Information patterns and optimal stochastic control problems

Michel DE LARA

CERMICS, École nationale des ponts et chaussées

and *Stochatic Optimization Working Group*

P. Carpentier, J.-P. Chancelier, G. Cohen, A. Dallagi, C. Strugarek, etc.

A linear control-oriented stochastic model

Controls: $u_1 \in \mathbb{U}_1 = \mathbb{R}$ and $u_2 \in \mathbb{U}_2 = \mathbb{R}$

Random issue: $\omega = (x_0, v) \in \Omega = \mathbb{R} \times \mathbb{R}$

State equations $$\begin{cases} x_1 = x_0 + u_0 \\ x_2 = x_1 - u_1 \,. \end{cases}$$

Output equations $$\begin{cases} y_0 = x_0 \\ y_1 = x_1 + v \,. \end{cases}$$

H. S. Witsenhausen. A counterexample in stochastic optimal control. *SIAM J. Control*, 6(1):131–147, 1968.

A LQG problem with linear solution

x_0 and v are Gaussian independent.

$$\inf \mathbb{E}(k^2 u_0^2 + x_2^2) \,,$$

$$\begin{cases} x_1 &= x_0 + u_0 \\ x_2 &= x_1 - u_1 \end{cases}$$

$$\begin{cases} u_0 & \text{measurable w.r.t. } \; y_0 = x_0 \\ u_1 & \text{measurable w.r.t. } \; \boxed{(y_0, y_1) = (x_0, x_1 + v)} \,. \end{cases}$$

Solution $u_0 = \psi_1(y_0)$, $\boxed{u_1 = \psi_2(y_0, y_1)}$, where ψ_1 and ψ_2 are affine functions.

Classical information pattern

Still LQG but... nonlinear solution!

x_0 and v are Gaussian independent.

$$\inf \mathbb{E}(k^2 u_0^2 + x_2^2)\,,$$

$$\begin{cases} x_1 &= x_0 + u_0 \\ x_2 &= x_1 - u_1 \end{cases}$$

$$\begin{cases} u_0 & \text{measurable w.r.t. } y_0 = x_0 \\ u_1 & \text{measurable w.r.t. } \boxed{y_1 = x_1 + v}\,. \end{cases}$$

Solution $u_0 = \psi_1(y_0)$, $\boxed{u_1 = \psi_2(y_1)}$ *is known to exist and to be (highly) nonlinear!*

The Witsenhausen counterexample

Classical information pattern

- Sequential and memory of past knowledge:
  1. agent 0 observes y_0;
  2. agent 1 observes y_0 and y_1.
- Stochastic dynamic programming, HJB…:
  value function parameterized by the state (usually finite dimensional);
  infimum taken over the controls.

Nonclassical information patterns

- In the sequential case, no memory of past knowledge: agent 1 observes y_1.
- Signaling/dual effect of control:
  direct cost minimization *versus*
  indirect effect on output available for control.
- Interaction between information and control.
- Stochastic dynamic programming due to sequentiality **but** value function parameterized by the distribution of the state (infinite dimensional); infimum taken over controls mappings (idem).

Witsenhausen H. S. A standard form for sequential stochastic control. *Mathematical Systems Theory*, **7**(1):5–11, (1973).

Plan

- TWO INTRODUCTORY REMARKS ON CONDITIONAL EXPECTATIONS
- ON VARIABLES AND RANDOM VARIABLES
- INFORMATION PATTERNS IN THE LINEAR-QUADRATIC GAUSSIAN EXAMPLE
- EQUIVALENT STOCHASTIC CONTROL PROBLEMS:
FROM DYNAMIC TO STATIC PROBLEMS
- DEFINING INFORMATION:
A BRIEF OVERVIEW

Functional dependence of $\mathbb{E}(X \mid Y)$ on the conditioning random variable Y

Let X and Y be two random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume X integrable.

$\mathbb{E}(X \mid Y)$ is $\sigma(Y)$-measurable, hence a function of Y:

$$\mathbb{E}(X \mid Y) = f(Y)\,, \quad \mathbb{P} - a.s.$$

However, f depends functionaly on Y:

$$\mathbb{E}(X \mid Y)(\omega) = f_Y(Y(\omega)) \text{ for } \mathbb{P} \text{ almost all } \omega$$

If X and Y are discrete random variables

$$\begin{aligned} &= \mathbb{E}(X \mid Y = y) \\ &= \sum_{x'} x' \mathbb{P}(X = x' \mid Y = y) \\ &= \sum_{x'} x' \frac{\int_\Omega \mathbf{1}_{x'}(X(\omega)) \mathbf{1}_y(Y(\omega)) \mathbb{P}(d\omega)}{\int_\Omega \mathbf{1}_y(Y(\omega)) \mathbb{P}(d\omega)} \end{aligned}$$

depends on the mapping $Y : \Omega \to \mathbb{R}$.

Infimum over variables *versus* over mappings

$$\inf_{\psi:\mathbb{X}\to\mathbb{U}} \sum_{x\in\mathbb{X}} \sum_{y\in\mathbb{Y}} J(\psi(x), x, y) = \sum_{x\in\mathbb{X}} \left(\inf_{u\in\mathbb{U}} \sum_{y\in\mathbb{Y}} J(u, x, y) \right)$$

is the elementary version of measurable selection theorems of the type

$$\inf_{\psi\preceq Z} \mathbb{E}[J(\cdot, \psi(\cdot))] = \mathbb{E}[\inf_{u\in\mathbb{U}} \mathbb{E}[J(\cdot, u) \mid Z]]$$

where $\boxed{\psi \preceq Z}$ means that the mapping $\psi : \Omega \to \mathbb{U}$ is *measurable* with respect to the random variable Z.

ON VARIABLES
AND
RANDOM VARIABLES

Primitive random variables

State and output equations: relations between simple variables

$$\begin{cases} x_1 &= x_0 + u_0 \\ x_2 &= x_1 - u_1 \\ y_0 &= x_0 \\ y_1 &= x_1 + v \, . \end{cases}$$

We endow $\mathbb{X}_0 = \mathbb{R}$ and $\mathbb{V} = \mathbb{R}$ with two probabilities with finite second moments. The coordinate mappings X_0 and V become independent random variables, following the tradition to label random variables by capital letters (x_0 and v are simple variables).

Policies, control designs, information patterns

At this stage, only X_0 and V are random variables.

u_0, u_1, x_1, x_2, y_0, y_1 are “variables” belonging to the spaces $\mathbb{U}_0 = \mathbb{U}_1 = \mathbb{X}_1 = \mathbb{X}_2 = \mathbb{Y}_0 = \mathbb{Y}_1 = \mathbb{R}$.

These (simple) variables have to be turned into *random* variables U_0, U_1, X_1, X_2, Y_0, Y_1 (capital letters) to give a meaning to $\mathbb{E}(k^2 U_0^2 + X_2^2)$.

There are two ways to do this, that we label by *policies* or by *control designs*.

Policies

A *policy* is a random variable $(U_0, U_1) : \Omega \to \mathbb{U}_0 \times \mathbb{U}_1$.

We write thus equalities between random variables as:

$$\begin{cases} X_1 &= X_0 + U_0 \\ X_2 &= X_1 - U_1 \\ Y_0 &= X_0 \\ Y_1 &= X_1 + V \,. \end{cases}$$

There remains to express the *information pattern*, namely how U_0 and U_1 may depend upon the other random variables, especially the observations Y_0 and Y_1, and possibly upon themselves.

Information patterns and policies

Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be two subfields of $\mathcal{F}$.

The following optimization problem is well defined:

$$\inf_{U_0 \preceq \mathcal{G}_0, U_1 \preceq \mathcal{G}_1} \mathbb{E}(k^2 U_0^2 + (X_1 - U_1)^2)$$

In doing so, we restrict the class of policies to those such that $U_0 \preceq \mathcal{G}_0, U_1 \preceq \mathcal{G}_1$.

For instance, the Witsenhausen counterexample is

$$\inf_{U_0 \preceq Y_0, U_1 \preceq Y_1} \mathbb{E}(k^2 U_0^2 + (X_1 - U_1)^2) \, .$$

It means that the first decision u_0 may only depend upon Y_0, and u_1 upon Y_1.

σ-fields *versus* signals

While policies are related to σ-fields, control designs make use of *signals*. A signal is a measurable mapping $Y : \Omega \rightarrow \mathbb{Y}$, where $\mathbb{Y}$ is a measurable space: in all generality, it is a random element, and usually a random variable.

To any signal is associated the subfield $\sigma(Y)$ generated by Y.

Not all σ-fields may be obtained as $\sigma(Y)$, where Y takes value in a Borel space. Counterexample is given by the σ-field consisting of countable subsets and their complements.

Control design

A *control design* is a measurable mapping
$(\psi_1, \psi_2) : \mathbb{X}_0 \times \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{U}_0 \times \mathbb{U}_1 \times \mathbb{Y}_0 \times \mathbb{Y}_1 \to \mathbb{U}_0 \times \mathbb{U}_1$.

We write thus relationships between random variables

$$\left\{ \begin{array}{rcl} U_0 & = & \psi_1(X_0, X_1, X_2, U_0, U_1, Y_0, Y_1) \\ U_1 & = & \psi_2(X_0, X_1, X_2, U_0, U_1, Y_0, Y_1) \\ X_1 & = & X_0 + U_0 \\ X_2 & = & X_1 - U_1 \\ Y_0 & = & X_0 \\ Y_1 & = & X_1 + V \; . \end{array} \right.$$

Information patterns and control designs

Notice that the first two equations are implicit ones and may not admit solutions: *intricacies* of stochastic control…(step by step solution in case of causality).

These equations $U_0 = \psi_1(X_0, \ldots)\ldots$ express the information pattern, namely how U_0 and U_1 depend *functionaly* upon the other random variables.

For instance, the Witsenhausen counterexample is

$$\inf_{\psi_1 \preceq y_0, \psi_2 \preceq y_1} \mathbb{E}(k^2 U_0^2 + (X_1 - U_1)^2)$$

It means that ψ_1 depends only upon y_0, and ψ_2 depends only upon y_1.

INFORMATION PATTERNS
IN THE
LINEAR-QUADRATIC GAUSSIAN EXAMPLE

Information patterns

Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be two subfields of $\mathcal{F}$, where

$$\mathcal{G}_0 = \sigma(Y_0) \quad \text{and} \quad \mathcal{G}_1 = \begin{cases} & \sigma(Y_0, U_0, Y_1) \\ \text{or} & \sigma(Y_0, Y_1) \\ \text{or} & \sigma(U_0, Y_1) \\ \text{or} & \sigma(Y_1) \end{cases}$$

Recall that

$$\begin{cases} Y_0 &= X_0 \\ X_1 &= X_0 + U_0 \\ Y_1 &= X_1 + V = X_0 + U_0 + V\,. \end{cases}$$

Exploiting sequentiality

$$\inf_{U_0 \preceq \mathcal{G}_0, U_1 \preceq \mathcal{G}_1} \mathbb{E}[k^2 U_0^2 + (X_1 - U_1)^2]$$

$$= \inf_{U_0 \preceq \mathcal{G}_0} \left(k^2 \mathbb{E}[U_0^2] + \inf_{U_1 \preceq \mathcal{G}_1} \mathbb{E}[(X_1 - U_1)^2] \right)$$

by sequentiality

$$= \inf_{U_0 \preceq \mathcal{G}_0} \left(k^2 \mathbb{E}[U_0^2] + \mathbb{E}[\big(X_1 - \mathbb{E}(X_1 \mid \mathcal{G}_1)\big)^2] \right)$$

by definition of $\mathbb{E}(X_1 \mid \mathcal{G}_1)$

$$= \inf_{U_0 \preceq \mathcal{G}_0} \left(k^2 \mathbb{E}[U_0^2] + \mathbb{E}[\mathsf{var}[X_1 \mid \mathcal{G}_1]] \right) .$$

The dual effect of U_0

$$\inf_{U_0 \preceq \mathcal{G}_0, U_1 \preceq \mathcal{G}_1} \mathbb{E}[k^2 U_0^2 + (X_1 - U_1)^2]$$

$$= \inf_{U_0 \preceq \mathcal{G}_0} \mathbb{E}[k^2 \underbrace{U_0^2}_{\text{pointwise dependence on } U_0} + \mathsf{var}[V \underbrace{\mid \mathcal{G}_1]}_{\text{functional dependence on } U_0}] \qquad (1)$$

$$= \inf_{U_0 \preceq \mathcal{G}_0} \mathbb{E}[k^2 U_0^2 + (V - \mathbb{E}(V \mid \mathcal{G}_1))^2] \,.$$

where

$$\mathcal{G}_1 = \sigma(Y_0, U_0, Y_1)\,, \quad \sigma(Y_0, Y_1)\,, \quad \sigma(U_0, Y_1)\,, \quad \sigma(Y_1)\,.$$

We used a trick...

$$
\begin{aligned}
-(X_1 - \mathbb{E}(X_1 \mid \mathcal{G}_1)) &= Y_1 - X_1 - \mathbb{E}(Y_1 - X_1 \mid \mathcal{G}_1) \\
&\quad \text{since } Y_1 \preceq \mathcal{G}_1 \\
&= V - \mathbb{E}(V \mid \mathcal{G}_1)
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbb{E}[(X_1 - \mathbb{E}(X_1 \mid \mathcal{G}_1))^2] &= \mathbb{E}[(V - \mathbb{E}(V \mid \mathcal{G}_1))^2] \\
&= \mathbb{E}[V^2] - (\mathbb{E}(V \mid \mathcal{G}_1))^2 .
\end{aligned}
$$

Strictly classical pattern: perfect recall plus control transmission

$$\mathcal{G}_0 = \sigma(Y_0) \quad \text{and} \quad \mathcal{G}_1 = \sigma(Y_0, U_0, Y_1)\,.$$

We have

$$\mathcal{G}_1 = \sigma(X_0, U_0, X_0 + U_0 + V) = \sigma(X_0, V)$$

Notice that $\mathcal{G}_1$, which seemed to depend upon policies U_0 and U_1 (or upon control designs ψ_1 and ψ_2), is in fact policy independent: absence of *dual effect*.

K. Barty, P. Carpentier, J-P. Chancelier, G. Cohen, M. De Lara, and T. Guilbaud. Dual effect free stochastic controls. *To be published in Annals of Operation Research*, 2005.

Thus,

$$\mathsf{var}[V \mid X_0, V] = \mathbb{E}[V^2 \mid X_0, V] - \mathbb{E}[V \mid X_0, V]^2$$
$$= V^2 - V^2 = 0$$

and (**??**) is

$$\inf_{U_0 \preceq \mathcal{G}_0, U_1 \preceq \mathcal{G}_1} \mathbb{E}[k^2 U_0^2 + (X_1 - U_1)^2]$$
$$= \inf_{U_0 \preceq \mathcal{G}_0} \mathbb{E}[k^2 U_0^2 + \mathsf{var}[V \mid X_0, V]] = k^2 \inf_{U_0 \preceq X_0} \mathbb{E}[U_0^2] \,.$$

The solution is $U_0 = 0$ and

$$U_1 = \mathbb{E}(X_1 \mid X_0, V) = X_1 = X_0 + U_0 = X_0 = Y_0 \,.$$

Classical pattern: perfect recall

$$\mathcal{G}_0 = \sigma(Y_0) \quad \text{and} \quad \mathcal{G}_1 = \sigma(Y_0, Y_1) = \sigma(X_0, U_0 + V)\,.$$

Since $U_0 \preceq \sigma(X_0)$, there exists a measurable ψ_1 such that $U_0 = \psi_1(X_0)$. Thus, $V = U_0 + V - \psi_1(X_0)$ is $\sigma(X_0, U_0 + V)$-measurable and we can write

$$\mathcal{G}_1 = \sigma(Y_0, Y_1) = \sigma(X_0, U_0 + V) = \sigma(X_0, V)\,.$$

This absence of dual effect is related to the linear structure of the problem.
Thus (**??**) has the same solution $U_0 = 0$ and $U_1 = Y_0$.

Policy independence of conditional expectation

H. S. Witsenhausen. On policy independence of conditional expectations. *Information and Control*, 28(1):65–75, 1975.

"If an observer of a stochastic control system observes both the decision taken by an agent in the system and the data that was available for this decision, then the conclusions that the observer can draw do not depend on the functional relation (policy, control law) used by this agent to reach his decision. "

A weaker form of absence of dual effect is when the conditional expectation $\mathbb{E}[V \mid \mathcal{G}_1]$ can be expressed without functional dependence upon U_0.

Nonclassical pattern: control recall

In this nonclassical pattern

$$\mathcal{G}_0 = \sigma(Y_0) \quad \text{and} \quad \mathcal{G}_1 = \sigma(U_0, Y_1)$$

$$\mathcal{G}_1 = \sigma(U_0, Y_1) = \sigma(U_0, X_0+U_0+V) = \sigma(U_0, X_0+V)$$

and thus

$$\mathbb{E}[V \mid \mathcal{G}_1] = \mathbb{E}[V \mid U_0, X_0 + V] \, .$$

The control variable enters a conditioning term: this is an example of the so called *dual effect* where the decision has an impact on future decisions by providing more or less information, in addition to contributing to cost minimization.

ε-optimal policies

Bismut, J., An example of interaction between information and control: the transparency of a game *Automatic Control, IEEE Transactions on* Volume 18, Issue 5, Oct. 1973, 518 - 522

For $\varepsilon > 0$, take

$$U_0 = \varepsilon Y_0 = \varepsilon X_0 \quad \text{and} \quad U_1 = \frac{U_0}{\varepsilon} = X_0$$

which yields the cost

$$k^2 U_0^2 + (X_0 + U_0 - U_1)^2 = \varepsilon^2 (k^2 + 1) X_0^2$$

Nonclassical pattern: no recall

$$\mathcal{G}_0 = \sigma(Y_0) \quad \text{and} \quad \mathcal{G}_1 = \sigma(Y_1) = \sigma(X_0 + U_0 + V)$$

and $\mathbb{E}[V \mid \mathcal{G}_1] = \mathbb{E}[V \mid X_0 + U_0 + V]$.

Notice that, since $U_0 \preceq X_0$, we may take $X_1 = X_0 + U_0$ instead of U_0 as “control variable”:

$$\inf_{U_0 \preceq \mathcal{G}_0, U_1 \preceq \mathcal{G}_1} \mathbb{E}[k^2 U_0^2 + (X_1 - U_1)^2] =$$

$$\inf_{X_1 \preceq X_0} \mathbb{E}[k^2 (X_1 - X_0)^2 + \mathsf{var}[V \mid X_1 + V]] .$$

Signalling

$$\inf_{X_1 \preceq X_0} \mathbb{E}[k^2(X_1 - X_0)^2 + \mathsf{var}[V \mid X_1 + V]]$$

Setting $X_1 = 0$ kills the $\mathsf{var}[V \mid X_1 + V]$ term, but has a cost $k^2 X_0^2$.

EQUIVALENT STOCHASTIC CONTROL PROBLEMS: FROM DYNAMIC TO STATIC PROBLEMS

H. S. Witsenhausen. Equivalent stochastic control problems. *Mathematics of Control, Signals, and Systems*, 1(1):3–7, 1988.

Criterion and observations

We write the criterion $k^2u_0^2 + (x_1 - u_1)^2$ as $J : \mathbb{Y}_0 \times \mathbb{U}_0 \times \mathbb{U}_1 \to \mathbb{R}$:

$$(y_0, u_0, u_1) \mapsto k^2u_0^2 + (y_0 + u_0 - u_1)^2 \, .$$

The observations under open-loop control $u \in \mathbb{U}_0 \times \mathbb{U}_1$ are given by

$$\Phi^u : \mathbb{X}_0 \times \mathbb{V} \to \mathbb{Y}_0 \times \mathbb{Y}_1 \, , \quad (x_0, v) \mapsto (x_0, x_0 + u_0 + v) \, .$$

Probabilities

$\mathbb{P}$ denotes the standard Gaussian probability on the probability space $\Omega = \mathbb{X}_0 \times \mathbb{V}$, that is

$$\mathbb{P}(dx_0 dv) = \frac{1}{2\pi} \exp(-\frac{x_0^2 + v^2}{2}) dx_0 dv \, .$$

$\mathbb{Q}_0(dy_0)$ and $\mathbb{Q}_1(dy_1)$ are standard Gaussian probabilities on $\mathbb{Y}_0$ and $\mathbb{Y}_1$ and

$$\begin{aligned} \mathbb{Q}(dy_0 dy_1) &= \frac{1}{2\pi} \exp(-\frac{y_0^2 + y_1^2}{2}) dy_0 dy_1 \\ &= \mathbb{Q}_0(dy_0) \otimes \mathbb{Q}_1(dy_1) \, . \end{aligned}$$

Control design

Let $\gamma_0 : \mathbb{Y}_0 \to \mathbb{U}_0$ and $\gamma_1 : \mathbb{Y}_0 \times \mathbb{Y}_1 \to \mathbb{U}_1$ be two measurable mappings, and denote $\gamma \stackrel{\text{def}}{=} (\gamma_0, \gamma_1)$.

With X_0 and V the coordinate mappings on $\mathbb{X}_0 \times \mathbb{V}$, we define the policies and the closed-loop observations by

$$
\begin{aligned}
Y_0 &= X_0 \\
U_0 &= \gamma_0(Y_0) \\
Y_1 &= X_0 + U_0 + V = X_0 + \gamma_0(Y_0) + V \\
U_1 &= \gamma_1(Y_0, Y_1) = \gamma_1(Y_0, X_0 + \gamma_0(Y_0) + V) .
\end{aligned}
$$

Thus defined, Y_0 and Y_1 are random variables which depend upon γ.

Cost criterion

The random cost is $J(Y_0, U_0, U_1)$, the expectation of which we try and minimize over all designs γ_0 and γ_1.

Noticing that

$$J(Y_0, U_0, U_1) = J(Y_0, \gamma_0(Y_0), \gamma_1(Y_0, Y_1)) ,$$

we define

$$J^\gamma : \mathbb{Y}_0 \times \mathbb{Y}_1 \to \mathbb{R} , \quad (y_0, y_1) \mapsto J(y_0, \gamma_0(y_0), \gamma_1(y_0, y_1)) .$$

Radon-Nikodym density

For all $u \in \mathbb{U}_0 \times \mathbb{U}_1$, we define $T^u : \mathbb{Y}_0 \times \mathbb{Y}_1 \longrightarrow \mathbb{R}_+$

$$(y_0, y_1) \mapsto \exp(-\frac{-y_1^2 + (y_1 - y_0 - u_0)^2}{2})$$

and $T^\gamma(y_0, y_1) = T^{\gamma_0(y_0)}(y_0, y_1)$.

We can show that the distribution $Y_\star(\mathbb{P})$ of the observations has density T^γ with respect to $\mathbb{Q}$:

$$\begin{aligned} Y_\star(\mathbb{P})(dy_0 dy_1) &= T^{\gamma_0(y_0)}(y_0, y_1) \exp(-\frac{y_0^2 + y_1^2}{2}) \frac{dy_0 dy_1}{2\pi} \\ &= T^\gamma(y_0, y_1) \mathbb{Q}(dy_0 dy_1) . \end{aligned}$$

Reduction to an equivalent static problem

$$\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\left(J(Y_0, U_0, U_1)\right) \\
= \; & \mathbb{E}_{\mathbb{P}}\left(J^{\gamma}(Y_0, Y_1)\right) \text{ by definition of } J^{\gamma} \\
= \; & \mathbb{E}_{Y_\star(\mathbb{P})}\left(J^{\gamma}\right) \text{ by definition of probability image} \\
= \; & \mathbb{E}_{\mathbb{Q}}\left(T^{\gamma} J^{\gamma}\right) \text{ by } Y_\star(\mathbb{P}) = T^{\gamma}\mathbb{Q} \\
= \; & \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \underbrace{J(y_0, \gamma_0(y_0), \gamma_1(y_0, y_1)) T^{\gamma_0(y_0)}(y_0, y_1)}_{new \; cost} \\
& \quad \mathbb{Q}_0(dy_0)\mathbb{Q}_1(dy_1) \, .
\end{aligned}$$

A static problem

Introducing a new cost

$$\widetilde{J}(u_0, u_1, y_0, y_1) \stackrel{\text{def}}{=} J(y_0, u_0, u_1) T^{u_0}(y_0, y_1)$$

the original optimization problem becomes now a *static problem*:

$$\inf_{\gamma_0, \gamma_1} \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \widetilde{J}(\gamma_0(y_0), \gamma_1(y_0, y_1), y_0, y_1) \mathbb{Q}_0(dy_0) \mathbb{Q}_1(dy_1) \, .$$

Indeed, the observations are just noise y_0 and y_1, not dynamical quantities affected by the controls.

Information structure

Let $g : \mathbb{Y}_0 \times \mathbb{Y}_1 \longrightarrow \mathbb{G}$ be a measurable mapping:

- $\mathbb{G} = \mathbb{Y}_0 \times \mathbb{Y}_1$ and $g(y_0, y_1) = (y_0, y_1)$;
- $\mathbb{G} = \mathbb{Y}_1$ and $g(y_0, y_1) = y_1$.

Let the information structure be captured by

$$u_0 = \phi_0(y_0) \quad \text{and} \quad u_1 = \phi_1(g(y_0, y_1))$$

where $\phi_0 : \mathbb{Y}_0 \longrightarrow \mathbb{U}_0$, $\phi_1 : \mathbb{G} \longrightarrow \mathbb{U}_1$.

$$\gamma_0 = \phi_0 \quad \text{and} \quad \gamma_1 = \phi_1 \circ g \,.$$

Exploiting sequentiality

$$\inf_{\substack{\phi_0 : \mathbb{Y}_0 \to \mathbb{U}_0, \\ \phi_1 : \mathbb{G} \to \mathbb{U}_1}} \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \widetilde{J}(\phi_0(y_0), \phi_1(g(y_0, y_1)), y_0, y_1) \mathbb{Q}(dy$$

$$= \inf_{\phi_0 : \mathbb{Y}_0 \to \mathbb{U}_0} \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \mathbb{Q}(dy_0 dy_1)$$

$$\inf_{u_1 \in \mathbb{U}_1} \mathbb{E}_{\mathbb{Q}} \left[\widetilde{J}(\phi_0(y_0), u_1, y_0, y_1) \mid g(y_0, y_1) \right]$$

Value function, state

$$
\begin{aligned}
& \inf_{u_1 \in \mathbb{U}_1} \mathbb{E}_{\mathbb{Q}} \left[\widetilde{J}(\phi_0(y_0), u_1, y_0, y_1) \mid g(y_0, y_1) \right] \\
= & \inf_{u_1 \in \mathbb{U}_1} \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \widetilde{J}(\phi_0(y_0'), u_1, y_0', y_1') \mathbb{Q}^g(dy_0' dy_1', g(y_0, y_1)) \\
= & W_1(\phi_0(\cdot), \mathbb{Q}^g(\cdot, g(y_0, y_1))
\end{aligned}
$$

where the value function $W_1(\phi_0(\cdot), \rho)$ is defined over

- $\phi_0 : \mathbb{Y}_0 \rightarrow \mathbb{U}_0$ and
- ρ distribution over $\mathbb{Y}_0 \times \mathbb{Y}_1$.

Policy independence of conditional expectations holds true when

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[\widetilde{J}(\phi_0(y_0), u_1, y_0, y_1) \mid g(y_0, y_1)\right] \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[\widetilde{J}(u_0, u_1, y_0, y_1) \mid g(y_0, y_1)\right]_{u_0=\phi_0(y_0)} \\
= \; & \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \widetilde{J}(\phi_0(y_0), u_1, y_0', y_1') \mathbb{Q}^g(dy_0' dy_1', g(y_0, y_1))
\end{aligned}
$$

so that the value function can be defined as

$$
W_1(u_0, \rho) = \inf_{u_1 \in \mathbb{U}_1} \int_{\mathbb{Y}_0 \times \mathbb{Y}_1} \widetilde{J}(u_0, u_1, y_0', y_1') \rho(dy_0' dy_1')
$$

over $u_0 \in \mathbb{U}_0$ and ρ distribution over $\mathbb{Y}_0 \times \mathbb{Y}_1$.

Classical pattern

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}\left[\widetilde{J}(\phi_0(y_0), u_1, y_0, y_1) \mid y_0, y_1\right] \\
= \; & \widetilde{J}(\phi_0(y_0), u_1, y_0, y_1) \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[\widetilde{J}(u_0, u_1, y_0, y_1) \mid y_0, y_1\right]_{u_0=\phi_0(y_0)}
\end{aligned}
$$

Nonclassical pattern

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\widetilde{J}(\phi_0(y_0), u_1, y_0, y_1) \mid y_1\right] \\
= &\int_{\mathbb{Y}_0} \widetilde{J}(\phi_0(y_0'), u_1, y_0', y_1)\mathbb{Q}_0(dy_0') \\
\neq &\ \mathbb{E}_{\mathbb{Q}}\left[\widetilde{J}(u_0, u_1, y_0, y_1) \mid y_1\right]_{u_0=\phi_0(y_0)} \\
= &\int_{\mathbb{Y}_0} \widetilde{J}(\phi_0(y_0), u_1, y_0', y_1)\mathbb{Q}_0(dy_0')
\end{aligned}$$

Comparison

Compared to the classical patter, there still is a backward stochastic dynamic programming principle but on a huger set consisting of mappings, and no longer on variables.

DEFINING INFORMATION:
A BRIEF OVERVIEW

Three options for a structure on information

Information is a collection $\mathcal{G}$ of subsets of Ω ($\mathcal{G} \subset 2^{\Omega}$) called *events*. The more events, the more information.

- $\mathcal{G}$ is an *algebra*: non empty, stable under complementation and finite union.
- $\mathcal{G}$ is a *partition field* (or π-field): non empty, stable under complementation and unlimited union.
- $\mathcal{G}$ is a σ*-algebra* (or σ-field, or even a field).

In any case, $\mathcal{G}$ is a lattice:

1. $\wedge$ is the intersection of fields;
2. $\vee$ is the smallest field generated by the union.

Alternative definitions

Information	equivalence relation on Ω	partition of Ω	function with domain Ω	partition fi eld on Ω
Notation	$\mathfrak{R}$	$(\Omega_i)_{i\in I}$	$X:\Omega\to\mathbb{X}$	$\mathcal{X}$
Example	$\Omega\times\Omega$ universal	(Ω) trivial	$X:\Omega\to\{0\}$ constant	$\{\emptyset,\Omega\}$ trivial
Example	$\{(\omega,\omega)\mid\omega\in\Omega\}$ equality	$(\{\omega\})_{\omega\in\Omega}$ discrete	$X = \mathrm{Id}_\Omega$ identity	2^Ω discrete
Correspond.	classes $\mathfrak{R}\omega\subset\Omega,\omega\in\Omega$	atoms $\{\Omega_i, i\in I\}$	singletons pre-images $X^{-1}(x), x\in\mathbb{X}$	atoms $B\in\mathcal{X}$ $\mathcal{X}\cap B=\{\emptyset,B\}$
	indistinguishable elements $\omega'\mathfrak{R}\omega$	indistinguishable elements $\exists i\in I, \{\omega,\omega'\}\subset\Omega_i$	same image $X(\omega)=X(\omega')$	belong to same atom

Operations on information

Information	equivalence relation on Ω	partition of Ω	function with domain Ω	partition field on Ω
Notation	$\mathfrak{R}$	$(\Omega_i)_{i \in I}$	$X : \Omega \to \mathbb{X}$	$\mathcal{X}$
Compare	$\mathfrak{R}_X \subset \mathfrak{R}_Y$ classes inclusion	$\Omega_i^X \cap \Omega_j^Y \in \{\emptyset, \Omega_i^X\}$	$\exists \varphi : \mathbb{Y} \to \mathbb{X}$ $X = \varphi(Y)$	$\mathcal{X} \subset \mathcal{Y}$
Or	$\mathfrak{R}_X \cap \mathfrak{R}_Y$ intersection of classes	$\Omega_i^X \cap \Omega_j^Y$ intersection of classes	(X, Y) $\Omega \to \mathbb{X} \times \mathbb{Y}$	$\mathcal{X} \vee \mathcal{Y}$ generated by $\mathcal{X} \cup \mathcal{Y}$
And	$(\mathfrak{R}_X \cup \mathfrak{R}_Y)^\infty$ closure			$\mathcal{X} \wedge \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}$ intersection

Problem: the lattice is *not* distributive

Equipped with the operators $\wedge$ and $\vee$, the set of all partition fields is a lattice. However, it is not distributive.

Indeed, the following inclusions

$$\begin{cases} \mathcal{G} \wedge (\mathcal{G}' \vee \mathcal{G}'') & \supset & (\mathcal{G} \wedge \mathcal{G}') \vee (\mathcal{G} \wedge \mathcal{G}'') \\ \\ \mathcal{G} \vee (\mathcal{G}' \wedge \mathcal{G}'') & \subset & (\mathcal{G} \vee \mathcal{G}') \wedge (\mathcal{G} \vee \mathcal{G}'') \end{cases}$$

hold true, but not the equalities in general.