Tutorial
Perturbation-Duality Scheme

Michel De Lara   Seta Rakotomandimby

June 24, 2024

Outline

Outline of the presentation

Duality for linear programs

Perturbation-duality scheme in convex analysis

Duality for pure integer linear programs

Generalized perturbation duality scheme

Developments and examples

Outline of the presentation

Illustration of the scheme in Linear Programming (LP)

- Constraint matrix $A \in \mathbb{R}^{m \times n}$
- Cost vector $k \in \mathbb{R}^n$
- Anchor $\bar{b} \in \mathbb{R}^m$

Initial minimization problem

$$\inf_{\substack{x \in \mathbb{R}^n \\ Ax = \bar{b} \\ x \geq 0}} \langle k \mid x \rangle$$

Step 1. Perturbation of the initial minimization problem

- Embedding the problem in a family of minimization problems

$$\forall b \in \mathbb{R}^m , \ \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle k \mid x \rangle$$

- Introducing a perturbation space: $\mathbb{R}^m$
- Perturbation function $\varphi : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \underbrace{\{-\infty\}}_{\text{unbounded}} \cup \underbrace{\{+\infty\}}_{\text{unfeasible}}$
- Value of the initial problem: $\varphi(\bar{b})$

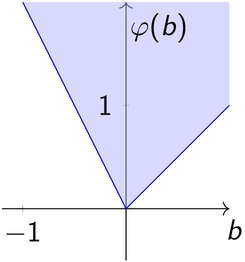
Examples of epigraphs of the perturbation functions for LP

Example

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\forall b \in \mathbb{R}\ , \quad \varphi(b) = \inf_{\substack{x \in \mathbb{R}^2 \\ x_1 - x_2 = b \\ x \geq 0}} \quad x_1 + 2x_2$$

Then $\varphi(b) = \max\{-2b, b\}\ , \ \ \forall b \in \mathbb{R}$



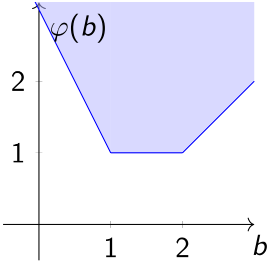
Examples of epigraphs of the perturbation functions for LP

Example

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(b) = \inf_{\substack{x \in \mathbb{R}^3 \\ x_1 + x_2 + 3x_3 = 1 \\ x_1 + 2x_2 + 4x_3 = b \\ x \geq 0}} x_1 + x_2 + x_3$$

Then $\varphi(b) = \max\{3 - 2b, 1, b - 1\}$, $\forall b \in \mathbb{R}$



Step 2. Dual space, coupling and conjugate function

- Perturbation function

$$\forall b \in \mathbb{R}^m , \ \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle k \mid x \rangle$$

- Introducing the bilinear coupling

$$\langle \cdot \mid \cdot \rangle : \overbrace{\mathbb{R}^m}^{\substack{\text{perturbation} \\ \text{space}}} \times \overbrace{\mathbb{R}^m}^{\text{dual space}} \to \mathbb{R}$$

- Conjugate function $\varphi^\star : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$\forall p \in \mathbb{R}^m , \ \varphi^\star(p) = \sup_{b \in \mathbb{R}^m} \left\{ \langle b \mid p \rangle - \varphi(b) \right\}$$

Conjugate function and Lagrangian

$$
\begin{aligned}
\varphi^\star(p) &= \sup_{b\in\mathbb{R}^m} \left\{ \langle b \mid p\rangle - \varphi(b) \right\} \\
&= \sup_{b\in\mathbb{R}^m} \left\{ \langle b \mid p\rangle - \inf_{\substack{Ax=b\\ x\geq 0}} \langle k \mid x\rangle \right\} \\
&= \sup_{b\in\mathbb{R}^m} \left\{ \langle b \mid p\rangle + \sup_{\substack{Ax=b\\ x\geq 0}} \langle -k \mid x\rangle \right\} \\
&= \sup_{x\geq 0} \left\{ \sup_{\substack{Ax=b\\ b\in\mathbb{R}^m}} \langle b \mid p\rangle + \langle -k \mid x\rangle \right\} \\
&= \sup_{x\geq 0} \left\{ \langle Ax \mid p\rangle - \langle k \mid x\rangle \right\} \\
&= \left\langle \bar{b} \mid p \right\rangle - \inf_{x\geq 0} \underbrace{\left\{ \left\langle \bar{b} - Ax \mid p \right\rangle + \langle k \mid x\rangle \right\}}_{\text{Lagrangian: } \mathcal{L}(x,p)}
\end{aligned}
$$

- $\inf_{x\geq 0} \mathcal{L}(x,p) = \left\langle \bar{b} \mid p \right\rangle - \varphi^\star(p)$

Step 3. Biconjugate and weak duality

- Biconjugate function $\varphi^{\star\star'} : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$\forall b \in \mathbb{R}^m \;, \;\; \varphi^{\star\star'}(b) = \sup_{p \in \mathbb{R}^m} \left\{ \langle b \mid \; p \rangle - \varphi^{\star}(p) \right\}$$

- We obtain weak duality for all $b \in \mathbb{R}^m$

$$\underbrace{\begin{array}{cc} \sup & \langle b \mid \; p \rangle \\ p \in \mathbb{R}^m & \\ p^T A \leq k & \end{array}}_{\text{dual problem}} = \varphi^{\star\star'}(b) \leq \varphi(b) = \begin{array}{cc} \inf & \langle k \mid \; x \rangle \\ x \in \mathbb{R}^n & \\ Ax = b & \\ x \geq 0 & \end{array}$$

- At the anchor $\bar{b}$

$$\varphi^{\star\star'}(\bar{b}) = \sup_{p \in \mathbb{R}^m} \{ \underbrace{\langle \bar{b} \mid \; p \rangle - \varphi^{\star}(p)}_{\inf_{x \geq 0} \mathcal{L}(x,p)} \}$$

Step 4. Conditions for strong duality

Proposition

Let $A \in \mathbb{R}^{m\times n}$ and $k \in \mathbb{R}^n$
If there exists $\bar{b} \in \mathbb{R}^m$ such that $\varphi(\bar{b}) \in \mathbb{R}$

(the corresponding LP is bounded and feasible)

then for all $b \in \mathbb{R}^m$

$$\left(\sup_{\substack{p \in \mathbb{R}^m \\ p^T A \leq k}} \langle b \mid p \rangle = \right) \underbrace{\varphi^{\star\star'}(b) = \varphi(b)}_{\text{strong duality}} \left(= \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle k \mid x \rangle \right)$$

Remark

1. *$\varphi(b)$ can take the value $+\infty$*
2. *If there is no $\bar{b} \in \mathbb{R}^m$ such that $\varphi(\bar{b}) \in \mathbb{R}$ then*
   - $\varphi^{\star\star'}(b) = -\infty \,, \quad \forall b \in \mathbb{R}^m$
   - $\varphi(b) = \begin{cases} -\infty \,, & \forall b \in \mathrm{dom}\, \varphi \\ +\infty \,, & \textit{otherwise} \end{cases}$

Proof of strong duality for LP. Sketch of the proof

- We assume there is a bounded and feasible program

$$\exists b \in \mathbb{R}^m \,, \;\; \varphi(b) \in \mathbb{R}$$

- We use the *properness lemma* to show that every program is bounded

φ is proper

- We show that $\mathrm{epi}\ \varphi$ is a closed convex set (by showing that $\mathrm{epi}\ \varphi$ is a polyhedron)

φ is a closed convex function

- We apply Rockafellar's result [Rockafellar, 1974, Theorem 5] to get strong duality

$$\varphi^{\star\star} = \varphi$$

Proof of strong duality for LP. Proper functions

Definition

Let $f : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

- $\mathrm{dom}\ f = \{b \in \mathbb{R}^m : f(b) < +\infty\}$
- The function f is said to be proper
  if $\mathrm{dom}\ f \neq \emptyset$ and $-\infty < f(b)\ ,\ \ \forall b \in \mathbb{R}^m$

Lemma

If there is $\bar{b} \in \mathbb{R}^m$ such that $-\infty < \varphi(\bar{b})$

the corresponding LP is bounded

then the value function φ is proper

all LPs are bounded

Proof of strong duality for LP. Closed convex functions

Definition

Let $f : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

- Epigraph of the function f is defined by

$$\mathrm{epi}\ f = \left\{(b,t) \in \mathbb{R}^m \times \mathbb{R} : f(b) \leq t\right\}$$

- The function f is said to be closed convex Rockafellar [1974] if [f is proper AND $\mathrm{epi}\ f$ is a closed convex set] OR $f = +\infty$ OR $f = -\infty$

Theorem

[Rockafellar, 1974, Theorem 5]

Let $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ be a function.
f is closed convex if and only if $f^{\star\star} = f$

Proof of strong duality for LP. Argmin lemma

Lemma

Let $b \in \mathbb{R}^m$ such that $\varphi(b) \in \mathbb{R}$ then for $t \in \mathbb{R}$

$$\varphi(b) \leq t \iff \exists x \in \mathbb{R}^n \text{ s.t. } \underbrace{Ax = b\,,\ \ x \geq 0}_{x \text{ is a feasible point}} \text{ and } \langle k \mid x \rangle \leq t$$

Remark

This Lemma is also true for $b \in \mathbb{R}^m$ such that $\varphi(b) = -\infty$

TODO proof using Minkowski-Weyl's theorem [?, Theorem 3.52]

Proof of strong duality for LP. $\mathrm{epi}\,\varphi$ is a polyhedron

Proposition

Let $A \in \mathbb{R}^{m \times n}$ and $k \in \mathbb{R}^n$ define the value function $\varphi : \mathbb{R}^m \to \overline{\mathbb{R}}$ by

$$\forall b \in \mathbb{R}^m \,, \quad \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle k \mid x \rangle$$

Then $\mathrm{epi}\,\varphi$ is a polyhedron

Proof that $\operatorname{epi} \varphi$ is a polyhedron

$A \in \mathbb{R}^{m \times n}$, $k \in \mathbb{R}^n$
Let $b \in \mathbb{R}^m$, we assume that $\varphi(b) < +\infty$

$$\varphi(b) \le t$$

$$\iff \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \ge 0}} \langle k \mid x \rangle \le t$$

Using argmin lemma

$$\iff \min_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \ge 0}} \langle k \mid x \rangle \le t$$

$$\iff \exists x \in \mathbb{R}^n \text{ s.t. } Ax = b\,,\ \ x \ge 0\,,\ \ \langle k \mid x \rangle - t \le 0$$

$$\iff \operatorname{epi} \varphi = \pi_{(b,t)} \left\{ (b, t, x) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : \begin{cases} Ax = b \\ x \ge 0 \\ \langle k \mid x \rangle - t \le 0 \end{cases} \right\}$$

Thus $\operatorname{epi} \varphi$ is the projection of a polyhedron.
So, $\operatorname{epi} \varphi$ is a polyhedron.

Proof of strong duality for LP. Sketch of the proof

- We assume there is a bounded and feasible program

$$\exists b \in \mathbb{R}^m \,, \;\; \varphi(b) \in \mathbb{R}$$

- We use the *properness lemma* to show that every program is bounded

φ is proper

- We show that $\mathrm{epi}\ \varphi$ is a closed convex set (by showing that $\mathrm{epi}\ \varphi$ is a polyhedron)

φ is a closed convex function

- We apply Rockafellar's result [Rockafellar, 1974, Theorem 5] to get strong duality

$$\varphi^{\star\star} = \varphi$$

Remark on weak and strong duality

- Weak duality is a global notion on a family of minimization problem and a family of maximization problems
- Strong duality is a local notion between a minimization problem and a maximization problem produced by the perturbation duality scheme

Example when strong duality is not achieve for LP

$$-\infty = \begin{pmatrix} \sup & p_1 + 2p_2 \\ p \in \mathbb{R}^2 & \\ p_1 + p_2 = -1 & \\ p_1 + p_2 = 0 & \end{pmatrix}$$

$$= \varphi^{\star\star}(1,2) < \varphi(1,2) =$$

$$\begin{pmatrix} \inf & -x_1 \\ x \in \mathbb{R}^2 & \\ x_1 + x_2 = 1 & \\ x_1 + x_2 = 2 & \\ x \geq 0 & \end{pmatrix} = +\infty$$

Outline of the presentation

Summary of the perturbation-duality scheme for LP

Rockafellar [1974]

1. We perturb a minimization problem

$$\forall b \in \mathbb{R}^m \ , \ \ \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle k \mid x \rangle$$

2. We pair the primal space $\mathbb{R}^m$ and a dual space $\mathbb{R}^m$

$$\langle \cdot \mid \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

3. We biconjugate the perturbation function φ

$$\Big(\sup_{\substack{p \in \mathbb{R}^m \\ p^T A \leq k}} \langle b \mid p \rangle = \Big) \underbrace{\varphi^{\star\star'}(b) \leq \varphi(b) \ , \ \ \forall b \in \mathbb{R}^m}_{\text{weak duality is guaranteed}}$$

4. Strong duality at the anchor $\overline{b}$
   when φ is proper or $\overline{b} \in \mathrm{dom}\ \varphi$

Back to the coupling

- Bilinear coupling $\langle \cdot \mid \cdot \rangle : \overbrace{\mathbb{R}^m}^{\text{perturbation space}} \times \overbrace{\mathbb{R}^m}^{\text{dual space}} \to \mathbb{R}$
- $\mathbb{R}^m$ can be identified to the functional space $L^m = \{F : \mathbb{R}^m \to \mathbb{R} | F \text{ is linear}\}$, thus the coupling

$$p \in \mathbb{R}^m \leftrightarrow F \in L^m$$
$$\langle b \mid p \rangle \leftrightarrow F(b)$$

- Conjugate function $\varphi^\star : L^m \to \overline{\mathbb{R}}$

$$\forall F \in L^m \,, \;\; \varphi^\star(F) = \sup_{b \in \mathbb{R}^m} \left\{ F(b) - \varphi(b) \right\}$$

Where we stand and where we go

- We have illustrated the perturbation duality scheme (PDS) on Linear Programs
- We will present the PDS in the convex case
- Then we will present the PDS in the generalized convex case

Outline of the presentation

Duality for linear programs

Perturbation-duality scheme in convex analysis

Duality for pure integer linear programs

Generalized perturbation duality scheme

Developments and examples

Outline of the presentation

The Fenchel conjugacy

Definition

Two vector spaces $\mathcal{X}$ and $\mathcal{Y}$, paired by a bilinear form $\langle \cdot \mid \cdot \rangle$ (in the sense of convex analysis),
give rise to the classic Fenchel conjugacy

$$f^{\star}(y) = \sup_{x \in \mathcal{X}} \Big(\langle x \mid y \rangle - f(x) \Big) \; , \; \forall y \in \mathcal{Y}$$

for any function $f : \mathcal{X} \to \overline{\mathbb{R}}$

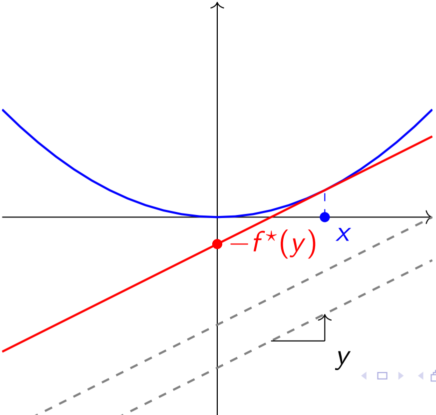
Fenchel conjugate	Fourier transform
$\sup \to +$	
$+ \to \times$	
$\sup_{x \in \mathcal{X}} \Big(\langle x \mid y \rangle - f(x) \Big)$	$\int_{\mathcal{X}} e^{\langle x \mid y \rangle} f(x) dx$

Representing Fenchel conjugacy

For $y \in \mathbb{R}^n$

$$f^\star(y) = \sup_{x \in \mathbb{R}^n} \Big(\langle x \mid y \rangle - f(x) \Big)$$

$$\iff \Big\langle x \mid \underbrace{y}_{\text{Slope}} \Big\rangle \underbrace{-f^\star(y)}_{\text{Intercept}} \leq f(x) \;, \quad \forall x \in \mathbb{R}^n$$



The biconjugate function is a minorant of the function

Definition

Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a function
Its biconjugate $f^{\star\star'} : \mathcal{X} \to \overline{\mathbb{R}}$ is defined by

$$f^{\star\star'}(x) = \sup_{y \in \mathcal{Y}} \langle x \mid y \rangle - f^{\star}(y)$$

Proposition

Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a function
Then $f^{\star\star'} : \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfies

$$f^{\star\star'} \leq f$$

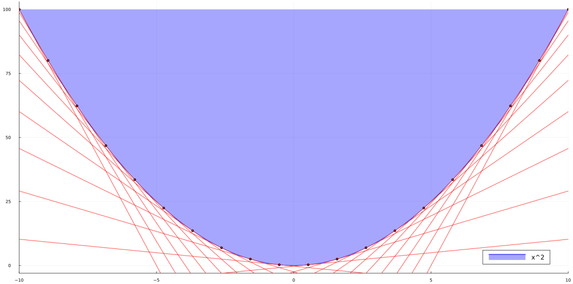
Fenchel-Moreau Theorem

[Bauschke and Combettes, 2017, Theorem 13.37]

Theorem

Let $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ be a real Hilbert space
Let $f : \mathcal{H} \to]-\infty, +\infty]$ proper
Then f is lower semicontinous and convex if and only if

$$f = f^{\star\star'}$$



Outline of the presentation

Dual problems: perturbation scheme [Rockafellar, 1974]

sets	optimization set $\mathbb{W}$	primal space $\mathbb{U}$	pairing $\mathbb{U} \overset{\langle \cdot, \cdot \rangle}{\leftrightarrow} \mathbb{V}$	dual space $\mathbb{V}$
variables	decision $w \in \mathbb{W}$	perturbation $u \in \mathbb{U}$	$\langle u, v \rangle \in \mathbb{R}$	sensitivity $v \in \mathbb{V}$
bivariate functions		Rockafellian $\mathcal{R} : \mathbb{W} \times \mathbb{U} \to \overline{\mathbb{R}}$		Lagrangian $\mathcal{L} : \mathbb{W} \times \mathbb{V} \to \overline{\mathbb{R}}$
definition				$\mathcal{L}(w, v) = \inf_{u \in \mathbb{U}} \{\mathcal{R}(w, u) - \langle u, v \rangle\}$
property				$-\mathcal{L}(w, \cdot) = \big(\mathcal{R}(w, \cdot)\big)^{\star}$
property				$-\mathcal{L}(w, \cdot)$ is $\star'$-convex (hence $\mathcal{L}(w, \cdot)$ is concave usc)
univariate functions		perturbation function $\varphi : \mathbb{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathbb{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{w \in \mathbb{W}} \mathcal{R}(w, u)$		$\psi(v) = \inf_{w \in \mathbb{W}} \mathcal{L}(w, v)$
property				$-\psi = \varphi^{\star}$

- Anchor $0 \in \mathcal{X}$ and dual maximization problem (weak duality) $\varphi^{\star\star'}(0) = \sup_{y \in \mathcal{Y}} \{-\psi(y)\} \leq \inf_{w \in \mathbb{W}} \mathcal{R}(w, 0) = \varphi(0)$
- Strong duality iff φ is $\star$-convex at 0 iff $\varphi^{\star\star'}(0) = \varphi(0)$

Dual problems given by Fenchel conjugacy

- Set $\mathbb{W}$, function $h : \mathbb{W} \to \overline{\mathbb{R}}$ and original minimization problem

$$\inf_{w \in \mathbb{W}} h(w)$$

- Embedding/perturbation scheme given by a nonempty set $\mathbb{U}$, an anchor $\overline{u} \in \mathbb{U}$ and a Rockafellian $\mathcal{R} : \mathbb{W} \times \mathbb{U} \to \overline{\mathbb{R}}$ such that

$$h(w) = \mathcal{R}(w, \overline{u}) \ , \ \ \forall w \in \mathbb{W}$$

- Paired spaces $\mathbb{U}$ and $\mathbb{V}$, and Lagrangian $\mathcal{L} : \mathbb{W} \times \mathbb{V} \to \overline{\mathbb{R}}$ given by

$$\mathcal{L}(w, v) = \inf_{u \in \mathbb{U}} \Big\{ \mathcal{R}(w, u) - \langle u - \overline{u} \mid \ v \rangle \Big\}$$

- Original minimization problem

$$\inf_{w \in \mathbb{W}} \sup_{v \in \mathbb{V}} \mathcal{L}(w, v) = \inf_{w \in \mathbb{W}} h(w)$$

Duality gap

- Dual maximization problem

$$\sup_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v)$$

- Weak duality always holds true

$$\sup_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v) \leq \inf_{w \in \mathbb{W}} h(w)$$

  When it exists, the duality gap is the nonnegative difference

- Strong duality holds true, or there is no duality gap, when

$$\sup_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v) = \inf_{w \in \mathbb{W}} h(w)$$

Outline of the presentation

Classic Lagrangian duality

- Let $\theta = (\theta_1, \ldots, \theta_p) : \mathbb{W} \to \mathbb{R}^p$ be a mapping, and $\overline{u} \in \mathbb{R}^p$
- We consider the optimization problem

$$\min_{\theta(w) \leq \overline{u}} h(w) = \min_{\substack{\theta_1(w) \leq \overline{u}_1 \\ \cdots \\ \theta_p(w) \leq \overline{u}_p}} h(w)$$

- In that case, take the perturbation scheme with $\mathbb{U} = \mathbb{R}^p$ and

$$\mathcal{R}(w, u) = h(w) \dotplus \iota_{\{\theta(w) \leq u\}} = h(w) \dotplus \sum_{j=1}^{p} \iota_{\{\theta_j(w) \leq u_j\}}$$

- which gives the Lagrangian $\mathcal{L} : \mathbb{W} \times \mathbb{V} \to \overline{\mathbb{R}}$, with $\mathbb{V} = \mathbb{R}^p$ and

$$\mathcal{L}(w, v) = h(w) + \langle \theta(w) - \overline{u} \mid v \rangle = h(w) + \sum_{j=1}^{p} v_j \big(\theta_j(w) - \overline{u} \big)$$

Outline of the presentation

Duality for linear programs

Perturbation-duality scheme in convex analysis

Duality for pure integer linear programs

Generalized perturbation duality scheme

Developments and examples

Outline of the presentation

Illustration of the scheme in PILP

- Constraint matrix $A \in \mathbb{R}^{m \times n}$
- Anchor $\bar{b} \in \mathbb{R}^m$

Initial minimization problem

$$\inf_{\substack{x \in \mathbb{Z}^n \\ Ax = \bar{b} \\ x \geq 0}} \langle k \mid x \rangle$$

Step 1. Perturbation of the initial minimization problem

$$\forall b \in \mathbb{R}^m \,, \quad \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle k \mid x \rangle$$

- Perturbation space: $\mathbb{R}^m$
- Perturbation function $\varphi : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \underbrace{\{-\infty\}}_{\text{unbounded}} \cup \underbrace{\{+\infty\}}_{\text{unfeasible}}$
- Value of the initial problem: $\varphi(\bar{b})$

Example of epigraph of the perturbation function for a PILP

Step 2. Dual space, coupling and conjugate function

- Set of subadditive functions

$$\mathcal{S}^m = \{F : \mathbb{R}^m \to \overline{\mathbb{R}} | F(b_1 + b_2) \leq F(b_1) \dot{+} F(b_2) \ , \ \ \forall b_1, b_2\}$$

- Subadditive evaluation coupling

$$c_{\mathcal{S}} : \mathbb{R}^m \times \mathcal{S}^m \to \overline{\mathbb{R}}$$
$$c_{\mathcal{S}}(b, F) = F(b)$$

- Conjugate function $\varphi^{c_{\mathcal{S}}} : \mathcal{S}^m \to \overline{\mathbb{R}}$

$$\forall F \in \mathcal{S}^m \ , \ \ \varphi^{c_{\mathcal{S}}}(F) = \sup_{b \in \mathbb{R}^m} \left\{ c_{\mathcal{S}}(b, F) \dot{+} \big(-\varphi(b)\big) \right\}$$

Step 3. Biconjugate and weak duality

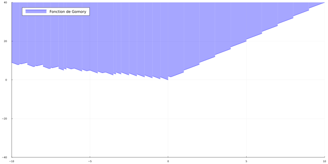
- Biconjugate function $\varphi^{c_{\mathcal{S}} c_{\mathcal{S}}'} : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$\forall b \in \mathbb{R}^m \ , \ \ \varphi^{c_{\mathcal{S}} c_{\mathcal{S}}'}(b) = \sup_{F \in \mathcal{S}^m} \left\{ c_{\mathcal{S}}(b, F) - \varphi^{c_{\mathcal{S}}}(F) \right\}$$

- We obtain weak duality

$$\underbrace{\sup_{F \in \mathcal{S}^m} \left\{ F(b) \dotplus \inf_{x \in \mathbb{Z}^n_+} \{ \langle k \mid x \rangle - F(Ax) \} \right\}}_{\text{dual problem}} = \varphi^{c_{\mathcal{S}} c_{\mathcal{S}}'}(b) \le \varphi(b) = \begin{array}{ll} \inf\limits_{x} & \langle k \mid x \rangle \\ Ax = b & \\ x \geq 0 & \\ x \in \mathbb{Z}^n & \end{array}$$

Step 4. Strong duality for the subadditive dual problem



- If $Ax_1 = b_1$ and $Ax_2 = b_2$, then $A\big(x_1 + x_2\big) = b_1 + b_2$
- So

$$\varphi(b_1 + b_2) \leq \varphi(b_1) \dotplus \varphi(b_2) \ , \quad \forall b_1, b_2 \in \mathbb{R}^m$$

- It results that the function φ is subadditive

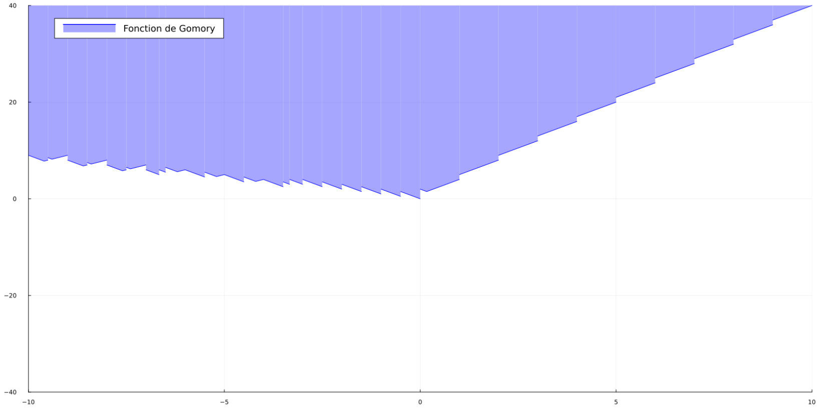
$$\varphi(b) \dotplus \inf_{x \in \mathbb{Z}_+^n} \underbrace{\{\langle k \mid x \rangle - \varphi(Ax)\}}_{\geq 0} \leq \varphi^{cs\,cs'}(b)$$

- Thus we have strong duality $\forall b \in \mathbb{R}^m$

$$\underbrace{\sup_{F \in \mathcal{S}^m} \Big\{ F(b) \dotplus \inf_{x \in \mathbb{Z}_+^n} \{\langle k \mid x \rangle - F(Ax)\} \Big\}}_{\text{dual problem}} = \varphi^{cs\,cs'}(b) = \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle k \mid x \rangle$$

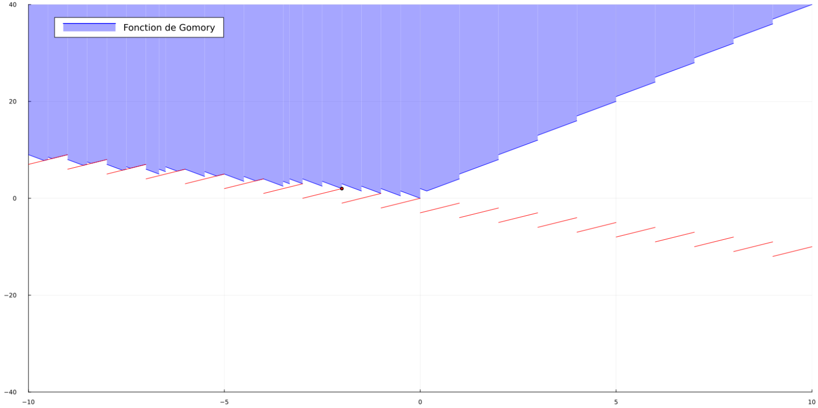
Epigraph of a perturbation function for a PILP

$$\varphi(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



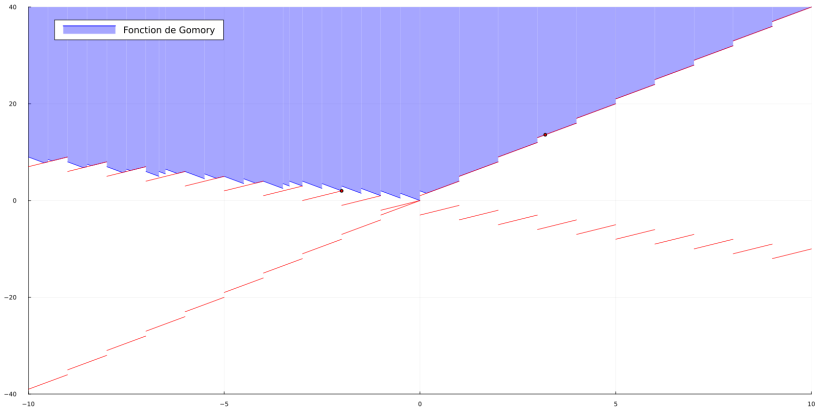
Epigraph of a perturbation function for a PILP

$$\varphi(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



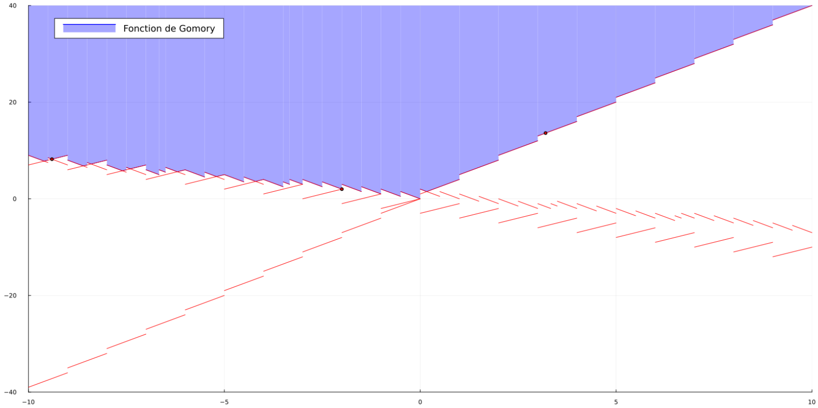
Epigraph of a perturbation function for a PILP

$$\varphi(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



Epigraph of a perturbation function for a PILP

$$\varphi(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



Outline of the presentation

Summary of the perturbation-duality scheme for PILP

1. We perturb a minimization problem

$$\forall b \in \mathbb{R}^m , \quad \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \in \mathbb{Z}^n_+}} \langle k \mid x \rangle$$

2. We pair the primal space $\mathbb{R}^m$ and a dual space $\mathcal{S}^m$

$$c_{\mathcal{S}}(\cdot,\cdot) : \mathbb{R}^m \times \mathcal{S}^m \to \overline{\mathbb{R}}$$
$$c_{\mathcal{S}}(b, F) = F(b)$$

Reminder: set of subadditive functions

$$\mathcal{S}^m = \{F : \mathbb{R}^m \to \overline{\mathbb{R}} | F(b_1 + b_2) \leq F(b_1) \dotplus F(b_2) , \ \forall b_1, b_2\}$$

3. We biconjugate the perturbation function φ

$$\underbrace{\varphi^{c_{\mathcal{S}} c_{\mathcal{S}}'}(b) \leq \varphi(b) , \ \forall b \in \mathbb{R}^m}_{\text{weak duality is guaranteed}}$$

4. Strong duality as φ is subadditive

Outline of the presentation

Duality for linear programs

Perturbation-duality scheme in convex analysis

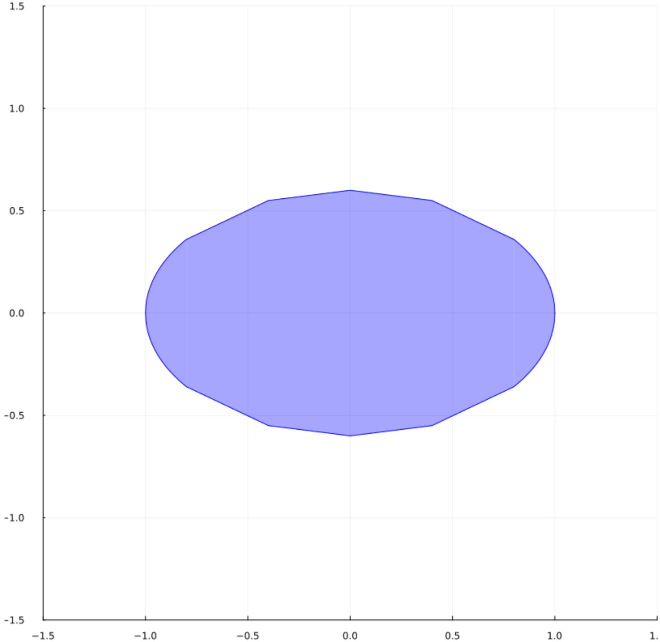
Duality for pure integer linear programs

Generalized perturbation duality scheme

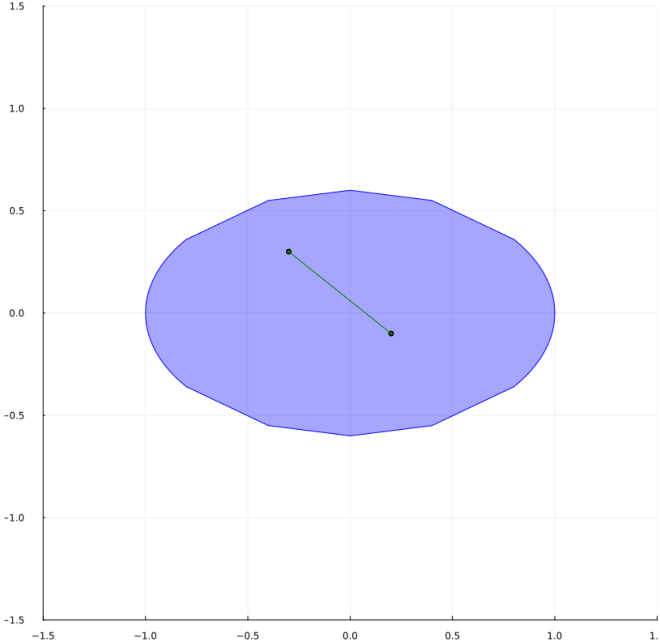
Developments and examples

Outline of the presentation

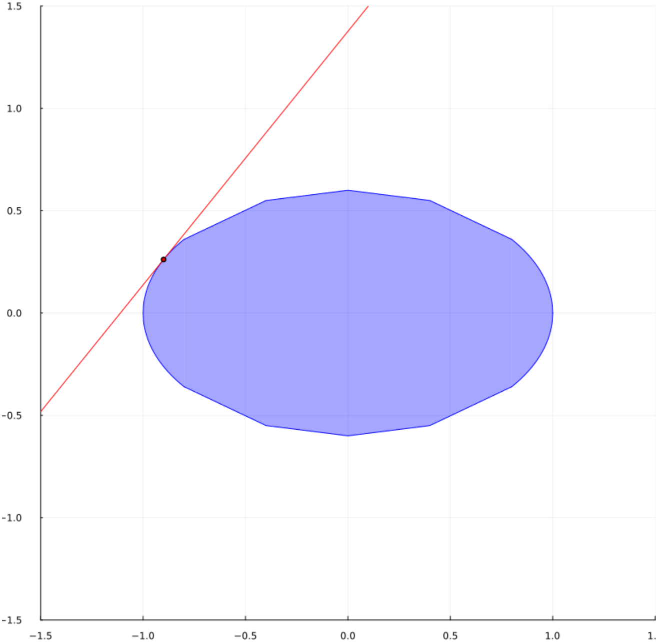
A closed convex set



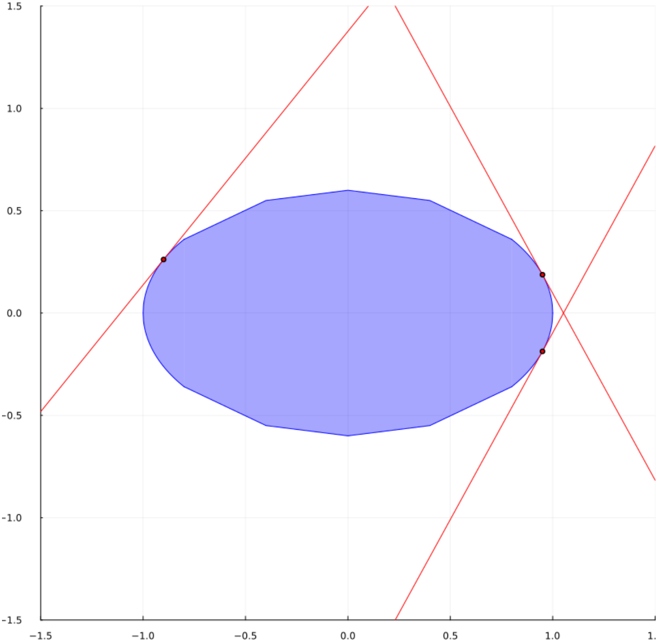
Usual definition of convexity by the interior



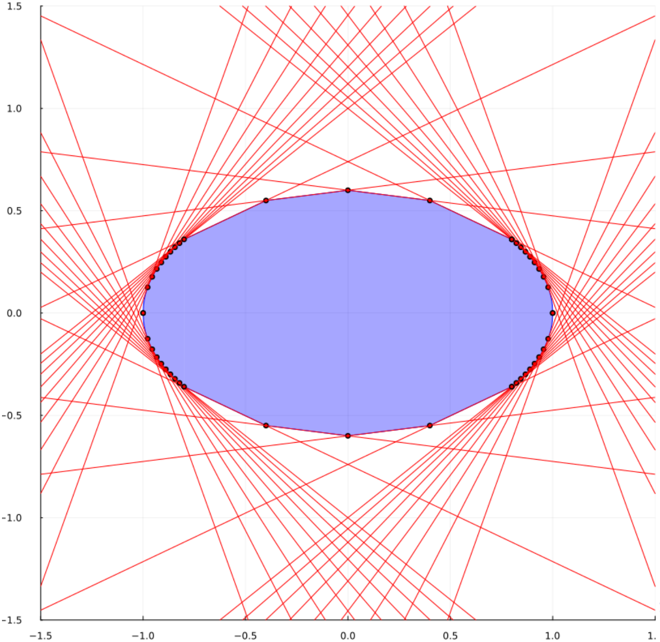
Equivalent definition for closed-convexity by the exterior



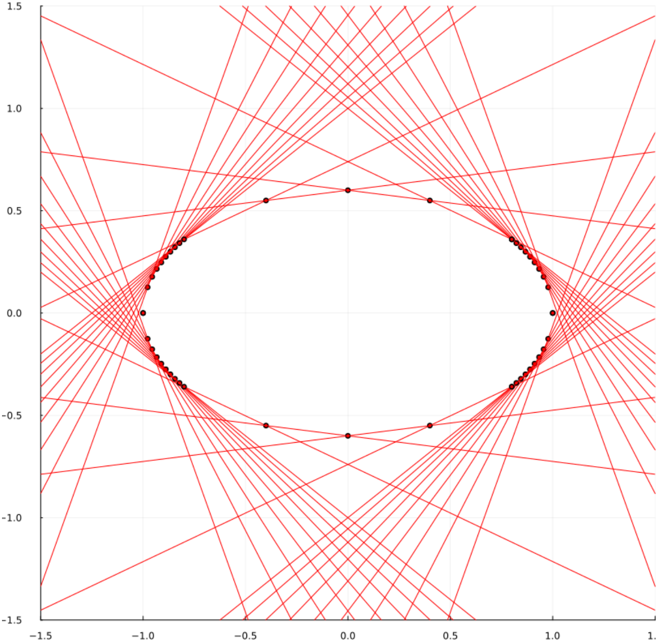
Equivalent definition for closed-convexity by the exterior



Equivalent definition for closed-convexity by the exterior

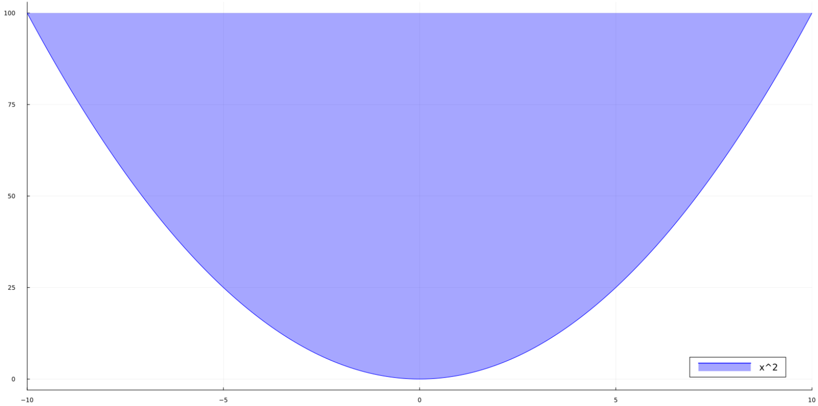


Approximation by finite number of cuts



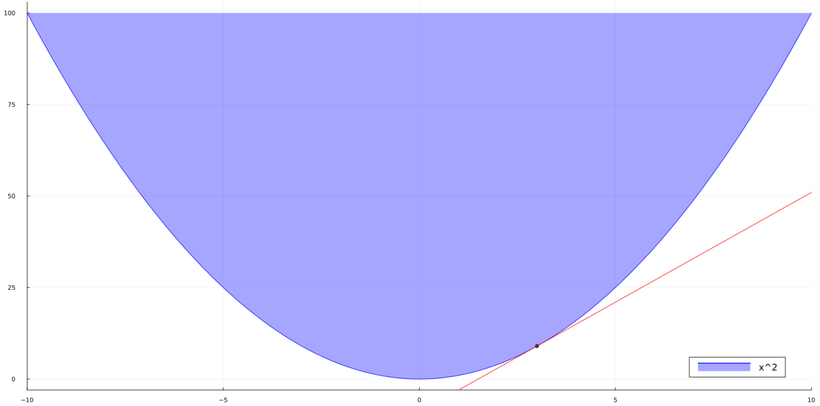
Epigraph of a closed-convex function

$y = x^2$



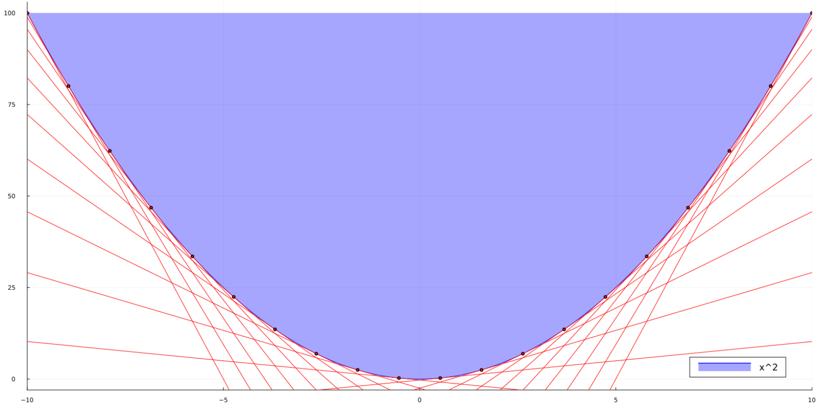
Epigraph of a closed-convex function

$y = x^2$



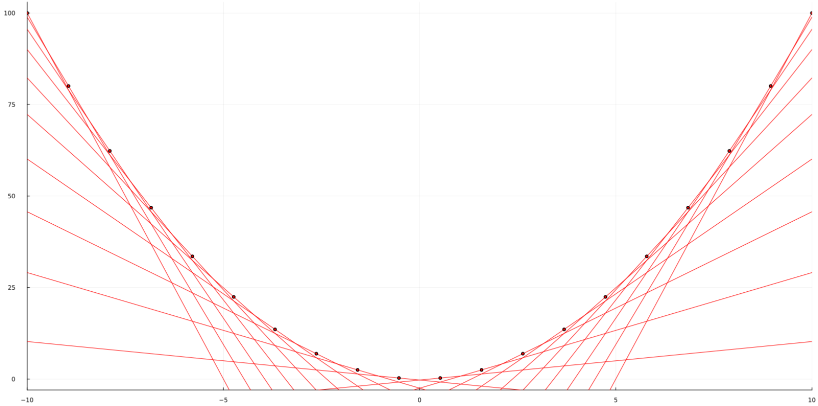
The epigraph is above its tangents

$y = x^2$

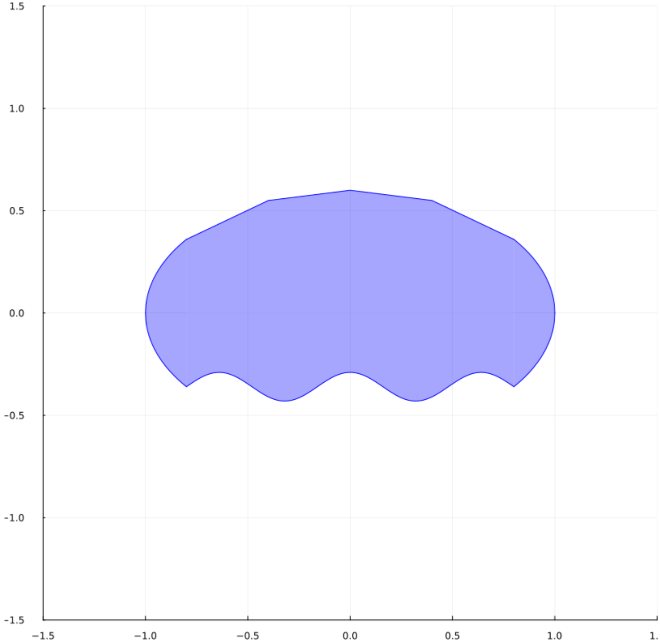


Approximation by a finite number of cuts

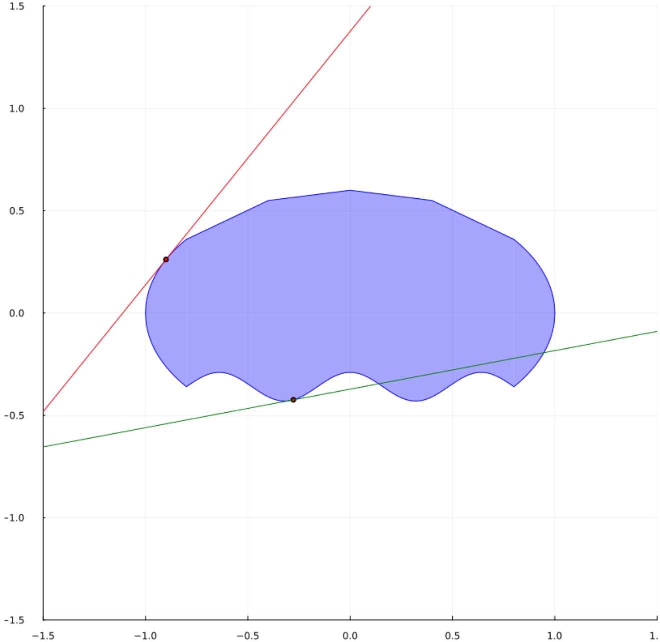
$y = x^2$



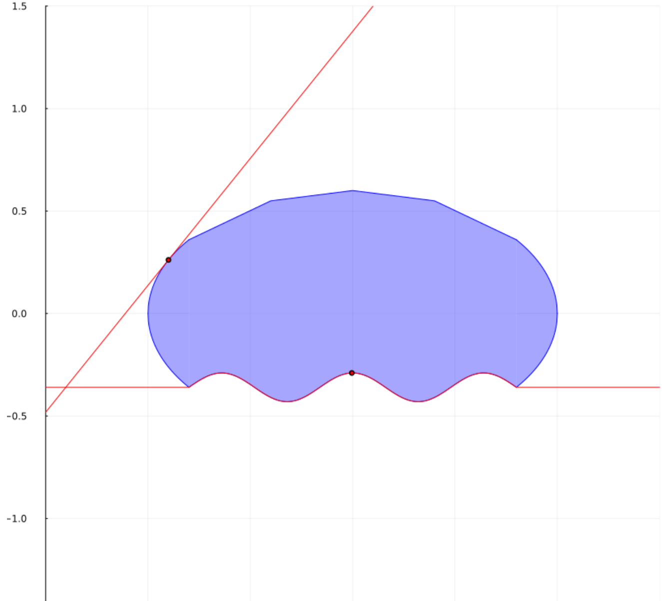
Example of a nonconvex set



Some tangents won't stay outside!



Generalized convexity: we change the shape of the tangents!



Generalized convexity: we change the shape of the tangents!

$$T(x) = \langle x \mid \alpha \rangle + \beta \ , \ \ \forall x \in \mathbb{R}^n$$

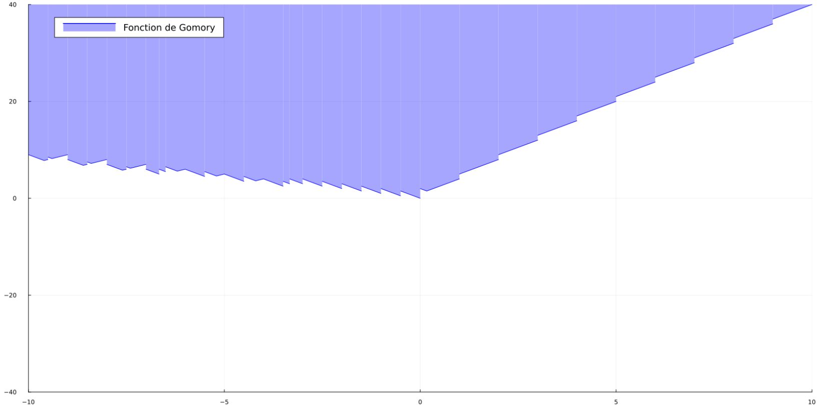
Scalar product $\langle \cdot \mid \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n$
Slope: $\alpha \in \mathbb{R}^n$
Intercept: $\beta \in \mathbb{R}$

$$S(u) = c(u, v) + \beta$$

Coupling $c : U \times V \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$
"Slope": $v \in V$
Intercept: $\beta \in \mathbb{R}$

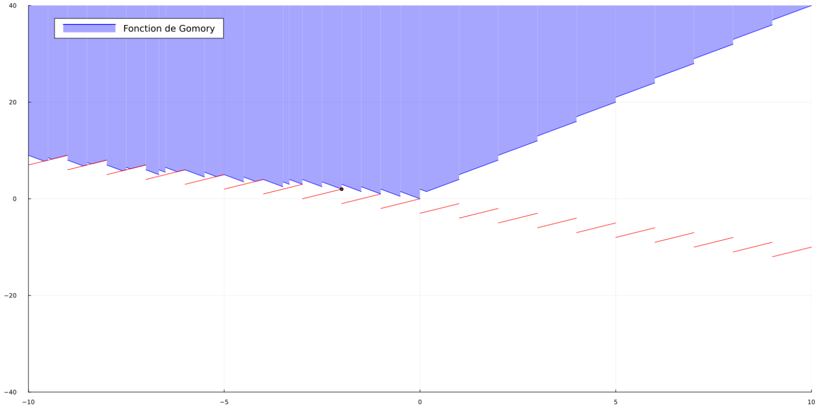
Epigraph of a perturbation function for a PILP

$$G(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



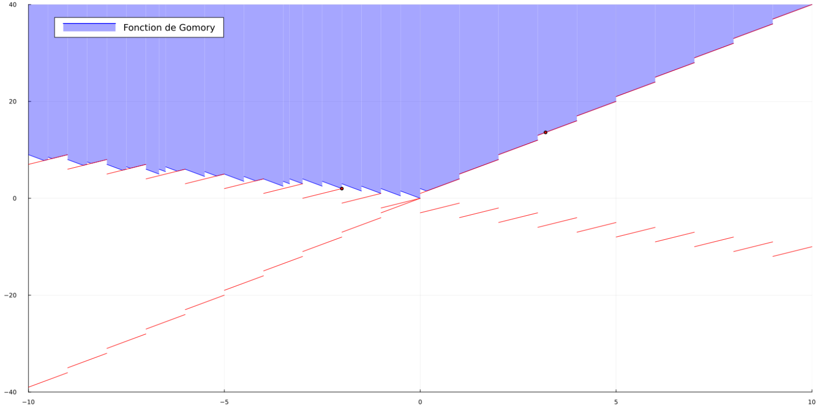
Epigraph of a perturbation function for a PILP

$$G(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



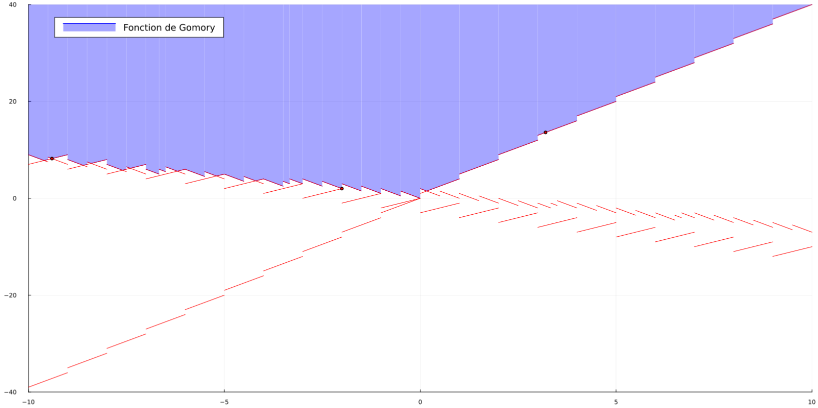
Epigraph of a perturbation function for a PILP

$$G(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \tfrac{3}{10} b \rceil\}$$



Epigraph of a perturbation function for a PILP

$$G(b) = \max\{3b + \lceil b \rceil, 2b + -3\lceil b \rceil, -3b + \lceil 2b \rceil + \lceil \frac{3}{10} b \rceil\}$$



Outline of the presentation

Moreau lower and upper additions

- The Moreau lower addition extends the usual addition with

$$(+\infty) \overset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty$$

- The Moreau upper addition extends the usual addition with

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$$

Background on couplings and Fenchel-Moreau conjugacies

- Let be given two sets $\mathcal{X}$ ("primal") and $\mathcal{Y}$ ("dual")
- Consider a coupling function $c : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}} = [-\infty, +\infty]$
- We also use the notation $\mathcal{X} \overset{c}{\leftrightarrow} \mathcal{Y}$ for a coupling

Definition

The c-Fenchel-Moreau conjugate of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$,
with respect to the coupling c,
is the function $f^c : \mathcal{Y} \to \overline{\mathbb{R}}$ defined by

$$f^c(y) = \sup_{x \in \mathcal{X}} \Big(c(x,y) \dot{+} \big(-f(x)\big) \Big) \ , \quad \forall y \in \mathcal{Y}$$

Fenchel-Moreau conjugate $(\max, +)$	Kernel transform $(+, \times)$
$\sup_{x \in \mathcal{X}} \Big(c(x,y) \dot{+} \big(-f(x)\big) \Big)$	$\int_{\mathcal{X}} c(x,y) f(x) dx$

Background on couplings and Fenchel-Moreau conjugacies

With the coupling c, we associate the reverse coupling c'

$$c' : \mathcal{Y} \times \mathcal{X} \to \overline{\mathbb{R}} \; , \;\; c'(y,x) = c(x,y) \; , \;\; \forall (y,x) \in \mathcal{Y} \times \mathcal{X}$$

- The c'-Fenchel-Moreau conjugate of a function $g : \mathcal{Y} \to \overline{\mathbb{R}}$, with respect to the coupling c', is the function $g^{c'} : \mathcal{X} \to \overline{\mathbb{R}}$

$$g^{c'}(x) = \sup_{y \in \mathcal{Y}} \Big(c(x,y) \dot{+} \big(-g(y) \big) \Big) \; , \;\; \forall x \in \mathcal{X}$$

- The c-Fenchel-Moreau biconjugate $f^{cc'} : \mathcal{X} \to \overline{\mathbb{R}}$ of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is given by

$$f^{cc'}(x) = \big(f^{c} \big)^{c'}(x) = \sup_{y \in \mathcal{Y}} \Big(c(x,y) \dot{+} \big(-f^{c}(y) \big) \Big) \; , \;\; \forall x \in \mathcal{X}$$

Dual problems: perturbation scheme [Rockafellar, 1974]

- Set $\mathbb{W}$, function $h : \mathbb{W} \to \overline{\mathbb{R}}$
  and original minimization problem

$$\inf_{w \in \mathbb{W}} h(w)$$

- Embedding/perturbation scheme given by
  a nonempty set $\mathcal{X}$ (perturbations), an element $\overline{x} \in \mathcal{X}$ (anchor)
  and a function (Rockafellian) $\mathcal{R} : \mathbb{W} \times \mathcal{X} \to \overline{\mathbb{R}}$ such that

$$h(w) = \mathcal{R}(w, \overline{x})$$

- Perturbation function

$$\phi(x) = \inf_{w \in \mathbb{W}} \mathcal{R}(w, x)$$

- Original minimization problem

$$\phi(\overline{x}) = \inf_{w \in \mathbb{W}} \mathcal{R}(w, \overline{x}) = \inf_{w \in \mathbb{W}} h(w)$$

Dual problems: conjugacy, weak and strong duality

- Coupling $\mathcal{X} \overset{c}{\leftrightarrow} \mathcal{Y}$, and Lagrangian $\mathcal{L} : \mathbb{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$ given by

$$\mathcal{L}(w, y) = \inf_{x \in \mathcal{X}} \left\{ \mathcal{R}(w, x) \dotplus \left(-c(x, y) \right) \right\}$$

- Dual function

$$\psi(y) = -\phi^{c}(y) = \inf_{w \in \mathbb{W}} \mathcal{L}(w, y)$$

- Dual maximization problem (weak duality)

$$\phi^{cc'}(\overline{x}) = \sup_{y \in \mathcal{Y}} \left\{ c(\overline{x}, y) \underset{\cdot}{+} \psi(y) \right\} \leq \inf_{w \in \mathbb{W}} h(w) = \phi(\overline{x})$$

- Strong duality holds true when ϕ is c-convex at $\overline{x}$, that is,

$$\phi^{cc'}(\overline{x}) = \sup_{y \in \mathcal{Y}} \left\{ c(\overline{x}, y) \underset{\cdot}{+} \psi(y) \right\} = \inf_{w \in \mathbb{W}} h(w) = \phi(\overline{x})$$

Dual problems with general couplings

Dual problems: perturbation scheme [Rockafellar, 1974]

sets	optimization set $\mathbb{W}$	primal set $\mathcal{X}$	coupling $\mathcal{X} \overset{c}{\leftrightarrow} \mathcal{Y}$	dual set $\mathcal{Y}$
variables	decision $w \in \mathbb{W}$	perturbation $x \in \mathcal{X}$	$c(x,y) \in \overline{\mathbb{R}}$	sensitivity $y \in \mathcal{Y}$
bivariate functions		Rockafellian $\mathcal{R} : \mathbb{W} \times \mathcal{X} \to \overline{\mathbb{R}}$		Lagrangian $\mathcal{L} : \mathbb{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$
definition				$\mathcal{L}(w,y) = \inf_{x \in \mathcal{X}} \left\{ \mathcal{R}(w,x) \dotplus \left(-c(x,y)\right) \right\}$
property				$-\mathcal{L}(w,\cdot) = \left(\mathcal{R}(w,\cdot)\right)^{c}$
property				$-\mathcal{L}(w,\cdot)$ is c'-convex
univariate functions		perturbation function $\phi : \mathcal{X} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{Y} \to \overline{\mathbb{R}}$
definition		$\phi(x) = \inf_{w \in \mathbb{W}} \mathcal{R}(w,x)$		$\psi(y) = \inf_{w \in \mathbb{W}} \mathcal{L}(w,y)$
property				$-\psi = \phi^{c}$

Anchor $\overline{x} \in \mathcal{X}$ and dual maximization problem (weak duality)
$\phi^{cc'}(\overline{x}) = \sup_{y \in \mathcal{Y}} \left\{ c(\overline{x},y) \dotplus \psi(y) \right\} \leq \inf_{w \in \mathbb{W}} h(w) = \phi(\overline{x})$
Strong duality iff ϕ is c-convex at $\overline{x}$ iff $\phi^{cc'}(\overline{x}) = \phi(\overline{x})$

Duality between Lagrangians and Rockafellians (work in progress)

$(-\mathcal{L}, \mathcal{R})$ is minimal in the inequality

$$\left(-\mathcal{L}(w,y)\right) \dotplus \mathcal{R}(w,x) \geq c(x,y)$$

$$\Longleftrightarrow$$

$$-\mathcal{L}(w,\cdot) = \left(\mathcal{R}(w,\cdot)\right)^{c} \text{ and } \mathcal{R}(w,\cdot) = \left(-\mathcal{L}(w,\cdot)\right)^{c'}$$

$$\Longleftrightarrow$$

$$-\mathcal{L}(w,\cdot) = \left(\mathcal{R}(w,\cdot)\right)^{c} \text{ and } \left(\mathcal{R}(w,\cdot)\right)^{cc'} = \mathcal{R}(w,\cdot)$$

$$\Longleftrightarrow$$

$$\mathcal{R}(w,\cdot) = \left(-\mathcal{L}(w,\cdot)\right)^{c'} \text{ and } \left(-\mathcal{L}(w,\cdot)\right)^{c'c} = -\mathcal{L}(w,\cdot)$$

The c-subdifferential is defined as the Rockafellar-Moreau subdifferential

Definition

For any function $f : \mathcal{X} \to \overline{\mathbb{R}}$ and $x \in \mathcal{X}$, the c-subdifferential is

$$\partial_c f(x) = \{y \in \mathcal{Y} \mid \begin{array}{rl} & c(x', y) \dotplus (-f(x')) \\ \leq & c(x, y) \dotplus (-f(x)) \,, \ \forall x' \in \mathcal{X}\} \end{array}$$

The following properties are satisfied

$$y \in \partial_c f(x) \iff f^c(y) = c(x, y) \dotplus (-f(x))$$

"à la" Fenchel-Young

$$\partial_c f(x) \neq \emptyset \Rightarrow f^{cc'}(x) = f(x)$$

Introducing generalized convexity

Balder [1977]

Fenchel conjugate	c-conjugate
$f^{\star}(v) = \sup_{u \in \mathbb{R}^m} \langle u \mid v \rangle - f(u)$	$g^{c}(v) = \sup_{u \in U} c(u,v) \dotplus \big(-g(u)\big)$
Fenchel biconjugate	**c-biconjugate**
$f^{\star\star'}(u) = \sup_{v \in \mathbb{R}^m} \langle u \mid v \rangle - f^{\star}(v)$	$g^{cc'}(u) = \sup_{v \in V} c(u,v) \dotplus \big(-g^{c}(v)\big)$
$\star -$ convex functions	**c-convex functions**
$\iff f = f^{\star\star'}$	$\iff g = g^{cc'}$

Perturbation-duality scheme with generalized convexity

1. We perturb a minimization problem

$$\varphi : \mathbb{R}^m \to \overline{\mathbb{R}}$$

2. We pair the primal space $\mathbb{R}^m$ and a dual space V

$$c : \mathbb{R}^m \times V \to \overline{\mathbb{R}}$$

3. We biconjugate the perturbation function φ

$$\underbrace{\varphi^{cc'}(b) \leq \varphi(b) \; , \;\; \forall b \in \mathbb{R}^m}_{\text{Weak duality is guaranteed!}}$$

4. Strong duality when φ is c-convex

Functions in the perturbation duality scheme

bivariate functions	univariate functions	definition	property
Rockafellian $\mathcal{R} : \mathbb{W} \times \mathcal{X} \to \overline{\mathbb{R}}$			
Lagrangian $\mathcal{L} : \mathbb{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$		$\mathcal{L}(w, y) = \inf_{x \in \mathcal{X}} \left\{ \mathcal{R}(w, x) \dotplus (-c(x, y)) \right\}$	$-\mathcal{L}(w, \cdot) = \left(\mathcal{R}(w, \cdot) \right)^c$
	perturbation $\varphi : \mathcal{X} \to \overline{\mathbb{R}}$	$\varphi(x) = \inf_{w \in \mathbb{W}} \mathcal{R}(w, x)$	
	dual $\Psi : \mathcal{Y} \to \overline{\mathbb{R}}$	$\Psi(y) = \inf_{w \in \mathbb{W}} \mathcal{L}(w, y)$	$-\Psi = \varphi^c$
	dual objective $\Phi_{\bar{x}} : \mathcal{Y} \to \overline{\mathbb{R}}$	$\Phi_{\bar{x}}(y) = c(\bar{x}, y) \dotplus (-\varphi^c(y))$	$\Phi_{\bar{x}}(y) = c(\bar{x}, y) \dotplus \Psi(y)$ $\varphi^{cc'}(\bar{x}) = \sup_{y \in \mathcal{Y}} \Phi_{\bar{x}}(y)$

Outline of the presentation

Outline of the presentation

Summary of the perturbation-duality scheme

1. We partially perturb

$$\forall b \in \mathbb{R}^{m_1}\ ,\ \ \varphi(b_1) = \inf_{\substack{x \\ A_1 x = b_1 \\ A_2 x = b_2 \\ x \geq 0 \\ x \in \mathbb{R}^n}} \langle k \mid x \rangle$$

2. We pair the primal space $\mathbb{R}^{m_1}$ and a dual space $\mathbb{R}^{m_1}$

$$\langle \cdot \mid \cdot \rangle : \mathbb{R}^{m_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$$

3. We biconjugate the perturbation function φ

$$\varphi^{\star\star'}(b_1) = \sup_{\lambda \in \mathbb{R}^{m_1}} \inf_{\substack{x \\ A_2 x = b_2 \\ x \geq 0 \\ x \in \mathbb{R}^n}} \langle k \mid x \rangle + \langle \lambda \mid b_1 - A_1 x \rangle \ \leq \varphi(b$$

4. Strong duality when φ is lsc convex

Outline of the presentation

Definition of Chvátal functions

Definition

The class of Chvátal functions $\mathcal{C}^m$
is the smallest class of functions $D \subset \{f | f : \mathbb{Q}^m \to \mathbb{Q}\}$ such that

$$b \in \mathbb{Q}^m \mapsto \lambda b \in D\ , \ \ \forall b \in \mathbb{Q}^m \qquad \text{(linear functions)}$$

$$\alpha F_1 + \beta F_2 \in D\ , \ \ \forall F_1, F_2 \in D\ , \ \ \alpha, \beta \in \mathbb{Q}_+ \qquad \text{(conic combination)}$$

$$\lceil F \rceil \in D\ , \ \ \forall F \in D \qquad \text{(round-up)}$$

Examples in 1D

- $b \mapsto \frac{3}{4} b$
- $b \mapsto \lceil b \rceil$
- $b \mapsto \frac{3}{4} b + \frac{7}{10} \lceil b \rceil$
- $b \mapsto 15b + \frac{39}{22} \lceil \frac{3}{4} b + \frac{7}{10} \lceil b \rceil \rceil + \lceil 16b \rceil$

Chvátal perturbation-duality scheme

- We define a perturbation function

$$\forall b \in \mathbb{Q}^m \,, \;\; \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \in \mathbb{Z}_+^n}} \langle k \mid x \rangle$$

- We define a coupling between primal and dual space

$$c_{\mathcal{C}} : \mathbb{Q}^m \times \mathcal{C}^m \to \mathbb{R}$$
$$c_{\mathcal{C}}(b, F) = F(b) \,, \;\; \forall b \in \mathbb{Q}^m \,, \;\; \forall F \in \mathcal{C}^m$$

- We biconjugate the perturbation functions

$$\underbrace{\varphi^{c_{\mathcal{C}} c_{\mathcal{C}}'}(b) \leq \varphi(b) \,, \;\; \forall b \in \mathbb{Q}^m}_{\text{weak duality}}$$

- We get strong duality $\varphi^{c_{\mathcal{C}} c_{\mathcal{C}}'}(\bar{b}) = \varphi(\bar{b})$

Obtained dual problems

Formulation 1:

$$\varphi^{cc\,cc'}(\bar{b}) = \sup_{F \in \mathcal{C}^m} \left\{ F(\bar{b}) + \inf_{b \in \mathbb{Q}^m} \{\varphi(b) - F(b)\} \right\}$$

Formulation 2:

$$\varphi^{cc\,cc'}(\bar{b}) = \sup_{F \in \mathcal{C}^m} \left\{ F(\bar{b}) + \inf_{x \in \mathbb{Z}^n_+} \{\langle k \mid x \rangle - F(Ax)\} \right\}$$

Reminder Jeroslow's dual problem

$$\begin{array}{cc} \sup\limits_{F} & F(\bar{b}) \\ F(A_j) \leq k_j & \\ F(0) \leq 0 & \\ F \in \mathcal{C}^m & \end{array}$$

Generalized subdifferential and complementary slackness

Proposition

- φ: bounded perturbation function of a MILP
- $A = \big(A_j\big)_{j=1,\dots,n} \in \mathbb{Q}^{m\times n}$ constraint matrix
- $\bar{b} \in \mathbb{Q}^n$ anchor

If $\hat{x} \in \{x \in \mathbb{Z}^n_+ | Ax = \bar{b}\}$ and $\widehat{F} \in \mathcal{C}^m$ are "primal"-dual optimal solutions then we have the equivalence

$$\widehat{F} \in \partial^{cc}\varphi(\bar{b})$$
$$\iff -k \in \partial\big(-\widehat{F} \circ A \dotplus \delta_{\mathbb{Z}^n_+}\big)(\hat{x})$$

Furthermore, if $\widehat{F}(A_j) \leq k_j, \forall j = 1, \dots, n$, then the following assertion is also equivalent

$$\widehat{F}(0) \leq 0\ ,\ \ \widehat{F}(\bar{b}) = \varphi(\bar{b}) \text{ and } \big(k_j - \widehat{F}(A_j)\big)\hat{x}_j = 0\ ,\ \ \forall j = 1, \dots, n\ .$$

Outline of the presentation

Subdifferential of the perturbation function (sensitivity analysis)

The perturbation function is

$$\varphi(u) = \inf_{w \in \mathbb{W}} \mathcal{R}(w, u) \ , \quad \forall u \in \mathbb{U}$$

Theorem [Rockafellar, 1974, Theorem 16, p. 40]

For $\overline{v} \in \mathbb{V}$, the following conditions are equivalent

1. $\overline{v} \in \arg\max_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v)$ and $\max_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v) = \inf_{w \in \mathbb{W}} h(w)$
2. $\overline{v} \in \partial\varphi(\overline{u})$
3. $\inf_{w \in \mathbb{W}} h(w) = \inf_{w \in \mathbb{W}} \mathcal{L}(w, \overline{v})$

Subdifferential of the perturbation function (sensitivity analysis)

The convex case

Theorem [Rockafellar, 1974, Theorem 18, p. 41]

Suppose that

- the function $\mathcal{R} : \mathbb{W} \times \mathbb{U} \to \overline{\mathbb{R}}$ is convex
- there exists $w \in \mathbb{W}$ such that the function $u \mapsto \mathcal{R}(w, u)$ is bounded above in a neighborhood of $\overline{u}$

Then there exists $\overline{v} \in \mathbb{V}$ such that

1. $\overline{v} \in \arg\max_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v)$ and $\max_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v) = \inf_{w \in \mathbb{W}} h(w)$
2. $\overline{v} \in \partial\varphi(\overline{u})$
3. $\inf_{w \in \mathbb{W}} h(w) = \inf_{w \in \mathbb{W}} \mathcal{L}(w, \overline{v})$

Slater qualification constraint

The convex case

Theorem [Rockafellar, 1974, p. 45]

Suppose that

- the functions h and $\theta_1, \ldots, \theta_p$ are is convex
- there exists $w \in \mathbb{W}$ such that

$$\theta_1(w) < \overline{u}_1, \ldots, \theta_p(w) < \overline{u}_p$$

Then there exists $\overline{v} \in \mathbb{V}$ such that

1. $\overline{v} \in \arg\max_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v)$ and $\max_{v \in \mathbb{V}} \inf_{w \in \mathbb{W}} \mathcal{L}(w, v) = \inf_{w \in \mathbb{W}} h(w)$
2. $\overline{v} \in \partial\varphi(\overline{u})$
3. $\inf_{w \in \mathbb{W}} h(w) = \inf_{w \in \mathbb{W}} \mathcal{L}(w, \overline{v})$

Outline of the presentation

Optimal transport

The optimal transport problem is

$$\inf_{\pi \in \Pi(P_0,P_1)} \int_{\mathbb{A}\times\mathbb{B}} k(a,b)d\pi(a,b)$$

where

- the sets $\mathbb{A}$ and $\mathbb{B}$ are two Polish spaces
- we denote by $\mathcal{P}(\mathbb{A}\times\mathbb{B})$, $\mathcal{P}(\mathbb{A})$ and $\mathcal{P}(\mathbb{B})$
  the corresponding probability spaces (rectangle and marginals)
- the set $\Pi(P_0, P_1)$ is made of
  probabilities $\pi \in \mathcal{P}(\mathbb{A}\times\mathbb{B})$ on the rectangle,
  whose marginals are $P_0 \in \mathcal{P}(\mathbb{A})$ and $P_1 \in \mathcal{P}(\mathbb{B})$
- the measurable cost function $k : \mathbb{A}\times\mathbb{B} \to]-\infty, +\infty]$,
  where $k(a,b)$ represents
  the cost to move from $a \in \mathbb{A}$ towards $b \in \mathbb{B}$

We introduce a suitable coupling between probabilities and functions

- We denote by $C_b^0(\mathbb{A})$ and $C_b^0(\mathbb{B})$ the spaces of continuous bounded functions
- We introduce the bilinear coupling

$$\mathcal{P}(\mathbb{A}) \times \mathcal{P}(\mathbb{B}) \overset{\beta}{\longleftrightarrow} C_b^0(\mathbb{A}) \times C_b^0(\mathbb{B})$$

$$\beta\big((P_0, P_1); (f, g)\big) = \int_{\mathbb{B}} g(b) dP_1(b) - \int_{\mathbb{A}} f(a) dP_0(a)$$

The k-conjugacy appears naturally as a sub-product of computations with the β-conjugacy

The optimal cost is, as function of marginals,

$$K(P_0, P_1) = \inf_{\pi \in \Pi(P_0, P_1)} \int_{\mathbb{A} \times \mathbb{B}} k(a, b) d\pi(a, b)$$

and its β-conjugate is

$$\begin{aligned} K^{\beta}(f, g) &= \sup_{a \in \mathbb{A}, b \in \mathbb{B}} \left[g(b) - f(a) - k(a, b) \right] \\ &= \sup_{b \in \mathbb{B}} \Big(g(b) + \underbrace{f^{-k}(b)}_{k\text{-conjugate}} \Big) \end{aligned}$$

Conjugacy properties in optimal transport

$$K(P_0, P_1) = \inf_{\pi \in \Pi(P_0,P_1)} \int_{\mathbb{A}\times\mathbb{B}} k(a,b) d\pi(a,b)$$

$$D(f,g) = \sup_{a\in\mathbb{A}, b\in\mathbb{B}} \left[g(b) - f(a) - k(a,b)\right] = \sup_{b\in\mathbb{B}} \left(g(b) + f^{-k}(b)\right)$$

We have the following conjugacy equalities and inequalities

$$K^{\beta}(f,g) = D(f,g) = D^{\beta'\beta}(f,g)$$

$$K(P_0,P_1) \geq K^{\beta\beta'}(P_0,P_1) = D^{\beta'}(P_0,P_1)$$

$$= \sup_{f,g} \left(\int_{\mathbb{B}} g(b) dP_1(b) - \int_{\mathbb{A}} f(a) dP_0(a) - D(f,g)\right)$$

$$\geq \sup_{g-f\leq k} \left(\int_{\mathbb{B}} g(b) dP_1(b) - \int_{\mathbb{A}} f(a) dP_0(a)\right)$$

Perturbation for optimal transport

$$\inf_{\pi\in\Pi(P_0,P_1)} \int_{\mathbb{A}\times\mathbb{B}} k(a,b)d\pi(a,b)$$

$$= \inf_{\pi\in\mathcal{P}(\mathbb{A}\times\mathbb{B})} \left(\int_{\mathbb{A}\times\mathbb{B}} k(a,b)d\pi(a,b) \dot{+} \delta_{\Pi(P_0,P_1)}(\pi)\right)$$

$$\geq \sup_{f\in C_b^0(\mathbb{A}),g\in C_b^0(\mathbb{B})} \left(-\left(\pi\mapsto\int_{\mathbb{A}\times\mathbb{B}} k(a,b)d\pi(a,b)\right)^{\alpha}(f,g) \quad \dot{+} \left(-\delta^{-\alpha}_{\Pi(P_0,\right.\right.$$

$$= \sup_{f\in C_b^0(\mathbb{A}),g\in C_b^0(\mathbb{B})} \left(-\sup_{a\in\mathbb{A},b\in\mathbb{B}}\left[f(a)-g(b)-k(a,b)\right] \quad \dot{+} \left(\int_{\mathbb{A}} f(a)dP_0(\right.\right.$$

$$= \sup_{f\in C_b^0(\mathbb{A}),g\in C_b^0(\mathbb{B})} \left(-\sup_{a\in\mathbb{A}}\left[f(a)+g^{-k}(a)\right]\right.$$

$$\left. + \int_{\mathbb{A}} f(a)dP_0(a) - \int_{\mathbb{B}} g(b)dP_1(b)\right)$$

$$=\left((f,g)\mapsto\sup_{a\in\mathbb{A}}\left[f(a)+g^{-k}(a)\right]\right)^{\alpha'}(P_0\otimes P_1)$$

Thank you for your attention !

E. J. Balder. An extension of duality-stability relations to nonconvex optimization problems. *SIAM Journal on Control and Optimization*, 15(2):329–343, 1977.

Heinz H. Bauschke and Patrick L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer-Verlag, New York, second edition, 2017.

R. T. Rockafellar and R. J-B. Wets. *Variational Analysis*. Springer-Verlag, Berlin, 1998.

R. Tyrrell Rockafellar. *Conjugate Duality and Optimization*. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 1974.