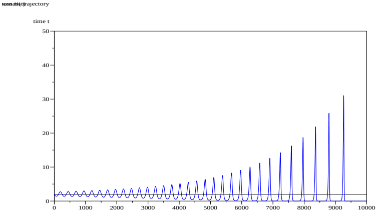
$$\begin{pmatrix} R(1 - qE_{\mathrm{E}}) & -RqB_{\mathrm{E}} \\ \alpha pqE_{\mathrm{E}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{Rc}{p} \\ \alpha p\frac{R-1}{R} & 1 \end{pmatrix}$$

- The two eigenvalues (λ_1, λ_2) are

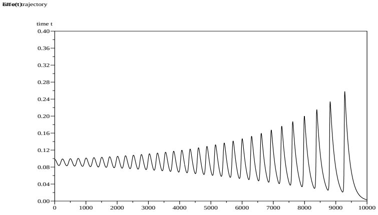
$$\begin{cases} \lambda_1 & = 1 - i\sqrt{\alpha c(R-1)} \\ \lambda_2 & = 1 + i\sqrt{\alpha c(R-1)} \end{cases}$$

- Stability cannot be guaranteed since $|\lambda_i|^2 > 1$

Biomass and effort trajectories display oscillations and divergence



(a) Biomass trajectory $B(t)$



(b) Effort trajectory $E(t)$

We consider a dynamical model of n species in competition for the same resource

$$
\begin{aligned}
N_i(t+1) &= N_i(t) + \Delta_t \Big(N_i(t) \overbrace{\big(f_i R(t) - d_i\big)}^{\text{growth rate}} \Big) \\
R(t+1) &= R(t) + \Delta_t \Big(\underbrace{S - aR(t)}_{\text{intrinsic growth}} - \sum_{i=1}^{n} \underbrace{w_i f_i R(t) N_i(t)}_{\text{interaction}} \Big)
\end{aligned}
$$

- Δ_t time unit
- $N_i(t)$ density of species i
- $R(t)$ resource for which all species compete
- $f_i R(t)$ growth, d_i death rates of species i
- $S - aR(t)$ natural growth rate of the resource (S a stationary input)

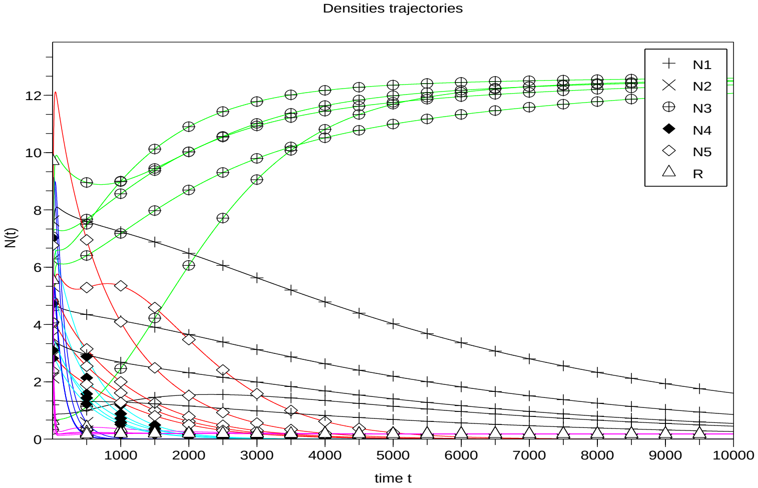
The competitive exclusion principle states that only one species survives

- If Δ_t is small enough, a positive stable equilibrium is (Tilman, 1988)

$$
\begin{cases}
R_{\mathrm{E}} = \min_{i=1,\dots,n} \frac{d_i}{f_i} = \frac{d_{i_{\mathrm{E}}}}{f_{i_{\mathrm{E}}}} \\
N_{i,\mathrm{E}} = \begin{cases} \frac{S_{\mathrm{E}} - R_{\mathrm{E}} a}{R_{\mathrm{E}} w_i f_i} & \text{if} \quad i = i_{\mathrm{E}} \\ 0 & \text{if} \quad i \neq i_{\mathrm{E}} \end{cases}
\end{cases}
$$

- Only species i_{E} survives: this is the competitive exclusion principle

Competitive exclusion principle



Summary

- Asymptotic stability tackles both transitories and asymptotics
- The stability of the nul equilibrium of a linear system
  is related to the location of eigenvalues with respect to the unit disk
- Local stability can often be deduced
  from linearization of the dynamics at the equilibrium

Outline of the presentation

Summary

- Equilibrium analysis is the basics of natural resource management relying upon biomass models: sustainability=equilibrium
- Though being strongly criticized, the maximum sustainable yield (MSY) remains the reference till now
- Three equilibria are worth being distinguished
    - maximum sustainable equilibrium (MSE)
    - private property equilibrium (PPE)
    - common property equilibrium (CPE)
- Stability is important for conservation issues, to avoid biomass collapse
- Stability analysis relies upon the study of the linearized dynamics at equilibrium