Optimization under risk constraints: an economic interpretation as maxmin over a family of utility functions

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Motivations

Liberalization of energy market:

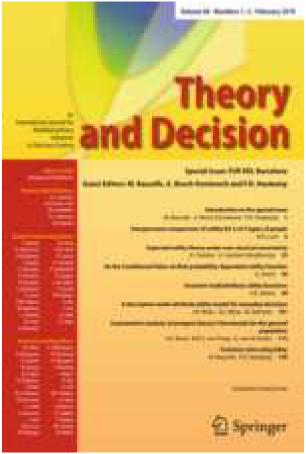
- new set of problems for electrical companies
- instrument for hedging
  - gap between production and demand
  - financial risk

Our objective:
integrate risk constraints in the historical problem which consists in managing the electrical generation at lowest expected cost.

Main result

Equivalence between
an expected profit maximization problem
under risk constraint
and a maxmin expected utility problem "à la Maccheroni".
$\Downarrow$
In the case of CVaR constraint,
interpretation of the corresponding
piecewise linear utility functions class
with the loss aversion notion "à la Kahneman et Tversky".

Babacar Seck, Laetitia Andrieu, Michel De Lara
Parametric multi-attribute utility functions for optimal profit under risk constraints. In Theory and Decision, 2011.



Outline of the presentation

1. Optimization under risk: two formulations
2. Conditional Value-at-Risk and loss aversion
3. Application to electrical portfolio management
4. Conclusion

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   - Two equivalents formulations
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   - The mathematical formulation
   - Numerical results
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We consider the maximisation of expected profit $\mathbb{E}\big[\texttt{Profit}(a,\xi)\big]$ subject to risk constraint

$$\begin{cases} \max_{a\in\mathbb{A}} \mathbb{E}\big[\texttt{Profit}(a,\xi)\big] \\ \text{s.t. } \texttt{Risk}\big(-\texttt{Profit}(a,\xi)\big) \leq \gamma, \end{cases}$$

where

- $\mathbb{A} \subset \mathbb{R}^n$ is a set of actions;
- ξ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $\texttt{Profit}$ is a mapping, such that for any action $a \in \mathbb{A}$, the random variable $\texttt{Profit}(a,\xi)$ represents the profit of the decision maker (DM);
- $\texttt{Risk}$ is the risk measure on the loss $-\texttt{Profit}(a,\xi)$, together with the level constraint $\gamma \in \mathbb{R}$.

Examples of risk constraints formulations

The risk constraint can be expressed by

- a mathematical expectation:

$$\mathbb{E}[L(a,\xi)] \leq \gamma$$

- a Value-at-Risk:

$$\mathtt{VaR}_p(L(a,\xi)) \leq \gamma$$

- a Conditional Value-at-Risk:

$$\mathtt{CVaR}_p(L(a,\xi)) \leq \gamma$$

Pros and cons

$$\begin{cases} \max_{a \in \mathbb{A}} \mathbb{E}\big[\texttt{Profit}(a,\xi)\big] \\ \text{subject to risk constraint}\,. \end{cases}$$

- Plus: explicit formulation of risk.
- Minus:
  - hard to solve or to formulate for some problems
  - no theoretical foundation

Expected utility theory

We consider the maximisation of the expected utility of the profit $\mathtt{Profit}(a, \xi)$

$$\max_{a \in \mathbb{A}} \mathbb{E}\Big[\mathtt{Util}\big(\mathtt{Profit}(a, \xi)\big)\Big],$$

where `Util` is a utility fonction.

The fonction `Util` captures more or less risk aversion of the DM.

Extensions: nonexpected utility theories.

Pros and cons

$$\max_{a \in \mathbb{A}} \mathbb{E}\Big[\mathtt{Util}(\mathtt{Profit}(a,\xi))\Big] .$$

- Plus: no additional constraint in the optimization problem
- Minus: how to choose a utility function `Util`?

The infinimum of expectations class of risk measures

We shall consider the risk measures which can be expressed as

$$\mathtt{Risk}_\rho(L) := \min_{\eta \in \mathbb{R}} \mathbb{E}\big[\rho\big(L, \eta\big)\big]$$

Risk measure $\mathtt{Risk}_\rho$	$\rho(x, \eta)$
Variance	$(x - \eta)^2$
Conditional Value-at-Risk	$\eta + \frac{1}{1-p}(x - \eta)_+$
Weighted Mean Deviation	$\max\big\{p(x-\eta), (1-p)(\eta - x)\big\}$
Optimized Certainty Equivalent (Ben-Tal and Teboulle)	$\eta - \mathtt{Util}(\eta - x)$

Two equivalents formulations

Under technical hypotheses,
the maximisation problem subject to risk constraint

$$\max_{a \in \mathbb{A}} \mathbb{E}\big[\texttt{Profit}(a,\xi)\big] \text{ s.t. } \texttt{Risk}_{\rho}\big(-\texttt{Profit}(a,\xi)\big) \leq \gamma$$

is equivalent to the maxmin expected utility problem
("à la Maccheroni")

$$\max_{(a,\eta) \in \mathbb{A} \times \mathbb{R}} \min_{\texttt{Util} \in \mathcal{U}} \mathbb{E}\left[\texttt{Util}\big(\texttt{Profit}(a,\xi),\eta\big)\right]$$

where the set of utility functions $\mathcal{U}$ is defined by:

$$\mathcal{U} := \Big\{\texttt{Util}^{(\lambda)} : \mathbb{R}^2 \to \mathbb{R}\,,\ \lambda \geq 0 \mid \texttt{Util}^{(\lambda)}(x,\eta) = x + \lambda\big(-\rho(-x,\eta)+\gamma\big)\Big\}\,.$$

Risk measure $\mathtt{Risk}_\rho$	$\mathtt{Util}^{(\lambda)}(x,\eta)$, $\lambda \geq 0$
$\rho(-x,\eta)$	$x - \lambda\rho(-x,\eta) + \lambda\gamma$
Variance	$x - \lambda(x+\eta)^2 + \lambda\gamma$
Conditional Value-at-Risk	$x - \frac{\lambda}{1-p}(-x-\eta)_+ - \lambda\eta + \lambda\gamma$
Weighted Mean Deviation	$x - \lambda \max\{ -p(x+\eta), (1-p)(\eta - x)\} + \lambda\gamma$
Optimized Certainty Equivalent	$x + \lambda\mathtt{Util}(\eta + x) + \lambda\eta + \lambda\gamma$

Table: Usual risk measures and their corresponding family of two-attributes utility functions.

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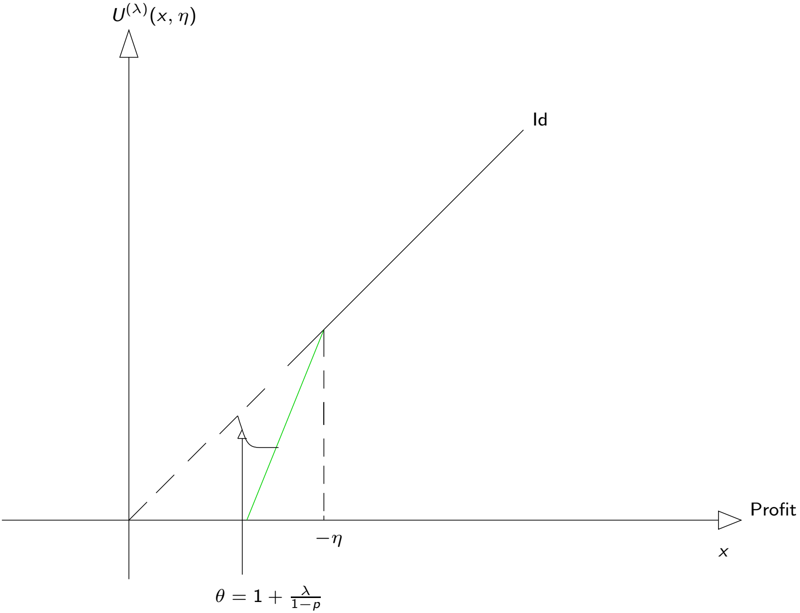
The Conditional Value-at-Risk case

When the risk measure $\texttt{Risk}_\rho$ is the $\texttt{CVaR}$, we have

$$\texttt{Util}^{(\lambda)}(x,\eta) = x - \frac{\lambda}{1-p}(-x-\eta)_+ - \lambda\eta + \lambda\gamma\,.$$

1. We consider only the argument x (profit) and we interpret η as a parameter.
2. Up to an additive constant, we obtain the utility function

$$x+\eta - \frac{\lambda}{1-p}(-x-\eta)_+ = \begin{cases} x+\eta & \text{if} \quad x+\eta \geq 0\,, \\ (1+\frac{\lambda}{1-p})(x+\eta) & \text{if} \quad x+\eta < 0\,. \end{cases}$$


$U^{(\lambda)}(x, \eta)$
Id
Profit
$-\eta$
x
$\theta = 1 + \frac{\lambda}{1-p}$

Loss aversion interpretation

We interpret the ratio of derivatives

$$\theta := 1 + \frac{\lambda}{1 - p}$$

as a loss aversion parameter, introduced by Kahneman et Tversky.

Loss aversion interpretation

The utility function

$$\texttt{Util}(x) = x + \eta + (1 - \theta)(-x - \eta)_+$$

is parameterized by

- anchorage $-\eta$,
- loss aversion θ,

and expresses the property that one monetary unit more than the anchorage gives one unit of utility, while one unit less gives $-\theta$.

Loss aversion framing

(Meyerowitz-Chaiken, 1987)

- You can gain several potential health benefits by spending only five minutes each month doing breast self-examination

$$\wedge$$

- You can lose several potential health benefits by failing to spend only five minutes each month doing breast self-examination
- Subjects who read a pamphlet with arguments framed in loss language manifested more positive breast self-examination attitudes, intentions, and behaviors ($57\% > 38\%$ at the 4-month follow-up)

A toy portfolio problem

- A portfolio with two assets

$$\texttt{Profit}(a,\xi) = a \underbrace{\xi^0}_{\text{sure}} + (1-a) \underbrace{\xi^1}_{\mathcal{N}(M,\Sigma)}$$

  - ξ^0=1 030 US$
  - M =1 113.3425 US$ and the standard deviation Σ=186.29 US$ (annual MSCI world developed market performance index between 1970 and 2009).

- Now, handling risk can be done by “bounding the gains below”. We choose the Conditional Value-at-Risk constraint

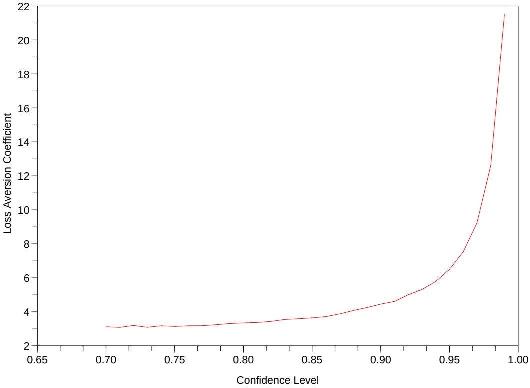
$$\texttt{CVaR}_p\big(-\texttt{Profit}(a,\xi)\big) \leq \gamma \; .$$

Loss aversion coefficient $\theta^\sharp$ as function of confidence level p

$$p \mapsto \theta^\sharp = 1 + \frac{\lambda^\sharp(p)}{1-p}$$

Notice that $\theta^\sharp$ does not depend on γ.

Loss aversion coefficient $\theta^\sharp$ as function of confidence level p



- We observe high values for loss aversion for $p \geq 0.9$, well above the empirical findings (median value of 2.25 in (Tversky-Kahneman, 1992)).
- Dealing with risk by controlling this portfolio Conditional Value-at-Risk at usual confidence levels $p = 0.95$ and $p = 0.99$ reveals high loss aversion.
- Controlling this portfolio Conditional Value-at-Risk at lower confidence levels between 0.7 and 0.8 reveals a quite acceptable loss aversion slightly higher than 3.

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Dynamical system

We consider the following dynamical system

$$X(t+1) = \mathtt{DYN}(X(t), a(t), W(t)) \,,\; t = t_0, \ldots, T-1 \,,$$

with

- $t = t_0, \ldots, T-1$ the time supposed to be discrete,
- $X(t) \in \mathbb{R}^n$ the state, with $X(t_0) = X_0$ given,
- $a(t) \in \mathbb{R}^m$ the decision variable,
- $W(t) \in \mathbb{R}^k$ the noise, with $W(t_0)$, …, $W(T-1)$ a sequence of i.i.d. random variables,
- DYN the dynamics (stocks variations, etc.).

Optimization problem

The optimization problem that we consider is

$$\max_{\texttt{Rule}(\cdot)} \mathbb{E}\Big[\sum_{t=t_0}^{T-1} \texttt{Profit}(X(t), a(t), W(t)) + K(X(T))\Big]$$

under dynamic constraints

$$\begin{aligned} X(t+1) &= \texttt{DYN}(X(t), a(t), W(t)) \\ a(t) &= \texttt{Rule}(t, X(t)) \quad (\texttt{Rule}\ \text{feedback}) \end{aligned}$$

and risk constraints

$$\texttt{Risk}_\rho\left[-\texttt{Profit}\big(X(t), a(t), W(t)\big)\right] \leq \gamma(t)\,,\ t = t_0, \ldots, T-1\,.$$

Two equivalent problems

Under technical hypotheses, the dynamic maximisation problem subject to risk constraint is equivalent to

$$\max_{(\mathtt{Rule}(\cdot),\eta(\cdot))} \min_{(\mathtt{Util}_{t_0},\ldots,\mathtt{Util}_{T-1})\in\mathcal{U}^{T-t_0}} \mathbb{E}\Big[\sum_{t=t_0}^{T-1}\mathtt{Util}_t\big(\mathtt{Profit}(X(t),a(t),W(t)),\eta(t)\big)+K(X(T))\Big]$$

where the set of utility functions $\mathcal{U}$ is defined by:

$$\mathcal{U} := \Big\{\mathtt{Util}^{(\lambda)}:\mathbb{R}^2\to\mathbb{R}\,,\ \lambda\geq 0 \mid \mathtt{Util}^{(\lambda)}(x,\eta)=x+\lambda\big(-\rho(-x,\eta)+\gamma\big)\Big\}\,.$$

Application

We aim at optimizing the electrical production when financial risk constraints are added.

Mathematically, the problem consists in minimizing the expected production cost under the following constraints

- energy balance
- dynamic on hydraulic generation
- admissible actions
- financial risk constraint

The electrical production

- Thermal generation: $c_i(t)$ is a unitary cost depending on the source (coal, fuel, gas, etc.), while $u_i(t)$ is a quantity

$$\text{cost of thermal generation } = \sum_{i=1}^{I} c_i(t) u_i(t) \,.$$

- Hydraulic generation takes into account
  - the random hydraulic inflows $A_j(t)$,
  - the release $w_j(t)$,

  with the following dynamic:

$$R_j(t+1) = R_j(t) + \Delta t \big(A_j(t) - w_j(t) \big) \,.$$

The energy market

- The forward market

$$\text{forward market cost } = \sum_{p \in \mathcal{P}} \big(t_e(p) - t_f(p)\big) F_p(t) q_p(t) \, ,$$

where $t_e(p)$ and $t_f(p)$ denote initial and final dates for delivering the future contract p.

- The spot market

$$\text{spot market cost } = v(t) S(t) \, .$$

Uncertainties affecting electrical production

- the hydraulic inflows $A(t)$,
- the demand $D(t)$,
- the prices $F_p(t)$ and $S(t)$,
- the breakdown $N_p(t)$ and $N_r(t)$ on the electrical production.

Production cost

At the end of the last period T, the cost is

$$L(T) = \sum_{t=t_0}^{T-1} \Delta L(t)$$

where we have denoted the instantaneous cost at time t by

$$\begin{aligned} \Delta L(t) &= \sum_{i=1}^{I} c_i(t)u_i(t) + v(t)S(t) \\ &+ \sum_{p \in \mathrm{P}(t)} \big(t_e(p) - t_f(p)\big)\big(q_p(t)F_p(t) + |q_p(t)|B(t)\big) \\ &+ \delta(t)d(t) \, . \end{aligned}$$

The optimization problem

The problem is

$$\min_{u(\cdot), v(\cdot), \mathbf{q}(\cdot), w(\cdot)} \mathbb{E}\Big[\sum_{t=t_0}^{T-1} \Delta L(t) \Big] \, ,$$

subject to

- financial risk constraints,

$$\mathtt{CVaR}_p\Big(\Delta L(t)\Big) \leq \gamma(t) \, , \quad \forall t = 0, \dots, T-1 \, ,$$

- dynamic on hydraulics and on forward contracts,
- bounds on the state and decision variables,
- non-anticipativity measurability constraints on decisions variables.

The reference problem: optimization without risk constraint

- Let us denote C^* the optimal cost without risk constraint.
- $\Delta\overline{L}(t)$ denotes the optimal instantaneous cost at time t.

Optimal cost with risk constraint

- We choose a constant constraint level

$$\gamma(t) = \gamma = \min_{s=0,\dots,T-1} \mathtt{CVaR}(\Delta\overline{L}(s)) \,.$$

so that the following constraint is binding:

$$\mathtt{CVaR}_p\Big(\Delta\overline{L}(t)\Big) \leq \gamma \;, \quad \forall t = 0, \dots, T-1 \;.$$

- Thus, the optimal cost with risk constraint is strictly higher than without risk constraint

optimal cost with risk constraint	$1.058C^*$

- Difficulties: the algorithm is delicate to tune and the computation is long; it makes use of dynamic programming.

Utility function approach

The risk is taken into account via a single utility function

$$\mathtt{Util}(x) = x + \eta + (1-\theta)(-x-\eta)_+ .$$

- We compute $\eta = \frac{1}{T}\sum_{t=0}^{T-1} \mathtt{VaR}(\Delta\overline{L}(t))$,
- and we fix the loss aversion $\theta = 4$,

we obtain

optimal cost with utility function	$0.93C^*$

Table: Optimal cost with utility function

Comparisons

method	cost $\mathbb{E}\left[\sum_{t=t_0}^{T-1} \Delta L(t)\right]$	satisfaction of constraints $\frac{\gamma - \max_{t=0,\ldots,T-1} \mathtt{CVaR}(\Delta \overline{L}(t))}{\gamma}$	loss aversion θ
without risk constraint	C^*	$\approx -125\%$	1
with risk constraint	$1.058C^*$	0	1 248
utility function	$0.93C^*$	$\approx -100\%$	4

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Conclusion

- Theoretical results
  - Economic interpretation of an "engineer" stochastic optimization problem under risk constraints
  - CVaR constraint associated to utility functions exhibiting loss aversion "à la Kahneman and Tversky"
- Application:
  taking risk into account in electrical portfolio management