Viable Sequential Decisions

Extended from Chapter 4 of
Sustainable Management of Natural Resources. Mathematical Models and Methods
by Luc DOYEN and Michel DE LARA

Michel DE LARA

École des Ponts ParisTech

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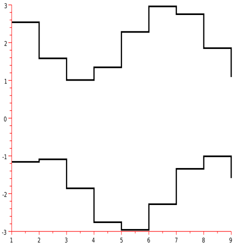
Outline of the presentation

1. Motivation
2. Resource management examples under viability constraints
3. The viability kernel and viable controls
4. Resource management by viability methods
5. Summary

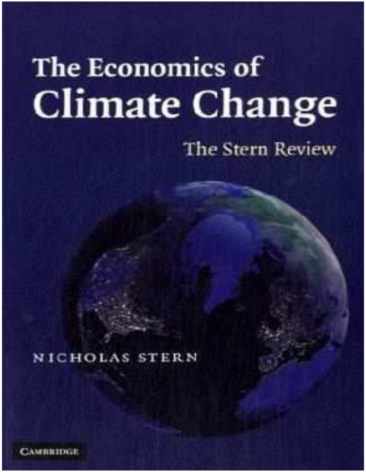
Outline of the presentation

Viability approaches can be found in various fields



- ▷ MVP: minimum viable population is a lower bound on the population of a species such that it can survive in the wild
- ▷ PVA: population viability analysis
- ▷ TWA: tolerable windows approach (set of guardrails)
- ▷ SMS: safe minimum standards

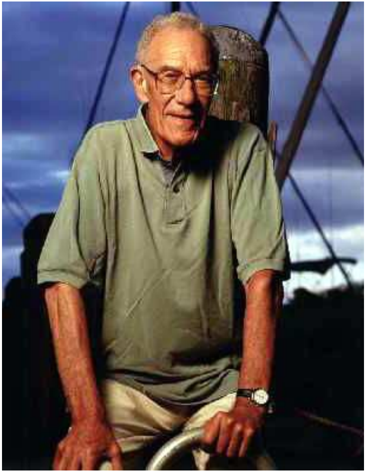
"Please leave the toilets clean for the next person to use" ;-)



> *The notion of "stewardship" can be seen as a special form of sustainability. It points to particular aspects of the world, which should themselves be passed on in a state at least as good as that inherited from the previous generation.*

Nicholas Stern, The Economics of Climate Change, Cambridge University Press, 2006

Solow's generalized capacity



If sustainability means anything more than a vague emotional commitment, it must require that something be conserved for the very long run. It is very important to understand what that thing is: I think it has to be a generalized capacity to produce economic well-being.

R. M. Solow. An almost practical step towards sustainability. *Resources Policy*, 19:162–172, 1993.

Report of the Brundtland Commission, *Our Common Future*, 1987

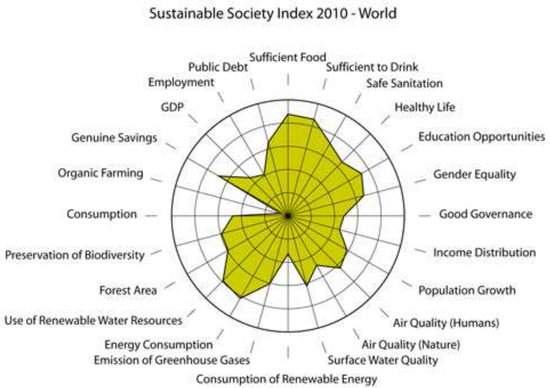
"Sustainable development is development that meets the needs of the present without compromising the ability of future generations to meet their own needs. It contains within it two key concepts:

- the concept of 'needs', in particular the essential needs of the world's poor, to which overriding priority should be given; and
- the idea of limitations imposed by the state of technology and social organization on the environment's ability to meet present and future needs."

Management of natural resources requires specific modeling options

- ▷ Take into account
  - Dynamics, that capture inertia, stock variations, interactions
  - Decisions, actions, controls
  - Uncertainties and information
- ▷ Deal with
  - Multi-criteria
    - Ecology: conservation
    - Economy: efficiency
  - Intergenerational equity:
    envisage alternatives to compensation between generations

Some economists recommend objectives to be expressed in their own units, without aggregation



The "Stiglitz-Sen-Fitoussi" Commission (2009) déconseille de privilégier un indicateur synthétique unique car, quel que soit l'indicateur envisagé, l'agrégation de données disparates ne va pas de soi

When dealing with economic and environmental objectives, this disaggregated approach is coined co-viability

- Co-viability when
  - environmental constraints: conservation, viability
  - economic constraints: production, efficiency
- C. Béné, L. Doyen, and D. Gabay
  A viability analysis for a bio-economic model.
  Ecological Economics, 36:385–396, 2001

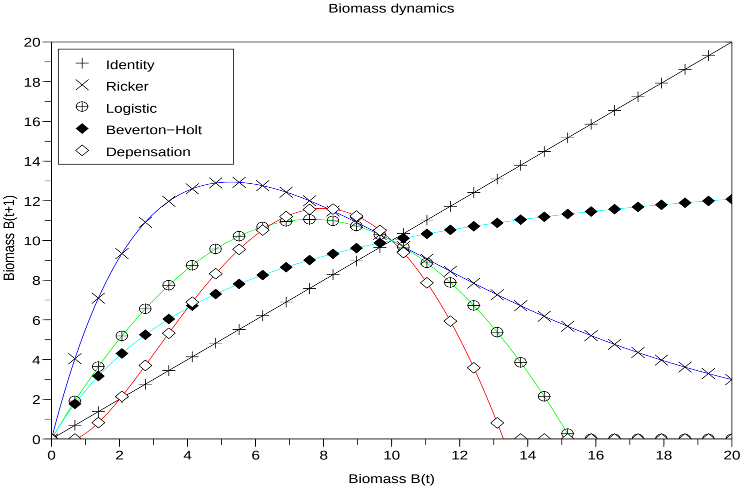
Outline of the presentation

Outline of the presentation

Viable management of an animal population

$$B(t+1) = \overbrace{\mathtt{Biol}}^{\text{dynamic}} \Big(\underbrace{B(t)}_{\text{biomass}} - \underbrace{h(t)}_{\text{catches}} \Big)$$

- $B(t)$ biomass
- $h(t)$ catch with $0 \leq h(t) \leq B(t)$
- Biol natural resource growth function (linear, logistic, etc.)



Distinct population dynamics `Biol` for $r = 1.9$, $K = 10$, $B^\flat = 2$

We define an ecological window by lower and upper bounds for the biomass

State constraints

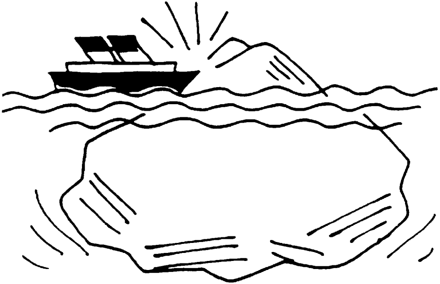
$$B^{\flat} \leq B(t) \leq B^{\sharp} \;, \quad t = t_0, \dots, T$$

- $B^{\flat}$ minimum viable population
- $B^{\sharp}$ maximal safety value (pest control, invasive species)

The problem is one of inertia

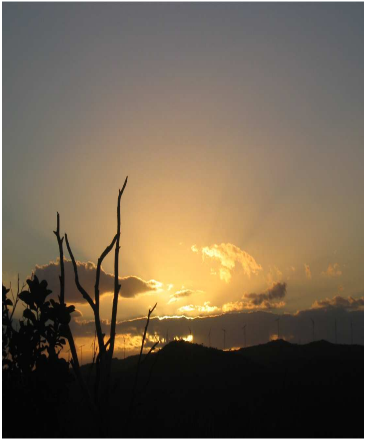
$$
\begin{array}{rcll}
B(t_0) & & & \in [B^\flat, B^\sharp] \\
B(t_0+1) & = & \mathtt{Biol}\big(B(t_0) - h(t_0)\big) & \in [B^\flat, B^\sharp] \\
B(t_0+2) & = & \mathtt{Biol}\big(B(t_0+1) - h(t_0+1)\big) & \\
 & = & \mathtt{Biol}\big(\mathtt{Biol}\big(B(t_0) - h(t_0)\big) - h(t_0+1)\big) & \in [B^\flat, B^\sharp]
\end{array}
$$

$B(t_0+s)$ depends on $B(t_0)$ and on past decisions $h(t_0), \ldots, h(t_0+s-1)$ because of the dynamic (inertia)



Outline of the presentation

Let us scout a very stylized model of the climate-economy system



We lay out a dynamical model with

- two state variables

  environmental: atmospheric CO_2 concentration level $M(t)$

  economic: gross world product GWP $Q(t)$

- one decision variable, the emission abatement rate $a(t)$

A carbon cycle model "à la Nordhaus" is an example of *decision model*

- Time index t in years
- Economic production $Q(t)$ (GWP)

$$Q(t+1) = \overbrace{(1+g)}^{\text{economic growth}} Q(t)$$

- CO_2 concentration $M(t)$

$$M(t+1) = M(t) \underbrace{-\delta(M(t) - M_{-\infty})}_{\text{natural sinks}} + \alpha \overbrace{\texttt{Emiss}(Q(t))}^{\text{emissions}} \underbrace{(1 - a(t))}_{\text{abatement}}$$

- Decision $a(t) \in [0, 1]$ is the abatement rate of CO_2 emissions

Data

- $M(t)$ $\mathrm{CO_2}$ atmospheric concentration, measured in ppm, parts per million (379 ppm in 2005)
- $M_{-\infty}$ pre-industrial atmospheric concentration (about 280 ppm)
- $\mathtt{Emiss}(Q(t))$ "business as usual" $\mathrm{CO_2}$ emissions (about 7.2 GtC per year between 2000 and 2005)
- $0 \leq a(t) \leq 1$ abatement rate reduction of $\mathrm{CO_2}$ emissions
- α conversion factor from emissions to concentration ($\alpha \approx 0.471$ ppm.GtC^{-1} sums up highly complex physical mechanisms)
- δ natural rate of removal of atmospheric $\mathrm{CO_2}$ to unspecified sinks ($\delta \approx 0.01$ year^{-1})

A concentration target is pursued to avoid danger

United Nations Framework Convention on Climate Change

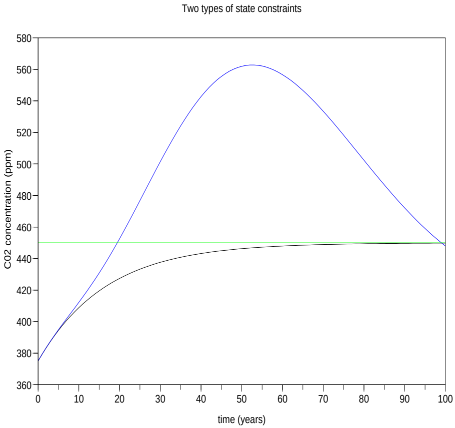
"to achieve, (. . .), stabilization of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system"

Limitation of concentrations of CO_2

- below a tolerable threshold $M^\sharp$ (say 350 ppm, 450 ppm)
- at a specified date $T > 0$ (say year 2050 or 2100)

$$\underbrace{M(T)}_{\text{concentration at horizon}} \leq \underbrace{M^\sharp}_{\text{threshold}}$$

Constraints capture different requirements



- The concentration has to remain below a tolerable level at the horizon T:

$$M(T) \leq M^{\sharp}$$

- More demanding: from the initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp}$$

$$t = t_0, \ldots, T$$

Constraints may be environmental, physical, economic

- The concentration has to remain below a tolerable level from initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp}\,, \qquad t = t_0, \ldots, T$$

- Abatements are expressed as fractions

$$0 \leq a(t) \leq 1\,, \qquad t = t_0, \ldots, T-1$$

- As with "cap and trade", setting a ceiling on CO_2 price amounts to cap abatement costs

$$\underbrace{\texttt{Cost}\big(a(t), Q(t)\big)}_{\text{costs}} \leq c^{\sharp}\,(100 \text{ euros / tonne } \mathrm{CO}_2)\,, \qquad t = t_0, \ldots, T-1$$

Mixing dynamics, optimization and constraints yields a cost-effectiveness problem

- Minimize abatement costs

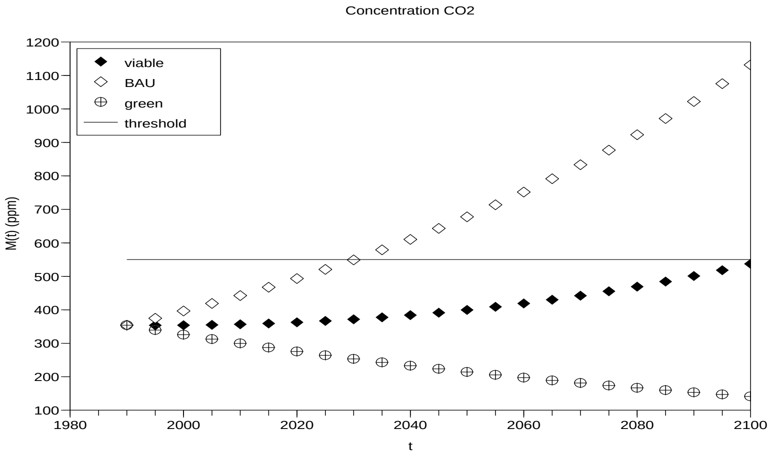
$$\min_{a(t_0),\ldots,a(T-1)} \sum_{t=t_0}^{T-1} \left(\frac{1}{1+r_e}\right)^{t-t_0} \underbrace{\texttt{Cost}\big(a(t), Q(t)\big)}_{\text{abatement costs}}$$

- under the GWP-CO_2 dynamics

$$\begin{cases} M(t+1) &= M(t) - \delta(M(t) - M_{-\infty}) + \alpha \texttt{Emiss}\big(Q(t)\big)\big(1 - a(t)\big) \\ Q(t+1) &= (1+g)Q(t) \end{cases}$$

- and under target constraint

$$\underbrace{M(T) \leq M^{\sharp}}_{\text{CO2 concentration}}$$


Concentration CO2
viable
BAU
green
threshold
M(t) (ppm)
1200
1100
1000
900
800
700
600
500
400
300
200
100
1980
2000
2020
2040
2060
2080
2100
t

Outline of the presentation

We consider an age-class forest dynamic linear model

- The forest is described by a vector $N(t)$ of abundances

$$N(t) = \begin{pmatrix} N_A(t) \\ N_{A-1}(t) \\ \vdots \\ N_1(t) \end{pmatrix} = \begin{pmatrix} \text{number of trees of age} & \geq A \\ \text{number of trees of age} & \in [A-1, A[\\ \vdots & \vdots \\ \text{number of trees of age} & \in [1, 2[\\ \text{number of trees of age} & \in [0, 1[\end{pmatrix}$$

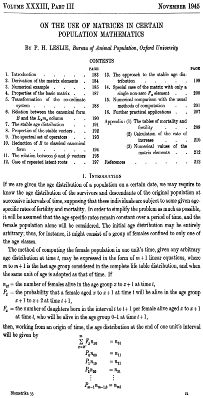
- The evolution from an age-class a to the next $a+1$ is described by

$$N_{a+1}(t+1) = (1 - \underbrace{\mu_a}_{\text{mortality}}) N_a(t)$$

- Young trees result from the offspring of the different age-classes

$$N_1(t+1) = \sum_{a=1}^{A} \underbrace{\gamma_a}_{\text{fertility}} N_a(t)$$

P. H. Leslie introduced mortality-natality matrix models in forestry


VOLUME XXXIII, PART III NOVEMBER 1945

ON THE USE OF MATRICES IN CERTAIN POPULATION MATHEMATICS

BY P. H. LESLIE, *Bureau of Animal Population, Oxford University*

CONTENTS

1. INTRODUCTION

If we are given the age distribution of a population on a certain date, we may require to know the age distribution of the survivors and descendants of the original population at successive intervals of time, supposing that these individuals are subject to some given age-specific rates of fertility and mortality. In order to simplify the problem as much as possible, it will be assumed that the age-specific rates remain constant over a period of time, and the female population alone will be considered. The initial age distribution may be entirely arbitrary; thus, for instance, it might consist of a group of females confined to only one of the age classes.

The method of computing the female population in one unit's time, given any arbitrary age distribution at time t, may be expressed in the form of $m+1$ linear equations, where m to $m+1$ is the last age group considered in the complete life table distribution, and when the same unit of age is adopted as that of time. If

n_{xt} = the number of females alive in the age group x to $x+1$ at time t,
P_x = the probability that a female aged x to $x+1$ at time t will be alive in the age group $x+1$ to $x+2$ at time $t+1$,
F_x = the number of daughters born in the interval t to $t+1$ per female alive aged x to $x+1$ at time t, who will be alive in the age group 0–1 at time $t+1$,

then, working from an origin of time, the age distribution at the end of one unit's interval will be given by

$$\begin{aligned} \sum_{x=0}^{m} F_x n_{x0} &= n_{01} \\ P_0 n_{00} &= n_{11} \\ P_1 n_{10} &= n_{21} \\ P_2 n_{20} &= n_{31} \\ \vdots \quad &\quad \vdots \\ P_{m-1} n_{m-1,0} &= n_{m1} \end{aligned}$$

$$N(t+1) = L\,N(t)$$

$$L = \begin{bmatrix} 1 - \overbrace{\mu_A}^{\text{mortality}} & 1-\mu_{A-1} & 0 & \cdots & 0 \\ 0 & 0 & 1-\mu_{A-2} & \ddots & 0 \\ & & & \ddots & 0 \\ 0 & \ldots & 0 & \ddots & 1-\mu_1 \\ \underbrace{\gamma_A}_{\text{fertility}} & \gamma_{A-1} & \cdots & \cdots & \gamma_1 \end{bmatrix}$$

Leslie, P.H. (1945)
"The use of matrices in certain population mathematics"
Biometrika, 33(3), 183–212

We suppose that only old trees are cut and that they are replaced by young ones

- Only trees of age A can be cut in quantity $h(t)$
- Each time a tree of age A is cut,
  it is immediately replaced by a tree of age 1

$$\begin{pmatrix} N_A(t+1) \\ N_{A-1}(t+1) \\ \vdots \\ N_2(t+1) \\ N_1(t+1) \end{pmatrix} = L\,N(t) + \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} h(t)$$

A. Rapaport, J.-P. Terreaux, and L. Doyen.
Sustainable management of renewable resource: a viability approach.
Mathematics and Computer Modeling, 43(5-6):466–484, March 2006.

We add a social objective of minimal harvesting

- One cannot plan to harvest more than will exist at the end of period $[t, t+1[$

$$0 \leq h(t) \leq \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix} LN(t)}_{\text{future old trees}} = N_A(t+1)$$

- A minimal guaranteed harvesting $h^\flat > 0$ is required (when $h(t)$ is associated with an income)

$$h^\flat \leq h(t)$$

This approach differs from the classical one of Faustmann optimal rotation problem, which attaches a value to harvesting and formulates an intertemporal optimization problem

Outline of the presentation

Detecting an epidemic outbreak by *corredor endémico (canal endémico)*

- success, security, alert, epidemics

Endemic channels form the core of a decision rule for dengue outbreak prevention

> *The epidemiological surveillance system should be able to differentiate between transient and seasonal increases in disease incidence and increases observed at the beginning of a dengue outbreak. One such approach is to track the occurrence of current (probable) cases and compare them with the average number of cases by week (or month) of the preceding 5–7 years, with confidence intervals set at two standard deviations above and below the average (± 2 SD). This is sometimes referred to as the "endemic channel". If the number of cases reported exceeds 2 SDs above the "endemic channel" in weekly or monthly reporting, an outbreak alert is triggered.*

Dengue. Guidelines for Diagnosis, Treatment, Prevention and Control.
A joint publication of the World Health Organization (WHO) and the Special Programme for Research and Training in Tropical Diseases (TDR), 2009

We consider an epidemiological model with vector control

- Basic variables and parameters are
  - time t, measured in weeks
  - M_t, the abundance of infected mosquitos (Aedes Aegypti adultos)
  - H_t, the abundance of infected humans
  - $\Delta\mu_t^M$, the additional mortality rate of mosquitos, a control variable
  - $\overline{M}$, $\overline{H}$, f^H, f^M, μ^M and μ^H, parameters
- The controlled dynamics is

$$
\begin{aligned}
M_{t+1} &= f^H H_t(\overline{M} - M_t) - (\mu^M + \Delta\mu_t^M) M_t \\
H_{t+1} &= f^M M_t(\overline{H} - H_t) - \mu^H H_t
\end{aligned}
$$

- The objective is to maintain infected humans at a low level

$$
H_t \leq H^\sharp , \quad \forall t = t_0, \ldots, T
$$

with limited resources $0 \leq \mu_t^M \leq \mu^\sharp , \quad \forall t = t_0, \ldots, T-1$

Summary

- ▷ We have seen examples of natural resources management problems where objectives are formulated as constraints
- ▷ We now present the mathematical control theory framework, and especially viability theory

Outline of the presentation

Outline of the presentation

A control system connects input and output variables

Input variables

Control	wood logs
Uncertainty	wood humidity
	metal conductivity

Output variables

soup quality
water vapor
temperature (internal state)

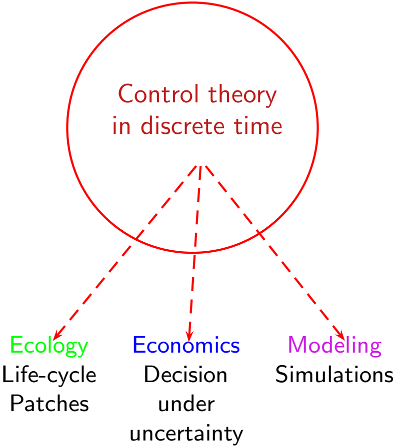
Discrete-time nonlinear state-control systems are special input-output systems

A specific output is distinguished, and is labeled state, when the system may be written as

$$x(t+1) = \mathtt{Dyn}(t, x(t), u(t)), \quad t \in \mathbb{T} = \{t_0, t_0+1, \ldots, T-1\}$$

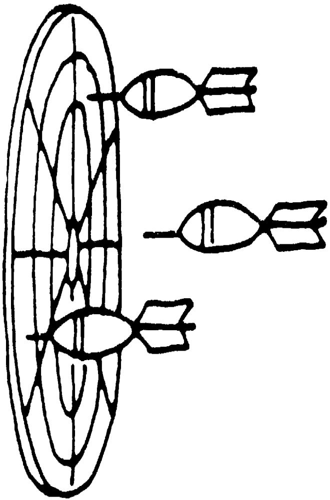
- the time $t \in \overline{\mathbb{T}} = \{t_0, t_0+1, \ldots, T-1, T\} \subset \mathbb{N}$ is discrete with initial time t_0 and horizon T ($T < +\infty$ or $T = +\infty$) (*the time period* $[t, t+1[$ *may be a year, a month, etc.*)
- the state variable $x(t)$ belongs to the finite dimensional *state space* $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$; (*stocks, biomasses, abundances, capital, etc.*)
- the control variable $u(t)$ is an element of the *control space* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ (*outflows, catches, harvesting effort, investment, etc.*)
- the dynamics $\mathtt{Dyn}$ maps $\mathbb{T} \times \mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$ (*storage, age-class model, population dynamics, economic model, etc.*)

We dress natural resources management issues in the formal clothes of control theory in discrete time



- Problems are framed as
  - find controls/decisions driving a dynamical system
  - to achieve various goals
- Three main ingredients are
  - controlled dynamics
  - constraints
  - criterion to optimize

We mathematically express the objectives pursued as control and state constraints



- For a state-control system, we cloth objectives as constraints
- and we distinguish
  control constraints (rather easy)
  state constraints (rather difficult)
- Viability theory deals with state constraints

Constraints may be explicit on the control variable

and are rather easily handled by reducing the decision set

Examples of control constraints

- ▷ Irreversibility constraints, physical bounds
  $0 \leq a(t) \leq 1 \;, \quad 0 \leq h(t) \leq B(t)$
- ▷ Tolerable costs $c\big(a(t), Q(t)\big) \leq c^{\sharp}$

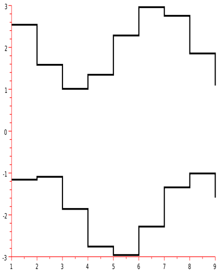
Control constraints / admissible decisions

$$\underbrace{u(t)}_{\text{control}} \in \underbrace{\mathbb{B}\big(t, x(t)\big)}_{\text{admissible set}} \;, \quad t = t_0, \ldots, T-1$$

Easy because control variables $u(t)$ are precisely those variables whose values the decision-maker can fix at any time within given bounds

Meeting constraints bearing on the state variable is delicate

due to the dynamics pipeline between controls and state



State constraints / admissible states

$$\underbrace{x(t)}_{\text{state}} \in \underbrace{\mathbb{A}(t)}_{\text{admissible set}} , \quad t = t_0, \ldots, T$$

Examples ("tipping points")

- CO_2 concentration $M(t) \leq M^\sharp$
- biomass $B^\flat \leq B(t) \leq B^\sharp$

State constraints are mathematically difficult because of "inertia"

$$x(t) = \underbrace{\text{function}}_{\text{iterated dynamics}} \big(\underbrace{u(t-1), \ldots, u(t_0)}_{\text{past controls}}, x(t_0) \big)$$

Target and asymptotic state constraints are special cases

▷ Final state achieves some target

$$\underbrace{x(T)}_{\text{final state}} \in \underbrace{\mathbb{A}(T)}_{\text{target set}}$$

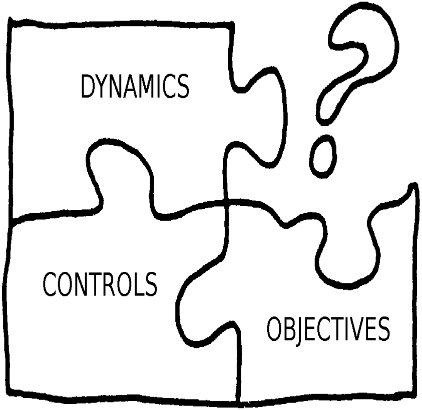
Example: CO_2 concentration

▷ State converges toward a target

$$\underbrace{\lim_{t \to +\infty} x(t)}_{\text{asymptotic state}} \in \underbrace{\mathbb{A}(\infty)}_{\text{target set}}$$

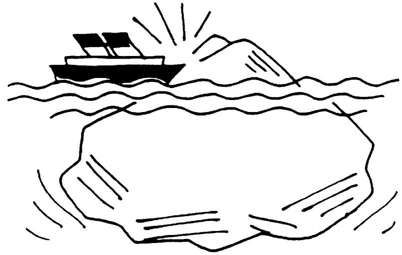
Example: convergence towards an endemic state in epidemiology

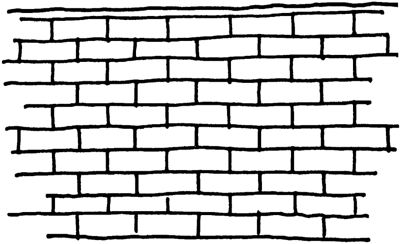
Can we solve the compatibility puzzle between dynamics and objectives by means of appropriate controls?



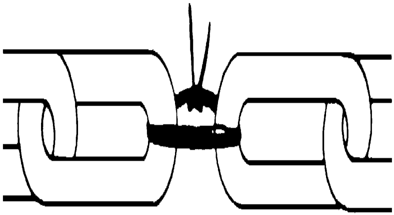
- Given a dynamics that mathematically embodies the causal impact of controls on the state
- Imposing objectives bearing on output variables (states, controls)
- Is it possible to find a control path that achieves the objectives for all times?

Crisis occurs when constraints are trespassed at least once





- An initial state is not viable if, whatever the sequence of controls, a crisis occurs
- There exists a time when one of the state or control constraints is violated

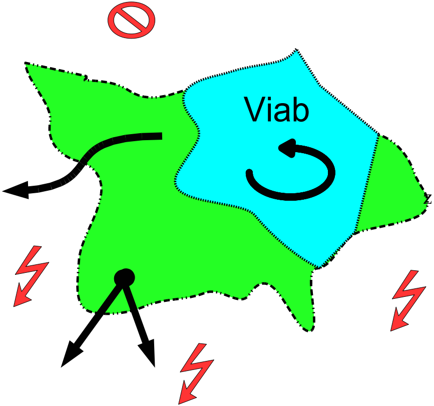


The compatibility puzzle can be solved when the initial viability kernel $\mathbb{V}\mathrm{iab}(t_0)$ is not empty

Viable initial states form the viability kernel (Jean-Pierre Aubin)

$$
\mathbb{V}\mathrm{iab}(t) := \left\{ \begin{array}{c} \text{initial} \\ \text{states} \\ x \in \mathbb{X} \end{array} \;\middle|\; \begin{array}{ll} \text{there exist a control path } u(\cdot) = & \\ \big(u(t), u(t+1), \ldots, u(T-1)\big) & \\ \text{and a state path } x(\cdot) = & \\ \big(x(t), x(t+1), \ldots, x(T)\big) & \\ \text{starting from } x(t) = x \text{ at time } t & \\ \text{satisfying for any time } s \in \{t, \ldots, T-1\} & \\ x(s+1) = \mathtt{Dyn}\big(s, x(s), u(s)\big) & \textit{dynamics} \\ u(s) \in \mathbb{B}(s, x(s)) & \textit{control constraints} \\ x(s) \in \mathbb{A}(s) & \textit{state constraints} \\ \text{and } x(T) \in \mathbb{A}(T) & \textit{target constraints} \end{array} \right\}
$$

The viability kernel is included in the state constraint set



- The largest set is the state constraint set $\mathbb{A}$
- It includes the smaller blue viability kernel $\mathbb{V}\mathrm{iab}(t_0)$
- The green set measures the incompatibility between dynamics and constraints: good start, but inevitable crisis!

The viability program aims at turning a priori constraints, with state constraints, into a posteriori constraints, without state constraints

- A priori constraints, with state constraints

$$\begin{cases} x(t_0) \in \mathbb{X} \\ x(t+1) = \mathtt{Dyn}\big(t, x(t), u(t)\big) \\ u(t) \in \mathbb{B}\big(t, x(t)\big) & \text{control constraints} \\ x(t) \in \mathbb{A}(t) & \text{state constraints} \end{cases}$$

- are turned into a posteriori constraints, without state constraints except for the initial state

$$\begin{cases} x(t_0) \in \mathbb{V}\mathrm{iab}(t_0) & \text{initial state constraint} \\ x(t+1) = \mathtt{Dyn}\big(t, x(t), u(t)\big) \\ u(t) \in \mathbb{B}^{\mathsf{viab}}\big(t, x(t)\big) & \text{control constraints} \end{cases}$$

Outline of the presentation

1. Motivation
2. Resource management examples under viability constraints
   - Viable management of an animal population
   - Mitigation for climate change
   - Forestry management
   - Viable epidemics control
3. The viability kernel and viable controls
   - Viability kernel
   - Dynamic programming equation and viable controls
   - Discussion on optimization, state constraints and multipliers
   - Example: viable control of an invasive species
   - Viability in the autonomous case
   - Approximation of viability kernels
4. Resource management by viability methods
   - A bioeconomic precautionary threshold
   - The anchovy–hake couple in the Peruvian upwelling ecosystem
5. Summary

The viability kernels satisfy a backward dynamic programming equation

Proposition

Assume that $T < +\infty$. The viability kernels $\mathbb{V}\mathrm{iab}(t)$ satisfy a backward induction, where t runs from $T-1$ down to t_0:

$$\mathbb{V}\mathrm{iab}(T) = \mathbb{A}(T)$$

$$\mathbb{V}\mathrm{iab}(t) = \{ \textit{admissible states } x \in \mathbb{A}(t) \mid \textit{there exists an admissible control } u \in \mathbb{B}(t,x) \textit{ such that the future state } \mathtt{Dyn}(t,x,u) \textit{ belongs to the next viability kernel } \mathbb{V}\mathrm{iab}(t+1) \}$$

The dynamic programming equation yields viable controls

- The following viable regulation set

$$\mathbb{B}^{\mathrm{viab}}(t,x) := \{u \in \mathbb{B}(t,x) \mid \mathtt{Dyn}(t,x,u) \in \mathbb{V}\mathrm{iab}(t+1)\}$$

is not empty if and only if $x \in \mathbb{V}\mathrm{iab}(t)$

$$\mathbb{B}^{\mathrm{viab}}(t,x) \neq \emptyset \iff x \in \mathbb{V}\mathrm{iab}(t)$$

- Any $u \in \mathbb{B}^{\mathrm{viab}}(t,x)$ is said to be a viable control
- A viable policy is a mapping $\mathtt{Pol} : \mathbb{T} \times \mathbb{X} \to \mathbb{U}$ such that

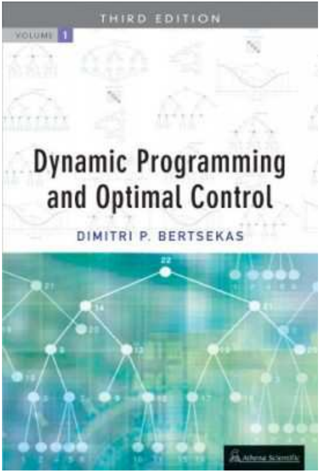
$$\mathtt{Pol}(t,x) \in \mathbb{B}^{\mathrm{viab}}(t,x)$$

for all $(t,x) \in \mathbb{T} \times \mathbb{X}$

Any viable control yields a viable trajectory

1. Initial state $x^{\star}(t_0) = x_0 \in \mathbb{V}\mathrm{iab}(t_0) \subset \mathbb{A}(t_0)$
2. Plug the state $x^{\star}(t_0)$ into the viable policy $\mathtt{Pol} \to$ initial decision $u^{\star}(t_0) = \mathtt{Pol}^{\star}\big(t_0, x^{\star}(t_0)\big) \in \mathbb{B}^{\mathrm{viab}}\big(t_0, x^{\star}(t_0)\big) \subset \mathbb{B}\big(t_0, x^{\star}(t_0)\big)$
3. Run the dynamics $\to$ second state $x^{\star}(t_0+1) = \mathtt{Dyn}\big(t_0, x^{\star}(t_0), u^{\star}(t_0)\big)$ $\in \mathbb{V}\mathrm{iab}(t_0+1) \subset \mathbb{A}(t_0+1)$
4. Second decision $u^{\star}(t_0+1) = \mathtt{Pol}^{\star}\big(t_0+1, x^{\star}(t_0+1)\big)$ $\in \mathbb{B}^{\mathrm{viab}}\big(t_0+1, x^{\star}(t_0+1)\big) \subset \mathbb{B}\big(t_0+1, x^{\star}(t_0+1)\big)$
5. And so on $x^{\star}(t_0+2) = \mathtt{Dyn}\big(t_0+1, x^{\star}(t_0+1), u^{\star}(t_0+1)\big)$
6. ...

"Life is lived forward but understood backward" (Søren Kierkegaard)



D. P. Bertsekas introduces his book *Dynamic Programming and Optimal Control* with a citation by Søren Kierkegaard

> *"Livet skal forstås baglaens, men leves forlaens"*
>
> *Life is to be understood backwards, but it is lived forwards*

- The viability kernels and the viable policies are computed backward and offline by means of the dynamic programming equation
- The viable trajectories are computed forward and online

Thanks to the dynamic programming equation, the viability program is achieved

- The a priori constraints, with state constraints

$$\begin{cases} x(t_0) \in \mathbb{X} \\ x(t+1) = \mathtt{Dyn}\big(t, x(t), u(t)\big) \\ u(t) \in \mathbb{B}\big(t, x(t)\big) & \text{control constraints} \\ x(t) \in \mathbb{A}(t) & \text{state constraints} \end{cases}$$

- have been turned into a posteriori constraints, without state constraints except for the initial state

$$\begin{cases} x(t_0) \in \mathbb{V}\mathrm{iab}(t_0) & \text{initial state constraint} \\ x(t+1) = \mathtt{Dyn}\big(t, x(t), u(t)\big) \\ u(t) \in \mathbb{B}^{\mathsf{viab}}\big(t, x(t)\big) & \text{control constraints} \end{cases}$$

Viable controls are not unique, in general

- Multiplicity and flexibility of viable decisions: the set $\mathbb{B}^{\mathrm{viab}}(t,x)$ is generally not a singleton.
- Selections
  - Random viable selection
  - Slow viable regulations: $\|\mathtt{Pol}(t,x)\| \in \arg\min_{u\in\mathbb{B}^{\mathrm{viab}}(t,x)} \|u\|$
  - Inertial viable selection: $\mathtt{Pol}(t,x) \in \arg\min_{u\in\mathbb{B}^{\mathrm{viab}}(t,x)} \|u-u^{*}\|$
  - Viable and optimal intertemporal selection

Outline of the presentation

Viability theory can help to turn cost-effectiveness problems into a standard form

1. Once obtained the true constraints $\mathbb{B}^{\text{viab}}(t,x)$ and $\mathbb{V}\text{iab}(t)$ from the dynamic and the a posteriori constraints
2. Optimize some intertemporal criterion

$$\max_{x(\cdot),u(\cdot)} \left(\sum_{t=t_0}^{T-1} \mathtt{L}\big(t,x(t),u(t)\big) + \mathtt{K}\big(T,x(T)\big) \right)$$

   under the constraints which now take the form

$$\begin{cases} x(t_0) \in \mathbb{V}\text{iab}(t_0) \\ x(t+1) = \mathtt{Dyn}(t,x(t),u(t)) \\ u(t) \in \mathbb{B}^{\text{viab}}(t,x) \end{cases}$$

3. There are no more state constraints! Only control constraints

Constraints in dynamic optimization problems can be hard or soft

- State constraints penalization (hard)

$$\max_{x(\cdot),u(\cdot)} \left(\sum_{t=t_0}^{T-1} \mathrm{L}\big(t, x(t), u(t)\big) - \sum_{t=t_0}^{T-1} \chi_{\mathbb{A}(t)}\big(x(t)\big) \right)$$

$$\text{where } \chi_{\mathbb{A}(t)}(x) = \begin{cases} +\infty & \text{if} \quad x \notin \mathbb{A}(t) \\ 0 & \text{if} \quad x \in \mathbb{A}(t) \end{cases}$$

- State constraints dualization (soft), with Lagrange multipliers $p(t)$, when constraints are given by inequalities $\mathcal{I}\big(t, x(t)\big) \geq 0 \iff x(t) \in \mathbb{A}(t)$

$$\max_{x(\cdot),u(\cdot)} \min_{p(\cdot)\geq 0} \left(\sum_{t=t_0}^{T-1} \mathrm{L}\big(t, x(t), u(t)\big) + \sum_{t=t_0}^{T-1} p(t)\mathcal{I}\big(t, x(t)\big) \right)$$

and then, interchange to obtain a $\min_{p(\cdot)\geq 0} \max_{x(\cdot),u(\cdot)}$ when a saddle point exists

More on the soft constraints and on interpreting a marginal variation of intertemporal utility as a price

- For $\epsilon(\cdot) = \big(\epsilon(t_0), \dots, \epsilon(T)\big)$, define

$$J\big(\epsilon(\cdot)\big) = \max_{x(\cdot),u(\cdot)} \Big(\sum_{t=t_0}^{T-1} \mathtt{L}\big(t, x(t), u(t)\big) \Big)$$

the optimal intertemporal payoff under the constraints that

$$\mathcal{I}\big(t, x(t)\big) \geq -\epsilon(t) \,, \quad t = t_0, \dots, T$$

- The Lagrange multiplier $p(t)$ attached to the constraint $\mathcal{I}\big(t, x(t)\big) \geq 0$ is the marginal variation of the intertemporal utility when the constraint is slightly modified

$$p(t) = \frac{\partial J}{\partial \epsilon(t)}(0)$$

- The Lagrange multiplier $p(t)$ is the price one is ready to pay for an extra unit of the "resource" $\mathcal{I}\big(t, x(t)\big)$

Outline of the presentation

We consider an invasive species biomass model driven by an effort control

- Consider a density-dependent linear dynamic

$$B(t+1) = R \underbrace{B(t)}_{\text{biomass}} \big(1 - \underbrace{E(t)}_{\text{effort}}\big)$$

  - where the invasive species is described by its biomass $B(t)$
  - where the control is exterted under the form of a harvesting effort $E(t)$
- The effort is constrained by

$$E^\flat \leq E(t) \leq E^\sharp$$

where

$$0 \leq E^\flat \leq E^\sharp \leq 1$$

We constrain the ultimate biomass to lie between conservation and maximal safety values

- Consider two thresholds
  - a conservation lower bound $B^\flat > 0$
  - a safety upper bound $B^\sharp > 0$
- We assume that the policy goal is to constrain the ultimate biomass $B(T)$ within the ecological window $[B^\flat, B^\sharp]$

$$B^\flat \leq B(T) \leq B^\sharp$$

- We will show that this target constraint is achieved whenever the initial biomass is sufficiently high, but not too high

We write a dynamic programming equation relating the viability kernels

- The abstract dynamic programming equation relating the viability kernels is

$$\mathbb{V}\mathrm{iab}(T) = \mathbb{A}(T)$$

$$\mathbb{V}\mathrm{iab}(t) = \{x \in \mathbb{A}(t) \mid \exists u \in \mathbb{B}(t,x), \mathtt{Dyn}(t,x,u) \in \mathbb{V}\mathrm{iab}(t+1)\}$$

- In our case, it materializes as

$$\mathbb{V}\mathrm{iab}(T) = [B^{\flat}, B^{\sharp}]$$

$$\mathbb{V}\mathrm{iab}(t) = \Big\{ B \in \mathbb{R}_+ \mid \text{ there exists an effort } E \in [E^{\flat}, E^{\sharp}] \text{ such that the future biomass } RB(1-E) \in \mathbb{V}\mathrm{iab}(t+1) \Big\}$$

Exercise: calculate the penultimate viability kernel

$$\mathbb{V}\mathrm{iab}(T-1) = \left\{ B \in \mathbb{R}_+ \mid \exists E \in [E^\flat, E^\sharp] \,, \quad RB\big(1-E\big) \in \underbrace{[B^\flat, B^\sharp]}_{\mathbb{V}\mathrm{iab}(T)} \right\}$$

- ▷ Fix a biomass $B \geq 0$
- ▷ Look for an effort $E \in [E^\flat, E^\sharp]$ such that

$$B^\flat \leq RB\big(1-E\big) \leq B^\sharp$$

- ▷ Observe that such an effort E exists if and only if the intersection of $[E^\flat, E^\sharp]$ with another interval (the bounds of which depend on the fixed biomass B) is not empty
- ▷ As a consequence, establish for which biomasses $B \geq 0$ such an effort E exists
- ▷ These biomasses $B \geq 0$ delineate the viability kernel

$$\mathbb{V}\mathrm{iab}(T-1) = [\frac{B^\flat}{R(1-E^\flat)}, \frac{B^\sharp}{R(1-E^\sharp)}]$$

Viability kernels are intervals and viable efforts belong to intervals

- The viability kernels are intervals

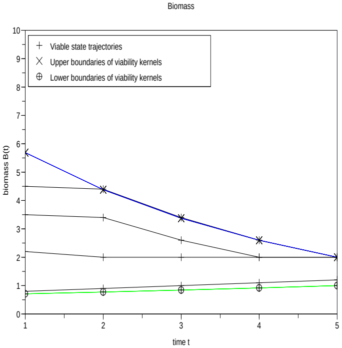
$$\mathbb{V}\mathrm{iab}(t) = [B^\flat(t), B^\sharp(t)]$$

whose viability biomass bounds are given by

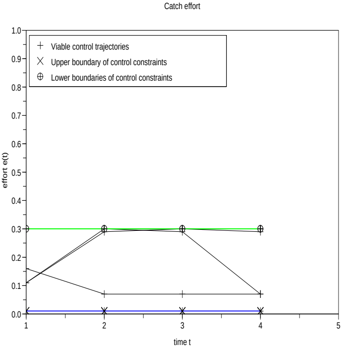
$$\begin{cases} B^\flat(t) & = & B^\flat \left(R(1-E^\flat)\right)^{t-T} \\ B^\sharp(t) & = & B^\sharp \left(R(1-E^\sharp)\right)^{t-T} \end{cases}$$

- Viable efforts E belong to the set

$$1 - \frac{B^\sharp(t)}{RB} \leq E \leq 1 - \frac{B^\flat(t)}{RB} \text{ and } E^\flat \leq E \leq E^\sharp$$



(a) Biomass $B(t)$



(b) Effort $E(t)$

Outline of the presentation

1. Motivation
2. Resource management examples under viability constraints
   - Viable management of an animal population
   - Mitigation for climate change
   - Forestry management
   - Viable epidemics control
3. The viability kernel and viable controls
   - Viability kernel
   - Dynamic programming equation and viable controls
   - Discussion on optimization, state constraints and multipliers
   - Example: viable control of an invasive species
   - **Viability in the autonomous case**
   - Approximation of viability kernels
4. Resource management by viability methods
   - A bioeconomic precautionary threshold
   - The anchovy–hake couple in the Peruvian upwelling ecosystem
5. Summary

We restrict to stationary constraints and dynamics

- Stationary state constraints

$$\mathbb{A}(t) = \mathbb{A}$$

- Stationary control constraints

$$\mathbb{B}(t, x) = \mathbb{B}(x)$$

- Stationary dynamics

$$\mathtt{Dyn}(t, x, u) = \mathtt{Dyn}(x, u)$$

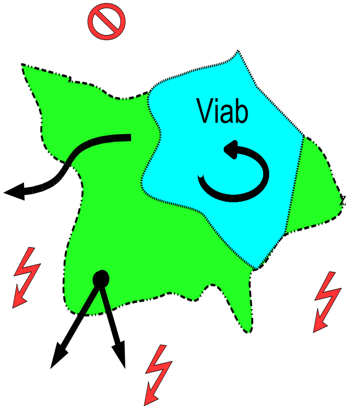
Stationary constraints can express intergenerational equity

- Consider the autonomous case of stationary constraints and dynamics

$$
\begin{aligned}
\mathbb{A}(t) &= \mathbb{A} \\
\mathbb{B}(t,x) &= \mathbb{B}(x) \\
\mathtt{Dyn}(t,x,u) &= \mathtt{Dyn}(x,u)
\end{aligned}
$$

- When the horizon $T = +\infty$ is infinite,
constraints to be satisfied for all times may be a way to
embody intergenerational equity, sustainability, stewardship

There are three relevant configurations for an autonomous viability problem



- Comfortable case: the viability kernel is the whole state constraint

$$\mathbb{V}\mathrm{iab}(t_0) = \mathbb{V}\mathrm{iab}(t) = \mathbb{A}$$

$\longrightarrow$ we say that $\mathbb{A}$ is viable

- Dangerous case:

$$\emptyset \subsetneq \mathbb{V}\mathrm{iab}(t_0) \subsetneq \mathbb{A}$$

$\longrightarrow$ crisis outside $\mathbb{V}\mathrm{iab}(t_0)$
$\longrightarrow$ security margins in $\mathbb{V}\mathrm{iab}(t_0)$

- Hopeless case: the viability kernel is empty $\mathbb{V}\mathrm{iab}(t_0) = \emptyset$

In the autonomous case, the viability kernel extends the concept of equilibrium

Proposition

In the autonomous case,
the admissible equilibria belong to the viability kernel $\mathbb{V}\mathrm{iab}(t)$
at any time t*:*

$$\{x_{\mathrm{E}} \in \mathbb{A} \mid \exists u_{\mathrm{E}} \in \mathbb{B}(x_{\mathrm{E}}) \ , \quad x_{\mathrm{E}} = \mathtt{Dyn}(x_{\mathrm{E}}, u_{\mathrm{E}})\} \subset \mathbb{V}\mathrm{iab}(t)$$

Indeed, the stationary control $u(t) = u_{\mathrm{E}} \in \mathbb{B}(x_{\mathrm{E}})$ makes that

$$x(t) = \mathtt{Dyn}(x(t), u(t)) = \mathtt{Dyn}(x_{\mathrm{E}}, u_{\mathrm{E}}) = x_{\mathrm{E}} \in \mathbb{A}$$

The viability kernels are increasing with respect to time

Proposition

- *In the autonomous case,*
  the viability kernels are increasing with respect to time:

$$\mathbb{V}\mathrm{iab}(t_0) \subset \mathbb{V}\mathrm{iab}(t_0+1) \subset \cdots \subset \mathbb{V}\mathrm{iab}(T) = \mathbb{A}$$

- *If, in addition, the horizon is infinite ($T = +\infty$),*
  the viability kernels are stationary
  and we write the common set $\mathbb{V}\mathrm{iab}$:

$$\mathbb{V}\mathrm{iab}(t_0) = \cdots = \mathbb{V}\mathrm{iab}(t) = \cdots = \mathbb{V}\mathrm{iab} \subset \mathbb{A}$$

Viable controls delineate the "true constraints"

- In the autonomous case and in the infinite horizon case, the time component vanishes and we obtain the viable controls as follows:

$$\mathbb{B}^{\mathsf{viab}}(x) := \{u \in \mathbb{B}(x) \mid \mathtt{Dyn}(x,u) \in \mathbb{V}\mathrm{iab}\}$$

- Hence, ensuring viability means remaining in the viability kernel:

$$\underbrace{\mathbb{A}}_{\text{state constraints}} \to \underbrace{\mathbb{V}\mathrm{iab}}_{\text{“true constraints”}} \subset \mathbb{A}$$

The notion of viability domain is relevant in the autonomous case

Definition

A subset $\mathbb{V} \subset \mathbb{X}$ of states is said to be a viability domain if

$$\forall x \in \mathbb{V} \qquad \underbrace{\exists u \in \mathbb{B}(x)}_{\text{admissible control}} \qquad \underbrace{\mathtt{Dyn}(x, u)}_{\text{future state}} \in \mathbb{V}$$

That is,

- for any state x in $\mathbb{V}$
- there exists an admissible control $u \in \mathbb{B}(x)$
- such that the future state $\mathtt{Dyn}(x, u)$ belongs to $\mathbb{V}$

Viability kernel and viability domains are tied sets

Theorem (J.-P. Aubin)

In the autonomous case with infinite horizon $T = +\infty$,
the viability kernel $\mathbb{V}\mathrm{iab}$ is, equivalently,

- *the largest viability domain $\mathbb{V}$ contained in the state constraint set $\mathbb{A}$*
- *the union of all viability domains in the state constraint set $\mathbb{A}$*

Any viability domain is a lower approximation of the viability kernel

$$\underbrace{\mathbb{V}}_{\text{viability domain}} \subset \underbrace{\mathbb{V}\mathrm{iab}}_{\text{viability kernel}} \subset \underbrace{\mathbb{A}}_{\text{state constraints}}$$

Outline of the presentation

We outline an alternative formulation with an acceptable configurations set

A decision maker describes *acceptable configurations of the system* through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the *acceptable set*

$$(x(t), u(t)) \in \mathbb{D}\,, \quad \forall t = t_0, t_0 + 1, \ldots$$

where $\mathbb{D}$ includes both system states and controls constraints

Upper sets

We say that a set $S \subset \mathbb{X}$ is an *upper set* (or is an *increasing set*) if it satisfies the following property:

$$\forall x \in S\,,\ \forall x' \in \mathbb{X}\,,\ \ x' \geq x \Rightarrow x' \in S$$

In the same way, a set $K \subset \mathbb{X} \times \mathbb{U}$ is said to be an *upper set* if

$$\forall (x, u) \in K\,,\ \forall x' \in \mathbb{X}\,,\ \ x' \geq x \Rightarrow (x', u) \in K$$

The notion of monotone harvest dynamics will prove useful for management

We say that the dynamic $\mathtt{Dyn} : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is

- *increasing with respect to the state* if it satisfies

$$\forall \, (x, x', u) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U} \, , \quad x' \geq x \Rightarrow \mathtt{Dyn}(x', u) \geq \mathtt{Dyn}(x, u)$$

- *decreasing with respect to the control* if

$$\forall (x, u, u') \in \mathbb{X} \times \mathbb{U} \times \mathbb{U} \, , \quad u' \geq u \Rightarrow \mathtt{Dyn}(x, u') \leq \mathtt{Dyn}(x, u)$$

Monotone harvest dynamic

We coin $\mathtt{Dyn} : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ a *monotone harvest dynamic* if $\mathtt{Dyn}$ is increasing with respect to the state and decreasing with respect to the control

More on approximation of viability kernels

Proposition

If $\mathbb{V}$ *is a viability domain of* $\mathtt{Dyn}$ *in* $\mathbb{D}$*, then*

$$\widetilde{\mathbb{V}} = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}\,,\ (x,u) \in \mathbb{D} \ \textit{and}\ \mathtt{Dyn}(x,u) \in \mathbb{V}\}$$

is a viability domain which contains $\mathbb{V}$*. As a consequence,*

1. *the induction*

$$\widetilde{\mathbb{V}}_0 = \mathbb{V} \ \textit{and}\ \widetilde{\mathbb{V}}_{k+1} = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}\,,\ (x,u) \in \mathbb{D} \ \textit{and}\ \mathtt{Dyn}(x,u) \in \widetilde{\mathbb{V}}_k\}$$

   generates an increasing sequence of viability domains

2. *and its limit is included in the viability kernel*

$$\bigcup_{k\in\mathbb{N}} \widetilde{\mathbb{V}}_k = \lim_{k\to+\infty} \uparrow \widetilde{\mathbb{V}}_k \subset \mathbb{V}(\mathtt{Dyn},\mathbb{D})$$

More on approximation of viability kernels

Proposition

Assume that

- *the desirable set* $\mathbb{D}$ *is increasing*
- *the dynamics* $\mathtt{Dyn}$ *is bounded below by an increasing* $\mathtt{Dyn}^\flat : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$

$$\mathtt{Dyn}^\flat(x, u) \leq \mathtt{Dyn}(x, u) , \quad \forall (x, u) \in \mathbb{X} \times \mathbb{U}$$

and

$$\mathtt{Dyn}^\flat \text{ is increasing with respect to the state}$$

Then, $\mathbb{V}(\mathtt{Dyn}^\flat, \mathbb{D})$ *is a viability domain associated with* $\mathtt{Dyn}$ *in* $\mathbb{D}$*, and thus*

$$\mathbb{V}(\mathtt{Dyn}^\flat, \mathbb{D}) \subset \mathbb{V}(\mathtt{Dyn}, \mathbb{D})$$

Summary

The viability program

- The compatibility puzzle between dynamics and objectives can be solved when the viability kernel is not empty
- The viability program aims at turning
  a priori constraints, with state constraints
  into a posteriori constraints, without state constraints

Summary

Dynamic programming and viable controls

- The viability kernels satisfy a backward dynamic programming equation
- The dynamic programming equation displays viable controls
- Therefore, the viability program is achieved
- Viable controls delineate the "true constraints",
  those that allow to satisfy the state constraints
- There is generally no uniqueness of viable controls and policies

Summary

Viability kernel in the autonomous case

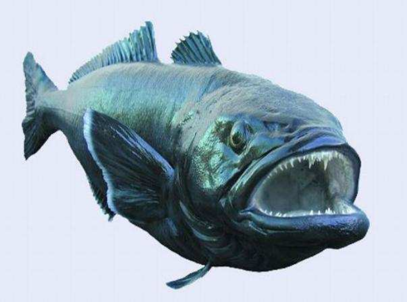
In the autonomous case,

- the viability kernel extends the concept of equilibrium
- The viability kernel is the union of all viability domains in the state constraint set
- Monotonicity properties of sets and dynamics provide approximations of the viability kernel

Outline of the presentation

Outline of the presentation

We consider a biomass model for a harvested renewable resource



$$B(t+1) = \overbrace{\texttt{Biol}}^{\text{dynamic}} \Big(\underbrace{B(t)}_{\text{biomass}} - \underbrace{h(t)}_{\text{catches}} \Big)$$

- $B(t)$ biomass
- $h(t)$ catch with $0 \leq h(t) \leq B(t)$
- $\texttt{Biol}$ natural resource growth function (Beverton-Holt, for instance)

A regulating agency aims to guarantee along time both a minimal harvesting and a minimal stock

- Consider a regulating agency whose policy goals are to guarantee at each time t
  - a minimal harvesting $h_{\text{LIM}} > 0$

$$h(t) \geq h_{\text{LIM}} \qquad \text{production}$$

  - a minimal biomass $B_{\text{LIM}} > 0$

$$B(t) \geq B_{\text{LIM}} \qquad \text{preservation}$$

- By a viability analysis, we will determine whether these goals can be achieved or not
- When possible, we will display viable policies to achieve these goals

We need and recall the notion of sustainable yield

- The sustainable yield function $\texttt{Sust}$ is defined by

$$h = \texttt{Sust}(B) \iff B = \texttt{Biol}(B - h) \text{ and } 0 \leq h \leq B$$

- The maximum sustainable biomass B_{MSE} and maximum sustainable yield h_{MSE} are defined by

$$h_{\mathrm{MSE}} = \texttt{Sust}(B_{\mathrm{MSE}}) = \max_{B \geq 0} \texttt{Sust}(B)$$

When the dynamic is increasing, the viability kernel is either empty or is an interval

Proposition

- *Assume that the dynamic* $B \mapsto \texttt{Biol}(B)$ *is increasing and continuous, and let* K *be the carrying capacity* ($\texttt{Biol}(K) = K$)
- *The viability kernel is either empty or has the form* $[B_{\mathrm{PA}}, K]$
- *Any interval* $[\underline{B}, K]$ *is a viability domain whenever*

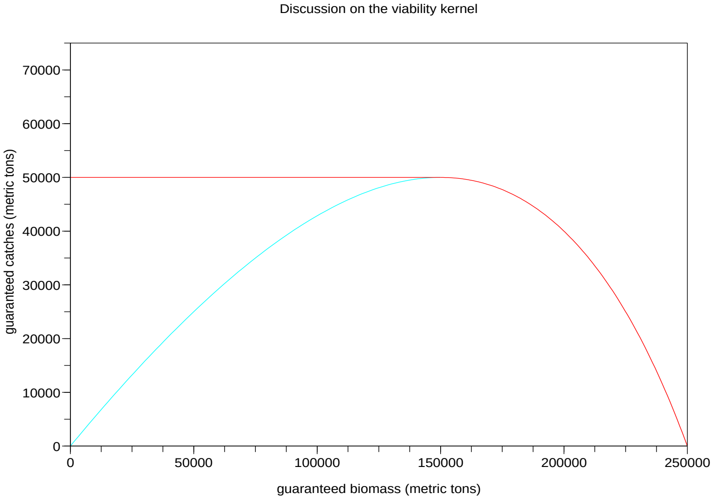
$$\texttt{Sust}(\underline{B}) \geq h_{\mathrm{LIM}}$$

- *The largest of the viability domains* $[\underline{B}, K]$ *included in the state constraint set* $[B_{\mathrm{LIM}}, K]$ *is*

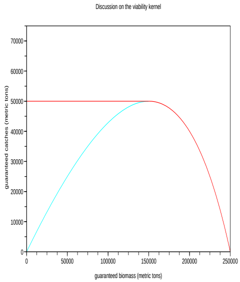
$$\mathbb{V}\mathrm{iab} = [B_{\mathrm{PA}}, K]$$

where $B_{\mathrm{PA}} = \min\{B \geq B_{\mathrm{LIM}} \mid \texttt{Sust}(B) \geq h_{\mathrm{LIM}}\}$

The expression of the viability kernel depends on the minimal guaranteed thresholds h_{LIM} and B_{LIM}



The expression of the viability kernel depends on the minimal guaranteed thresholds h_{LIM} and B_{LIM}

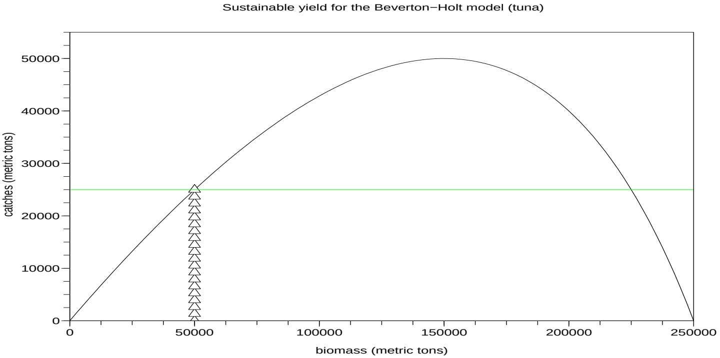


$$\mathbb{V}\mathrm{iab} = \begin{cases} \emptyset & \text{if } h_{\mathrm{LIM}} > h_{\mathrm{MSE}} \\ \emptyset & \text{if } B_{\mathrm{LIM}} > B_{\mathrm{MSE}} \\ & \text{and } h_{\mathrm{LIM}} > \mathtt{Sust}(B_{\mathrm{LIM}}) \\ [B_{\mathrm{LIM}}, K] & \text{if } h_{\mathrm{LIM}} \leq \mathtt{Sust}(B_{\mathrm{LIM}}) \\ [\mathtt{Sust}^{-1}(h_{\mathrm{LIM}}), K] & \text{if } B_{\mathrm{LIM}} \leq B_{\mathrm{MSE}} \text{ and} \\ & \mathtt{Sust}(B_{\mathrm{LIM}}) < h_{\mathrm{LIM}} \leq h_{\mathrm{MSE}} \end{cases}$$

where

$$\mathtt{Sust}^{-1}(h) := \min\{B \mid \mathtt{Sust}(B) = h\}$$

$\texttt{Sust}^{-1}(h) := \min\{B \mid \texttt{Sust}(B) = h\}$



The green horizontal line $h = h_{\mathrm{LIM}}$ intersects the sustainable yield curve in two points. The smaller abcisse is $\texttt{Sust}^{-1}(h_{\mathrm{LIM}})$.
For $h_{\mathrm{LIM}} = 25\,000$ tonnes, $\texttt{Sust}^{-1}(h_{\mathrm{LIM}}) = 50\,000$ tonnes.

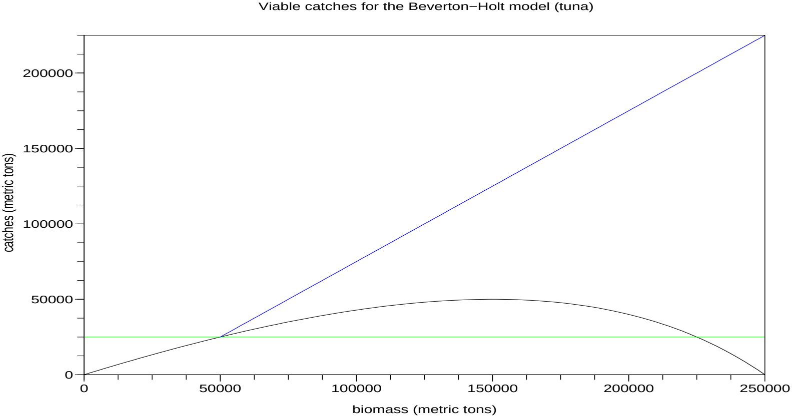
Viable controls belong to an interval

- For any stock $B \in \mathbb{V}\mathrm{iab}$, the viable catches lie within the set

$$\mathbb{B}^{\mathsf{viab}}(B) = [h_{\mathrm{LIM}}, \mathtt{Catch}_{\mathrm{PA}}(B)]$$

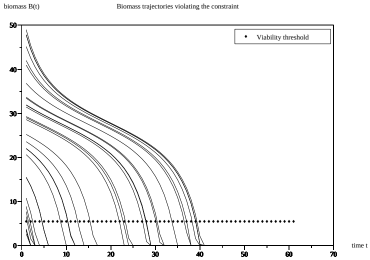
- The ceiling viable catch is given by

$$\mathtt{Catch}_{\mathrm{PA}}(B) = B + \mathtt{Sust}(B_{\mathrm{PA}}) - B_{\mathrm{PA}}$$

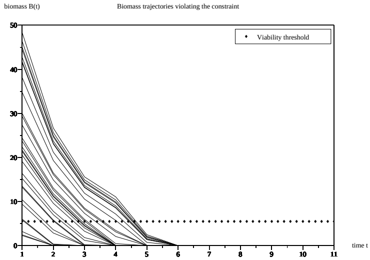


The green horizontal line is the lower limit for viable catches
The blue line is the upper limit for viable catches

The unsustainable case: $\mathbb{V}\text{iab} = \emptyset$ or $h_{\text{LIM}} > h_{\text{MSE}}$

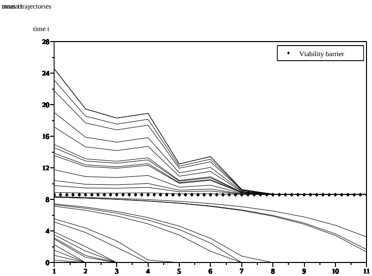


(c) Biomass $B(t)$ for stationary harvesting $h(t) = h_0$

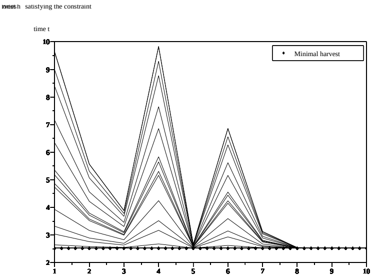


(d) Biomass $B(t)$ for random harvesting $h(t)$

The sustainable case: $\mathbb{V}\mathrm{iab} \neq \emptyset$ or $h_{\mathrm{LIM}} \leq h_{\mathrm{MSE}}$



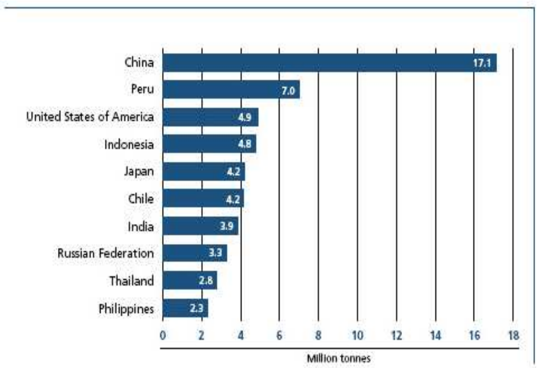
(e) Viable biomass $B(t) \geq B_{\mathrm{PA}}$. Non viable biomass $B(t) < B_{\mathrm{PA}}$

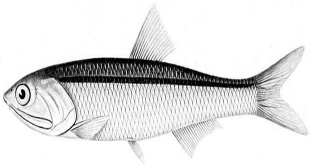


(f) Viable catch $h(t) \geq h_{\mathrm{LIM}}$

Outline of the presentation

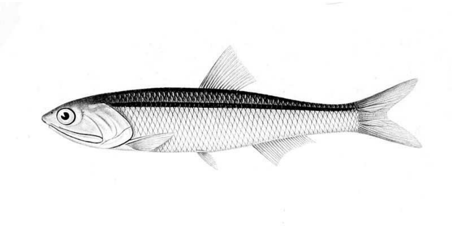
Perú is World 2nd for marine and inland capture fisheries

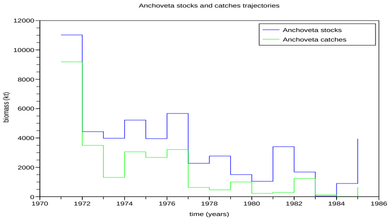


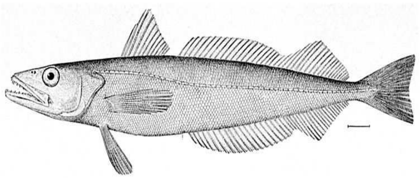


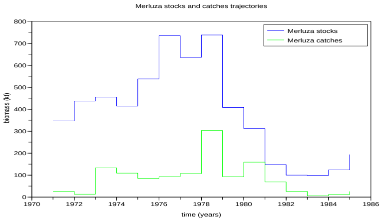
The northern Humboldt current system off Perú covers less than 0.1% of the world ocean but presently sustains about 10% of the world fish catch

We were lucky enough that IMARPE entrusted us yearly data of anchoveta and merluza stock and catches from 1971 to 1985









We consider two species targeted by two fleets in a biomass ecosystem dynamic

We embody stocks and fishing interactions
in a two-dimensional dynamical model

$$
\begin{aligned}
\overbrace{A(t+1)}^{\text{future biomass}} &= A(t)\,\overbrace{\mathcal{R}_A\big(A(t),H(t)\big)}^{\text{growth factor}}\,\big(1-\underbrace{E_A(t)}_{\text{effort}}\big) \\
H(t+1) &= H(t)\mathcal{R}_H\big(A(t),H(t)\big)\big(1-\overbrace{E_H(t)}^{\text{control}}\big)
\end{aligned}
$$

- ▷ State vector $(A(t), H(t))$ represents biomasses
- ▷ Control vector $(E_A(t), E_H(t))$ is fishing effort of each species
- ▷ Catches are $E_A(t)\mathcal{R}_A\big(A(t),H(t)\big)A(t)$ and $E_H(t)\mathcal{R}_H\big(A(t),H(t)\big)H(t)$ (measured in biomass)

Our objectives are twofold: conservation and production

The viability kernel is the set of initial species biomasses $\big(A(t_0), H(t_0)\big)$ from which appropriate effort controls $\big(E_A(t), E_H(t)\big),\ t = t_0, t_0 + 1, \ldots$ produce a trajectory of biomasses $\big(A(t), H(t)\big),\ t = t_0, t_0 + 1, \ldots$ such that the following goals are satisfied

- ▷ preservation (minimal biomass thresholds)

$$A \text{ \underline{stocks}}: \qquad A(t) \geq S_A^\flat$$

$$H \text{ \underline{stocks}}: \qquad H(t) \geq S_H^\flat$$

- ▷ economic/social requirements (minimal catch thresholds)

$$A \text{ \underline{catches}}: \qquad E_A(t)\mathcal{R}_A\big(A(t), H(t)\big)A(t) \geq C_A^\flat$$

$$H \text{ \underline{catches}}: \qquad E_H(t)\mathcal{R}_H\big(A(t), H(t)\big)H(t) \geq C_H^\flat$$

We provide an explicit expression for the viability kernel under rather weak assumptions

Proposition

If the thresholds $S_A^\flat, S_H^\flat$ and $C_A^\flat, C_H^\flat$ meet the inequalities

$$\underbrace{S_A^\flat \mathcal{R}_A(S_A^\flat, S_H^\flat) - S_A^\flat}_{\text{surplus}} \geq C_A^\flat \quad \text{and} \quad \underbrace{S_H^\flat \mathcal{R}_H(S_A^\flat, S_H^\flat) - S_H^\flat}_{\text{surplus}} \geq C_H^\flat$$

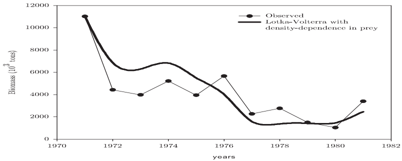
the viability kernel is given by

$$\left\{(A, H) \mid A \geq S_A^\flat,\ H \geq S_A^\flat,\ A\mathcal{R}_A(A, H) - S_A^\flat \geq C_A^\flat,\ H\mathcal{R}_H(A, H) - S_H^\flat \geq C_H^\flat\right\}$$

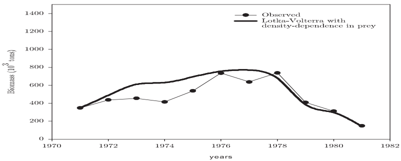
We taylor a Lotka-Volterra *decision model* to hake-anchovy Peruvian fisheries scarce data

Hake-anchovy Peruvian fisheries data between 1971 and 1981, in thousands of tonnes (10^3 tons)

- ▷ anchoveta_stocks= [11019 4432 3982 5220 3954 5667 2272 2770 1506 1044 3407]
- ▷ merluza_stocks= [347 437 455 414 538 735 636 738 408 312 148]
- ▷ anchoveta_captures= [9184 3493 1313 3053 2673 3211 626 464 1000 223]
- ▷ merluza_captures= [26 13 133 109 85 93 107 303 93 159 69]



(g) Anchovy



(h) Hake

Figure : Comparison of observed and simulated biomasses of anchovy and hake using a Lotka-Volterra model with density-dependence in the prey. Model parameters are $R = 2.25$, $L = 0.945$, $\kappa = 67\ 113 \times 10^3$ t ($K = 37\ 285 \times 10^3$ t), $\alpha = 1.22 \times 10^{-6}\ t^{-1}$, $\beta = 4.845 \times 10^{-8}\ t^{-1}$.

Here is the Lotka-Volterra *decision model*

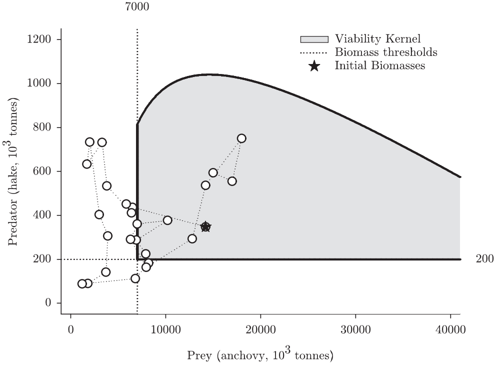
- A is the prey biomass (anchovy)
- H is the predator biomass (hake)
- The discrete-time Lotka-Volterra system is

$$
\begin{aligned}
A(t+1) &= A(t)\overbrace{\Big(R - \frac{R}{\kappa}A(t) - \alpha H(t)\Big)}^{\mathcal{R}_A\big(A(t),H(t)\big)}\big(1 - E_A(t)\big) \\
H(t+1) &= H(t)\underbrace{\big(L + \beta A(t)\big)}_{\mathcal{R}_H\big(A(t),H(t)\big)}\big(1 - E_H(t)\big),
\end{aligned}
$$

- The associated deterministic viability kernel is

$$
\mathbb{V}(t_0) = \Big\{ \; (A,H) \mid A \geq S_A^\flat, \frac{1}{\alpha}[R - \frac{R}{\kappa}A - \frac{S_A^\flat + C_A^\flat}{A}] \geq H \geq \max\{\frac{S_H^\flat + C_H^\flat}{L + \beta A}, S_H^\flat\} \; \Big\}
$$

For given biomasses and catches thresholds, we display the associated viability kernel



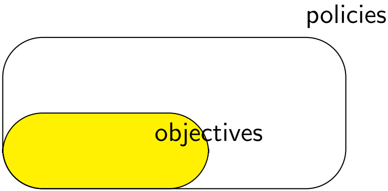
- Minimal biomasses thresholds
  - $S_A^\flat = 7\ 000\ kt$ (anchovy)
  - $S_H^\flat = 200\ kt$ (hake)
- Minimal catches thresholds
  - $C_A^\flat = 2\ 000\ kt$ (anchovy)
  - $C_H^\flat = 5\ kt$ (hake)

First acid test: plotting years of observed biomasses

- The range of values for viable states fits with measured biomasses
- Theoretically, a viable management with guaranteed biomasses and catches would have been possible since the initial state $\star$ is viable

Outline of the presentation

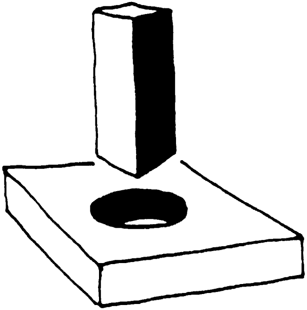
Resource managers often design policies contingent on implicit objectives



In practice, we observe that resource managers generally

- design policies
- which directly incorporate objectives
- with confusion between
  - objectives
  - and decision rules

Control theory draws an explicit line between objectives and policies



- We can observe a mismatch between proposed policies and implicit objectives (ICES precautionary approach)
- Control theory makes a clear distinction between objectives and policies

$$\text{objectives} \Rightarrow \text{adapted policies}$$

- More specifically, viability theory puts emphasis on consistency between dynamics and objectives

$$\text{objectives} + \text{dynamics} \Rightarrow \text{policies}$$