Resilience, Viability and Stochastic Optimization

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Formal ingredients for an operational definition of resilience

[Holling, 1973] C. S. Holling. Resilience and stability of ecological systems. *Annual Review of Ecology and Systematics*, 4:1–23, 1973.

Resilience is the capacity of a system to continually change and adapt yet remain within critical thresholds (Stockholm Resilience Centre)

- "continually change", "remain"
  $\rightarrow$ time variable (continuous, discrete)
- "system", "change"
  $\rightarrow$ states, dynamics, dynamical system
- "adapt"
  $\rightarrow$ actions, controls, decisions, strategies, policies, decision rules
- "remain within critical thresholds"
  $\rightarrow$ constraint set, admissibility, viable set, viability

To make a long story short . . .

Mathematical control theory, viability and stochastic optimization
offer material for an operational definition of resilience

Theory. Mathematics provides concepts, tools and methods

- states, controls, uncertainties, dynamics (control theory)
- scenarios, policies, critical thresholds
- (stochastic, robust) viability kernel = viable states
- minimal time of crisis, cost-efficiency (optimization)

Answers. Geometry + Optimization

- Viable states = resilient states
- Measuring resilience as the inverse of the minimal cost (expected, robust) to reach a viability kernel

Tribute to
Jean-Pierre Aubin, Patrick Saint-Pierre, Luc Doyen, Sophie Martin

Our emphasis on the treatment of uncertainties:
stochastic and robust viability, and possible extensions

Outline of the presentation

Critical thresholds *versus* smooth trade-offs

The viability approach and resilience

Measures of resilience and extensions

"Self-promotion, nobody will do it for you" ;-)

Outline of the presentation

Critical thresholds *versus* smooth trade-offs
- Critical thresholds in ecology, environment, sustainability
- Synthetic indicators
- Giving prices as a way to deal with multiple goals
- Economics of risk and time vs. catastrophe insurance
- Conclusion and roadmap

The viability approach and resilience
- A few words on the purpose of modelling
- (Deterministic) viability in a nutshell
- Handling uncertainty in control theory
- Stochastic viability in a nutshell
- Robust viability in a nutshell

Measures of resilience and extensions
- How to measure resilience?
- From viable states to viable random paths

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Prices vs. Quantities [Weitzman, 1974]

- In ecology, environment, sustainability, etc. critical thresholds are often put forward to control impacts of human activities
- Whereas economists lean on the sense of control by prices, although they consider control by quantities as equivalent by a duality argument (under convexity)
- However, in the presence of uncertainties, one instrument may prove superior to the other depending on the situation [Weitzman, 1974]
  For instance,
  Our intuitive feeling, which is confirmed by the formal analysis, is that it doesn't pay to "fool around" with prices in such situations [emergencies or natural calamities].

M. L. Weitzman. Prices vs. quantities. *Review of Economic Studies*, 41 (4):477–491, Oct. 1974

Our roadmap

- Showcase the assessment frameworks supposed to tackle multiple goals and risks, with exogenous critical thresholds
- Take note of the difficulty to agree on trade-offs
  for many issues in sustainable management:
  future generations, uncertainties, ecosystems
- Showcase the economics standpoint of smooth trade-offs (continuity assumption)
- Discuss critical thresholds *versus* smooth trade-offs in the climate change economic debate

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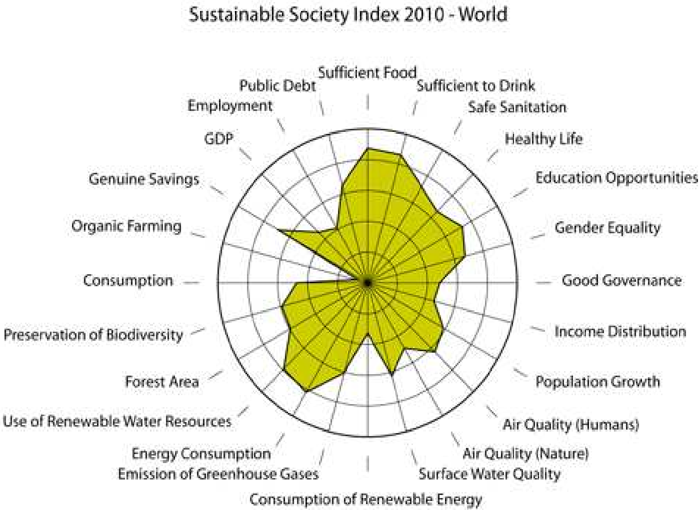

Sustainable Society Index 2010 - World
Sufficient Food
Sufficient to Drink
Safe Sanitation
Healthy Life
Education Opportunities
Gender Equality
Good Governance
Income Distribution
Population Growth
Air Quality (Humans)
Air Quality (Nature)
Surface Water Quality
Consumption of Renewable Energy
Emission of Greenhouse Gases
Energy Consumption
Use of Renewable Water Resources
Forest Area
Preservation of Biodiversity
Consumption
Organic Farming
Genuine Savings
GDP
Employment
Public Debt

A profusion of indicators compete to capture sustainability issues

En 2002, l'OCDE a dénombré 22 batteries d'indicateurs de développement durable

- 155 indicateurs de la stratégie européenne de développement durable, hiérachisés en 3 niveaux
- 800 indicateurs de la Banque Mondiale
- 99 indicateurs de la Suisse ventilés entre 24 thèmes
- 68 indicateurs de la Grande Bretagne
- 138 indicateurs des Nations unies
- les objectifs du millénaire du PNUD des Nations unies

A battery of assessment frameworks have been concocted to gauge policies w.r.t. risk and ecological impact

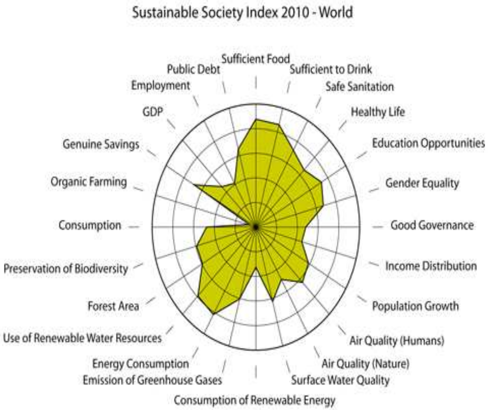
- Integrated Ecosystem Assessment (IEA)
  (National Oceanic and Atmospheric Administration)
- Ecological Risk Assessment
- Ecosystem-based Management (EBM)
- Ecosystem Approach to Management
- Driver Pressure State Impact Response (DPSIR) Approach
- Management strategy evaluation (MSE)

The a priori modelling of trade-offs between time, risk, economy, ecology, etc. is a delicate task

The contradictions of the Alliance of Small Island States w.r.t. shipping and aviation carbon tax

- The Alliance of Small Island States (AOSIS) is a coalition of small island and low-lying coastal countries that share similar development challenges and concerns about the environment, especially their vulnerability to the adverse effects of global climate change
- It functions primarily as an ad hoc lobby and negotiating voice for small island developing States (SIDS) within the United Nations system
- AOSIS was concerned about a proposal for a shipping and aviation tax as a way to mitigate against carbon emission, saying such a concept will discriminate against remote and far-flung members who already suffer from poor shipping and airline connections

Some economists recommend objectives to be expressed in their own units, without aggregation



The "Stiglitz-Sen-Fitoussi" Commission (2009) déconseille de privilégier un indicateur synthétique unique car, quel que soit l'indicateur envisagé, l'agrégation de données disparates ne va pas de soi

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Synthetic indicators remain numerous

- produit intérieur brut par tête (PIB)
- indice de soutenabilité environnementale du Forum Economique Mondial (ISE)
- indicateur de développement humain de l'ONU (IDH)
- empreinte écologique (EE)
- épargne nette ajustée de la Banque Mondiale
- indicateur de bien-être (IB)

No consistency emerges from such synthetic indicators

- Une étude de 2007 compare différents indicateurs en reprenant l'ensemble des performances de la plupart des pays du monde
- Le classement des différents pays apparaît alors comme fortement dépendant de l'indicateur choisi
- Le Canada
  - figure dans les huit premiers pays du monde en terme de PIB, d'IDH, d'ISE et d'IB
  - mais il est l'antépénultième en terme d'EE

Rapport CAS-LERNA, Préparation Grenelle de l'environnement,
La responsabilisation des entreprises, 2007

How Reliable an Indicator is the Ecological Footprint?

- *L'empreinte écologique : un indicateur ambigu*, Frédéric Paul Piguet, Isabelle Blanc, Tourane Corbiere-Nicollier et Suren Erkman, *Futuribles*, No 334, octobre 2007
- The concept of the ecological footprint has become well known as a composite indicator that is supposed to inform us about the space that human beings occupy in order to produce the resources they consume and the waste they create
- This is then set against the ecological capacity of the Earth (its biocapacity), and hence one can work out the environmental income that humanity has at its disposal
- The authors discuss various dead-ends that GFN reached deliberately, and the criticisms already made of the indicator, which - precisely because of its composite nature - aggregates disparate data and the proceeds to make calculations involving somewhat risky weightings
- The resulting conclusions are then open to challenge, for example when it is suggested that some countries should cut down their forests in order to increase the area for growing crops, whereas the increase of built-up areas (which also encroach on land under cultivation) is not questioned at all

The Ecological Footprint weighs in “space”, whereas Economics does it in “numeraire”

As Marcel Boiteux – Honorary President of Électricité de France, and famous economist – expresses it

- to decide is to choose
- to choose is to balance
- and to balance is to give prices to all things
  - material or immaterial
  - tradable or not tradable

[Boiteux, 1977] Marcel Boiteux, Du Culte de l’énergie, Foi et Vie, n. 23, avril 1977, 76e année

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Three seems a trifle too much...

Comme le dit un général Français

Pour décider,
il faut être un nombre impair.
Et trois, ça me paraît beaucoup...

A French general saying

To decide, you must be an odd number.
And three seems a bit too much...

Georges Clémenceau (1841–1929)

Une commission d'enquête pour être efficace
ne doit compter que trois membres,
dont deux sont absents

Décider c'est choisir, choisir c'est pondérer et pondérer c'est donner des prix à toute chose [Boiteux, 1977]

- Décider c'est choisir
- choisir c'est pondérer
- et pondérer c'est donner des prix à toute chose,
  - matérielle ou immatérielle,
  - marchande ou non marchande

"Pondération de chacune des raretés primaires dans leur infinie diversité, bilan consolidé de tous les cheminements, les uns dans les autres imbriqués, jusqu'à remonter à chacune de ces ressources rares, cela parait a priori tout à fait inextricable"

The economic posture: defining a social optimum respecting that you and I do not have the same tastes

Martin L. Weitzman [Weitzman, 2007]

> *An enormously important part of the "discipline" of economics is supposed to be that economists understand the difference between their own personal preferences for apples over oranges and the preferences of others for apples over oranges*

[Weitzman, 2007] Martin L. Weitzman. A review of the Stern review on the economics of climate change. *Journal of Economic Literature*, 45(3):703–724, 2007

The "invisible hand", the "tâtonnement de Walras" are supposed to adjust prices so as to decentralize a Pareto optimum

- Question: how to achieve a Pareto allocation?
- An economic answer: by means of a price system (a price for any good)
- Suppose that each agent has a budget (social issue)
- There exists a price system such that
  - if every agent selects the most preferable basket of good under his/her budget constraint
  - the resulting allocation is Pareto optimal
- Thus, prices are the coefficient weights that make decentralized decisions compatible with economic scarcity

Un vieux “truc” qui ne marche pas si mal [Boiteux, 1977]

- “Et pourtant, il y a, pour ce faire, un vieux ’truc’ que l’on utilise depuis des siècles et qui ne marche pas si mal.
- Cela consiste à affecter à chaque ressource élémentaire un coefficient plus ou moins élevé suivant sa rareté. . . coefficient que l’on appelle un prix.
- En multipliant par ce coefficient-prix la quantité de telle ressource rare que l’on mobilise, on obtient un coût ;
- ces coûts se cumulent tout le long des processus de fabrication pour aboutir au *prix de revient* du produit final. . .
- et la solution la meilleure, celle qui épargne au mieux les raretés élémentaires pondérées par leur importance relative, c’est celle qui coûte le moins cher !”
- “Je suis un peu confus d’avoir retenu votre attention jusqu’à maintenant pour en arriver à une telle banalité”.

Here are the ingredients for a multicriteria optimization problem

- A set $\mathbb{U}^{ad} \subset \mathbb{U}$ comprising decisions (over which there will be bargaining)
- A finite set $\mathbb{A}$ (stakeholders, viewpoints, multiple selves)
- Each stakeholder expresses her/his objective, need, preference by means of an indicator, criterion, objective function

$$\mathbb{U} \ni u \mapsto J_a(u) \in \mathbb{R} \; , \quad \forall a \in \mathbb{A}$$

  - Each criterion $J_a : \mathbb{U} \to \mathbb{R}(\cup\{+\infty\})$ takes (possibly extended) real numerical values, but expressed in its own unit
  - A large value is bad

Blanket assumption: when needed, the set $\mathbb{U}^{ad}$ is a convex subset of $\mathbb{R}^d$, all functions J_a are convex and qualification of constraints holds true

The space of outcomes

- In multicriteria optimization, stakeholders $a \in \mathbb{A}$ bargain over a common decision $u \in \mathbb{U}$
- For this purpose, they consider the image of the mapping

$$\{J_a\}_{a \in \mathbb{A}} : \mathbb{U}^{ad} \to \mathbb{R}^{\mathbb{A}}$$

in the space $\mathbb{R}^{\mathbb{A}}$ of joint outcomes

Pareto optima can be obtained by (monocriterion) optimization in two ways

- Weights (prices)
  Pick a family $\{\lambda_a\}_{a \in \mathbb{A}} \in \mathbb{R}_+^{\mathbb{A}}$ of weights,
  and then solve the optimization problem

$$\min_{u \in \mathbb{U}^{ad}} \sum_{a \in \mathbb{A}} \lambda_a J_a(u)$$

- Focal agent and thresholds (quantities)
  - Pick a focal agent $\bar{a} \in \mathbb{A}$ (whatever)
  - Pick a family $\theta_{-\bar{a}} = \{\theta_a\}_{a \in \mathbb{A} \setminus \{\bar{a}\}} \in \mathbb{R}^{\mathbb{A} \setminus \{\bar{a}\}}$ of thresholds
    (each in its own unit)

  and then solve the optimization problem

$$\begin{aligned} J_{\bar{a}}^*(\theta_{-\bar{a}}) = & \quad \min_{u \in \mathbb{U}^{ad}} J_{\bar{a}}(u) \\ \text{under the constraints} & \quad J_a(u) \leq \theta_a \; , \quad \forall a \in \mathbb{A} \setminus \{\bar{a}\} \end{aligned}$$

From thresholds to weights

- Solving the optimization problem (cost-effectiveness)

$$J^*_{\bar{a}}(\theta_{-\bar{a}}) = \begin{array}{l} \min_{u \in \mathbb{U}^{ad}} J_{\bar{a}}(u) \\ J_a(u) \leq \theta_a \ , \ \ \forall a \in \mathbb{A} \setminus \{\bar{a}\} \end{array}$$

one obtains

  - an optimal solution $u^* \in \mathbb{U}$
  - a family $\lambda^*_{-\bar{a}} = \{\lambda^*_a\}_{a \in \mathbb{A} \setminus \{\bar{a}\}} \in \mathbb{R}_+^{\mathbb{A} \setminus \{\bar{a}\}}$ of Lagrange multipliers (provided as multipliers of the constraints)

- The optimal solution $u^* \in \mathbb{U}$ also solves

$$\min_{u \in \mathbb{U}^{ad}} \underbrace{1 \times J_{\bar{a}}(u) + \sum_{a \neq \bar{a}} \lambda^*_a \times J_a(u)}_{\text{socio-economic costs}}$$

Weights are (shadow) prices

- Starting from thresholds expressed in their own units, we obtain Lagrange multipliers, that is, dual variables in the duality between quantities and prices
- Historically, dual variables have moved from geometric (Lagrange) to economic (Kantorovich) flavor
  - Lagrange multipliers of inequality constraints are geometric dual variables
  - Kantorovich "resolving multipliers" of constrained primal quantities (or "objectively determined estimators") are economic dual variables
- The price of a resource is the sensitivity of the optimal payoff with respect to marginal changes $\theta_a \to \theta_a + \epsilon_a$

$$\lambda_a^* = \frac{\partial J_{\bar{a}}^*(\theta_{-\bar{a}})}{\partial \theta_a} \; , \;\; \forall a \in \mathbb{A} \setminus \{\bar{a}\}$$

Are prices proper weights?

- Brûler du pétrole, c'est comme brûler sa commode Louis XV (Marcel Boiteux)
- "Les prix qui règnent dans nos économies traduisent-ils correctement, et durablement, tous les aspects des raretés dont la menace pèse sur l'humanité ?" [Boiteux, 1977]
- "l'application obtuse de l'actualisation, à prix constants et sur les seules valeurs marchandes, trahit les réalités et les aspirations profondes de nos sociétés" [Boiteux, 1976]
- "pour les modèles à long terme, l'approche par les prix n'est pas la meilleure (mieux vaut travailler sur les quantités et trouver les prix par dualité pour orienter ensuite les choix décentralisés des acteurs)" [Boiteux, 1976]
- Debate on economic valuation of externalities

M. Boiteux. À propos de la "critique de la théorie de l'actualisation telle qu'employée en France". *Revue d'Économie Politique*, 5, 1976

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"Self-promotion, nobody will do it for you" ;-)

How policy-makers aggregate flows of goods or services over time, risk and within generations is crucial to policy design and choice [Stern, 2006]

[Stern, 2006] Nicholas Stern. *The Economics of Climate Change*, 2006

- Flows of goods or services over time
  - consumption C_t
  - environment E_t
- *How policy-makers aggregate over consequences*
  - *(i) within generations*
  - *(ii) over time*
  - *(iii) according to risk*

  will be crucial to policy design and choice

The discount rate materializes trade-offs between distant time periods

$$\sum_{t=t_0}^{+\infty} \left(\frac{1}{1+r_e}\right)^{t-t_0} \overbrace{L(C_t, E_t)}^{\text{utility}}$$

"In fact, it is not an exaggeration to say that the biggest uncertainty of all in the economics of climate change is the uncertainty about which interest rate to use for discounting. In one form or another this little secret is known to insiders in the economics of climate change, but it needs to be more widely appreciated by economists at large."

[Weitzman, 2007]

Expected intertemporal discounted utility is grounded in smooth trade-offs

$$\mathbb{E}\left[\sum_{t=t_0}^{+\infty}\left(\frac{1}{1+r_e}\right)^{t-t_0}\overbrace{L(C_t,E_t)}^{\text{utility}}\right]$$

Expected intertemporal discounted utility is built upon two well axiomatized theories,
where "continuity of preferences" plays a major role

- the discounted intertemporal utility
  T. Koopmans. On the concept of optimal economic growth. *Academia Scientiarium Scripta Varia*, 28:225–300, 1965
- the expected utility
  J. von Neuman and O. Morgenstern. *Theory of games and economic behaviour*. Princeton University Press, Princeton, 1947. 2nd edition

This approach is widely used; it displays time consistency

Catastrophe insurance vs. consumption smoothing

[Weitzman, 2007]

But I think progress begins by recognizing
that the hidden core meaning of Stern vs. Critics may be about $(\cdots)$

- *catastrophe insurance*

$$\max \mathbb{P}\left[\underbrace{C_t \geq C^\flat,\ E_t \geq E^\flat}_{\text{indicators} \,\geq\, \text{thresholds}} \ , \quad \forall t = t_0, \ldots, +\infty \right]$$

- *versus consumption smoothing*

$$\max \mathbb{E}\left[\sum_{t=t_0}^{+\infty} \left(\frac{1}{1+r_e}\right)^{t-t_0} \underbrace{L\big(C_t, E_t\big)}_{\text{utility}} \right]$$

“Please leave the toilets clean for the next person to use” ;-)

> *The notion of “stewardship” can be seen as a special form of sustainability. It points to particular aspects of the world, which should themselves be passed on in a state at least as good as that inherited from the previous generation.*

[Stern, 2006]

> *If sustainability means anything more than a vague emotional commitment, it must require that something be conserved for the very long run. It is very important to understand what that thing is: I think it has to be a generalized capacity to produce economic well-being.*

R. M. Solow. An almost practical step towards sustainability. *Resources Policy*, 19:162–172, 1993.

A summary table

	time compensatory	time non-compensatory
deterministic	discounted utility	Rawls, viability
risk compensatory	expected discounted utility	expected Rawls, stochastic viability
risk non-compensatory	robust discounted utility	robust Rawls, robust viability

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- "Self-promotion, nobody will do it for you" ;-)

Conclusion

Tension between
critical thresholds (non-compensatory)
and
smooth trade-offs (compensatory)

Maybe critical thresholds (quantities)
for long term issues or risky situations

Our roadmap

- Showcase control theory as a panoply of concepts and tools to handle
  - time and dynamics
  - multiple objectives
  - uncertainty

  (Display examples)
- Prepare the ground for a “geometric” approach (acceptable sets) to handle sustainability and resilience issues
  (Display examples and the role of stochastic optimization)
- Recover trade-offs thanks to cost-efficiency, by measuring a “cost distance” to proper sets

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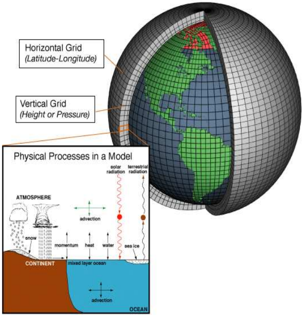
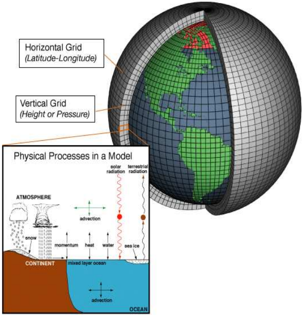
We distinguish two polar classes of models: knowledge models *versus* decision models



Knowledge models:
$1/1\ 000\ 000 \to 1/1\ 000 \to 1/1$
maps

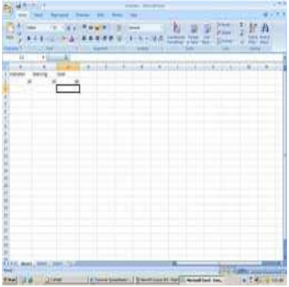
Office of Oceanic and Atmospheric Research (OAR) climate model

We distinguish two polar classes of models: knowledge models *versus* decision models



Knowledge models:
1/1 000 000 → 1/1 000 → 1/1
maps

Office of Oceanic and Atmospheric Research (OAR) climate model



Action/decision models:
economic models are fables designed to provide insight

William Nordhaus economic-climate model

Crafting a model is a trade-off between, on the one hand, realism and complexity, and, on the other hand, mathematical tractabilility

- System: Greek *systema*, arrangement, organized whole
- Complex: Greek *complexus*, composed of parts
  - *com-* "with"
  - *plectere* "to weave, braid, twine"

This talk is *not* about crafting dynamical models

- Elaborating a dynamical model is a delicate venture
  - Peter Yodzis, *Predator-Prey Theory and Management of Multispecies Fisheries*, Ecological Applications 4:51–58, 1994

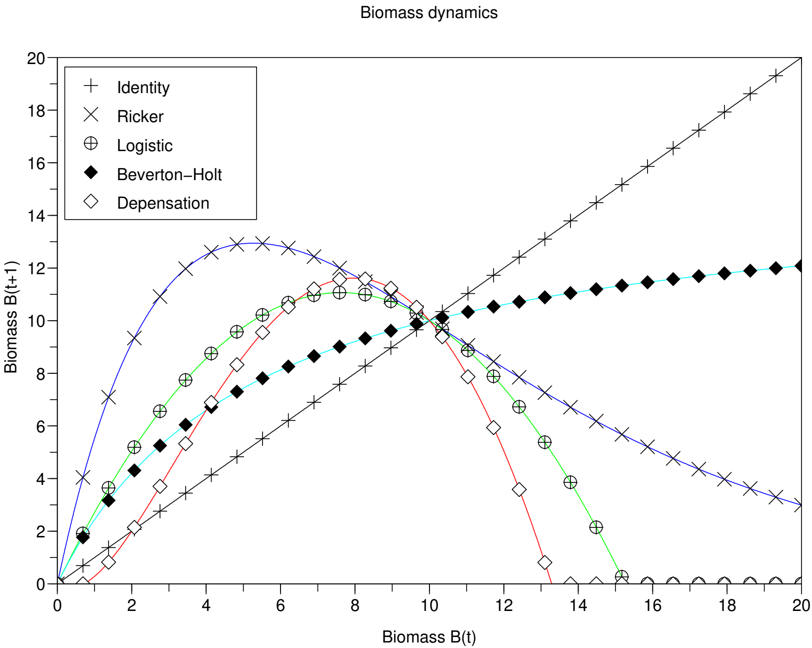
  In population modelling the functional forms of models are at least as important as are parameter values in expressing the underlying biology and in determining the outcome. (...) For instance, May et al. (1979) assumed, without comment, a particular form of predator-prey interaction; and this particular form was carried over, again without comment, by Flaaten. It turns out that this "invisible" but powerful assumption is responsible in large part for the conclusion reached by Flaaten (1988). (...) Flaaten's work is controversial because of his conclusion that "sea mammals should be heavily depleted to increase the surplus production of fish resources for man" (Flaaten 1988:114).
- Our starting point will be a mathematical dynamical model that captures how sequences of decisions affect a "piece of reality"
- Then, we will use such a model to frame a decision problem

Population management

Viable management of an animal population [De Lara and Doyen, 2008]

$$B(t+1) = \overbrace{\texttt{Biol}}^{\text{dynamic}} \Big(\underbrace{B(t)}_{\text{biomass}} - \underbrace{h(t)}_{\text{catches}} \Big)$$

- $B(t)$ biomass
- $h(t)$ catch with $0 \leq h(t) \leq B(t)$
- `Biol` natural resource growth function
  (linear, logistic, etc.)



Distinct population dynamics `Biol` for $r = 1.9$, $K = 10$, $B^\flat = 2$

We define an ecological window by lower and upper bounds for the biomass

State constraints

$$B^{\flat} \leq B(t) \leq B^{\sharp} \,, \quad t = t_0, \ldots, T$$

- $B^{\flat}$ minimum viable population
- $B^{\sharp}$ maximal safety value
  (pest control, invasive species)

Epidemics control

Endemic channel forms the core of a decision rule for dengue outbreak prevention

> *The epidemiological surveillance system should be able to differentiate between transient and seasonal increases in disease incidence and increases observed at the beginning of a dengue outbreak. One such approach is to track the occurrence of current (probable) cases and compare them with the average number of cases by week (or month) of the preceding 5–7 years, with confidence intervals set at two standard deviations above and below the average (± 2 SD). This is sometimes referred to as the "endemic channel". If the number of cases reported exceeds 2 SDs above the "endemic channel" in weekly or monthly reporting, an outbreak alert is triggered.*

Dengue. Guidelines for Diagnosis, Treatment, Prevention and Control. A joint publication of the World Health Organization (WHO) and the Special Programme for Research and Training in Tropical Diseases (TDR), 2009

“Canal Endémico” stands as the reference to control dengue

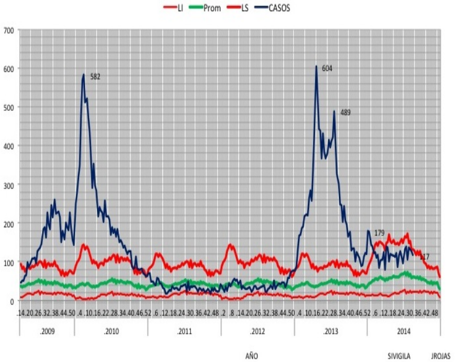


Figure: Cases of dengue between 2009 and 2014. Source: Secretaría Municipal de Salud de Cali.

Program "Dengue Control" of SMS

Control mosquito breeding sites

Capping the human infected population with the Ross-Macdonald model [De Lara and Sepulveda, 2016]

- The dynamics of the system is given by

$$\text{infected mosquito proportion} \quad \frac{dm}{dt} = A_m h(t)(1-m(t)) - u(t)m(t)$$

$$\text{infected human proportion} \quad \frac{dh}{dt} = A_h m(t)(1-h(t)) - \gamma h(t)$$

- Determine, if it exists, a piecewise continuous function (fumigation policy rates) $u(\cdot)$,

$$u(\cdot) : t \mapsto u(t) \,, \;\; \underline{u} \leq u(t) \leq \overline{u} \,, \;\; \forall t \geq 0$$

such that the following so-called viability constraint is satisfied

$$h(t) \leq \overline{H} \,, \;\; \forall t \geq 0$$

To deal with uncertainties, we sample the controlled Ross–Macdonald model [Sepulveda Salcedo and De Lara, 2019]

$$\big(M_{t+1}, H_{t+1}\big) = f\big(M_t, H_t, u_t, \underbrace{A_t^M, A_t^H}_{\text{uncertainties}}\big)$$

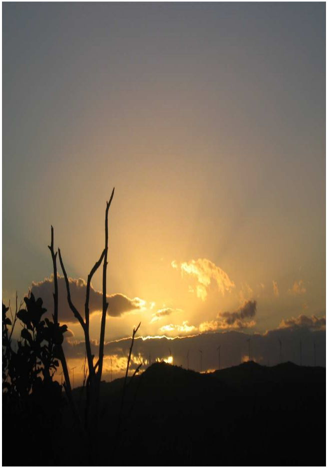
- Basic variables and parameters are
  - time $t = t_0, t_0 + 1 \ldots, T - 1, T$, measured in days
  - M_t, the proportion of infected mosquitos (Aedes Aegypti adultos) at the beginning of the day $[t, t+1[$
  - H_t, the proportion of infected humans at the beginning of the day $[t, t+1[$
  - $u(t)$, the mosquito mortality rate (application of chemical control) applied during all day $[t, t+1[$
- The objective is to maintain infected humans at a low level

$$H_t \leq \overline{H}\ , \ \ \forall t = t_0, \ldots, T$$

with limited resources $\underline{u} \leq u_t \leq \overline{u}\ , \ \ \forall t = t_0, \ldots, T - 1$

Climate change mitigation

Let us scout a very stylized model of the climate-economy system [De Lara and Doyen, 2008]



We lay out a dynamical model with

- two state variables

  environmental: atmospheric CO_2 concentration level $M(t)$

  economic: gross world product GWP $Q(t)$

- one decision variable, the emission abatement rate $a(t)$

A carbon cycle model "à la Nordhaus" is an example of *decision model*

- Time index t in years
- Economic production $Q(t)$ (GWP)

$$Q(t+1) = \overbrace{(1+g)}^{\text{economic growth}} Q(t)$$

- CO_2 concentration $M(t)$

$$M(t+1) = M(t) \underbrace{-\delta(M(t) - M_{-\infty})}_{\text{natural sinks}} + \alpha \overbrace{\texttt{Emiss}\big(Q(t)\big)}^{\text{emissions}} \underbrace{\big(1 - a(t)\big)}_{\text{abatement}}$$

- Decision $a(t) \in [0, 1]$ is the abatement rate of CO_2 emissions

Data

- $M(t)$ CO_2 atmospheric concentration, measured in ppm, parts per million
  (379 ppm in 2005)
- $M_{-\infty}$ pre-industrial atmospheric concentration
  (about 280 ppm)
- $\texttt{Emiss}(Q(t))$ "business as usual" CO_2 emissions
  (about 7.2 GtC per year between 2000 and 2005)
- $0 \leq a(t) \leq 1$ abatement rate reduction of CO_2 emissions
- α conversion factor from emissions to concentration
  ($\alpha \approx 0.471$ ppm.GtC^{-1} sums up highly complex physical mechanisms)
- δ natural rate of removal of atmospheric CO_2 to unspecified sinks
  ($\delta \approx 0.01$ year^{-1})

A concentration target is pursued to avoid danger

United Nations Framework Convention on Climate Change

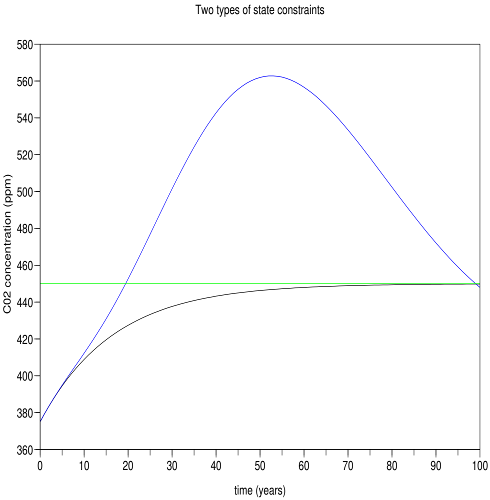
"to achieve, (...), stabilization of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system"

Limitation of concentrations of CO_2

- below a tolerable threshold $M^\sharp$ (say 350 ppm, 450 ppm)
- at a specified date $T > 0$ (say year 2050 or 2100)

$$\underbrace{M(T)}_{\text{concentration at horizon}} \leq \underbrace{M^\sharp}_{\text{threshold}}$$

Constraints capture different requirements



- The concentration has to remain below a tolerable level at the horizon T:

$$M(T) \leq M^{\sharp}$$

- More demanding: from the initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp}$$

$$t = t_0, \ldots, T$$

Constraints may be environmental, physical, economic

- The concentration has to remain below a tolerable level from initial time t_0 up to the horizon T

$$M(t) \leq M^{\sharp} , \quad t = t_0, \ldots, T$$

- Abatements are expressed as fractions

$$0 \leq a(t) \leq 1 , \quad t = t_0, \ldots, T-1$$

- As with "cap and trade", setting a ceiling on $\mathrm{CO_2}$ price amounts to cap abatement costs

$$\underbrace{C\big(a(t), Q(t)\big)}_{\text{costs}} \leq c^{\sharp} \, (100 \text{ euros / tonne } \mathrm{CO_2}) , \quad t = t_0, \ldots, T-1$$

Mixing dynamics, optimization and constraints yields a cost-effectiveness problem

- Minimize abatement costs

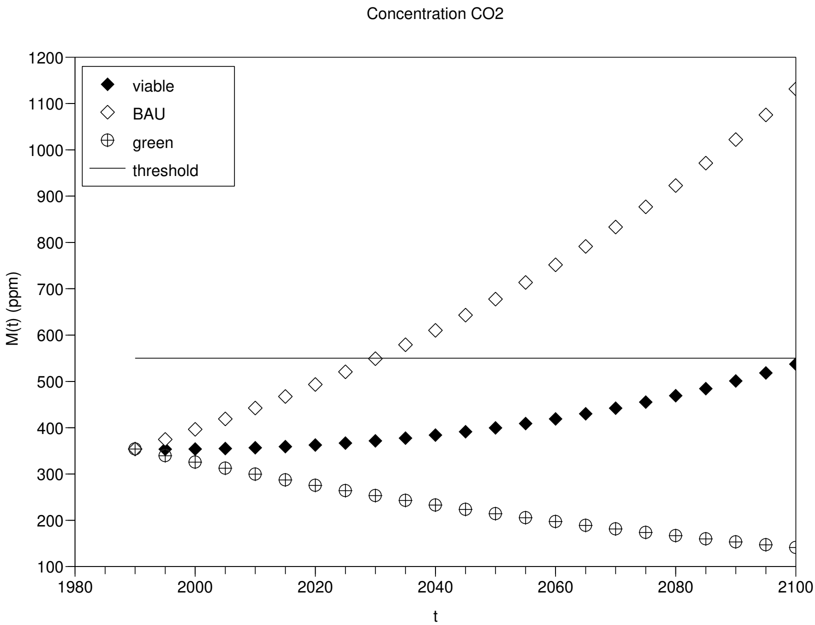
$$\min_{a(t_0),\dots,a(T-1)} \sum_{t=t_0}^{T-1} \left(\frac{1}{1+r_e}\right)^{t-t_0} \underbrace{C\big(a(t),Q(t)\big)}_{\text{abatement costs}}$$

- under the GWP-CO_2 dynamics

$$\begin{cases} M(t+1) & = M(t) - \delta(M(t) - M_{-\infty}) + \alpha \mathtt{Emiss}\big(Q(t)\big)(1-a(t)) \\ Q(t+1) & = (1+g)Q(t) \end{cases}$$

- and under target constraint

$$\underbrace{M(T) \leq M^{\sharp}}_{\text{CO2 concentration}}$$


Concentration CO2
viable
BAU
green
threshold
M(t) (ppm)
1200
1100
1000
900
800
700
600
500
400
300
200
100
1980
2000
2020
2040
2060
2080
2100
t

Fishery management (I)

Populations can be described by abundances at ages

Jack Mackrel abundances (Chilean data) are measured in thousand of individuals

13651022 thousand of age < 1 (recruits)
7495888 thousand of age $\in [1, 2[$
6804151
4191318
4582943
2500338
1139182
523261
269328
166390
95606 thousand of age ≥ 11

We now line up the ingredients of a harvested population age-class dynamical model

- Time $t \in \mathbb{N}$ measured in years
- Abundances at age $N = (N_a)_{a=1,\dots,A} \in \mathbb{X} = \mathbb{R}_+^A$
- $a \in \{1, \dots, A\}$ age class index
  - $A = 3$ for anchovy
  - $A = 8$ for hake
  - $A = 40$ for bacalao
- Control variable $\lambda \in \mathbb{U} = \mathbb{R}_+$ is fishing effort

One year older every year...

Except for the recruits ($a = 1$) and the last age class ($a = A$),

$$N_a(t+1) = e^{-\overbrace{(\underbrace{M_{a-1}}_{\text{natural}} + \underbrace{\lambda(t)F_{a-1}}_{\text{fishing}})}^{\text{mortality}}} N_{a-1}(t), \quad a = 2, \ldots, A-1$$

where

- M_a stands for the natural mortality-at-age a
- F_a is the harvesting mortality rate of individuals of age a, also called exploitation pattern-at-age a, related to the mesh size for instance
- the control variable $\lambda(t)$ is the fishing effort, or the exploitation pattern multiplier

The last age-class may comprise a plus-group

- N_A is the abundance of individuals of age above $A-1$ (and not equal, like for other classes)
- To account for this specificity, one considers the dynamics

$$\begin{aligned} N_A(t+1) &= N_{A-1}(t)\exp\big(-(M_{A-1}+\lambda(t)F_{A-1})\big) \\ &+ \underbrace{\pi}_{0 \text{ or } 1} N_A(t)\exp\big(-(M_A+\lambda(t)F_A)\big) \end{aligned}$$

- The parameter $\pi \in \{0,1\}$ is related to the existence of a so-called plus-group
  - if we neglect the survivors older than age A, then $\pi = 0$ (an example is anchovy)
  - if we consider the survivors older than age A, then $\pi = 1$, and the last age class is a plus group (an example is hake)

The stock-recruitment function mathematically turns spawning stock biomass into future recruits abundance

- The spawning stock biomass is

$$\mathtt{SSB}(N) = \sum_{a=1}^{A} \underbrace{\gamma_a}_{\text{proportion}} \overbrace{\mu_a}^{mass} \underbrace{N_a}_{\text{abundance}}$$

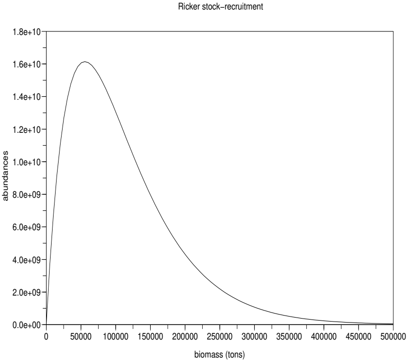
  - γ_a proportion of matures-at-age a
  - μ_a weight-at-age a

- The stock-recruitment relationship $\mathtt{S/R}$ turns biomass into abundance

$$\underbrace{N_1(t+1)}_{\text{future recruits}} = \mathtt{S/R}\Big(\underbrace{\mathtt{SSB}\big(N(t)\big)}_{\text{spawning biomass}} \Big)$$

Here are traditional examples of stock-recruitment functions

Recruitment involves complex biological and environmental processes that fluctuate in time, and are difficult to integrate into a population model



- constant: $\mathtt{S/R}(B) = R$
- linear: $\mathtt{S/R}(B) = rB$
- Beverton-Holt: $\mathtt{S/R}(B) = \frac{B}{\alpha+\beta B}$
- Ricker: $\mathtt{S/R}(B) = \alpha B \mathrm{e}^{-\beta B}$

And here are the state vector and the control

- The state vector $N(t)$ is forged with abundances at age

$$N(t) = \begin{pmatrix} N_1(t) \\ N_2(t) \\ \vdots \\ N_{A-1}(t) \\ N_A(t) \end{pmatrix} \in \mathbb{R}_+^A$$

- The scalar control $\lambda(t)$ is the fishing effort multiplier

A harvested population age-class model is an A—dimensional controlled dynamical system

$$
\begin{aligned}
N_1(t+1) &= \mathtt{S/R}\Big(\overbrace{\mathtt{SSB}\big(N(t)\big)}^{\text{spawning biomass}} \Big) \quad \text{recruitment} \\
N_2(t+1) &= e^{-(M_1+\lambda(t)F_1)} N_1(t) \\
N_a(t+1) &= e^{-\overbrace{(\underbrace{M_{a-1}}_{\text{natural}} + \underbrace{\lambda(t)F_{a-1}}_{\text{fishing}})}^{\text{mortality}}} N_{a-1}(t), \quad a = 2, \ldots, A-1 \\
N_{A-1}(t+1) &= e^{-(M_{A-2}+\lambda(t)F_{A-2})} N_{A-2}(t) \\
N_A(t+1) &= e^{-(M_{A-1}+\lambda(t)F_{A-1})} N_{A-1}(t) + \underbrace{\pi e^{-(M_A+\lambda(t)F_A)}}_{\text{plus group}} N_A(t)
\end{aligned}
$$

The ICES precautionary approach uses indicators and reference points to tackle ecological objectives [De Lara, Doyen, Guilbaud, and Rochet, 2007]

International Council for the Exploration of the Sea precautionary approach

- keeping (or restoring) spawning stock biomass SSB indicator above a threshold reference point B_{lim}
- restricting fishing effort to have mean fishing mortality F indicator below a threshold reference point F_{lim}

Definition	**Notation**	**Anchovy**	**Hake**
F limit RP	F_{lim}	/	0.35
SSB limit RP (t)	B_{lim}	21 000	100 000

Spawning biomass and fishing mortality are outputs of the harvested population age-class model

- Spawning stock biomass

$$\mathtt{SSB}(N) = \sum_{a=1}^{A} \underbrace{\gamma_a}_{\text{proportion}} \overbrace{\mu_a}^{\text{mass}} \underbrace{N_a}_{\text{abundance}}$$

with reference point $\mathtt{SSB}(N) \geq B_{lim}$

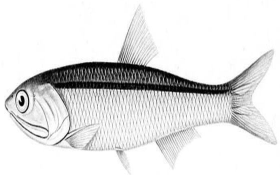
- Mean fishing mortality over age range from a_r to A_r

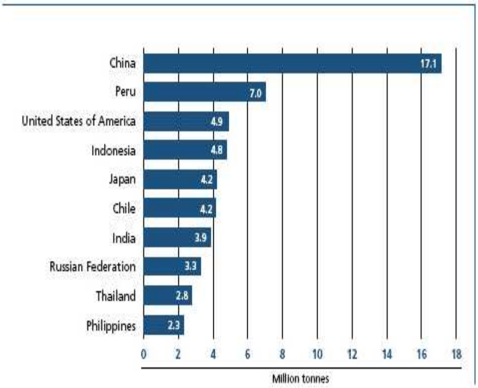
$$F(\lambda) = \frac{\lambda}{A_r - a_r + 1} \sum_{a=a_r}^{a=A_r} F_a$$

with reference point $F(\lambda) \leq F_{lim}$

Fishery management (II)

Perú is World 2nd for marine and inland capture fisheries





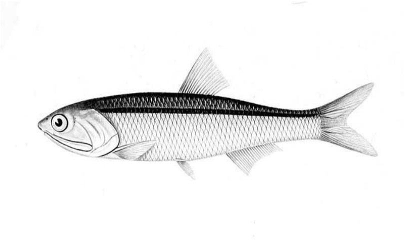
The northern Humboldt current system off Perú covers

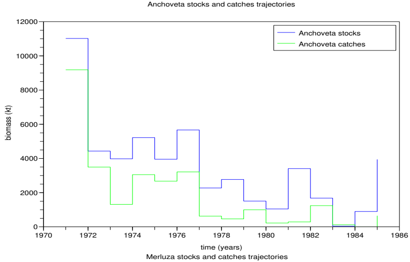
less than 0.1% of the world ocean

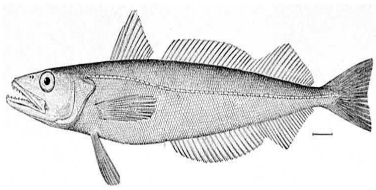
but presently sustains

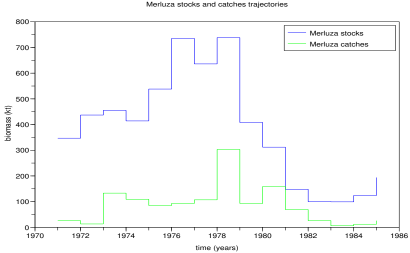
about 10% of the world fish catch

We were lucky enough that IMARPE entrusted us yearly data of anchoveta and merluza stock and catches from 1971 to 1985









We consider two species targeted by two fleets in a biomass ecosystem dynamic [De Lara, Ocaña Anaya, Oliveros-Ramos, and Tam, 2012]

We embody stocks and fishing interactions
in a two-dimensional dynamical model

$$
\begin{aligned}
\overbrace{A_{t+1}}^{\text{future biomass}} &= A_t \overbrace{\mathcal{R}^A(A_t, H_t)}^{\text{growth factor}} \big(1 - \underbrace{E_t^A}_{\text{effort}}\big) \\
H_{t+1} &= H_t \mathcal{R}^H(A_t, H_t) \big(1 - \overbrace{E_t^H}^{\text{control}}\big)
\end{aligned}
$$

- State vector (A_t, H_t) represents biomasses
- Control vector (E_t^A, E_t^H) is fishing effort of each species
- Catches are $E_t^A \mathcal{R}^A(A_t, H_t) A_t$ and $E_t^H \mathcal{R}^H(A_t, H_t) H_t$ (measured in biomass)

Our objectives are twofold: conservation and production

The viability kernel is the set of initial species biomasses (A_{t_0}, H_{t_0}) from which appropriate effort controls $(E_t^A, E_t^H),\ t = t_0, t_0+1, \ldots$ produce a trajectory of biomasses $(A_t, H_t),\ t = t_0, t_0+1, \ldots$ such that the following goals are satisfied

- preservation (minimal biomass thresholds)

$$A \text{ \underline{stocks}:} \qquad A_t \geq \underline{S^A}$$

$$H \text{ \underline{stocks}:} \qquad H_t \geq \underline{S^H}$$

- economic/social requirements (minimal catch thresholds)

$$A \text{ \underline{catches}:} \qquad E_t^A \mathcal{R}^A(A_t, H_t) A_t \geq \underline{C^A}$$

$$H \text{ \underline{catches}:} \qquad E_t^H \mathcal{R}^H(A_t, H_t) H_t \geq \underline{C^H}$$

Outline of the presentation

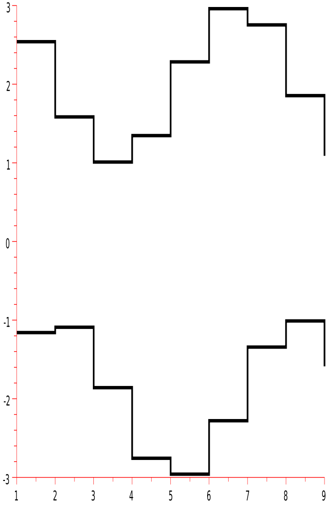
Critical thresholds *versus* smooth trade-offs
- Critical thresholds in ecology, environment, sustainability
- Synthetic indicators
- Giving prices as a way to deal with multiple goals
- Economics of risk and time vs. catastrophe insurance
- Conclusion and roadmap

The viability approach and resilience
- A few words on the purpose of modelling
- (Deterministic) viability in a nutshell
- Handling uncertainty in control theory
- Stochastic viability in a nutshell
- Robust viability in a nutshell

Measures of resilience and extensions
- How to measure resilience?
- From viable states to viable random paths

"Self-promotion, nobody will do it for you" ;-)

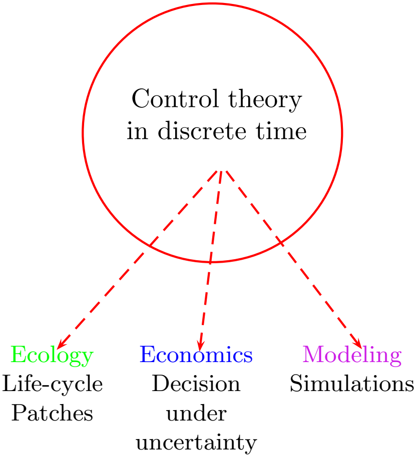
What is resilience?



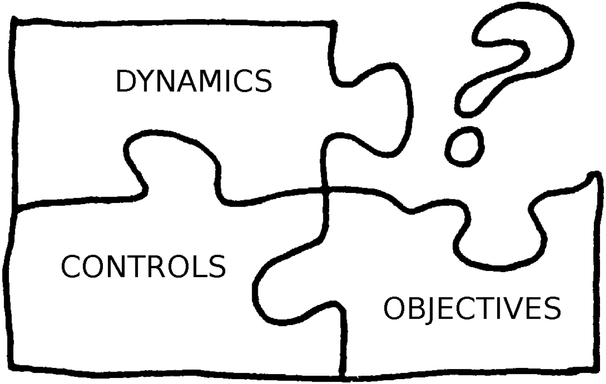
Resilience is the capacity of a system
to continually change and adapt
yet remain within critical thresholds

Stockholm Resilience Centre

We showcase control theory in discrete time as a proper vehicle for problem formulation [De Lara and Doyen, 2008]



Viability is relevant to address the compatibility puzzle

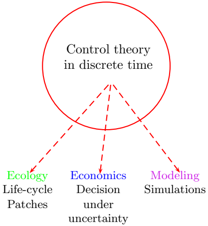


Discrete time nonlinear state-control system

$$x_{t+1} = f_t\big(x_t, u_t\big) \ , \quad t \in \mathbb{T} = \{t_0, t_0+1, \ldots, T-1\}$$

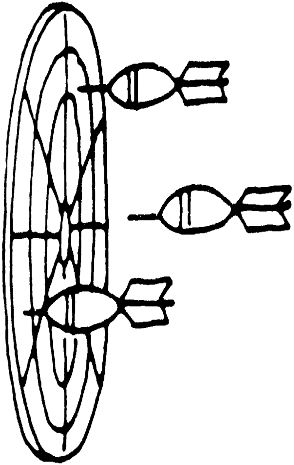
- the time t (stage) $\in \overline{\mathbb{T}} = \{t_0, t_0+1, \ldots, T-1, T\} \subset \mathbb{N}$ is discrete with initial time t_0 and horizon T ($T < +\infty$ or $T = +\infty$) (*the time period* $[t, t+1[$ *may be a year, a month, etc.*)
- the state variable x_t belongs to the *state space* $\mathbb{X} = \mathbb{R}^{n_\mathbb{X}}$ (*stocks, biomasses, abundances, capital*)
- the control variable u_t is an element of the *control space* $\mathbb{U} = \mathbb{R}^{n_\mathbb{U}}$ (*inflows, outflows, catches, harvesting effort, investment*)
- the dynamics f_t maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$ (*storage, age-class model, population dynamics, economic model*)

We dress natural resources management issues in the formal clothes of control theory in discrete time



- Problems are framed as
  - find controls/decisions driving a dynamical system
  - to achieve various goals
- Three main ingredients are
  - controlled dynamics
  - constraints
  - criterion to optimize

We mathematically express the objectives pursued as control and state constraints



- For a state-control system,
  we cloth objectives as constraints
- and we distinguish

  control constraints (rather easy)
  state constraints (rather difficult)
- Viability theory deals with
  state constraints

Constraints may be explicit on the control variable

and are rather easily handled by reducing the decision set

Examples of control constraints

- Irreversibility constraints, physical bounds
  $0 \leq a_t \leq 1 \; , \quad 0 \leq h_t \leq B_t$
- Tolerable costs $c\big(a_t, Q_t\big) \leq c^\sharp$

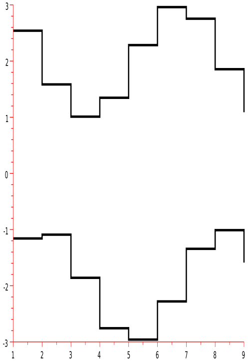
Control constraints / admissible decisions

$$\underbrace{u_t}_{\text{control}} \in \underbrace{\mathbb{B}_t\big(x_t\big)}_{\text{admissible set}} \quad , \quad t = t_0, \ldots, T-1$$

Easy because control variables u_t are precisely those variables
whose values the decision-maker can fix at any time within given bounds

Meeting constraints bearing on the state variable is delicate

due to the dynamics pipeline between controls and state



State constraints / admissible states

$$\underbrace{x_t}_{\text{state}} \in \underbrace{\mathbb{A}_t}_{\text{admissible set}} \quad , \quad t = t_0, \ldots, T$$

Examples ("tipping points")

- CO_2 concentration $M_t \leq M^\sharp$
- biomass $B^\flat \leq B_t \leq B^\sharp$

State constraints are mathematically difficult because of "inertia"

$$x_t = \underbrace{\text{function}}_{\text{iterated dynamics}} \big(\underbrace{u_{t-1}, \ldots, u_{t_0}}_{\text{past controls}}, x_{t_0} \big)$$

Target and asymptotic state constraints are special cases

- Final state achieves some target

$$\underbrace{x_T}_{\text{final state}} \in \underbrace{\mathbb{A}_T}_{\text{target set}}$$

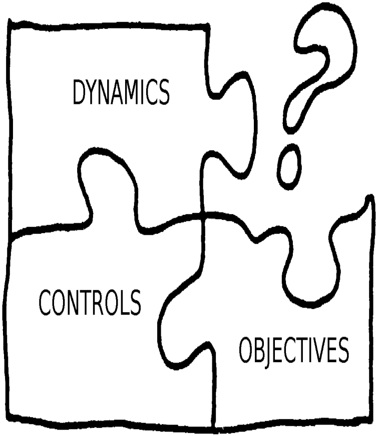
Example: $\mathrm{CO_2}$ concentration

- State converges toward a target

$$\underbrace{\lim_{t \to +\infty} x_t}_{\text{asymptotic state}} \in \underbrace{\mathbb{A}_\infty}_{\text{target set}}$$

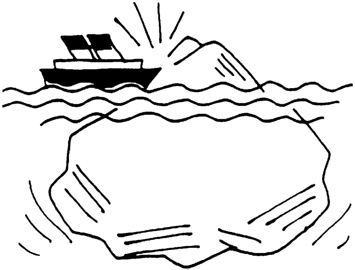
Example: in mathematical epidemiology,
convergence towards an endemic state

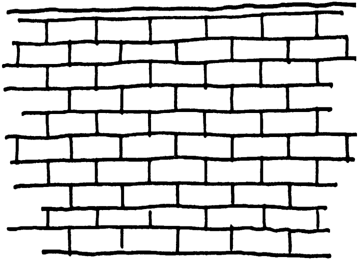
Can we solve the compatibility puzzle between dynamics and objectives by means of suitable controls?



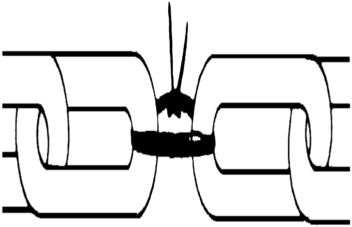
- Given a dynamics that mathematically embodies the causal impact of controls on the state
- Imposing objectives bearing on output variables (states, controls)
- Is it possible to find a control path that achieves the objectives for all times?

Crisis occurs when constraints are trespassed at least once





- An initial state is not viable if, whatever the sequence of controls, a crisis occurs
- There exists a time when one of the state or control constraints is violated



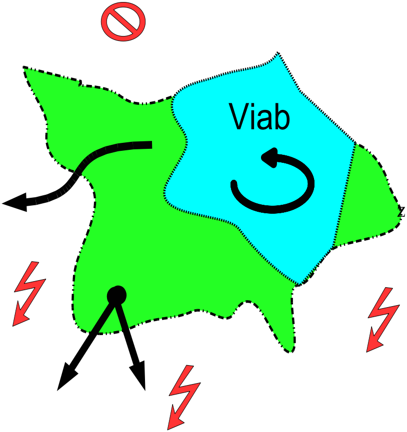
The compatibility puzzle can be solved when the initial viability kernel $\mathbb{V}\mathrm{iab}_{t_0}$ is not empty [Aubin, 1991]

Viable initial states form the viability kernel

$$
\mathbb{V}\mathrm{iab}_t = \left\{ \begin{array}{c} \text{initial} \\ \text{states} \\ x \in \mathbb{X} \end{array} \;\middle|\; \begin{array}{lr} \text{there exist a control path } u(\cdot) = & \\ \big(u_t, u_{t+1}, \ldots, u_{T-1}\big) & \\ \text{and a state path } x(\cdot) = & \\ \big(x_t, x_{t+1}, \ldots, x_T\big) & \\ \text{starting from } x_t = x \text{ at time } t & \\ \text{satisfying for any time } s \in \{t, \ldots, T-1\} & \\ x_{s+1} = f_s(x_s, u_s) & \textit{dynamics} \\ u_s \in \mathbb{B}_s(x_s) & \textit{control constraints} \\ x_s \in \mathbb{A}_s & \textit{state constraints} \\ x_T \in \mathbb{A}_T & \textit{target constraints} \end{array} \right\}
$$

J.-P. Aubin. *Viability Theory*. Birkhäuser, Boston, 1991.

The viability kernel is included in the state constraint set



- The largest set is the state constraint set $\mathbb{A}$
- It includes the smaller blue viability kernel $\mathbb{V}\mathrm{iab}_{t_0}$
- The green set measures the incompatibility between dynamics and constraints: good start, but inevitable crisis!

The viability program aims at turning a priori constraints, with state constraints, into a posteriori constraints, without state constraints

- A priori constraints, with state constraints

$$\begin{cases} x_{t_0} \in \mathbb{X} \\ x_{t+1} = f_t(x_t, u_t) \\ u_t \in \mathbb{B}_t(x_t) & \text{control constraints} \\ x_t \in \mathbb{A}_t & \text{state constraints} \end{cases}$$

- are turned into a posteriori constraints, without state constraints except for the initial state

$$\begin{cases} x_{t_0} \in \mathbb{V}\mathrm{iab}_{t_0} & \text{initial state constraint} \\ x_{t+1} = f_t\big(x_t, u_t\big) \\ u_t \in \mathbb{B}_t^{\mathrm{viab}}\big(x_t\big) & \text{control constraints} \end{cases}$$

Fishery management (II)
[De Lara, Ocaña Anaya, Oliveros-Ramos, and Tam, 2012]

We provide an explicit expression for the viability kernel under rather weak assumptions

Proposition

If the *thresholds* $\underline{S}^A, \underline{S}^H$ *and* $\underline{C}^A, \underline{C}^H$ *meet the inequalities*

$$\underbrace{\underline{S}^A \mathcal{R}^A(\underline{S}^A, \underline{S}^H) - \underline{S}^A}_{\text{surplus}} \geq \underline{C}^A \quad \text{and} \quad \underbrace{\underline{S}^H \mathcal{R}^H(\underline{S}^A, \underline{S}^H) - \underline{S}^H}_{\text{surplus}} \geq \underline{C}^H$$

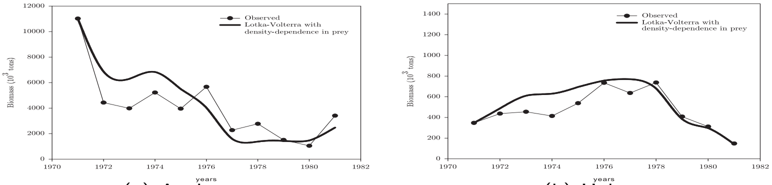
the *viability kernel* *is given by*

$$\{(A, H) \mid A \geq \underline{S}^A,\ H \geq \underline{S}^A,\ A\mathcal{R}^A(A, H) - \underline{S}^A \geq \underline{C}^A,\ H\mathcal{R}^H(A, H) - \underline{S}^H \geq \underline{C}^H\}$$

We tailor a Lotka-Volterra *decision model* to hake-anchovy Peruvian fisheries scarce data

Hake-anchovy Peruvian fisheries data between 1971 and 1981, in thousands of tonnes (10^3 tons)

- anchoveta_stocks= [11019 4432 3982 5220 3954 5667 2272 2770 1506 1044 3407]
- merluza_stocks= [347 437 455 414 538 735 636 738 408 312 148]
- anchoveta_captures= [9184 3493 1313 3053 2673 3211 626 464 1000 223]
- merluza_captures= [26 13 133 109 85 93 107 303 93 159 69]



(a) Anchovy (b) Hake

Figure: Comparison of observed and simulated biomasses of anchovy and hake using a Lotka-Volterra model with density-dependence in the prey. Model parameters are $R = 2.25$, $L = 0.945$, $\kappa = 67\ 113\ \times 10^3$ t ($K = 37\ 285\ \times 10^3$ t), $\alpha = 1.22 \times 10^{-6}\ t^{-1}$, $\beta = 4.845 \times 10^{-8}\ t^{-1}$.

Here is the Lotka-Volterra *decision model*

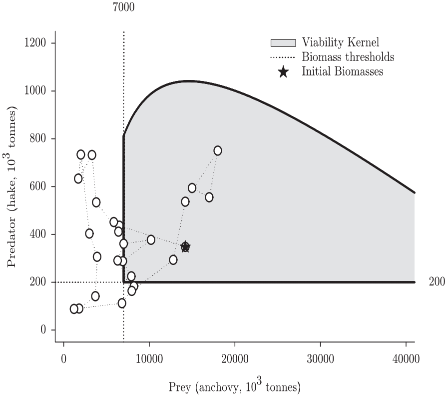
- A is the prey biomass (anchovy)
- H is the predator biomass (hake)
- The discrete-time Lotka-Volterra system is

$$
\begin{aligned}
A_{t+1} &= A_t \overbrace{\left(R - \frac{R}{\kappa} A_t - \alpha H_t\right)}^{\mathcal{R}^A\left(A_t, H_t\right)} \left(1 - E_t^A\right) \\
H_{t+1} &= H_t \underbrace{\left(L + \beta A_t\right)}_{\mathcal{R}^H\left(A_t, H_t\right)} \left(1 - E_t^H\right),
\end{aligned}
$$

- The associated deterministic viability kernel is $\mathbb{V}(t_0) =$

$$
\left\{ (A, H) \mid A \geq \underline{S}^A, \frac{1}{\alpha}[R - \frac{R}{\kappa} A - \frac{\underline{S}^A + \underline{C}^A}{A}] \geq H \geq \max\{\frac{\underline{S}^H + \underline{C}^H}{L + \beta A}, \underline{S}^H\} \right\}
$$

For given biomasses and catches thresholds, we display the associated viability kernel

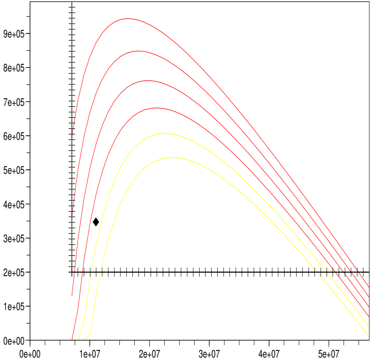


- Minimal biomasses thresholds
  - $S^A = 7\ 000\ kt$ (anchovy)
  - $S^H = 200\ kt$ (hake)
- Minimal catches thresholds
  - $C^A = 2\ 000\ kt$ (anchovy)
  - $C^H = 5\ kt$ (hake)

First acid test: plotting years of observed biomasses

- The range of values for viable states fits with measured biomasses
- Theoretically, a viable management with guaranteed biomasses and catches would have been possible since the initial state $\star$ is viable

Ecosystem viable yields



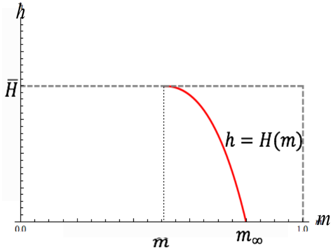
1. Considering that first are given minimal biomass conservation thresholds $S^{A^\flat} \geq 0$, $S^{H^\flat} \geq 0$
2. for initial biomasses $A_0 \geq S^{A^\flat}$ and $H_0 \geq S^{H^\flat}$, the following catch levels, if positive, can be sustainably maintained

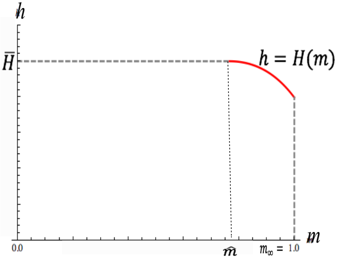
$$C^{A^\flat,\star}(A_0, H_0) = \min\left\{S^{A^\flat}\mathcal{R}^A(S^{A^\flat}, S^{H^\flat}) - S^{A^\flat}; A_0\mathcal{R}^A(A_0, H_0) - S^{A^\flat}\right\}$$

$$C^{H^\flat,\star}(A_0, H_0) = \min\left\{S^{H^\flat}\mathcal{R}^H(S^{A^\flat}, S^{H^\flat}) - S^{H^\flat}; H_0\mathcal{R}^H(A_0, H_0) - S^{H^\flat}\right\}$$

Epidemics control
[De Lara and Sepulveda, 2016]

Capping the human infected population with the Ross-Macdonald model: viability kernels [De Lara and Sepulveda, 2016]





Outline of the presentation

Critical thresholds *versus* smooth trade-offs

- Critical thresholds in ecology, environment, sustainability
- Synthetic indicators
- Giving prices as a way to deal with multiple goals
- Economics of risk and time vs. catastrophe insurance
- Conclusion and roadmap

The viability approach and resilience

- A few words on the purpose of modelling
- (Deterministic) viability in a nutshell
- Handling uncertainty in control theory
- Stochastic viability in a nutshell
- Robust viability in a nutshell

Measures of resilience and extensions

- How to measure resilience?
- From viable states to viable random paths

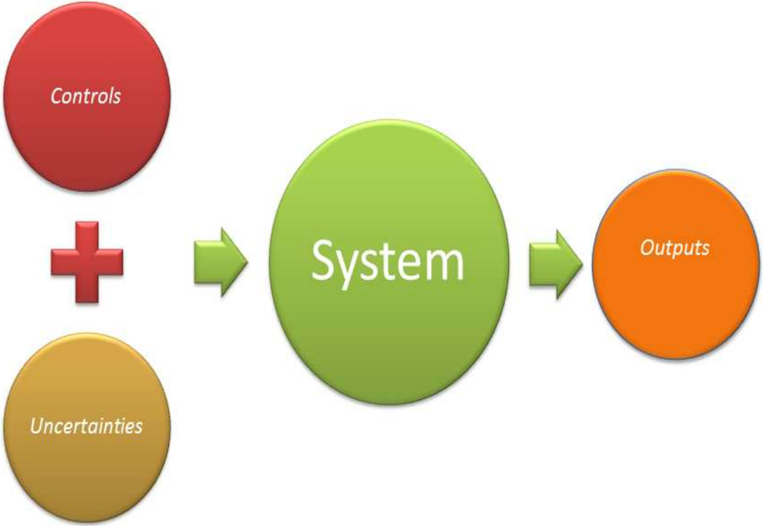
"Self-promotion, nobody will do it for you" ;-)

Discrete time nonlinear state-control system

$$x_{t+1} = f_t\big(x_t, u_t, w_{t+1}\big)\,, \quad t \in \mathbb{T} = \{t_0, t_0 + 1, \ldots, T - 1\}$$

- the time t (stage) $\in \overline{\mathbb{T}} = \{t_0, t_0 + 1, \ldots, T - 1, T\} \subset \mathbb{N}$ is discrete with initial time t_0 and horizon T ($T < +\infty$ or $T = +\infty$) (*the time period* $[t, t+1[$ *may be a year, a month, etc.*)
- the state variable x_t belongs to the *state space* $\mathbb{X} = \mathbb{R}^{n_\mathbb{X}}$ (*stocks, biomasses, abundances, capital*)
- the control variable u_t is an element of the *control space* $\mathbb{U} = \mathbb{R}^{n_\mathbb{U}}$ (*inflows, outflows, catches, harvesting effort, investment*)
- the uncertainty $w_t \in \mathbb{W} = \mathbb{R}^{n_\mathbb{W}}$ (*recruitment or mortality uncertainties, climate fluctuations*)
- the dynamics f_t maps $\mathbb{X} \times \mathbb{U} \times \mathbb{W}$ into $\mathbb{X}$ (*storage, age-class model, population dynamics, economic model*)

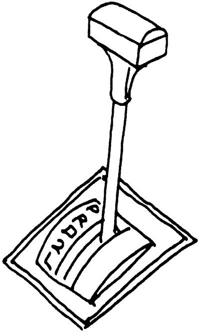
By contrast with control variables, uncertainty variables are exogenous input variables



Input control variables are in the hands of the decision-maker at successive stages

Control variables $u_t \in \mathbb{U}$

The decision-maker can choose the values of control variables u_t at any stage within given bounds

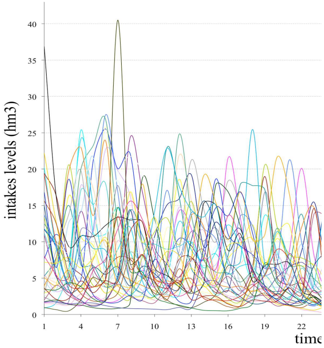


- at successive stages or time periods
  - annual catches
  - years, months: starting of energy units like nuclear plants
  - weeks, days, intra-day: starting of hydropower units
- within given bounds
  - fishing quotas
  - turbined capacity

Input uncertain variables are exogenous, that is, out of the control of the decision-maker

Uncertain variables $w_t \in \mathbb{W}$ are variables

- that take more than one single value (else they are deterministic)
- and over which the decision-maker (DM) has no control whatsoever



- Stationary parameters:
  unitary cost of $\mathrm{CO_2}$ emissions
- Trends or seasonal effects:
  energy consumption pathway, mean temperatures, mean prices
- Stochastic processes:
  rain inputs in a dam, energy demand, prices
- Else (set membership):
  costs of climate change damage, water inflows in a dam

What have we covered so far?

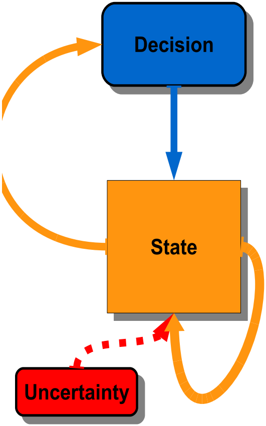
Uncertainty variables are new input variables

$$x_{t+1} = f_t\big(x_t, u_t, \underbrace{w_{t+1}}_{\text{uncertainty}}\big)$$

- The future state x_{t+1} is no longer predictable
- because of the uncertain term w_{t+1},
- but the current state x_t carries information relevant for decision-making,
- and we shed light on the notion of policy

$$u_t = \lambda_t\big(x_t\big)$$

"Policies" are closed-loop controls



- Deterministic control theory appeals to open-loop control, ⌚
  that is, a time-dependent sequence (planning, scheduling)

$$u : \underbrace{t \in \mathbb{T}}_{\text{time}} \mapsto \underbrace{u_t \in \mathbb{U}}_{\text{control}}$$

- Another notion of solution is a decision rule, ⌚ × 👁 a policy, that is, a mapping

$$\lambda : \underbrace{(t, x) \in \mathbb{T} \times \mathbb{X}}_{\text{(time, state)}} \mapsto u = \underbrace{\lambda_t(x) \in \mathbb{U}}_{\text{control}}$$

  which "closes the loop" between time t–state x and control u
  (and is especially relevant in presence of uncertainties)

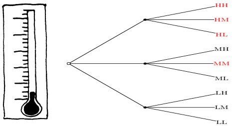
Scenarios
stochastic *vs* robust

We call scenario a temporal sequence of uncertainties

Scenarios are special cases of "states of Nature"

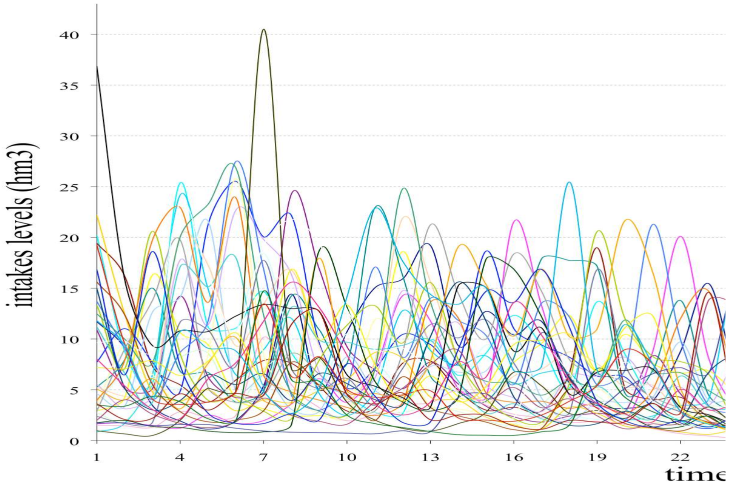
A scenario (pathway, chronicle) is a sequence of uncertainties

$$w(\cdot) = \big(w_{t_0}, \ldots, w_{T-1}\big) \in \mathbb{S} = \mathbb{W}^{T-t_0}$$

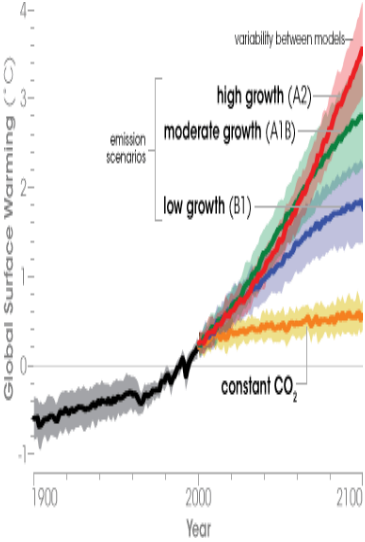


El tiempo se bifurca perpetuamente hacia innumerables futuros
(Jorge Luis Borges, *El jardín de senderos que se bifurcan*)

Water inflows historical scenarios



Beware! Scenario holds a different meaning in other scientific communities



- In practice, what modelers call a "scenario" is a mixture of
  - a sequence of uncertain variables (also called a pathway, a chronicle)
  - a policy
  - and even a static or dynamical model
- In what follows

  scenario = pathway = chronicle

Choosing a set of scenarios is excluding "things we don't know we don't know"

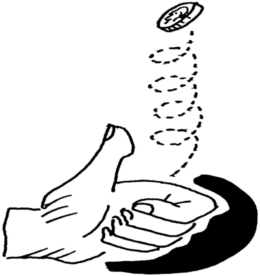
> *Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.*

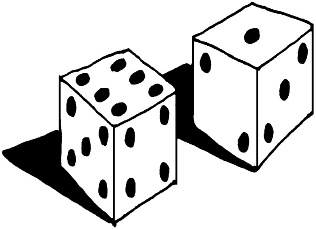
Donald Rumsfeld, former United States Secretary of Defense.
From Department of Defense news briefing, February 12, 2002

Probabilistic and set-membership approaches are ways to translate a priori / off-line information as illustrated in nuclear accidents prevention

- Three Mile Island accident:
  before the fact, the core meltdown was considered as excluded
- Nuclear accidents with probability per reactor per year
  - between 10^{-6} and 10^{-4} are considered as hypothetical,
  - whereas below 10^{-6} they are not envisaged
- Fukushima nuclear plants had a 10^{-9}
  nuclear accident probability per reactor per year

In the stochastic approach, the set of scenarios is equipped with a known probability





A priori information on the scenarios may be probabilistic

- A probability distribution $\mathbb{P}$ on $\mathbb{S}$
- In practice, one often assumes that the components $\big(w_{t_0}, \ldots, w_{T-1}\big)$ form
  - an independent and identically distributed sequence
  - a Markov chain, a time series, etc.

Water inflows in a dam

Water inflows in a dam may be modelled as time series (ARMA, etc.)

Probabilistic assumptions and expected value

- The domain of scenarios $\mathbb{S} = \mathbb{W}^{T+1-t_0} = \mathbb{R}^q \times \cdots \times \mathbb{R}^q$ is equipped with the σ-field $\mathcal{F} = \bigotimes_{t=t_0}^{T} \mathcal{B}(\mathbb{R}^q)$ and a probability $\mathbb{P}$
- The sequences $w(\cdot) = (w_{t_0}, w_{t_0+1}, \ldots, w_{T-1}, w_T)$ now become the primitive random variables
- The notation $\mathbb{E}_{\mathbb{P}}$ refers to the mathematical expectation over $\mathbb{S}$ under probability $\mathbb{P}$

$$\mathbb{E}[A(w(\cdot))] = \sum_{w(\cdot)\in\mathbb{S}} \mathbb{P}\{w(\cdot)\} A(w(\cdot))$$

- The expectation operator $\mathbb{E}_{\mathbb{P}}$ enjoys linearity in the $(+, \times)$ algebra:

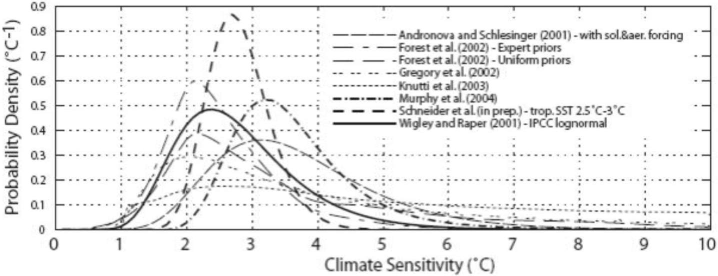
$$\mathbb{E}_{\mathbb{P}}(A + B) = \mathbb{E}_{\mathbb{P}}(A) + \mathbb{E}_{\mathbb{P}}(B)$$

- The random variables $(w_{t_0}, w_{t_0+1}, \ldots, w_{T-1}, w_T)$ are independent under $\mathbb{P}$ if probability $\mathbb{P}$ can be decomposed as a product

$$\mathbb{P} = \mathbb{P}_{w_{t_0}} \otimes \cdots \otimes \mathbb{P}_{w_T}$$

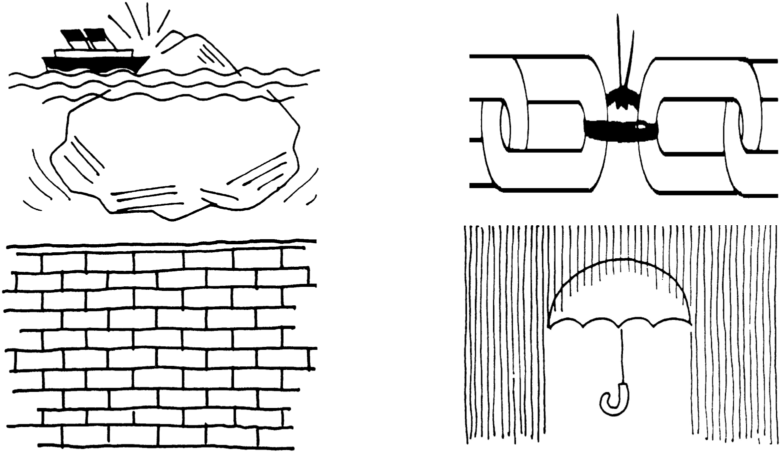
Equipping the set $\mathbb{S}$ of scenarios with a probability $\mathbb{P}$ is a delicate issue!

- The probabilistic distribution of the climate sensitivity parameter in climate models differs according to authors



- In the multi-prior approach, the a priori information consists of different probabilities (*beliefs*, *priors*), belonging to a set $\mathcal{P}$ of admissible probabilities on $\mathbb{S}$

In the set-membership approach,
only a subset of the set of scenarios is known

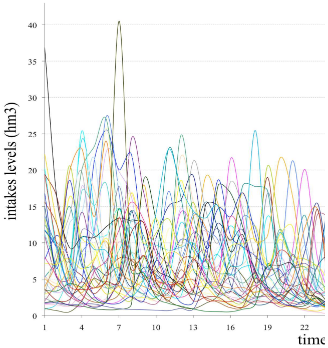


A priori information on the scenarios may be set membership

The general case

- Selected scenarios may belong to any subset $\overline{\mathbb{S}}$

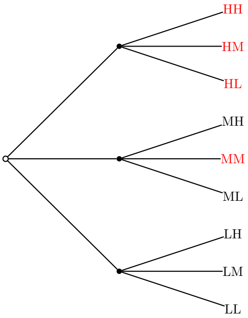
$$w(\cdot) \in \overline{\mathbb{S}} \subset \mathbb{S}$$



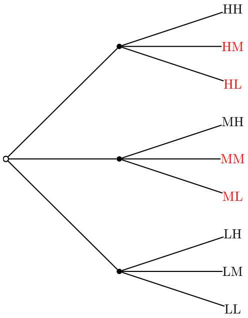
Historical water inflows scenarios in a dam

We can represent offline information by the observed historical water inflows scenarios

Specific subsets correspond to time independence



NO time independence because the range of values of w_{t+1} depends on the value of w_t:
$w_t = H \Rightarrow w_{t+1} \in \{H, M, L\}$
$w_t = M \Rightarrow w_{t+1} \in \{M\}$



Time independence because $\overline{\mathbb{S}} = \{H, M\} \times \{M, L\} \subset \mathbb{S}$ is a product set

A priori information on the scenarios may be set membership

The product case

- Uncertain variables may be restricted to subsets, period by period

$$w_t \in \mathbb{S}_t$$

so that some scenarios are selected and the rest are excluded

$$w(\cdot) \in \mathbb{S}_{t_0} \times \cdots \times \mathbb{S}_T \subset \mathbb{S} = \mathbb{W}^{T+1-t_0}$$

Bounded water inflows in a dam

If only an upper bound on water inflows is known,
we represent offline information by

$$0 \leq a_t \leq a^\sharp$$

A priori information on the scenarios may be softer than set membership thanks to plausibility functions

- The counterpart of a probability $\mathbb{P}$ is a plausibility function $\mathbb{Q}$
- Plausibility function $\mathbb{Q} : \mathbb{S} \to \mathbb{R}_- \cup \{-\infty\}$ such that (normalization)

$$\inf_{w(\cdot)\in\mathbb{S}} \left[-\mathbb{Q}\big(w(\cdot)\big)\right] = - \sup_{w(\cdot)\in\mathbb{S}} \left[\mathbb{Q}\big(w(\cdot)\big)\right] = 0$$

  can "soften" the above set membership approach
  - the higher $\mathbb{Q}\big(w(\cdot)\big)$, the more plausible the scenario $w(\cdot)$
  - totally implausible scenarios are those for which $\mathbb{Q}\big(w(\cdot)\big) = -\infty$

Historical water inflows scenarios in a dam

Attribute the value $\mathbb{Q}\big(w(\cdot)\big) = -\infty$ for all the scenarios $w(\cdot)$ which do not belong to the observed historical water inflows scenarios

Plausability and fear operator [Bernhard, 1995] are the robust counterparts of probability and expectation operator

- Let $\mathbb{Q} : \mathbb{S} \to \mathbb{R}_- \cup \{-\infty\}$ be a a plausibility function
- The feared value of a function $A : \mathbb{S} \to \mathbb{R}$ is defined by

$$\mathbb{F}_{\mathbb{Q}}(A) = \inf_{w(\cdot)\in\mathbb{S}} \big[A\big(w(\cdot)\big) - \mathbb{Q}\big(w(\cdot)\big)\big]$$

- The fear operator $\mathbb{F}_{\mathbb{Q}}$ enjoys linearity in the $(\min, +)$ algebra:

$$\mathbb{F}_{\mathbb{Q}}(\min\{A, B\}) = \min\{\mathbb{F}_{\mathbb{Q}}(A), \mathbb{F}_{\mathbb{Q}}(B)\}$$

- In the $(\min, +)$ algebra, the plausibility function $\mathbb{Q}$ plays the role of a weight, paralleling a probability distribution
- The uncertainties $\big(w_{t_0}, w_{t_0+1}, \ldots, w_{T-1}, w_T\big)$ are independent under $\mathbb{Q}$ if plausibility $\mathbb{Q}$ can be decomposed as a sum

$$\mathbb{Q} = \mathbb{Q}_{w_{t_0}} + \cdots + \mathbb{Q}_{w_T}$$

More on plausability and fear operator

- The Moreau upper addition extends the usual addition with

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$$

- The feared value of a function $A : \mathbb{S} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by

$$\mathbb{F}_{\mathbb{Q}}(A) = \inf_{w(\cdot) \in \mathbb{S}} \Big[A\big(w(\cdot)\big) \dot{+} \Big(-\mathbb{Q}\big(w(\cdot)\big) \Big) \Big]$$

- With a subset $\mathbb{S}' \subset \mathbb{S}$ of scenarios, we associate the characteristic function $\delta_{\mathbb{S}'} : \mathbb{S} \to \{0, +\infty\} \subset \mathbb{R} \cup \{+\infty\}$

$$\delta_{\mathbb{S}'}\big(w(\cdot)\big) = \begin{cases} 0 & \text{if } w(\cdot) \in \mathbb{S}' \\ +\infty & \text{if } w(\cdot) \notin \mathbb{S}' \end{cases}$$

Two possible definitions of the plausability of a subset

Either

$$\begin{aligned}\mathbb{Q}^-(\mathbb{S}') &= \mathbb{F}_{\mathbb{Q}}(\delta_{\mathbb{S}'}) \\ &= \inf_{w(\cdot)\in\mathbb{S}} \Big[\delta_{\mathbb{S}'}(w(\cdot)) \dot{+} \Big(-\mathbb{Q}\big(w(\cdot)\big)\Big)\Big] \\ &= \inf_{w(\cdot)\in\mathbb{S}'} \big[-\mathbb{Q}\big(w(\cdot)\big)\big] \geq 0\end{aligned}$$

or

$$\begin{aligned}\mathbb{Q}^+(\mathbb{S}') &= \mathbb{F}_{\mathbb{Q}}(-\delta_{\mathbb{S}'}) \\ &= \inf_{w(\cdot)\in\mathbb{S}} \Big[-\delta_{\mathbb{S}'}(w(\cdot)) \dot{+} \Big(-\mathbb{Q}\big(w(\cdot)\big)\Big)\Big]\end{aligned}$$

The classic robust case corresponds to uniform plausability

With a subset $\overline{\mathbb{S}} \subset \mathbb{S}$ of scenarios,
we associate the uniform plausability function $\mathbb{Q} = -\delta_{\overline{\mathbb{S}}}$, that is,

$$\mathbb{Q}\big(w(\cdot)\big) = \begin{cases} 0 & \text{if } w(\cdot) \in \overline{\mathbb{S}} \\ -\infty & \text{if } w(\cdot) \notin \overline{\mathbb{S}} \end{cases}$$

for which we have that

$$\mathbb{F}_{\mathbb{Q}}(A) = \inf_{w(\cdot)\in\mathbb{S}} \Big[A\big(w(\cdot)\big) \dot{+} \Big(-\mathbb{Q}\big(w(\cdot)\big)\Big)\Big] = \inf_{w(\cdot)\in\overline{\mathbb{S}}} A\big(w(\cdot)\big)$$

Uniform plausability, intersections and inclusions

- The $\mathbb{Q}^+$ uniform plausability is suitable to detect intersections

$$\mathbb{Q}^+(\mathbb{S}') = \mathbb{F}_{\mathbb{Q}}(\delta_{\mathbb{S}'}) = \inf_{w(\cdot)\in\mathbb{S}} \left[\delta_{\mathbb{S}'}\big(w(\cdot)\big) + \delta_{\overline{\mathbb{S}}}\big(w(\cdot)\big)\right] = \delta_{\mathbb{S}'\cap\overline{\mathbb{S}}}$$

- whereas the $\mathbb{Q}^-$ uniform plausability is suitable to detect inclusions (<u>to be checked!</u>)

$$\mathbb{Q}^-(\mathbb{S}') = \mathbb{F}_{\mathbb{Q}}(-\delta_{\mathbb{S}'}) = \inf_{w(\cdot)\in\mathbb{S}} \left[-\delta_{\mathbb{S}'}\big(w(\cdot)\big) \dotplus \delta_{\overline{\mathbb{S}}}\big(w(\cdot)\big)\right] = \inf_{w(\cdot)\in\overline{\mathbb{S}}} \left[-\delta_{\mathbb{S}'}\big(w(\cdot)\big)\right]$$

so that (<u>to be checked!</u>)

$$\mathbb{Q}^-(\mathbb{S}') = \mathbb{F}_{\mathbb{Q}}(-\delta_{\mathbb{S}'}) = 0 \iff \overline{\mathbb{S}} \subset \mathbb{S}'$$

Summary

- A priori information is carried by the scenarios set, and may be
  - probabilistic (probability and expectation operator)
  - set membership (plausibility and fear operator)
- This will be useful to mathematically express
  objectives and constraints
  in a decision problem under uncertainty

A scenario is said to be viable for a given policy if the state and control trajectories satisfy the constraints

Viable scenario under given policy

A scenario $w(\cdot) \in \mathbb{S}$ is said to be viable under policy $\lambda : \mathbb{T} \times \mathbb{X} \to \mathbb{U}$ if the trajectories $x(\cdot)$ and $u(\cdot)$ generated by the dynamics

$$x_{t+1} = f_t\big(x_t, u_t, w_{t+1}\big)\ , \quad t = t_0, \ldots, T-1$$

with the policy

$$u_t = \lambda_t\big(x_t\big)$$

satisfy the state and control constraints

$$\underbrace{u_t \in \mathbb{B}_t\big(x_t\big)}_{\text{control constraints}} \quad \text{and} \quad \underbrace{x_t \in \mathbb{A}_t}_{\text{state constraints}} \quad , \quad \forall t = t_0, \ldots, T$$

The set of viable scenarios is denoted by $\mathbb{S}^{\lambda}_{t_0, x_0}$

We look after policies that make the corresponding set of viable scenarios "large"

Set of viable scenarios

$$\mathbb{S}^{\lambda}_{t_0,x_0} = \{ w(\cdot) \in \mathbb{S} \mid \quad \text{the state constraints} \quad x_t \in \mathbb{A}_t \quad \text{and the control constraints} \quad u_t = \lambda_t\big(x_t\big) \in \mathbb{B}_t\big(x_t\big) \quad \text{are satisfied for all times } t = t_0, \ldots, T \}$$

- The larger set $\mathbb{S}^{\lambda}_{t_0,x_0}$ of viable scenarios, the better, because the policy λ is able to maintain the system within constraints for a large "number" of scenarios
- But "large" in what sense? Probabilistic (stochastic)? Robust?

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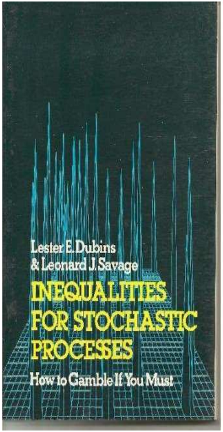
"Self-promotion, nobody will do it for you" ;-)

Maximizing the probability of success may be an objective



How to gamble if you must, L.E. Dubbins and L.J. Savage, 1965

Imagine yourself at a casino with \$1,000. For some reason, you desperately need \$10,000 by morning; anything less is worth nothing for your purpose.

The only thing possible is to gamble away your last cent, if need be, in an attempt to reach the target sum of \$10,000.

- The question is how to play, not whether. What ought you do? How should you play?
  - Diversify, by playing 1 \$ at a time?
  - Play boldly and concentrate, by playing 1,000 \$ only one time?
- What is your decision criterion?

We extend viability kernels to
stochastic viability kernels

Stochastic viability kernels

In stochastic viability, state constraints are to be met along time with a given confidence level $\beta \in [0,1]$

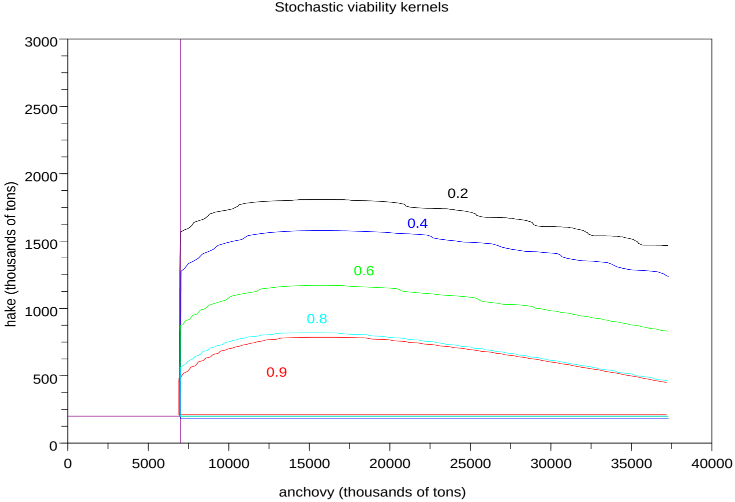
$$\mathbb{P}\Big(w(\cdot) \in \mathbb{S} \mid x_t \in \mathbb{A}_t \ , \ \ u_t = \lambda_t(x_t) \in \mathbb{B}_t(x_t) \ \text{for} \ t = t_0, \ldots, T\Big) \geq \beta$$

Stochastic viability kernels

The stochastic viability kernel at confidence level $\beta \in [0,1]$ is

$$\mathbb{V}\mathrm{iab}_{t_0}^{\beta} = \left\{ x_0 \in \mathbb{X} \ \middle| \ \begin{array}{l} \text{there exists a policy } \lambda \text{ such that} \\ \mathbb{P}\Big(w(\cdot) \in \mathbb{S} \mid x_t \in \mathbb{A}_t \ , \ \ u_t = \lambda_t(x_t) \in \mathbb{B}_t(x_t) \\ \ \text{for} \ t = t_0, \ldots, T\Big) \geq \beta \end{array} \right\}$$

Stochastic viability kernels $\mathbb{V}\mathrm{iab}_{t_0}^{\beta}$ for a hake-anchovy fisheries model [De Lara, Martinet, and Doyen, 2015]



Stochastic viability kernels
can be obtained by
dynamic programming
[Doyen and De Lara, 2010]

The viability probability is the probability of satisfying constraints under a policy

Viability probability

The viability probability associated with
the initial time t_0, the initial state x_0 and the policy λ
is the probability $\mathbb{P}\left[\mathbb{S}^{\lambda}_{t_0,x_0}\right]$ of the set $\mathbb{S}^{\lambda}_{t_0,x_0}$ of viable scenarios

$$\mathbb{P}\left[\mathbb{S}^{\lambda}_{t_0,x_0}\right] = \mathbb{P}\{w(\cdot) \in \mathbb{S} \mid \text{the state constraints } x_t \in \mathbb{A}_t \text{ and the control constraints } u_t = \lambda_t(x_t) \in \mathbb{B}_t(x_t) \text{ are satisfied for all times } t = t_0, \ldots, T\}$$

The maximal viability probability is the upper bound for the probability of satisfying constraints

Maximal viability probability and optimal viable policy

The maximal viability probability is

$$\max_{\lambda} \mathbb{P}\left[\mathbb{S}^{\lambda}_{t_0,x_0}\right]$$

An optimal viable policy λ^* satisfies

$$\mathbb{P}\left[\mathbb{S}^{\lambda^*}_{t_0,x_0}\right] \geq \mathbb{P}\left[\mathbb{S}^{\lambda}_{t_0,x_0}\right]$$

In a sense, any optimal viable policy makes the set of viable scenarios the "largest" possible

Let us introduce the stochastic viability Bellman function

Suppose that the primitive random variables
$\big(w_{t_0}, w_{t_0+1}, \ldots, w_{T-2}, w_{T-1}\big)$
are independent under the probability $\mathbb{P}$

Bellman function / stochastic viability value function

Define the probability-to-go as

$$V_t(x) =$$

$$\max_{\lambda} \mathbb{P}\Big(w(\cdot) \in \mathbb{S} \mid \overbrace{\lambda_s\big(x_s\big) \in \mathbb{B}_s\big(x_s\big)}^{\text{control constraints}} \text{ and } \overbrace{x_s \in \mathbb{A}_s}^{\text{state constraints}} \quad \text{for } s \geq t \Big)$$

where $x_{s+1} = f_s\big(x_s, \lambda_s\big(x_s\big), w_{s+1}\big)$ and $x_t = x$

- The function $V_t(x)$ is called stochastic viability value function (Bellman function)
- The original problem is $V_{t_0}\big(x_0\big)$

The dynamic programming equation is a backward equation satisfied by the stochastic viability value function

Proposition

If the *primitive random variables* $\big(w_{t_0}, w_{t_0+1}, \dots, w_{T-2}, w_{T-1}\big)$ *are* *independent* *under the probability* $\mathbb{P}$,
the stochastic viability value functions $V_{t_0}, \dots, V_T$
satisfy the following backward induction

$$
\begin{aligned}
V_T(x) &= 1_{\mathbb{A}_T}(x) \\
V_t(x) &= 1_{\mathbb{A}_t}(x) \max_{u \in \mathbb{B}_t(x)} \mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]
\end{aligned}
$$

for all $x \in \mathbb{X}$, *and where* t *runs from* $T-1$ *down to* t_0

Algorithm for the Bellman functions and the stochastic viable controls

initialization $V_T(x) = 1_{\mathbb{A}_T}(x)$;

for $t = T, T-1, \ldots, t_0$ **do**

  forall $x \in \mathbb{X}$ **do**

    forall $u \in \mathbb{B}_t(x)$ **do**

      $\mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]$

    $\max_{u \in \mathbb{B}(t,x)} \mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]$

  $V_t(x) = 1_{\mathbb{A}_t}(x) \max_{u \in \mathbb{B}_t(x)} \mathbb{E}_{w_{t+1}}\Big[V_{t+1}\Big(f_t\big(x, u, w_{t+1}\big)\Big)\Big]$

The stochastic viable dynamic programming equation yields stochastic viable policies

For any time t and state x, let us assume that the set

$$\mathbb{B}_t^{\text{viab}}(x) = \arg\max_{u \in \mathbb{B}_t(x)} \Big(1_{\mathbb{A}_t}(x) \mathbb{E}_{w_{t+1}} \Big[V_{t+1}\Big(f_t(x, u, w_{t+1}) \Big) \Big] \Big)$$

of viable controls is not empty

Proposition

Then, any (measurable) policy λ such that $\lambda_t^*(x) \in \mathbb{B}_t^{viab}(x)$ is an optimal viable policy which achieves the maximal viability probability

$$V_{t_0}(x_0) = \max_{\lambda} \mathbb{P}\left[\mathbb{S}_{t_0, x_0}^{\lambda} \right]$$

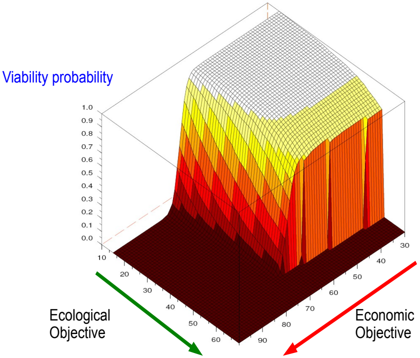
The dynamic programming equation yields the viability kernels

The viability kernel at confidence level β turns out to coincide with the section of level β of the stochastic value function:

$$V_{t_0}(x_0) \geq \beta \iff x_0 \in \mathbb{V}\mathrm{iab}^{\beta}_{t_0}$$

Displaying trade-offs between critical thresholds and risk [De Lara and Martinet, 2009]

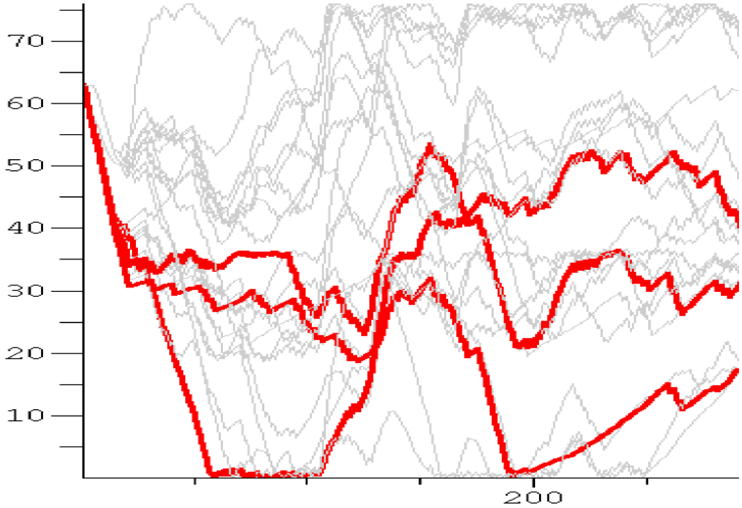
$$\mathbb{P}\left[\underbrace{C_t \geq C^\flat,\ E_t \geq E^\flat}_{\text{indicators} \,\geq\, \text{thresholds}}\ ,\ \forall t\right]$$



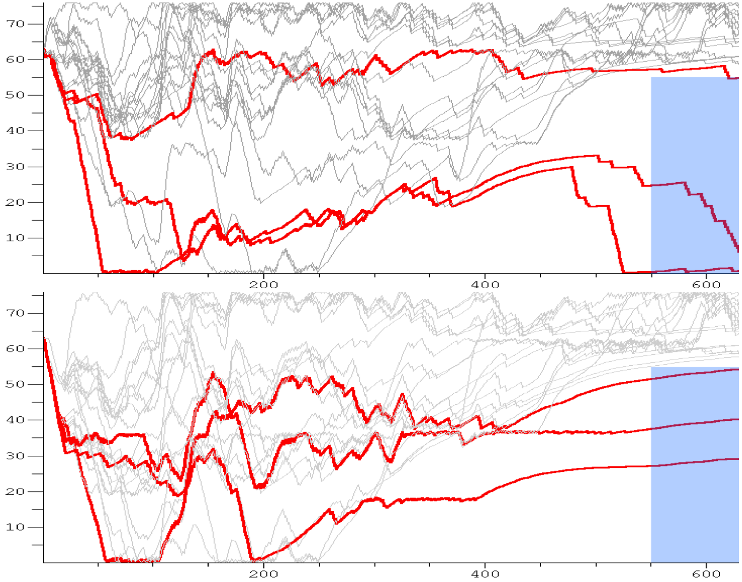
Tourism issues impose constraints upon traditional economic management of a hydro-electric dam [Alais, Carpentier, and De Lara, 2017]

- Maximizing the revenue from turbinated water
- under a tourism constraint of having enough water in July and August

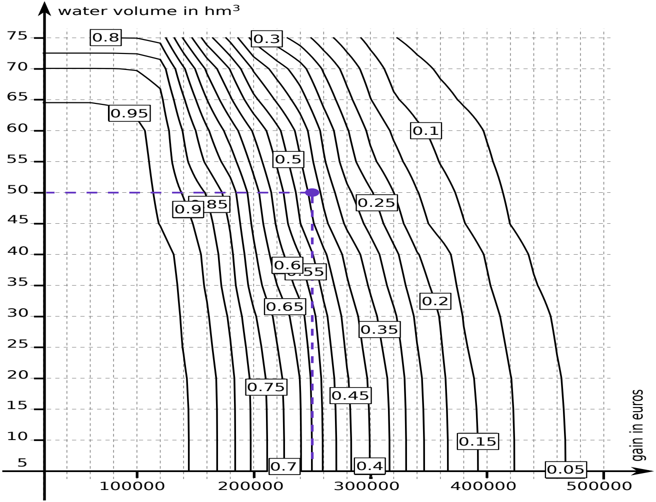
The red stock trajectories fail to meet the tourism constraint in July and August



90% of the stock trajectories meet the tourism constraint in July and August



We plot iso-values for the maximal viability probability as a function of guaranteed thresholds $S^\flat$ and $P^\flat$



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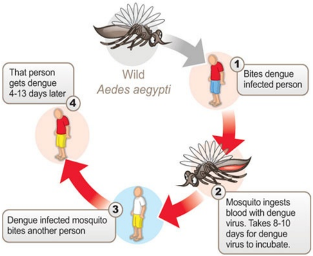
Fishery management
robust viability
[Regnier and De Lara, 2015]

Epidemics control
robust viability
[Sepulveda Salcedo and De Lara, 2019]

Sources of uncertainty abound



Uncertainties are captured by $\begin{cases} \text{mosquitoes transmission rate} & A_t^M \\ \text{human transmission rate} & A_t^H \end{cases}$

in the forthcoming model

New variables

- Time
  - Discrete-time $t = 0, 1, \ldots, T$
    with interval $[t, t+1[$ representing one day
- State variables
  - M_t denotes the proportion of infected mosquitoes
    at the beginning of the interval $[t, t+1[$
  - H_t denotes the proportion of infected humans
    at the beginning of the interval $[t, t+1[$
- Control variable
  - U_t denotes the mosquito mortality due to fumigation
    during the interval $[t, t+1[$

Discrete-time dynamic control model with uncertainties

- Let us denote by $f(M, H, U, A^M, A^H)$ the solution, at time $s = 1$, of the deterministic differential system with initial condition $\big(m_0, h_0\big) = (M, H)$ and stationary control U
- We obtain the sampled and controlled Ross–Macdonald model

$$\big(M_{t+1}, H_{t+1}\big) = f\big(M_t, H_t, U_t, A_t^M, A_t^H\big)$$

- The control constraints capture limited fumigation resources

$$\underline{U} \leq U_t \leq \overline{U}\ , \ \ \forall t = 0, \ldots, T-1$$

during a day

Viability problem statement

- We impose that the viability constraint

$$H_t \leq \overline{H} \ , \ \ \forall t = 0, \ldots, T$$

- holds true whatever the scenario (sequence of uncertainties)

$$(A^M(\cdot), A^H(\cdot)) = \Big((A^M_0, A^H_0), \ldots, (A^M_{T-1}, A^H_{T-1}) \Big)$$

belonging to a subset $\overline{\mathbb{S}} \subset (\mathbb{R}^2)^T$

In the robust framework, we need a new definition of solution

- A policy Λ is defined as a sequence of mappings

$$\Lambda = \{\Lambda_t\}_{t=0,\dots,T-1} , \quad \text{with} \quad \Lambda_t : [0,1]^2 \to \mathbb{R}$$

  where each Λ_t maps state (M, H) towards control U
- A policy induces a sequence of controls by

$$U_t = \Lambda_t\big(M_t, H_t\big)$$

- A policy Λ is said to be admissible
  if it satisfies the control constraints

$$\Lambda_t : [0,1]^2 \to [\underline{U}, \overline{U}]$$

Robust viability problem statement

The robust viability kernel is the set of initial conditions $\big(M_0, H_0\big)$ from which at least one admissible policy Λ gives infected mosquitoes and infected humans trajectories by the dynamics

$$\big(M_{t+1}, H_{t+1}\big) = f\big(M_t, H_t, U_t, A_t^M, A_t^H\big)$$

with input controls

$$U_t = \Lambda_t\big(M_t, H_t\big)$$

so that

$$H_t \leq \overline{H}\ , \ \ \forall t = 0, \ldots, T$$

for all the scenarios

$$\Big(\Big(A_0^M, A_0^H\Big), \ldots, \Big(A_{T-1}^M, A_{T-1}^H\Big)\Big) \in \overline{\mathbb{S}} \subset \left(\mathbb{R}^2\right)^T$$

We make a tough assumption on the set of scenarios

- A scenario is a time sequence of uncertainty couples

$$\big(A^M(\cdot), A^H(\cdot)\big) = \Big((A_0^M, A_0^H), \ldots, (A_{T-1}^M, A_{T-1}^H) \Big)$$

- We make the strong independence assumption that

$$\big(A_t^M(\cdot), A^H(\cdot)\big) \in \overline{\mathbb{S}} = \mathbb{S}_0 \times \mathbb{S}_1 \times \cdots \times \mathbb{S}_{T-1}$$

- Therefore, from one time t to the next $t+1$, uncertainties can be drastically different since (A_t^M, A_t^H) is not related to (A_{t+1}^M, A_{t+1}^H)
- Such an assumption makes it possible to write a dynamic programming equation with (M, H) as state variable
- For the sake of simplicity, we take

$$\mathbb{S}_0 = \mathbb{S}_1 = \cdots = \mathbb{S}_{T-1} = \mathbb{S}$$

Numerical resolution of the dynamic programming equation

initialization $\mathrm{V}_T(M,H) = 1_{[0,1]\times[0,\overline{H}]}(M,H)$;

for $t = T, T-1, \dots, 0$ **do**

- **forall** $(M,H) \in [0,1]\times[0,\overline{H}]$ **do**
  - **forall** $U \in [\underline{U}, \overline{U}]$ **do**
    - **forall** $(A^M, A^H) \in \mathbb{S}$ **do**
      - $\mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$
    - $\min_{(A^M,A^H)\in\mathbb{S}} \mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$
  - $\max_{U\in[\underline{U},\overline{U}]} \min_{(A^M,A^H)\in\mathbb{S}} \mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$
- $\mathrm{V}_t\big(t,M,H\big) = 1_{[0,1]\times[0,\overline{H}]}(M,H) \times \mathrm{V}_{t+1}\big(f(M,H,U,A^M,A^H)\big)$

Uncertainty sets

We consider three nested sets of uncertainties

$$\mathbb{S}_L \subset \mathbb{S}_M \subset \mathbb{S}_H \subset \mathbb{R}_+^2$$

L) deterministic case

$$\mathbb{S}_L = \left\{ \widehat{A^M} \right\} \times \left\{ \widehat{A^H} \right\}$$

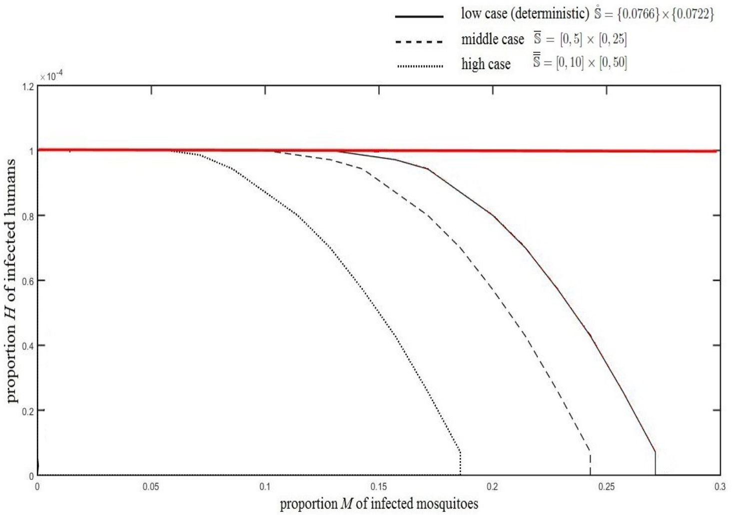
M) medium case

$$\mathbb{S}_M = \left[\underline{A^M}, \overline{A^M} \right] \times \left[\underline{A^H}, \overline{A^H} \right]$$

H) high case

$$\mathbb{S}_H = \left[\underline{\underline{A^M}}, \overline{\overline{A^M}} \right] \times \left[\underline{\underline{A^H}}, \overline{\overline{A^H}} \right]$$

Robust viability kernels shrink when uncertainties expand



Conclusion on robust viability analysis

The numerical results show that the viability kernel without uncertainties is highly sensitive to the variability of parameters such as

- biting rate
- probability of infection to mosquitoes and humans
- proportion of female mosquitoes per person

Maybe we should focus the effort on reducing
these three sources of uncertainty

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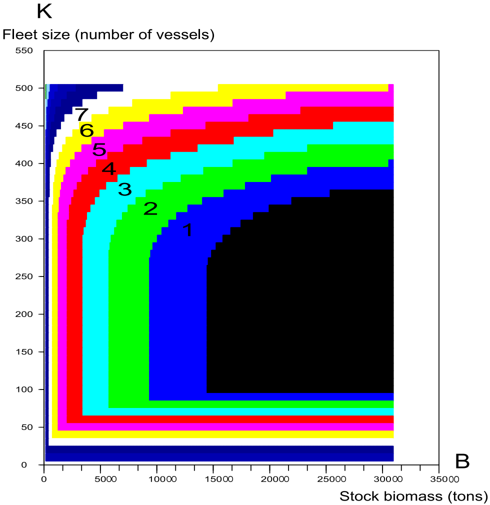
Exposure, vulnerability, resilience?

- Acceptable set/viability constraints:
  - possible values for output variables + critical thresholds
- Adaptive capacity: set of viable policies?
  - = policies depending on available observations and enabling the system to remain within the acceptable set for a certain number of scenarios (expressing the level of risk tolerated)
  - exist only in a viable state
- Exposure: exposure is high when
  - the current variables are close to the acceptable set boundary?
- Vulnerability: acceptable set/viability constraints + adaptive capacity?
- Resilience:
  - the more resilient, the lower the costs to reach a viable state
  - the less resilient, the farther from a robust or stochastic viability kernel

The minimal time of crisis and recovery measures the distance to a viability kernel in terms of time units [Doyen and Saint-Pierre, 1997]



[Martinet, Doyen, and Thébaud, 2007]

Relaxing some constraints to try and enter into the viability kernel

L. Doyen and P. Saint-Pierre. Scale of viability and minimum time of crisis. *Set-valued Analysis*, 5: 227–246, 1997.

From time units to cost units

- [Martin, 2005]
  La résilience est définie comme
  l'inverse du coût des perturbations envisagées
- Resilience as the inverse of minimal expected or robust costs to reach a stochastic or robust viability kernel

S. Martin. *La résilience dans les modèles de systèmes écologiques et sociaux*. Thèse École normale supérieure de Cachan - ENS Cachan, Juin 2005

The three Rs of resilience [Grafton, Doyen, Béné, Borgomeo, Brooks, Chu, Cumming, Dixon, Dovers, Garrick, Helfgott, Jiang, Katic, Kompas, Little, Matthews, Ringler, Squires, Steinshamn, Villasante, Wheeler, Williams, and Wyrwoll, 2019]

The '3Rs' of resilience

- resistance
- recovery
- robustness/reliability

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Dynamics and policies induce state-control random processes

Given a policy λ, we define a random process

$$w(\cdot) \mapsto \big(x(\cdot), u(\cdot)\big)_{\lambda}[w(\cdot)]$$

between scenarios towards state/control trajectories

$$\underbrace{\mathbb{S} = \mathbb{W}^{T-t_0}}_{\text{uncertainty trajectories}} \to \underbrace{\mathbb{X}^{T-t_0+1}}_{\text{state trajectories}} \times \underbrace{\mathbb{U}^{T-t_0}}_{\text{control trajectories}}$$

by the closed-loop dynamics

$$\begin{aligned} x_{t+1} &= f_t\big(x_t, \lambda_t(x_t), w_{t+1}\big)\,, \quad t = t_0, \ldots, T-1 \\ u_t &= \lambda_t(x_t) \end{aligned}$$

Stochastic and robust viability correspond to controlling a random process within a product acceptable set

We consider an acceptable set

$$\mathcal{A} = \{ \big(x(\cdot), u(\cdot)\big) \mid u_t \in \mathbb{B}_t\big(x_t\big) \text{ and } x_t \in \mathbb{A}_t \; , \quad \forall t = t_0, \ldots, T \} \subset \mathbb{X}^{T-t_0+1} \times \mathbb{U}^{T-t_0}$$

which has a product structure

$$\mathcal{A} = \prod_{t=t_0}^{T-1} \{ \big(x_t, u_t\big) \mid u_t \in \mathbb{B}_t\big(x_t\big) \text{ and } x_t \in \mathbb{A}_t \} \times \mathbb{A}_T \subset \prod_{t=t_0}^{T-1} (\mathbb{X} \times \mathbb{U}) \times \mathbb{X}$$

Stochastic and robust viability correspond to controlling a random process within a product acceptable set

Find a policy λ such that

- stochastic viability
  the probability that the random process $\big(x(\cdot), u(\cdot)\big)_\lambda$
  takes values in the acceptable set $\mathcal{A}$ is high enough

$$\mathbb{P}\left\{w(\cdot) \mid \big(x(\cdot), u(\cdot)\big)_\lambda[w(\cdot)] \in \mathcal{A}\right\} \geq p$$

- robust viability
  the random process $\big(x(\cdot), u(\cdot)\big)_\lambda$
  restricted to a subset $\overline{\mathbb{S}} \subset \mathbb{S}$ of the set $\mathbb{S}$ of scenarios
  takes values in the acceptable set $\mathcal{A}$

$$\big(x(\cdot), u(\cdot)\big)_\lambda[w(\cdot)] \in \mathcal{A}\,, \quad \forall w(\cdot) \in \overline{\mathbb{S}}$$

Extension to more general acceptable sets of random processes [De Lara, 2018]

Move to acceptable sets of random processes

$$\mathcal{A} \subset \mathbb{X}^{T-t_0+1} \times \mathbb{U}^{T-t_0} \longrightarrow \mathcal{A} \subset \left(\mathbb{X}^{T-t_0+1} \times \mathbb{U}^{T-t_0}\right)^{\mathbb{S}}$$

defined by vectorial risk measures? (one measure by relevant output)

- in mathematical finance, risk is often measured as a minimal capital requirement $\rho(\mathrm{X})$ to make a position X "acceptable" to a regulator
  thus, it is a form of minimal distance (gauge) to an acceptance set
- convex risk measures (diversification of risk)
  ex. tail value at risk (expected loss above a critical threshold)
- the stochastic and robust cases appear as special (extreme) cases of risk measures (built with expectation and fear operators)
  in a jungle to be explored and used (distributionaly robust, etc.)

Steps towards an operational definition of resilience

- Dynamical model
  - stages, decision steps
  - possible actions, controls, decisions, together with their restrictions
  - uncertainties, scenarios
  - states, dynamics, system
  - policies, decision rules
- Objectives
  - critical thresholds
  - risk measures (stochastic, robust, distributionaly robust, etc.)
  - acceptable sets of random processes
- Compute
  - (robust, stochastic) viability kernel = viable states for which policies exist that can keep the system within critical thresholds, despite of uncertainties
  - minimal cost to reach a viability kernel = inverse of resilience
  - 3Rs

Outline of the presentation

Critical thresholds *versus* smooth trade-offs

Critical thresholds in ecology, environment, sustainability

Synthetic indicators

Giving prices as a way to deal with multiple goals

Economics of risk and time vs. catastrophe insurance

Conclusion and roadmap

The viability approach and resilience

A few words on the purpose of modelling

(Deterministic) viability in a nutshell

Handling uncertainty in control theory

Stochastic viability in a nutshell

Robust viability in a nutshell

Measures of resilience and extensions

How to measure resilience?

From viable states to viable random paths

"Self-promotion, nobody will do it for you" ;-)

“Nul n’est mieux servi que par soi-même” “Self-promotion, nobody will do it for you” ;-)

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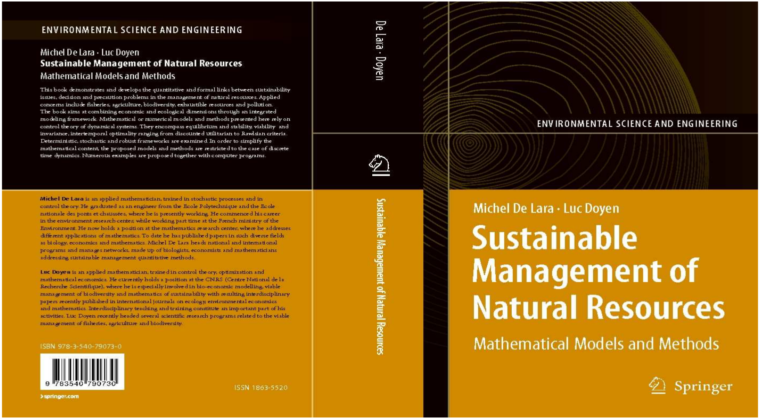

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