Local stabilization of a fluid–structure system around a stationary state with a structure given by a finite number of parameters[*]

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Abstract. We study the stabilization of solutions to a 2d fluid–structure system by a feedback control law acting on the acceleration of the structure. The structure is described by a finite number of parameters. The modelling of this system and the existence of strong solutions have been previously studied in [11]. We consider an unstable stationary solution to the problem. We assume a unique continuation property for the eigenvectors of the adjoint system. Under this assumption, the nonlinear feedback control that we propose stabilizes the whole fluid–structure system around the stationary solution at any chosen exponential decay rate for small enough initial perturbations. Our method reposes on the analysis of the linearized system and the feedback operator is given by a Riccati equation of small dimension.

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1 Introduction

The goal of this study is to stabilize a 2d fluid–structure interaction problem. The fluid is modelled by the incompressible Navier–Stokes equations and the structure, immersed in the fluid, is governed by two scalar parameters denoted θ_1 and θ_2. More precisely, the structure is a smooth approximation of a steering gear depending on two angles of deformation θ_1 and θ_2. Such a kind of structure can be found for instance in aeronautics [18]. Our goal is to design a finite dimensional feedback controller which stabilizes locally the system around a given stationary state at any prescribed exponential decay rate.

Even if we considered only two parameters, the present study can easily be extended to the case of a structure depending on a finite number N (≥ 1) of scalar parameters (see Remark 1.1). The extension of all proofs is indeed straightforward.

1.1 Modelling of the problem

The fluid–structure configuration considered in this paper has already been investigated in [11] where existence of strong solutions has been proven. We consider a bounded domain $\Omega = (0, L) \times (0, 1)$ (see Fig.1). The volume occupied by the structure depends on two parameters denoted (θ_1, θ_2), it is a closed subset of Ω that we denote $S(\theta_1, \theta_2) \subset \Omega$. The volume filled by the fluid is denoted $\mathscr{F}(\theta_1, \theta_2) = \Omega \backslash S(\theta_1, \theta_2)$.

The boundary $\partial\Omega$ can be decomposed into $\partial\Omega = \overline{\Gamma_i} \cup \overline{\Gamma_w} \cup \overline{\Gamma_N}$, where $\Gamma_i = \{0\} \times (0,1)$, $\Gamma_w = (0,L) \times \{0,1\}$ and $\Gamma_N = \{L\} \times (0,1)$. We also denote $\Gamma_D = \overline{\Gamma_i} \cup \Gamma_w$ the part of $\partial\Omega$ where Dirichlet conditions are imposed. We now introduce the equations modelling this system.

1.1.1 The equations of the fluid

The velocity of the fluid is assumed to fulfil the incompressible Navier–Stokes equations

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x}) - \operatorname{div} \sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t \in (0,\infty), \quad \mathbf{x} \in \mathscr{F}(\theta_1(t),\theta_2(t)), \\
\operatorname{div} \mathbf{u}(t,\mathbf{x}) = 0, & t \in (0,\infty), \quad \mathbf{x} \in \mathscr{F}(\theta_1(t),\theta_2(t)), \\
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t \in (0,\infty), \quad \mathbf{x} \in \Gamma_i, \\
\mathbf{u}(t,\mathbf{x}) = 0, & t \in (0,\infty), \quad \mathbf{x} \in \Gamma_w, \\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t \in (0,\infty), \quad \mathbf{x} \in \Gamma_N, \\
\mathbf{u}(t,\mathbf{x}) = \mathbf{v}_s(t,\mathbf{x}), & t \in (0,\infty), \quad \mathbf{x} \in \partial S(\theta_1(t),\theta_2(t)), \\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \mathscr{F}(\theta_{1,0},\theta_{2,0}),
\end{cases}
\tag{1.1}
$$

where $\mathbf{u}(t,\mathbf{x})$ and $p(t,\mathbf{x})$ are the velocity and the pressure of the fluid at point $\mathbf{x}$ and time t,

$$
\sigma_F(\mathbf{u},p) = \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - p\mathrm{I},
$$

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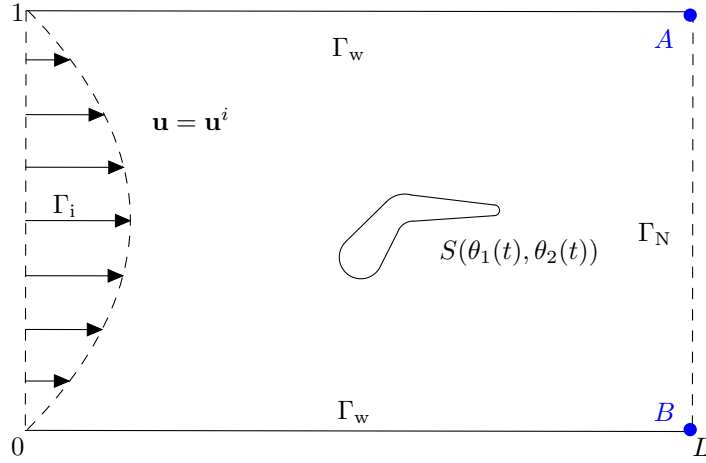
Figure 1: The geometrical configuration.

is the Cauchy stress tensor of the fluid and $\nu > 0$ is the kinematic viscosity. The term $\mathbf{f}_{\mathscr{F}}(t, \mathbf{x})$ in $(1.1)_1$ is a force per unit mass exerted on the fluid, $\mathbf{u}^i(t, \mathbf{x})$ is a nonhomogeneous boundary datum on Γ_i, $\mathbf{v}_s(t, \mathbf{x})$ denotes the velocity of the structure and $\mathbf{n}(\mathbf{x})$ is the outward unit normal to Ω. Dirichlet boundary conditions are imposed on Γ_D and Neumann type (free output) boundary conditions are imposed on Γ_N. We also consider an initial datum $\mathbf{u}_0(\mathbf{x})$ for the fluid velocity.

1.1.2 Equations of the structure

We consider that the couple of parameters (θ_1, θ_2) lies in an admissible domain $\mathbb{D}_\Theta$ which is an open connected subset of $\mathbb{R}^2$ containing $(0, 0)$. These parameters stand here for structure deformation angles (see Fig. 2).

We consider a function $\mathbf{X}$ defined on $\mathbb{D}_\Theta \times S(0, 0)$ that computes the position of a point of the structure according to its reference position in $S(0, 0)$ and the value of the parameters $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$.

Let us list below the assumptions that we make

Modelling Assumptions.

- For every $\mathbf{y} \in S(0, 0)$, $\mathbf{X}(0, 0, \mathbf{y}) = \mathbf{y}$. (1.2)
- The set $S(0, 0)$ is a smooth simply connected closed subset of Ω. (1.3)
- For every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1, \theta_2, S(0, 0)) = S(\theta_1, \theta_2) \subset \Omega$ and $\inf_{(\theta_1, \theta_2) \in \mathbb{D}_\Theta} \mathrm{d}(S(\theta_1, \theta_2), \partial\Omega) > 0$. (1.4)
- For every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1, \theta_2, .)$ is a $\mathscr{C}^\infty$ diffeomorphism from $S(0, 0)$ to its image $S(\theta_1, \theta_2)$. (1.5)
- The function $\mathbf{X}$ is $\mathscr{C}^\infty$ on $\mathbb{D}_\Theta \times S(0, 0)$. (1.6)
- The functions $\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .)$ and $\partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .)$ form
  a free family in $\mathbf{L}^2(\partial S(0, 0))$ for every (θ_1, θ_2) in $\mathbb{D}_\Theta$. (1.7)
- No friction and no elastic energy are considered in the structure. (1.8)

More information about these assumptions can be found in [11]. The inverse diffeomorphism of $\mathbf{X}(\theta_1, \theta_2, .)$, whose existence is guaranteed by (1.5), is denoted $\mathbf{Y}(\theta_1, \theta_2, .)$ and we have

$$\forall(\theta_1, \theta_2) \in \mathbb{D}_\Theta, \quad \forall\mathbf{y} \in S(0, 0), \quad \mathbf{Y}(\theta_1, \theta_2, \mathbf{X}(\theta_1, \theta_2, \mathbf{y})) = \mathbf{y}. \quad (1.9)$$

The diffeomorphisms $\mathbf{X}(\theta_1, \theta_2, .)$ and $\mathbf{Y}(\theta_1, \theta_2, .)$ are illustrated in Fig. 2.

In the sequel, we denote $\dot{\theta}_j$ and $\ddot{\theta}_j$ the first and second time derivatives of θ_j. The equations that are satisfied by the structure read on a matrix form

$$\mathcal{M}_{\theta_1, \theta_2} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \mathbf{M_I}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) + \mathbf{M_A}(\theta_1, \theta_2, -\sigma_F(\mathbf{u}, p)\mathbf{n}_{\theta_1, \theta_2}) + \mathbf{f_s} + \mathbf{h} \quad \text{on } (0, T), \quad (1.10)$$

where $\mathbf{f_s}$ is a source term, $\mathbf{h}$ a control function,

$$\mathcal{M}_{\theta_1, \theta_2} = \begin{pmatrix} (\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .), \partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .), \partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .))_S \\ (\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .), \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .), \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .))_S \end{pmatrix} \in \mathbb{R}^{2 \times 2}, \quad (1.11)$$

$$\mathbf{M_I}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{pmatrix} -(\dot{\theta}_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1, \theta_2, .) + 2\dot{\theta}_1\dot{\theta}_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1, \theta_2, .) + \dot{\theta}_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1, \theta_2, .)), \partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .))_S \\ -(\dot{\theta}_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1, \theta_2, .) + 2\dot{\theta}_1\dot{\theta}_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1, \theta_2, .) + \dot{\theta}_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1, \theta_2, .)), \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .))_S \end{pmatrix} \in \mathbb{R}^2, \quad (1.12)$$

where $(., .)_S$ is the scalar product

$$(\mathbf{\Phi}, \mathbf{\Psi})_S = \int_{S(0, 0)} \rho\mathbf{\Phi}(\mathbf{y}) \cdot \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \quad (1.13)$$
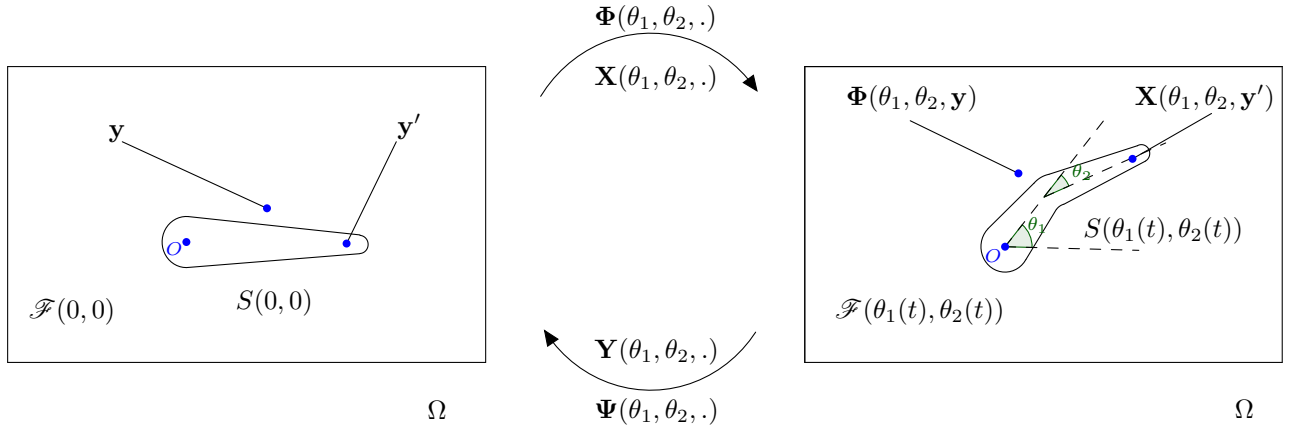
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Figure 2: Correspondence between real and reference configurations.

with $\rho > 0$ the mass per unit volume of the structure and

$$\mathbf{M_A}(\theta_1, \theta_2, -\sigma_F(\mathbf{u}, p)\mathbf{n}_{\theta_1, \theta_2}) = \begin{pmatrix} \displaystyle\int_{\partial S(\theta_1, \theta_2)} -\sigma_F(\mathbf{u}, p)\mathbf{n}_{\theta_1, \theta_2}(\gamma_x) \cdot \partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, \mathbf{Y}(\theta_1, \theta_2, \gamma_x)) \, \mathrm{d}\gamma_x \\ \displaystyle\int_{\partial S(\theta_1, \theta_2)} -\sigma_F(\mathbf{u}, p)\mathbf{n}_{\theta_1, \theta_2}(\gamma_x) \cdot \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, \mathbf{Y}(\theta_1, \theta_2, \gamma_x)) \, \mathrm{d}\gamma_x \end{pmatrix} \in \mathbb{R}^2, \quad (1.14)$$

where $\mathbf{n}_{\theta_1, \theta_2}$ is the outward unit normal to $\mathscr{F}(\theta_1, \theta_2)$ on $\partial S(\theta_1, \theta_2)$.

Moreover the velocity of the structure can be written

$$\forall t \in [0, \infty), \quad \forall x \in S(\theta_1(t), \theta_2(t)), \quad \mathbf{v}_s(t, \mathbf{x}) = \sum_{k=1}^{2} \dot{\theta}_k(t)\partial_{\theta_k}\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{Y}(\theta_1(t), \theta_2(t), \mathbf{x})).$$

More information about the derivation of these equations can be found in [11].

Note that the matrix $\mathcal{M}_{\theta_1, \theta_2}$ in (1.11) is the Gram matrix of the family $(\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2), \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2))$ with respect to the scalar product $(.,.)_S$. It is thus invertible due to Assumption (1.7) (if two $\mathscr{C}^\infty$ functions are not collinear in $\mathbf{L}^2(\partial S(0,0))$ then they are not collinear in $\mathbf{L}^2(S(0,0))$).

Remark 1.1. The proposed framework can be used to model other problems. For instance, in the case of a rigid solid whose center of mass is given by (a_1, a_2) and corresponds to $(0, 0)$ in the reference configuration and whose angle of rotation is given by θ (so that three parameters are considered), the diffeomorphism $\mathbf{X}$ now depends on three parameters and is given by

$$\mathbf{X}(a_1, a_2, \theta, \mathbf{y}) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + R_\theta\mathbf{y},$$

where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Moreover, we have

$$\mathcal{M}_{a_1, a_2, \theta} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \mathbf{M_I}(a_1, a_2, \theta, \dot{a}_1, \dot{a}_2, \dot{\theta}) = 0, \quad \mathbf{M_A}(a_1, a_2, \theta, \mathbf{f}) = \int_{\partial S(a_1, a_2, \theta)} \begin{pmatrix} \mathbf{f} \cdot \mathbf{e}_1 \\ \mathbf{f} \cdot \mathbf{e}_2 \\ \mathbf{f} \cdot R_{\theta + \frac{\pi}{2}}\mathbf{y} \end{pmatrix} \, \mathrm{d}\mathbf{x},$$

where $m = \displaystyle\int_{S(0,0,0)} \rho \, \mathrm{d}\mathbf{y}$ denotes the mass of the solid and $I = \displaystyle\int_{S(0,0,0)} \rho\mathbf{y}^2 \, \mathrm{d}\mathbf{y}$ its moment of inertia. Hence equation (1.10) corresponds to the usual Newton's law.

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1.1.3 The complete set of equations

The final system that we consider is given by the following set of equations

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x}) - \operatorname{div}\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\[2mm]
\operatorname{div}\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,\infty),\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\[2mm]
\mathbf{u}(t,\mathbf{x}) = \displaystyle\sum_{j=1}^{2}\dot\theta_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,\infty),\ \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)), \\[2mm]
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\mathrm{i}}, \\
\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\mathrm{w}}, \\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\mathrm{N}}, \\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}), \\[2mm]
\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = \mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) + \mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) + \mathbf{f_s} + \mathbf{h}, & t\in(0,\infty), \\[2mm]
\theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, \\
\dot\theta_1(0) = \omega_{1,0}, \quad \dot\theta_2(0) = \omega_{2,0}.
\end{cases}
\tag{1.15}
$$

Note that the fluid domain $\mathscr{F}(\theta_1(t),\theta_2(t))$ changes over time. The control $\mathbf{h}$ can be understood as a force acting on the structure. The data $(\theta_{1,0},\theta_{2,0})$ and $(\omega_{1,0},\omega_{2,0})$ respectively describe the initial position and velocity of the structure.

In the sequel, since we consider the stabilization of these equations around a stationary state, the data $\mathbf{f}_{\mathscr{F}}$, $\mathbf{u}^i$ and $\mathbf{f_s}$ will be considered as time independent.

1.2 Statement of the main result

Existence of strong solutions to (1.15) locally in time has been proven in [11]. The goal of the present study is to prove that, given a stationary state, we can choose $\mathbf{h}$ under a feedback form such that a solution to (1.15) stabilizes exponentially around that stationary state when t tends to infinity. In this section we present our stabilization result.

The stationary state. Let $(\mathbf{w},p_{\mathbf{w}},\eta_1,\eta_2)$ be a stationary state of (1.15) associated to stationary source terms $\mathbf{f}_{\mathscr{F}}$, $\mathbf{f_s}$ and boundary datum $\mathbf{u}^i$, i.e.

$$
\begin{cases}
(\mathbf{w}(\mathbf{x})\cdot\nabla)\mathbf{w}(\mathbf{x}) - \operatorname{div}\sigma_F(\mathbf{w}(\mathbf{x}),p_{\mathbf{w}}(\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\eta_1,\eta_2), \\
\operatorname{div}\mathbf{w}(\mathbf{x}) = 0, & \mathbf{x}\in\mathscr{F}(\eta_1,\eta_2), \\
\mathbf{w}(\mathbf{x}) = 0, & \mathbf{x}\in\partial S(\eta_1,\eta_2), \\
\mathbf{w}(\mathbf{x}) = \mathbf{u}^i(\mathbf{x}), & \mathbf{x}\in\Gamma_{\mathrm{i}}, \\
\mathbf{w}(\mathbf{x}) = 0, & \mathbf{x}\in\Gamma_{\mathrm{w}}, \\
\sigma_F(\mathbf{w}(\mathbf{x}),p_{\mathbf{w}}(\mathbf{x}))\mathbf{n} = 0, & \mathbf{x}\in\Gamma_{\mathrm{N}}, \\
0 = \mathbf{M_I}(\eta_1,\eta_2,0,0) + \mathbf{M_A}(\eta_1,\eta_2,-\sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n}_{\eta_1,\eta_2}) + \mathbf{f_s}.
\end{cases}
\tag{1.16}
$$

Note that $\mathbf{M_I}(\eta_1,\eta_2,0,0) = 0$ and it can thus be withdrawn from (1.16).

In the sequel, we take $(\eta_1,\eta_2) = (0,0)$ to simplify the notations. This choice is not restrictive as a change of variables can bring the stationary parameters to $(0,0)$. We denote respectively $\mathscr{F}_s$ and S_s the fluid and solid domains associated to the stationary solution,

$$
\mathscr{F}_s = \mathscr{F}(0,0) \qquad \text{and} \qquad S_s = S(0,0).
$$

Rewriting (1.16), we consider nonhomogeneous terms $\mathbf{f}_{\mathscr{F}}$, $\mathbf{u}^i$, $\mathbf{f_s}$ and a velocity–pressure profile $(\mathbf{w},p_{\mathbf{w}}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ fulfilling the equations

$$
\begin{cases}
-\operatorname{div}\sigma_F(\mathbf{w},p_{\mathbf{w}}) = -(\mathbf{w}\cdot\nabla)\mathbf{w} + \mathbf{f}_{\mathscr{F}} & \text{in } \mathscr{F}_s, \\
\operatorname{div}\mathbf{w} = 0 & \text{in } \mathscr{F}_s, \\
\mathbf{w} = 0 & \text{on } \partial S_s, \\
\mathbf{w} = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\
\mathbf{w} = 0 & \text{on } \Gamma_{\mathrm{w}}, \\
\sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}, \\
(f_s)_j = \displaystyle\int_{\partial S_s}(\sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n}_s)(\gamma_y)\cdot\partial_{\theta_j}\mathbf{X}(0,0,\gamma_y)\,\mathrm{d}\gamma_y,
\end{cases}
\tag{1.17}
$$

where $\mathbf{n}_s$ is the outward unit normal to $\mathscr{F}_s$ on ∂S_s,

$$
\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega) \quad \text{and} \quad \mathbf{u}^i \in \mathbf{U}^i = \left\{ \begin{array}{l} \mathbf{u}^i \in \mathbf{H}^{3/2}(\Gamma_{\mathrm{i}}) \quad | \quad \mathbf{u}^i_{|\partial\Gamma_{\mathrm{i}}} = 0, \ \displaystyle\int_0^{1/4}\frac{|\partial_{y_2}u^i_2(y_2)|^2}{y_2}\,\mathrm{d}y_2 < +\infty, \\[3mm] \displaystyle\int_{3/4}^1\frac{|\partial_{y_2}u^i_2(y_2)|^2}{1-y_2}\,\mathrm{d}y_2 < +\infty \end{array} \right\}. \tag{1.18}
$$

Remark 1.2. The regularity of the source term $\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega)$ is used for the estimation of some nonlinear terms in Appendix C.

Remark 1.3. We do not consider that (1.17) admits a unique solution $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ for every set of regular data $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f_s})$ satisfying some compatibility conditions. However, we consider that there exist a stationary solution $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ and some associated data $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f_s})$ such that (1.17) is fulfilled. More information about stationary solutions can be found in [23, Appendix].

The diffeomorphism $\boldsymbol{\Phi}$. A classical difficulty in fluid–structure problems is that the fluid domain changes over time. The classical way of getting rid of this difficulty is to use a change of variables on $\mathbf{u}$ and p in order to bring the study back into a fixed domain. This procedure uses a diffeomorphism that we have to define properly.

When the state of the structure depends only on a finite number of parameters, it is convenient to construct this diffeomorphism as an extension of the structure deformation into the fluid domain. The diffeomorphism used is defined as an extension of the diffeomorphism $\mathbf{X}$ given for the structure. For that reason, we use the following extension operator.

Lemma 1.4. *There exists a linear extension operator* $\mathcal{E} : \mathbf{W}^{3,\infty}(S_s) \to \mathbf{W}^{3,\infty}(\Omega) \cap \mathbf{H}_0^1(\Omega)$ *such that for every* $\varphi \in \mathbf{W}^{3,\infty}(S_s)$,

$$
\begin{aligned}
(i) \quad & \mathcal{E}(\varphi) = \varphi \text{ in } S_s, \\
(ii) \quad & \mathcal{E}(\varphi) \text{ has support within } \Omega_\varepsilon = \{\mathbf{x} \in \Omega \mid \mathrm{d}(x, \partial\Omega) > \varepsilon\} \text{ for some } \varepsilon > 0 \\
& \text{such that } \mathrm{d}(S(\theta_1, \theta_2), \partial\Omega) > 2\varepsilon \text{ for all } (\theta_1, \theta_2) \in \mathbb{D}_\Theta, \\
(iii) \quad & \|\varphi\|_{\mathbf{W}^{3,\infty}(\Omega)} \le C\|\varphi\|_{\mathbf{W}^{3,\infty}(S_s)}, \text{ for some } C > 0.
\end{aligned}
$$

Proof. Extension results are classical, we can for instance find an extension result for smooth domains in [19, Lemma 12.2]. We can get the present result by multiplying the extension function of [19, Lemma 12.2] by a cut–off function in $\mathcal{D}(\Omega_\varepsilon)$. Note that the existence of $\varepsilon > 0$ fulfilling (ii) is a consequence of Assumption (1.4). $\qquad\square$

Let us denote Id the identity function, we then define the following function

$$
\boldsymbol{\Phi}(\theta_1, \theta_2, \mathbf{y}) = \mathbf{y} + \mathcal{E}\Big(\mathbf{X}(\theta_1, \theta_2, .) - \mathrm{Id}\Big)(\mathbf{y}), \quad \forall(\theta_1, \theta_2) \in \mathbb{D}_\Theta, \quad \forall\mathbf{y} \in \Omega. \tag{1.19}
$$

We have $\nabla\boldsymbol{\Phi}(0, 0, \mathbf{y}) = \mathrm{I}$, the identity matrix in $\mathbb{R}^{2\times 2}$, for every $\mathbf{y} \in \Omega$, hence $\det(\nabla\boldsymbol{\Phi}(0, 0, \mathbf{y})) = 1$. Then, we can restrict $\mathbb{D}_\Theta$ such that for every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, the function $\boldsymbol{\Phi}(\theta_1, \theta_2, .)$ is a diffeomorphism close to the identity function. We denote $\boldsymbol{\Psi}(\theta_1, \theta_2, .)$ the inverse diffeomorphism of $\boldsymbol{\Phi}(\theta_1, \theta_2, .)$

$$
\forall(\theta_1, \theta_2) \in \mathbb{D}_\Theta, \quad \forall\mathbf{y} \in \Omega, \quad \boldsymbol{\Psi}(\theta_1, \theta_2, \boldsymbol{\Phi}(\theta_1, \theta_2, \mathbf{y})) = \mathbf{y}. \tag{1.20}
$$

If needed, we can once more reduce $\mathbb{D}_\Theta$ to prove that $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ belong to $\mathscr{C}^\infty(\mathbb{D}_\Theta, \mathbf{W}^{3,\infty}(\Omega))$. These diffeomorphisms are represented in Fig. 2.

The properties of $\mathcal{E}$ imply that

$$
\text{for every } (\theta_1, \theta_2) \in \mathbb{D}_\Theta, \quad \boldsymbol{\Phi}(\theta_1, \theta_2, S_s) = S(\theta_1, \theta_2) \quad \text{and} \quad \forall\mathbf{y} \in \Omega\backslash\Omega_\varepsilon, \quad \boldsymbol{\Phi}(\theta_1, \theta_2, \mathbf{y}) = \mathbf{y}, \tag{1.21}
$$

where Ω_ε is defined in Lemma 1.4.

The stabilization problem. In order to prove a stabilization result on the nonlinear problem, we first study the linearized problem around $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$ and prove its stabilizability. It requires the technical hypothesis $(\mathcal{H})_\delta$ that is presented hereafter.

In the sequel, $\mathbf{v}$ can be thought of as the difference between the state $\mathbf{u}$ and the stationary state $\mathbf{w}$ of the problem (see (3.1) for its precise definition). The linearized term in $\mathbf{v}$ in the fluid equation is the usual Oseen term $(\mathbf{v} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v}$. The linearized term in $(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ in the fluid equation is denoted $\mathbf{L_F}$. In the same way, we denote $\mathbf{L_S}$ the linearized term in (θ_1, θ_2) in the structure equation. Then we have

$$
\mathbf{L_F}(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2, \mathbf{y}) = \mathbf{L}_1(\mathbf{y})\theta_1 + \mathbf{L}_2(\mathbf{y})\theta_2 + \mathbf{L}_3(\mathbf{y})\dot\theta_1 + \mathbf{L}_4(\mathbf{y})\dot\theta_2, \quad \forall\mathbf{y} \in \mathscr{F}_s, \tag{1.22}
$$

and

$$
\mathbf{L_S}(\theta_1, \theta_2) = \mathbf{L}_5\theta_1 + \mathbf{L}_6\theta_2, \tag{1.23}
$$

where the exact expressions of the coefficients $\mathbf{L}_1 - \mathbf{L}_6$ are given in Appendix A. The coefficients $\mathbf{L}_1 - \mathbf{L}_4$ are functions and $\mathbf{L}_5 - \mathbf{L}_6$ are constant vectors of $\mathbb{R}^2$. They all depend on the non–null stationary state $(\mathbf{w}, p_{\mathbf{w}})$ which is solution of (1.17), on the diffeomorphism $\boldsymbol{\Phi}$ and on its derivatives taken in $(\theta_1, \theta_2) = (0, 0)$.

Let $\delta > 0$ be a prescribed exponential decay rate in time for the difference between the solution and the stationary state. In order to prove the main result of the study, we need the following assumption that depends on δ and corresponds to a Hautus test.

Hypothesis $(\mathcal{H})_\delta$ (A unique continuation property)**.** Every eigenvector $(\mathbf{v}, q, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathrm{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ of the adjoint problem associated to the eigenvalue $\overline{\lambda}$ with $\mathcal{R}e(\overline{\lambda}) \geq -\delta$, i.e. every solution of

$$\begin{cases} \mathrm{div}\, \sigma_F(\mathbf{v}, q) - (\nabla \mathbf{w})^T \mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{v} = \overline{\lambda} \mathbf{v} & \text{in } \mathscr{F}_s, \\ \mathrm{div}\, \mathbf{v} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v} = \omega_1 \partial_{\theta_1} \mathbf{\Phi}(0,0,.) + \omega_2 \partial_{\theta_2} \mathbf{\Phi}(0,0,.) & \text{on } \partial S_s, \\ \mathbf{v} = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_F(\mathbf{v}, q)\mathbf{n} + (\mathbf{w} \cdot \mathbf{n})\mathbf{v} = 0 & \text{on } \Gamma_{\mathrm{N}}, \\ \displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_1(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \\ \mathbf{L}_2(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} + \begin{pmatrix} \mathbf{L}_5 \cdot \left\{ \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\} \\ \mathbf{L}_6 \cdot \left\{ \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\} \end{pmatrix} = \overline{\lambda} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \\ \displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_3(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \\ \mathbf{L}_4(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} - \int_{\partial S_s} \begin{pmatrix} \sigma_F(\mathbf{v}, q)\mathbf{n}_s(\gamma_y) \cdot \partial_{\theta_1} \mathbf{\Phi}(0,0,\gamma_y) \\ \sigma_F(\mathbf{v}, q)\mathbf{n}_s(\gamma_y) \cdot \partial_{\theta_2} \mathbf{\Phi}(0,0,\gamma_y) \end{pmatrix} \mathrm{d}\gamma_y + \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \overline{\lambda} \mathcal{M}_{0,0} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \end{cases}$$

that belongs to the kernel of the adjoint of the control operator, i.e. that satisfies

$$\begin{cases} \omega_1 = 0, \\ \omega_2 = 0, \end{cases}$$

is necessarily null, i.e. $(\mathbf{v}, q, \theta_1, \theta_2, \omega_1, \omega_2) = (0,0,0,0,0,0)$.

This hypothesis is a unique continuation property for the adjoint system. Such a property is proven for some problems, in particular for the Stokes problem with localized observation [12]. However, in our case of study, the observation is nonlocal, and to our knowledge the corresponding unique continuation property is not available in the literature. In order to lead the study of the stabilization of our problem, we assume this unique continuation property to be valid. Although we do not know how to prove it, we can reasonably think that it is generically valid. Besides, it can be checked numerically on each particular instance.

Remark 1.5. Hypothesis $(\mathcal{H})_\delta$ is independent from the choice of the extension operator $\mathcal{E}$ used in Lemma 1.4 to construct the diffeomorphism $\mathbf{\Phi}$, see Appendix B.

In the sequel, $\mathcal{J}_{\mathbf{\Phi}}(\theta_1, \theta_2, .)$ denotes the Jacobian matrix of $\mathbf{\Phi}(\theta_1, \theta_2, .)$ and $\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1, \theta_2, .))$ its cofactor matrix. The goal of the study is to prove the following theorem.

Theorem 1.6 (Main result)**.** *Let $\delta > 0$ and assume that $(\mathcal{H})_\delta$ is fulfilled. Let $\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega)$, $\mathbf{u}^i \in \mathbf{U}^i$, $\mathbf{f_s} \in \mathbb{R}^2$, and $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ fulfil (1.17). Then, there exists $\varepsilon > 0$ such that for every $(\mathbf{u}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}(\theta_{1,0}, \theta_{2,0})) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ satisfying the compatibility conditions*

$$\begin{cases} \mathrm{div}\, \mathbf{u}_0 = 0 & \text{in } \mathscr{F}(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0(.) = \displaystyle\sum_{j=1}^2 \omega_{j,0} \partial_{\theta_j} \mathbf{X}(\theta_{1,0}, \theta_{2,0}, \mathbf{Y}(\theta_{1,0}, \theta_{2,0}, .)) & \text{on } \partial S(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0 = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\ \mathbf{u}_0 = 0 & \text{on } \Gamma_{\mathrm{w}}, \end{cases} \tag{1.24}$$

and

$$\|\mathbf{u}_0(\mathbf{\Phi}(\theta_{1,0}, \theta_{2,0}, .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \leq \varepsilon,$$

there exists a control $\mathbf{h}$ given under the feedback form

$$\mathbf{h}(t) = \mathcal{K}_\delta \left(\left[\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1(t), \theta_2(t), .))^T \mathbf{u}(t, \mathbf{\Phi}(\theta_1(t), \theta_2(t), .)) - \mathbf{w} \right], \theta_1(t), \theta_2(t), \dot{\theta}_1(t), \dot{\theta}_2(t) \right), \tag{1.25}$$

for some linear operator $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$ such that a solution $(\mathbf{u}, p, \theta_1, \theta_2)$ to problem (1.15) fulfils for all t in $(0, \infty)$

$$\|\mathbf{u}(t, \mathbf{\Phi}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot{\theta}_1(t)| + |\dot{\theta}_2(t)| \leq C e^{-\delta t},$$

for some $C > 0$ depending on the geometry, on δ and on the initial and nonhomogeneous data.

Theorem 1.6 is proven in Sections 2 and 3.

Remark 1.7. The feedback law proposed in (1.25) does not depend linearly on the state $(\mathbf{u}, p, \theta_1, \theta_2)$.

1.3 The functional framework

In this section we present the functional setting used in the sequel. We denote by $\mathscr{C}^0([0,\infty);\mathbb{X})$ the set of functions that are continuous on $[0,\infty)$ and valued in $\mathbb{X}$.

Sobolev spaces. We denote $H^r(\mathscr{F}_s)$ the usual Sobolev space of order $r \geq 0$. We identify $L^2(\mathscr{F}_s)$ with $H^0(\mathscr{F}_s)$. We will denote $\mathbf{L}^2(\mathscr{F}_s) = (L^2(\mathscr{F}_s))^2$, $\mathbf{H}^r(\mathscr{F}_s) = (H^r(\mathscr{F}_s))^2$ and so on.

Corners issues. The domain considered for the fluid has four corners of angle $\pi/2$. The ones that are located between Dirichlet and Neumann boundary conditions induce singularities, we denote them $A = (L, 1)$ and $B = (L, 0)$ (see Fig. 1). We also denote $\mathscr{J}_{d,n} = \{A, B\}$ the set of these corners and we define the distance of a point $\mathbf{x}$ from these corners

$$\text{for } j \in \mathscr{J}_{d,n}, \quad \text{and for } \mathbf{x} \in \Omega, \quad r_j(\mathbf{x}) = \mathrm{d}(\mathbf{x}, j). \tag{1.26}$$

Note that corners between two Dirichlet boundary conditions do not induce singularities as soon as suitable compatibility conditions are satisfied. We report to [20, Chapter 9] for more details.

Weighted Sobolev spaces. The solution to the Stokes problem in the domain with corners A and B and with a source term in $\mathbf{L}^2(\mathscr{F}_s)$ belongs to a classical Sobolev space of lower order than the one we usually have in smooth domains. In order to get the usual gain of regularity between solutions and source terms, we have to study the solution in adapted Sobolev spaces that are suitably weighted near corners A and B. The weighted Sobolev spaces are then defined for $\beta > 0$ by

$$\mathbf{H}^2_\beta(\mathscr{F}_s) = \{\mathbf{u} \text{ with } \|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)} < +\infty\}, \qquad H^1_\beta(\mathscr{F}_s) = \{p \text{ with } \|p\|_{H^1_\beta(\mathscr{F}_s)} < +\infty\},$$

where the norms $\|.\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)}$ and $\|.\|_{H^1_\beta(\mathscr{F}_s)}$ are given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\beta(\mathscr{F}_s)} = \sum_{|\alpha|=0}^{2} \sum_{i=1}^{2} \int_{\mathscr{F}_s} \left(\prod_{j \in \mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y}) \right) |\partial^\alpha u_i(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}, \tag{1.27}$$

and

$$\|p\|^2_{H^1_\beta(\mathscr{F}_s)} = \sum_{|\alpha|=0}^{1} \int_{\mathscr{F}_s} \left(\prod_{j \in \mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y}) \right) |\partial^\alpha p(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}. \tag{1.28}$$

Here the sum is on every multi–index α of length $|\alpha| \leq 2$ for (1.27), $|\alpha| \leq 1$ for (1.28) and r_j is defined in (1.26).

Steady Stokes problem with corners. The following lemma from [23] explains how and why the spaces $\mathbf{H}^2_\beta(\mathscr{F}_s)$ and $H^1_\beta(\mathscr{F}_s)$ appear in the presence of corners. It gives the expected result for the steady Stokes problem in $\mathscr{F}_s$ with weigthed Sobolev spaces and the regularity obtained in the classical Sobolev spaces.

Lemma 1.8. *[23, Theorem 2.5.] Let us assume that $\mathbf{f}_{\mathscr{F}} \in \mathbf{L}^2(\mathscr{F}_s)$. The unique solution $(\mathbf{u}, p)$ to the Stokes problem*

$$\begin{cases} -\mathrm{div}\,\sigma_F(\mathbf{u}, p) = \mathbf{f}_{\mathscr{F}} & \text{in } \mathscr{F}_s, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{u} = 0 & \text{on } \Gamma_{\mathrm{D}} \cup \partial S_s, \\ \sigma_F(\mathbf{u}, p)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{1.29}$$

belongs to $\mathbf{H}^2_\beta(\mathscr{F}_s) \times H^1_\beta(\mathscr{F}_s)$ for some $\beta \in (0, 1/2)$ and to $\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s) \times H^{1/2+\varepsilon_0}(\mathscr{F}_s)$ for some $\varepsilon_0 \in (0, 1/2)$. Moreover, we have the following estimate

$$\|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s) \cap \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s)} + \|p\|_{H^1_\beta(\mathscr{F}_s) \cap H^{1/2+\varepsilon_0}(\mathscr{F}_s)} \leq C\|\mathbf{f}_{\mathscr{F}}\|_{\mathbf{L}^2(\mathscr{F}_s)}. \tag{1.30}$$

Remark 1.9. A consequence of Lemma 1.8 is that the solution $(\mathbf{w}, p_{\mathbf{w}})$ of (1.17)–(1.18) belongs to $\mathbf{H}^2_\beta(\mathscr{F}_s) \times H^1_\beta(\mathscr{F}_s)$. This is the regularity that we will use for $(\mathbf{w}, p_{\mathbf{w}})$ in the sequel. A proof of this statement can be achieved by lifting the datum $\mathbf{u}^i$, which is done in [11, Lemma 2.10].

Keep in mind that $\mathbf{n}_s$ is the outward unit normal to $\mathscr{F}_s$. Note that, according to the regularity proven in Lemma 1.8, the traces $p_{|\partial\mathscr{F}_s}$ and $\partial_{\mathbf{n}_s}\mathbf{u}_{|\partial\mathscr{F}_s}$ are well defined, which gives a meaning to all integrations by parts.

Also note that according to [17, Theorem 1.4.3.1], there exists a continuous extension operator from $\mathbf{H}^s(\mathscr{F}_s)$ to $\mathbf{H}^s(\mathbb{R}^2)$ for every $s > 0$. This implies that all the classical Sobolev embeddings and interpolations are valid despite the presence of corners.

1.4 Scientific context

There are several works providing stabilization results in the context of the Navier–Stokes equations. For instance, the stabilization of a viscous fluid is treated for the wake of a cylinder in [14, 15, 16, 23] and for a cavity in [21]. A first strategy used in [15, 16, 21] reposes on the Proper Orthogonal Decomposition (POD) approach. Another approach consists in constructing a feedback operator by means of a Riccati equation making the closed–loop system stable [23, 25, 26]. If needed, it is possible to use dynamical controllers to meet compatibility conditions between the fluid initial datum and the initial control value, the control is then computed as the solution of an ODE [1, 2].

When we consider fluid–structure interaction problems the same strategies can be used. The reader can refer for instance to [28] for the stabilization by a POD approach of a fluid around an airfoil. It is also possible to build a stabilizing feedback control that uses only the state of the structure, see [7] for a 1D and [29] for a 2D fluid–solid interaction problems.

In the present study, we use a stabilizing control that is given under a feedback form and uses the state of the fluid and the structure. The feedback operator is computed via the solution of a finite dimensional Riccati equation. This is helpful when treating numerical simulations which are not the point of the current paper and are a work currently in progress. The same strategy has already been used to prove stabilization of strong solutions, which is what we aim for, and more recently stabilization of weak solutions to a fluid–beam interaction problem [5]. In the literature, the feedback control can be a Dirichlet datum imposed to the fluid on some part of the boundary [3, 22], it can be a change in the shape of the structure [8, 9] or a force acting on the structure [22, 27].

Although the control that we use in the current study acts on the structure, the study [3] is the closest one from what we want to prove. It treats the stabilization of a fluid–rigid body system by a feedback control law acting on the boundary of the fluid domain. In the current study, we follow its framework and account for the deformability of the structure and the control acting on the acceleration of the structure. Additional difficulties are induced by the corners on $\partial\Omega$, more information about them can be found in [20, 23].

1.5 Outline of the paper

In Section 2, we prove the existence of a feedback control law that stabilizes the solution of the linearized system in the fixed domain $\mathscr{F}_s$ around the stationary state $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$. The proof relies on the analysis of the properties of the linearized system. In Section 3, we extend successively the previous result to the full nonlinear system in the fixed domain and in the moving domain. The former is proven via a fixed point argument and the latter uses a change of variables. The linearized terms are summed up in Appendix A. An idea of the proof of Remark 1.5 is given in Appendix B. The proof of the technical estimates of the nonlinear terms can be found in Appendix C.

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2 Stabilization of the linearized problem

In the whole Section 2 we consider stationary nonhomogeneous terms $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f_s}) \in \mathbf{W}^{1,\infty}(\Omega) \times \mathbf{U}^i \times \mathbb{R}^2$ and a stationary state $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^2_\beta(\mathscr{F}_s) \times \mathrm{H}^1_\beta(\mathscr{F}_s)$ that fulfil (1.17). Our goal is to find a control law $\mathbf{h}$ under a feedback form that stabilizes the linearized problem with a given exponential decay δ in time.

2.1 The linearized problem

We study the linearized system associated to (1.15). The variables $(\mathbf{v}, q)$ correspond roughly to the difference between $(\mathbf{u}, p)$ written in the fixed domain $\mathscr{F}_s$ and $(\mathbf{w}, p_{\mathbf{w}})$. A proper definition of these variables can be found in (3.1). Here is the linearized system around $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{w} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{w} - \mathbf{L_F}(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2, \mathbf{y}) - \nu\Delta\mathbf{v} + \nabla q = \mathbf{f} & \text{in } (0, \infty) \times \mathscr{F}_s, \\[4pt] \text{div } \mathbf{v} = 0 & \text{in } (0, \infty) \times \mathscr{F}_s, \\[4pt] \mathbf{v} = \dot\theta_1 \partial_{\theta_1}\mathbf{\Phi}(0,0,.) + \dot\theta_2 \partial_{\theta_2}\mathbf{\Phi}(0,0,.) + \mathbf{g} & \text{on } (0, \infty) \times \partial S_s, \\[4pt] \mathbf{v} = 0 & \text{on } (0, \infty) \times \Gamma_{\mathrm{D}}, \\[4pt] \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 & \text{on } (0, \infty) \times \Gamma_{\mathrm{N}}, \\[4pt] \mathbf{v}(0,.) = \mathbf{v}_0 & \text{in } \mathscr{F}_s, \\[4pt] \mathcal{M}_{0,0}\begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} [q\mathrm{I} - \nu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)]\mathbf{n}_s \cdot \partial_{\theta_1}\mathbf{\Phi}(0,0,\gamma_y)\, \mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_s} [q\mathrm{I} - \nu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)]\mathbf{n}_s \cdot \partial_{\theta_2}\mathbf{\Phi}(0,0,\gamma_y)\, \mathrm{d}\gamma_y \end{pmatrix} \\ \hspace{9cm} + \mathbf{L_S}(\theta_1, \theta_2) + \mathbf{s} + \mathbf{h} \quad \text{on } (0, \infty), \\[4pt] \theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, \\[4pt] \dot\theta_1(0) = \omega_{1,0}, \quad \dot\theta_2(0) = \omega_{2,0}, \end{cases} \tag{2.1}$$

where $(\mathbf{f}, \mathbf{g}, \mathbf{s})$ are nonhomogeneous terms and $\mathbf{v}_0$ an initial datum for $\mathbf{v}$. Here, $\mathbf{L_F} \in \mathrm{L}^2(\mathscr{F}_s, \mathcal{L}(\mathbb{R}^4, \mathbb{R}^2))$ and $\mathbf{L_S} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ depend on the stationary state $(\mathbf{w}, p_\mathbf{w})$ and on the diffeomorphism $\mathbf{\Phi}$, they are given by

$$\mathbf{L_F}(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2, \mathbf{y}) = \mathbf{L}_1(\mathbf{y})\theta_1 + \mathbf{L}_2(\mathbf{y})\theta_2 + \mathbf{L}_3(\mathbf{y})\dot\theta_1 + \mathbf{L}_4(\mathbf{y})\dot\theta_2, \quad \forall \mathbf{y} \in \mathscr{F}_s, \tag{2.2}$$

$$\mathbf{L_S}(\theta_1, \theta_2) = \mathbf{L}_5\theta_1 + \mathbf{L}_6\theta_2, \tag{2.3}$$

where the coefficients $\mathbf{L}_1$–$\mathbf{L}_6$ are properly defined in Appendix A ((A.1)–(A.6)).

For any fixed $\delta > 0$ such that $(\mathcal{H})_\delta$ holds, we use the following spaces

$$\mathbb{U}_\delta^\infty = \{ \mathbf{v} \text{ with } e^{\delta t}\mathbf{v} \in \mathrm{L}^2(0, \infty; \mathbf{H}_\beta^2(\mathscr{F}_s)) \cap \mathscr{C}^0([0, \infty); \mathbf{H}^1(\mathscr{F}_s)) \cap \mathrm{H}^1(0, \infty; \mathbf{L}^2(\mathscr{F}_s)) \}, \tag{2.4}$$

$$\mathbb{P}_\delta^\infty = \{ q \text{ with } e^{\delta t}q \in \mathrm{L}^2(0, \infty; \mathrm{H}_\beta^1(\mathscr{F}_s)) \}, \tag{2.5}$$

$$\Theta_\delta^\infty = \{ (\theta_1, \theta_2) \text{ with } e^{\delta t}(\theta_1, \theta_2) \in \mathrm{H}^2(0, \infty; \mathbb{R}^2) \}, \tag{2.6}$$

$$\mathbb{F}_\delta^\infty = \{ \mathbf{f} \text{ with } e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0, \infty; \mathbf{L}^2(\mathscr{F}_s)) \}, \tag{2.7}$$

$$\mathbb{G}_\delta^\infty = \{ \mathbf{g} \text{ with } e^{\delta t}\mathbf{g} \in \mathrm{H}^1(0, \infty; \mathbf{H}^{3/2}(\partial S_s)) \}, \tag{2.8}$$

$$\mathbb{S}_\delta^\infty = \{ \mathbf{s} \text{ with } e^{\delta t}\mathbf{s} \in \mathrm{L}^2(0, \infty; \mathbb{R}^2) \}. \tag{2.9}$$

All these spaces are equipped with their natural norms, e.g. for Θ_δ^∞,

$$\|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty} = \|(\theta_1, \theta_2)e^{\delta t}\|_{\mathrm{H}^2(0,\infty;\mathbb{R}^2)}.$$

The goal of Section 2 is to prove the following result.

Proposition 2.1. *Let $\delta > 0$ and assume that $(\mathcal{H})_\delta$ is fulfilled. There exists a feedback operator $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$, such that for every $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4$, $\mathbf{f} \in \mathbb{F}_\delta^\infty$, $\mathbf{g} \in \mathbb{G}_\delta^\infty$ and $\mathbf{s} \in \mathbb{S}_\delta^\infty$ fulfilling the compatibility conditions*

$$\begin{cases} \text{div } \mathbf{v}_0 = 0 & \text{in } \mathscr{F}_s, \\[4pt] \mathbf{v}_0 = \displaystyle\sum_{j=1}^2 \omega_{j,0} \partial_{\theta_j}\mathbf{\Phi}(0,0,.) + \mathbf{g}(0) & \text{on } \partial S_s, \\[4pt] \mathbf{v}_0 = 0 & \text{on } \Gamma_{\mathrm{D}}, \end{cases} \tag{2.10}$$

problem (2.1) with the control taken as $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ admits a unique solution $(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ with the following estimate,

$$\|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty} \leq C(\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty}), \tag{2.11}$$

where C does not depend on the initial conditions and on the source terms.
A consequence is that for every $t \in (0, \infty)$,

$$\begin{aligned} \|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot\theta_1(t)| + |\dot\theta_2(t)| \leq C\Big(&\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \\ &+ \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty} \Big)e^{-\delta t}. \end{aligned} \tag{2.12}$$

We first work on the homogeneous system associated to (2.1). In Section 2.2 we develop the functional framework used to write the homogeneous system under a semigroup formulation. In Section 2.3 we study the adjoint operator. In Section 2.4 we exhibit a feedback operator $\mathcal{K}_\delta$ that stabilizes the homogeneous problem. We then prove Proposition 2.1 in Section 2.5.

2.2 Functional framework for the semigroup formulation

In this section, the linear problem considered in Section 2.1 is rewritten under a semigroup formulation. This enables us to use the classical strategy to derive a feedback operator $\mathcal{K}_\delta$ stabilizing the system (2.1) (see [6, Part V] and [10, Section 5.2] for a full presentation of this method). In a similar way to [11], we use the spaces

$$\mathbb{H} = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, & \operatorname{div} \mathbf{v} = 0 \text{ in } \mathscr{F}_s, \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D, \\ & \mathbf{v} \cdot \mathbf{n}_s = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}(0, 0, .) \cdot \mathbf{n}_s \text{ on } \partial S_s \end{array} \right\}, \qquad (2.13)$$

and

$$\mathbb{V} = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4, & \operatorname{div} \mathbf{v} = 0 \text{ in } \mathscr{F}_s, \quad \mathbf{v} = 0 \text{ on } \Gamma_D, \\ & \mathbf{v} = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}(0, 0, .) \text{ on } \partial S_s \end{array} \right\}. \qquad (2.14)$$

The spaces $\mathbb{H}$ and $\mathbb{V}$ are respectively endowed with the scalar products $(.,.)_0$ of $\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ and $(.,.)_1$ of $\mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4$ defined by

$$\forall (\mathbf{v}^j, \theta_1^j, \theta_2^j, \omega_1^j, \omega_2^j) \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4,$$
$$\left((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a), (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \right)_0 = \int_{\mathscr{F}_s} \mathbf{v}^a \cdot \mathbf{v}^b \, \mathrm{d}\mathbf{y} + \sum_j \theta_j^a \theta_j^b + (\omega_1^a \ \omega_2^a) \mathcal{M}_{0,0} \left(\begin{array}{c} \omega_1^b \\ \omega_2^b \end{array} \right),$$

and

$$\forall (\mathbf{v}^j, \theta_1^j, \theta_2^j, \omega_1^j, \omega_2^j) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4,$$
$$\left((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a), (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \right)_1 = \int_{\mathscr{F}_s} (\mathbf{v}^a \cdot \mathbf{v}^b + \nabla \mathbf{v}^a : \nabla \mathbf{v}^b) \, \mathrm{d}\mathbf{y} + \sum_j \theta_j^a \theta_j^b + (\omega_1^a \ \omega_2^a) \mathcal{M}_{0,0} \left(\begin{array}{c} \omega_1^b \\ \omega_2^b \end{array} \right),$$

where $\mathcal{M}_{0,0}$ is defined in (1.11) for $\theta_1 = \theta_2 = 0$.

In the sequel, for a better readability, we use the notation

$$(f_j)_{j=1,2} = \left(\begin{array}{c} f_1 \\ f_2 \end{array} \right).$$

Lemma 2.2. *The orthogonal space to $\mathbb{H}$ with respect to the scalar product $(.,.)_0$ is*

$$(\mathbb{H})^\perp = \left\{ \left(\nabla p, 0, 0, -\mathcal{M}_{0,0}^{-1} \left(\int_{\partial S_s} p \mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}(0, 0, \gamma_y) \, \mathrm{d}\gamma_y \right)_{j=1,2} \right) \text{ with } p \in \mathrm{H}^1(\mathscr{F}_s), \ p = 0 \text{ on } \Gamma_N \right\}.$$

Proof. See [11, Lemma 2.3]. $\qquad \square$

We adapt Rellich's compact embedding Theorem to our functional framework.

Lemma 2.3. *The embedding from $\mathbb{V}$ into $\mathbb{H}$ is compact.*

Proof. The proof is an easy consequence of Rellich's compact embedding Theorem [17, Theorem 1.4.3.2.]. $\qquad \square$

We define the operator $(A, D(A))$ on $\mathbb{H}$ by

$$D(A) = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, \ \mathbf{v} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s), \ \exists q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s) \text{ such that} \\ \operatorname{div} \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q) \mathbf{n} = 0 \text{ on } \Gamma_N \end{array} \right\}, \qquad (2.15)$$

$$A \left(\begin{array}{c} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{array} \right) = \Pi_{\mathbb{H}} \left(\begin{array}{c} \operatorname{div} \sigma_F(\mathbf{v}, q) + \mathbf{L_F}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v} \cdot \nabla) \mathbf{w} - (\mathbf{w} \cdot \nabla) \mathbf{v} \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q) \mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}(0, 0, \gamma_y) \, \mathrm{d}\gamma_y \right)_{j=1,2} + \mathbf{L_S}(\theta_1, \theta_2) \right) \end{array} \right), \qquad (2.16)$$

where ε_0 is introduced in Lemma 1.8 and $\Pi_{\mathbb{H}}$ denotes the orthogonal projection of $\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ onto $\mathbb{H}$. The next lemmas state some properties of $(A, D(A))$.

Remark 2.4. The use of q in the definition of $(A, D(A))$ is useful to guarantee that $\operatorname{div} \sigma_F(\mathbf{v}, q)$ belongs to $\mathbf{L}^2(\mathscr{F}_s)$ and then that the application of $\Pi_{\mathbb{H}}$ in the right hand-side of (2.16) makes sense.

Lemma 2.5. *The operator A is uniquely defined.*

Proof. A similar proof is presented in [11, Lemma 2.5], we need to slightly adapt it in order to take into account the terms coming from the linearization around a stationary solution.

Let $(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in D(A)$ and consider two functions p, $q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)$ satisfying the conditions appearing into the definition of $D(A)$. Then, $\operatorname{div} \sigma_F(0, p - q) = -\nabla(p - q) \in \mathbf{L}^2(\mathscr{F}_s)$ implies $p - q \in \mathrm{H}^1(\mathscr{F}_s)$, and $\sigma_F(0, p - q)\mathbf{n} = 0$ on Γ_{N} implies $p - q = 0$ on Γ_{N}.

Now,

$$
\begin{pmatrix}
\operatorname{div} \sigma_F(\mathbf{v}, p) + \mathbf{L}_{\mathbf{F}}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v} \\
\omega_1 \\
\omega_2 \\
\mathcal{M}_{0,0}^{-1}\left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, p)\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}(0, 0, \gamma_y)\mathrm{d}\gamma_y\right)_{j=1,2} + \mathbf{L}_{\mathbf{S}}(\theta_1, \theta_2)\right)
\end{pmatrix}
-
\begin{pmatrix}
\operatorname{div} \sigma_F(\mathbf{v}, q) + \mathbf{L}_{\mathbf{F}}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v} \\
\omega_1 \\
\omega_2 \\
\mathcal{M}_{0,0}^{-1}\left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}(0, 0, \gamma_y)\mathrm{d}\gamma_y\right)_{j=1,2} + \mathbf{L}_{\mathbf{S}}(\theta_1, \theta_2)\right)
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\nabla(p - q) \\
0 \\
0 \\
-\mathcal{M}_{0,0}^{-1}\left(\displaystyle\int_{\partial S_s} (p - q)\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}(0, 0, \gamma_y)\mathrm{d}\gamma_y\right)_{j=1,2}
\end{pmatrix},
$$

which belongs to $\mathbb{H}^{\perp}$ according to Lemma 2.2. Therefore A is uniquely defined. $\qquad\square$

Before going further, let us point out that $D(A)$ can be characterized as follows.

Lemma 2.6. *We have*

$$
D(A) = \left\{
\begin{array}{l}
(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, \ \mathbf{v} \in \mathbf{H}_{\beta}^2(\mathscr{F}_s), \ \exists q \in \mathrm{H}_{\beta}^1(\mathscr{F}_s) \text{ such that} \\
\qquad \operatorname{div} \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{N}}
\end{array}
\right\}.
$$

Proof. See [11, Lemma 2.6]. $\qquad\square$

We define the bilinear form a_1 on $\mathbb{V} \times \mathbb{V}$ for every $(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2)$ and $(\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)$ in $\mathbb{V}$ by

$$
\mathrm{a}_1((\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2), (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)) = \frac{\nu}{2}\int_{\mathscr{F}_s} (\nabla\mathbf{v} + \nabla\mathbf{v}^T) : (\nabla\mathbf{v}^b + (\nabla\mathbf{v}^b)^T)\,\mathrm{d}\mathbf{y}
$$
$$
+ \int_{\mathscr{F}_s} ((\mathbf{v} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v}) \cdot \mathbf{v}^b\,\mathrm{d}\mathbf{y}.
\tag{2.17}
$$

We define the operator $(A_1, D(A_1))$ on $\mathbb{H}$ by

$$
D(A_1) = \{\mathbf{z} \in \mathbb{V} \text{ with } \widetilde{\mathbf{z}} \mapsto \mathrm{a}_1(\mathbf{z}, \widetilde{\mathbf{z}}) \text{ is } \mathbb{H}-\text{continuous }\},
$$

and

$$
\forall \mathbf{z} \in D(A_1), \ \forall \widetilde{\mathbf{z}} \in \mathbb{V}, \ \left(A_1\mathbf{z}, \widetilde{\mathbf{z}}\right)_0 = -\mathrm{a}_1(\mathbf{z}, \widetilde{\mathbf{z}}).
$$

Lemma 2.7. *We have $D(A_1) = D(A)$ and*

$$
A_1
\begin{pmatrix}
\mathbf{v} \\
\theta_1 \\
\theta_2 \\
\omega_1 \\
\omega_2
\end{pmatrix}
= \Pi_{\mathbb{H}}
\begin{pmatrix}
\operatorname{div} \sigma_F(\mathbf{v}, q) - (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v} \\
0 \\
0 \\
\mathcal{M}_{0,0}^{-1}\left(\displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}(0, 0, \gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2}
\end{pmatrix}.
$$

Proof. This is a straightforward adaptation of [11, Lemma 2.7]. $\qquad\square$

Lemma 2.8. *The operator A generates an analytic semigroup on $\mathbb{H}$ and has compact resolvent.*

Proof. According to [23, p. 3015], there exists a constant $c > 0$ such that for $\lambda > 0$ large enough, we have

$$
\forall \mathbf{z} \in \mathbb{V}, \qquad \mathrm{a}_1(\mathbf{z}, \mathbf{z}) + \lambda\|\mathbf{z}\|_{\mathbb{H}}^2 \geq c\|\mathbf{z}\|_{\mathbb{V}}^2.
\tag{2.18}
$$

Moreover, according to [6, Theorem 2.12, p. 115], the estimate (2.18) implies that the operator A_1 generates an analytic semigroup on $\mathbb{H}$.

Now, as $A - A_1 \in \mathcal{L}(\mathbb{H})$, according to [24, Corollary 2.2], A generates an analytic semigroup on $\mathbb{H}$.

We have $D(A) \subset \mathbb{V}$ and, according to Lemma 2.3, the embedding from $\mathbb{V}$ into $\mathbb{H}$ is compact. The operator A then has compact resolvent. This ends this proof. $\qquad\square$

2.3 Study of the adjoint operator

In order to simplify the notations, in the sequel we do not write $d\mathbf{y}$ or $d\gamma_y$ anymore in the integrals.

Lemma 2.9. *The adjoint operator A_1^* of A_1 with respect to the scalar product $(.,.)_0$ is given by*

$$D(A_1^*) = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, & \mathbf{v} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s), \quad \exists q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s) \text{ such that} \\ & \operatorname{div} \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} + (\mathbf{w} \cdot \mathbf{n})\mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\},$$

and

$$A_1^* \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\mathbf{v}, q) - (\nabla \mathbf{w})^T \mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{v} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1} \left(\displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \right)_{j=1,2} \end{pmatrix}.$$

Proof. The adjoint A_1^* of A_1 is defined by

$$D(A_1^*) = \{ \, \tilde{\mathbf{z}} \in \mathbb{V} \text{ with } \mathbf{z} \mapsto \mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) \text{ is } \mathbb{H}\text{–continuous} \, \},$$

and

$$\forall \mathbf{z} \in \mathbb{V}, \quad \forall \tilde{\mathbf{z}} \in D(A_1^*), \quad \left(\mathbf{z}, A_1^* \tilde{\mathbf{z}} \right)_0 = -\mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}).$$

Let us denote

$$E = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, & \mathbf{v} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s), \quad \exists q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s) \text{ such that} \\ & \operatorname{div} \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} + (\mathbf{w} \cdot \mathbf{n})\mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\}.$$

For every $\tilde{\mathbf{z}} = (\tilde{\mathbf{v}}, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\omega}_1, \tilde{\omega}_2) \in E$, we have

$$\forall \mathbf{z} \in \mathbb{V}, \quad -\mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) = \left(\begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{v}}, \tilde{q}) - (\nabla \mathbf{w})^T \tilde{\mathbf{v}} + (\mathbf{w} \cdot \nabla)\tilde{\mathbf{v}} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1} \left(\displaystyle\int_{\partial S_s} -\sigma_F(\tilde{\mathbf{v}}, \tilde{q})\mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}(0,0,\gamma_y) \right)_{j=1,2} \end{pmatrix}, \mathbf{z} \right)_0.$$

Then, we have $E \subset D(A_1^*)$.

To prove the reverse inclusion, the same arguments as in the proof of [11, Lemma 2.7] lead to the existence of $\tilde{q} \in \mathrm{L}^2(\mathscr{F}_s)$ such that

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma_F(\tilde{\mathbf{v}}, \tilde{q}) & \in \mathbf{L}^2(\mathscr{F}_s), \\ \operatorname{div} \tilde{\mathbf{v}} = 0 & \text{in } \mathscr{F}_s, \\ \tilde{\mathbf{v}} = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \tilde{\mathbf{v}} = \sum_j \tilde{\omega}_j \partial_{\theta_j} \mathbf{\Phi}(0,0,.) & \text{on } \partial S_s, \\ \sigma_F(\tilde{\mathbf{v}}, \tilde{q})\mathbf{n} + (\mathbf{w} \cdot \mathbf{n})\tilde{\mathbf{v}} = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{array} \right.$$

Then $\sigma_F(\tilde{\mathbf{v}}, \tilde{q})\mathbf{n}$ belongs to $\mathbf{H}^{1/2}(\Gamma_{\mathrm{N}})$, this implies that $(\tilde{\mathbf{v}}, \tilde{q})$ belongs to $\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s) \times \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)$. This regularity is a consequence of [20, Theorem 9.4.5] and the arguments used in the proof of [23, Theorem 2.5]. This ends the proof. $\qquad \square$

Lemma 2.10. *The adjoint of A with respect to $(.,.)_0$ is given by*

$$D(A^*) = D(A_1^*),$$

and

$$A^* \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\mathbf{v}, q) + (\mathbf{w} \cdot \nabla)\mathbf{v} - (\nabla \mathbf{w})^T \mathbf{v} \\ \displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_1 \cdot \mathbf{v} \\ \mathbf{L}_2 \cdot \mathbf{v} \end{pmatrix} + \left(\mathbf{L}_{k+4} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right) \Bigg)_{k=1,2} \\ \mathcal{M}_{0,0}^{-1} \left[\displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_3 \cdot \mathbf{v} \\ \mathbf{L}_4 \cdot \mathbf{v} \end{pmatrix} - \int_{\partial S_s} \begin{pmatrix} \sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_1} \mathbf{\Phi}(0,0,\gamma_y) \\ \sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_2} \mathbf{\Phi}(0,0,\gamma_y) \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \right] \end{pmatrix}.$$

Proof. We have $A - A_1 \in \mathcal{L}(\mathbb{H})$, then $A^* - A_1^* \in \mathcal{L}(\mathbb{H})$ and $D(A^*) = D(A_1^*)$.

A computation of $(A\mathbf{z}, \tilde{\mathbf{z}})_0$ for $\mathbf{z} \in D(A)$ and $\tilde{\mathbf{z}} \in D(A^*)$ gives the explicit expression of A^*. $\qquad \square$

2.4 Construction of a feedback operator

We define the control operator $B \in \mathcal{L}(\mathbb{R}^2, \mathbb{H})$ by

$$B\mathbf{h} = \Pi_{\mathbb{H}}(0, 0, 0, \mathcal{M}_{0,0}^{-1}\mathbf{h}). \tag{2.19}$$

The linear system (2.1) with no source terms ($\mathbf{f} = 0$, $\mathbf{g} = 0$, $\mathbf{s} = 0$) can be rewritten under the form

$$\begin{cases} \mathbf{z}'(t) = A\mathbf{z}(t) + B\mathbf{h}(t), & t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{2.20}$$

where $\mathbf{z} = (\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ and $\mathbf{z}_0 = (\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$.

We want to exhibit a control $\mathbf{h}$ under a feedback form that stabilizes problem (2.20). In order to guarantee a decay rate $\delta > 0$ of the solution to this problem, we consider the stabilization of $\mathbf{z}_\delta = e^{\delta t}\mathbf{z}$, solution to the problem

$$\begin{cases} \mathbf{z}_\delta'(t) = (A + \delta\mathrm{I})\mathbf{z}_\delta(t) + B\mathbf{h}_\delta(t), & t > 0, \\ \mathbf{z}_\delta(0) = \mathbf{z}_0, \end{cases} \tag{2.21}$$

where

$$\mathbf{h}_\delta(t) = e^{\delta t}\mathbf{h}(t), \qquad t > 0.$$

Our goal is to find a control $\mathbf{h}$ providing the stabilization of (2.21).

Lemma 2.11. *The adjoint operator of B with respect to $(.,.)_0$ is bounded, $B \in \mathcal{L}(\mathbb{H}, \mathbb{R}^2)$, and is given by*

$$B^*(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Proof. The operator B is bounded, this is then a straightforward computation. $\qquad\square$

We define the unstable space associated to the operator $A + \delta\mathrm{I}$. Let J_u be the set of eigenvalues λ_j of A such that $\mathcal{R}e(\lambda_j) \geq -\delta$. The set J_u is exactly the set of all $\lambda \in \mathbb{C}$ such that $\lambda + \delta$ is an unstable eigenvalue of $A + \delta\mathrm{I}$. According to Lemmas 2.10 and 2.11, Hypothesis $(\mathcal{H})_\delta$ can be rewritten as the following unique continuation property:

$$\text{For every } \lambda \in J_u \text{ and all } \phi \in \mathbb{V} \text{ that obey } (A^* - \overline{\lambda}\mathrm{I})\phi = 0 \text{ and } B^*\phi = 0, \quad \text{we have } \phi = 0, \tag{2.22}$$

where $\overline{\lambda}$ denotes the conjugate of λ.

The goal of this section is to construct a feedback control operator which uses only a finite number of scalar data to determine a control law that stabilizes problem (2.21), i.e.

Proposition 2.12. *Let $\delta > 0$ such that hypothesis $(\mathcal{H})_\delta$ is fulfilled. There exists $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$ such that the operator $A + \delta\mathrm{I} + B\mathcal{K}_\delta$ generates a stable analytic semigroup on $\mathbb{H}$.*

Lemma 2.8 implies that the spectrum of the operator A consists of isolated eigenvalues with finite algebraic multiplicities, moreover it has no finite cluster point. In addition, the operator A generates an analytic semigroup, the control operator B belongs to $\mathcal{L}(\mathbb{R}^2, \mathbb{H})$ and we assume (2.22). Proposition 2.12 is then a consequence of the Fattorini criterion [4, Theorem 1.6].

In order to make the strategy clear, we provide a proof of Proposition 2.12.

Proof. We denote $\mathrm{G}(\lambda_j)$ the generalized eigenspace of A associated to the eigenvalue λ_j and we define the unstable space for $A + \delta\mathrm{I}$ by

$$\mathbb{Z}_u = \bigoplus_{\lambda \in J_u} \mathrm{G}(\lambda).$$

As $A + \delta\mathrm{I}$ generates an analytic semigroup on $\mathbb{H}$ and has compact resolvent (see Lemma 2.8), J_u is finite and every $\mathrm{G}(\lambda_j)$ is finite dimensional. Then the space $\mathbb{Z}_u$ is finite dimensional and we denote $d_u = \dim(\mathbb{Z}_u)$.

Besides, we can construct in the same way a space $\mathbb{Z}_s$ that contains all the stable eigenvectors of $A + \delta\mathrm{I}$, that is invariant under $(e^{tA})_{t \geq 0}$ and such that

$$\mathbb{H} = \mathbb{Z}_u \bigoplus \mathbb{Z}_s.$$

We denote Π_u the projection of $\mathbb{H}$ onto $\mathbb{Z}_u$ along $\mathbb{Z}_s$ and Π_s the projection of $\mathbb{H}$ onto $\mathbb{Z}_s$ along $\mathbb{Z}_u$. In the same way, we denote $\mathbb{Z}_u^*$ (respectively $\mathbb{Z}_s^*$) the direct sum of the generalized eigenspaces of A^* associated to an eigenvalue belonging (respectively not belonging) to J_u.

According to [13, Lemma 6.2], there exist respectively two biorthogonal families $(\mathbf{z}_i) = (\mathbf{v}_i, \theta_{1,i}, \theta_{2,i}, \omega_{1,i}, \omega_{2,i})$ and $(\tilde{\mathbf{z}}_j) = (\tilde{\mathbf{v}}_j, \tilde{\theta}_{1,j}, \tilde{\theta}_{2,j}, \tilde{\omega}_{1,j}, \tilde{\omega}_{2,j})$ of $\mathbb{Z}_u$ and $\mathbb{Z}_u^*$ satisfying the condition

$$\left(\mathbf{z}_i, \tilde{\mathbf{z}}_j\right)_0 = \delta_{ij}, \tag{2.23}$$

and $\Pi_u(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) = \sum_{i=1}^{d_u} \left((\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2), (\tilde{\mathbf{v}}_i, \tilde{\theta}_{1,i}, \tilde{\theta}_{2,i}, \tilde{\omega}_{1,i}, \tilde{\omega}_{2,i}) \right)_0 (\mathbf{v}_i, \theta_{1,i}, \theta_{2,i}, \omega_{1,i}, \omega_{2,i})$. We can also choose $\mathbf{z}_i$ and $\tilde{\mathbf{z}}_j$ to be respectively generalized eigenvectors of A and A^*.

The adjoint Π_u^* of Π_u is the projection of $\mathbb{H}$ onto $\mathbb{Z}_u^*$ along $\mathbb{Z}_s^*$ as a straightforward calculation gives

$$\forall \tilde{\mathbf{z}} \in \mathbb{H}, \ \Pi_u^* \tilde{\mathbf{z}} = \sum_{i=1}^{d_u} \left(\mathbf{z}_i, \tilde{\mathbf{z}}\right)_0 \tilde{\mathbf{z}}_i.$$

Denoting $\mathbf{z}_u = \Pi_u \mathbf{z}$ for every $\mathbf{z} \in \mathbb{H}$, the projection of (2.21) onto $\mathbb{Z}_u$ reads

$$\begin{cases} \mathbf{z}_u'(t) = A_u \mathbf{z}_u(t) + \delta \mathbf{z}_u(t) + B_u \mathbf{h}_u(t) & t > 0, \\ \mathbf{z}_u(0) = \Pi_u \mathbf{z}_0, \end{cases} \tag{2.24}$$

where $A_u = \Pi_u A$, $B_u = \Pi_u B$ and $\mathbf{h}_u = \mathbf{h}_\delta$. Note that this formulation uses $A_u((I - \Pi_u)\mathbf{z}) = 0$, which holds since $\mathbb{Z}_s$ is stable under $A + \delta I$.

At this point, we use the following lemma that is proven just after the end of the current proof.

Lemma 2.13. *Under Hypothesis $(\mathcal{H})_\delta$, problem (2.24) is controllable on $\mathbb{Z}_u$.*

Lemma 2.13 implies that problem (2.24) is stabilizable. Then the feedback control law

$$\mathbf{h}_u = -B_u^* \mathcal{R}_\delta \mathbf{z}_u,$$

stabilizes problem (2.24), where $\mathcal{R}_\delta \in \mathcal{L}(\mathbb{H})$ is the unique positive self–adjoint solution in $\mathcal{L}(\mathbb{H})$ to the Riccati equation

$$\mathcal{R}_\delta(A_u + \delta I) + (A_u^* + \delta I)\mathcal{R}_\delta + I - \mathcal{R}_\delta B_u B_u^* \mathcal{R}_\delta = 0,$$

(for more information see [10, Section 6.2]).

Now we can construct a feedback law

$$\tilde{\mathcal{K}}_\delta(\mathbf{z}_\delta) = \mathbf{h}_u = -B_u^* \mathcal{R}_\delta \Pi_u \mathbf{z}_\delta,$$

that stabilizes problem (2.21). The feedback operator $\tilde{\mathcal{K}}_\delta$ belongs to $\mathcal{L}(\mathbb{H}, \mathbb{R}^2)$ and can be extended to an element $\mathcal{K}_\delta$ of $\mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$,

$$\mathcal{K}_\delta \mathbf{z}_\delta = \tilde{\mathcal{K}}_\delta \Pi_\mathbb{H} \mathbf{z}_\delta = -B_u^* \mathcal{R}_\delta \Pi_u \Pi_\mathbb{H} \mathbf{z}_\delta, \ \text{for all } \mathbf{z}_\delta \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4. \tag{2.25}$$

The operator $A_u + \delta I + B_u \mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{Z}_u$.

Finally, let us prove that the operator $A + \delta I + B\mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{H}$. We can use the decomposition $\mathbf{z}_\delta = \Pi_u \mathbf{z}_\delta + \Pi_s \mathbf{z}_\delta$, where $\Pi_u \mathbf{z}_\delta$ and $\Pi_s \mathbf{z}_\delta$ fulfil

$$\begin{cases} \Pi_u \mathbf{z}_\delta' = (A_u + \delta I)\Pi_u \mathbf{z}_\delta + B_u \mathcal{K}_\delta \Pi_u \mathbf{z}_\delta & t > 0, \\ \Pi_u \mathbf{z}_\delta(0) = \Pi_u \mathbf{z}_0, \end{cases}$$

$$\begin{cases} \Pi_s \mathbf{z}_\delta' = (A_s + \delta I)\Pi_s \mathbf{z}_\delta + B_s \mathcal{K}_\delta \Pi_u \mathbf{z}_\delta & t > 0, \\ \Pi_s \mathbf{z}_\delta(0) = \Pi_s \mathbf{z}_0, \end{cases}$$

where $A_s = \Pi_s A$, $B_s = \Pi_s B$ and where we have used $\Pi_s A \Pi_u = \Pi_u A \Pi_s = 0$.

As the operator $A_u + \delta I + B\mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{Z}_u$, we have $\|\Pi_u \mathbf{z}_\delta\|_{\mathrm{L}^2(0,\infty;\mathbb{H})} \leq C$. Moreover, according to the definition of J_u, the operator $A_s + \delta I$ generates an analytic stable semigroup on $\mathbb{Z}_s$, then according to [6, Theorem 3.1.(i), p. 143], we have $\|\Pi_s \mathbf{z}_\delta\|_{\mathrm{L}^2(0,\infty;\mathbb{H})} \leq C$. Hence $A + \delta I + B\mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{H}$. This ends the proof of Proposition 2.12. $\qquad\square$

Proof of Lemma 2.13. According to the Hautus test (see for instance [30, Proposition 1.5.1]), this result is equivalent to $\mathrm{Ker}(A_u^* + \delta I - \bar{\lambda} I) \cap \mathrm{Ker}(B_u^*) = \{0\}$ for every eigenvalue λ of $A_u + \delta I$, i.e. $\mathrm{Ker}(A_u^* - \bar{\lambda} I) \cap \mathrm{Ker}(B_u^*) = \{0\}$ for every eigenvalue λ of A_u.

The definition of $\mathbb{Z}_u$ implies that this space is stable under A. Hence, for every $\mathbf{z}_u \in \mathbb{Z}_u$, $A_u \mathbf{z}_u = A\mathbf{z}_u$. Then, the eigenvalues of A_u are the elements of J_u.

Now, we need to compute A_u^* and B_u^*, the adjoint operators of A_u and B_u with respect to $\mathbb{H}$. Let $\mathbf{z}_u^* \in \mathbb{Z}_u^*$ and $\mathbf{z} \in \mathbb{H}$, we have

$$\left(A_u^* \mathbf{z}_u^*, \mathbf{z}\right)_0 = \left(\mathbf{z}_u^*, \Pi_u A\mathbf{z}\right)_0 = \left(A^* \mathbf{z}_u^*, \mathbf{z}\right)_0,$$

and

$$\left(B_u^* \mathbf{z}_u^*, \mathbf{z}\right)_0 = \left(\mathbf{z}_u^*, \Pi_u B\mathbf{z}\right)_0 = \left(B^* \mathbf{z}_u^*, \mathbf{z}\right)_0,$$

then $\forall \mathbf{z}_u^* \in \mathbb{Z}_u^*$, we have $A_u^* \mathbf{z}_u^* = A^* \mathbf{z}_u^*$ and $B_u^* \mathbf{z}_u^* = B^* \mathbf{z}_u^*$.

Let us now prove the Hautus test on (A_u^*, B_u^*). Let $\lambda \in \mathbb{C}$ be an eigenvalue of A_u, then $\lambda \in J_u$. Let $\mathbf{z}_u^* \in \mathbb{Z}_u^*$ and assume that $\mathbf{z}_u^* \in \mathrm{Ker}(A_u^* - \overline{\lambda}\mathrm{I}) \cap \mathrm{Ker}(B_u^*)$. We have $(A^* - \overline{\lambda}\mathrm{I})\mathbf{z}_u^* = 0$ and $B^* \mathbf{z}_u^* = 0$. Thus, $\mathbf{z}_u^* \in D(A^*)$ and the property (2.22) implies that $\mathbf{z}_u^* = 0$ and this ends the proof. $\qquad\square$

A consequence of Proposition 2.12 is the following lemma.

Lemma 2.14. *Let* $\mathbf{z}_0 \in \mathbb{V}$ *and* $\mathbf{F} \in \{\mathbf{f}$ *with* $e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0, \infty; \mathbb{H})\}$. *The solution* $\mathbf{z}$ *to the closed–loop problem*

$$\begin{cases} \mathbf{z}' = A\mathbf{z} + B\mathcal{K}_\delta \mathbf{z} + \mathbf{F} & t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{2.26}$$

belongs to $\{\mathbf{z}$ *with* $e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0, \infty; D(A)) \cap \mathrm{H}^1(0, \infty; \mathbb{H}) \cap \mathscr{C}^0([0, \infty); \mathbb{V})\}$. *Moreover, we have the estimate*

$$\|e^{\delta t}\mathbf{z}\|_{\mathrm{L}^2(0,\infty;D(A)) \cap \mathrm{H}^1(0,\infty;\mathbb{H}) \cap \mathscr{C}^0([0,\infty);\mathbb{V})} \le C(\|\mathbf{z}_0\|_{\mathbb{V}} + \|e^{\delta t}\mathbf{F}\|_{\mathrm{L}^2(0,\infty;\mathbb{H})}). \tag{2.27}$$

Proof. We have $\delta\mathrm{I} + B\mathcal{K}_\delta \in \mathcal{L}(\mathbb{H})$, then $D(A + \delta\mathrm{I} + B\mathcal{K}_\delta) = D(A)$ and by interpolation for $\lambda > 0$ large enough, $D((\lambda\mathrm{I} - A - \delta\mathrm{I} - B\mathcal{K}_\delta)^{1/2}) = D((\lambda\mathrm{I} - A)^{1/2}) = \mathbb{V}$, for the definition of these spaces see (2.14)–(2.15). Moreover, $e^{\delta t}\mathbf{F} \in \mathrm{L}^2(0, \infty; \mathbb{H})$.

In addition, according to Proposition 2.12, $A + \delta\mathrm{I} + B\mathcal{K}_\delta$ generates an analytic semigroup that is stable on $\mathbb{H}$, hence according to [6, Theorem 3.1.(i), p. 143], there exists $\mathbf{z}_\delta \in \mathrm{H}^1(0, +\infty; \mathbb{H}) \cap \mathrm{L}^2(0, +\infty; D(A))$ satisfying

$$\begin{cases} \mathbf{z}_\delta' = (A + \delta\mathrm{I} + B\mathcal{K}_\delta)\mathbf{z}_\delta + e^{\delta t}\mathbf{F} & t > 0, \\ \mathbf{z}_\delta(0) = \mathbf{z}_0. \end{cases}$$

By interpolation, $\mathbf{z}_\delta$ also belongs to $\mathscr{C}^0([0, \infty); \mathbb{V})$. Now, $\mathbf{z} = e^{-\delta t}\mathbf{z}_\delta$ belongs to $\{\mathbf{z}$ with $e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0, \infty; D(A)) \cap \mathrm{H}^1(0, \infty; \mathbb{H}) \cap \mathscr{C}^0([0, \infty); \mathbb{V})\}$ and is solution to (2.26). Moreover, we have the estimate (2.27) as a consequence of [6, Theorem 3.1.(i), p. 143]. $\qquad\square$

2.5 Stabilization of the linear system (2.1)

In this section we prove that the feedback operator $\mathcal{K}_\delta$ stabilizes the linear problem (2.1). The velocity is decomposed as $\mathbf{v} = \widetilde{\mathbf{v}} + \mathbf{v_g}$ where $\mathbf{v_g}$ is a lifting of $\mathbf{g}$. We first prove this result for distributed source terms only (i.e. on $\widetilde{\mathbf{v}}$) and then for boundary source terms (i.e. for $\widetilde{\mathbf{v}} + \mathbf{v_g}$). The first part is a consequence of the study of the semigroup.

2.5.1 Stabilization of the problem with nonhomogeneous distributed source terms

In this section we prove a stabilization result for the following system that corresponds to (2.1) with $\mathbf{g} = 0$,

$$\begin{cases} \dfrac{\partial \widetilde{\mathbf{v}}}{\partial t} + (\mathbf{w} \cdot \nabla)\widetilde{\mathbf{v}} + (\widetilde{\mathbf{v}} \cdot \nabla)\mathbf{w} - \mathbf{L_F}(\widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2, \mathbf{y}) - \nu\Delta\widetilde{\mathbf{v}} + \nabla\widetilde{q} = \widetilde{\mathbf{f}} & \text{in } (0, \infty) \times \mathscr{F}_s, \\ \mathrm{div}\,\widetilde{\mathbf{v}} = 0 & \text{in } (0, \infty) \times \mathscr{F}_s, \\ \widetilde{\mathbf{v}} = \dot{\widetilde{\theta}}_1 \partial_{\theta_1}\mathbf{\Phi}(0, 0, .) + \dot{\widetilde{\theta}}_2 \partial_{\theta_2}\mathbf{\Phi}(0, 0, .) & \text{on } (0, \infty) \times \partial S_s, \\ \widetilde{\mathbf{v}} = 0 & \text{on } (0, \infty) \times \Gamma_\mathrm{D}, \\ \sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n} = 0 & \text{on } (0, \infty) \times \Gamma_\mathrm{N}, \\ \widetilde{\mathbf{v}}(0, .) = \widetilde{\mathbf{v}}_0 & \text{in } \mathscr{F}_s, \\ \mathcal{M}_{0,0} \begin{pmatrix} \ddot{\widetilde{\theta}}_1 \\ \ddot{\widetilde{\theta}}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} [\widetilde{q}\mathrm{I} - \nu(\nabla\widetilde{\mathbf{v}} + (\nabla\widetilde{\mathbf{v}})^T)]\mathbf{n}_s \cdot \partial_{\theta_1}\mathbf{\Phi}(0, 0, \gamma_y)\,\mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_s} [\widetilde{q}\mathrm{I} - \nu(\nabla\widetilde{\mathbf{v}} + (\nabla\widetilde{\mathbf{v}})^T)]\mathbf{n}_s \cdot \partial_{\theta_2}\mathbf{\Phi}(0, 0, \gamma_y)\,\mathrm{d}\gamma_y \end{pmatrix} \\ \hspace{6cm} + \mathbf{L_S}(\widetilde{\theta}_1, \widetilde{\theta}_2) + \widetilde{\mathbf{s}} + \mathbf{h} & \text{on } (0, \infty), \\ \widetilde{\theta}_1(0) = \theta_{1,0}, \quad \widetilde{\theta}_2(0) = \theta_{2,0}, \\ \dot{\widetilde{\theta}}_1(0) = \omega_{1,0}, \quad \dot{\widetilde{\theta}}_2(0) = \omega_{2,0}. \end{cases} \tag{2.28}$$

Proposition 2.15. *Let* $\delta > 0$ *and let* $(\boldsymbol{\mathcal{H}})_\delta$ *be fulfilled. For every* $(\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4$, *fulfilling the compatibility conditions*

$$\begin{cases} \mathrm{div}\,\widetilde{\mathbf{v}}_0 = 0 & \text{in } \mathscr{F}_s, \\ \widetilde{\mathbf{v}}_0 = \displaystyle\sum_{j=1}^2 \omega_{j,0} \partial_{\theta_j}\mathbf{\Phi}(0, 0, .) & \text{on } \partial S_s, \\ \widetilde{\mathbf{v}}_0 = 0 & \text{on } \Gamma_\mathrm{D}, \end{cases} \tag{2.29}$$

and every source terms $\widetilde{\mathbf{f}} \in \mathbb{F}_\delta^\infty$ and $\widetilde{\mathbf{s}} \in \mathbb{S}_\delta^\infty$ problem (2.28) with the control taken as $\mathbf{h} = \mathcal{K}_\delta(\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2)$ admits a unique solution $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ with the following estimate

$$\|\widetilde{\mathbf{v}}\|_{\mathbb{U}_\delta^\infty} + \|\widetilde{q}\|_{\mathbb{P}_\delta^\infty} + \|(\widetilde{\theta}_1, \widetilde{\theta}_2)\|_{\Theta_\delta^\infty} \le C(\|\widetilde{\mathbf{v}}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\widetilde{\mathbf{f}}\|_{\mathbb{F}_\delta^\infty} + \|\widetilde{\mathbf{s}}\|_{\mathbb{S}_\delta^\infty}), \quad (2.30)$$

where C does not depend on the initial data and on the source terms.

Proof. The initial data $(\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ fulfil the compatibility conditions (2.29), that is why $\mathbf{z}_0 = (\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ belongs to $\mathbb{V}$. Moreover, as $\widetilde{\mathbf{f}} \in \mathbb{F}_\delta^\infty$ and $\widetilde{\mathbf{s}} \in \mathbb{S}_\delta^\infty$,

$$\mathbf{F} = \begin{pmatrix} \widetilde{\mathbf{f}} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\widetilde{\mathbf{s}} \end{pmatrix},$$

belongs to $\{\mathbf{f} \text{ with } e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0, \infty; \mathbb{H})\}$. Then, according to Lemma 2.14, the solution to (2.26) belongs to $\{\mathbf{z} \text{ with } e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0, \infty; D(A)) \cap \mathrm{H}^1(0, \infty; \mathbb{H}) \cap \mathscr{C}^0([0, \infty); \mathbb{V})\}$.

Now, we use the identity $\mathbf{z} = (\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2)$. According to Lemmas 2.2 and 2.6, $\widetilde{\mathbf{v}} \in \mathbb{U}_\delta^\infty$, $(\widetilde{\theta}_1, \widetilde{\theta}_2) \in \Theta_\delta^\infty$ and there exists $\widetilde{q} \in \mathbb{P}_\delta^\infty$ such that problem (2.26) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \widetilde{\mathbf{v}} \\ \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \\ \widetilde{\omega}_1 \\ \widetilde{\omega}_2 \end{pmatrix} = \begin{pmatrix} \operatorname{div} \sigma_F(\widetilde{\mathbf{v}}, \widetilde{q}) + \mathbf{L_F}(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2, \mathbf{y}) - (\widetilde{\mathbf{v}} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\widetilde{\mathbf{v}} + \widetilde{\mathbf{f}} \\ \widetilde{\omega}_1 \\ \widetilde{\omega}_2 \\ \mathcal{M}_{0,0}^{-1}\left(\begin{pmatrix} \displaystyle\int_{\partial S_s} -\sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n}_s \cdot \partial_{\theta_1}\boldsymbol{\Phi}(0, 0, \gamma_y) \\ \displaystyle\int_{\partial S_s} -\sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n}_s \cdot \partial_{\theta_2}\boldsymbol{\Phi}(0, 0, \gamma_y) \end{pmatrix} + \mathbf{L_S}(\widetilde{\theta}_1, \widetilde{\theta}_2) + \mathcal{K}_\delta(\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2) + \widetilde{\mathbf{s}}\right) \end{pmatrix},$$

where $\mathbf{L_F}$ and $\mathbf{L_S}$ are defined in (2.2)–(2.3). Then, $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2)$ is solution to (2.28).

Finally, the estimate (2.30) is a consequence of (2.27). $\qquad\square$

2.5.2 A first stabilization result for the problem with a nonhomogeneous boundary datum

In the sequel, for $\mathbf{g} \in \mathbb{G}_\delta^\infty$ we consider $\mathbf{v_g} \in \{\mathbf{v} \text{ with } e^{\delta t}\mathbf{v} \in \mathrm{H}^1(0, \infty; \mathbf{H}^2(\mathscr{F}_s))\}$ that satisfies

$$\begin{cases} \mathbf{v_g} = \mathbf{g} & \text{on } (0, \infty) \times \partial S_s, \\ \operatorname{div} \mathbf{v_g} = 0 & \text{in } (0, \infty) \times \mathscr{F}_s, \\ \mathbf{v_g} = 0 & \text{on } (0, \infty) \times \Gamma_\mathrm{D}, \\ (\nabla\mathbf{v_g} + (\nabla\mathbf{v_g})^T)\mathbf{n} = 0 & \text{on } (0, \infty) \times \Gamma_\mathrm{N}, \end{cases} \quad (2.31)$$

and

$$\|\mathbf{v_g}e^{\delta t}\|_{\mathrm{H}^1(0, \infty; \mathbf{H}^2(\mathscr{F}_s))} \le C\|\mathbf{g}\|_{\mathbb{G}_\delta^\infty}, \quad (2.32)$$

see [23, Theorem 2.16] for a proof of the existence of $\mathbf{v_g}$.

The following proposition enables the stabilization of problem (2.1) with $\mathbf{g} \ne 0$. However, contrary to Proposition 2.1, the feedback control is $\mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ instead of $\mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$.

Proposition 2.16. *Let $\delta > 0$ and let $(\boldsymbol{\mathcal{H}})_\delta$ be fulfilled. For every $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4$, $\mathbf{f} \in \mathbb{F}_\delta^\infty$, $\mathbf{g} \in \mathbb{G}_\delta^\infty$ and $\mathbf{s} \in \mathbb{S}_\delta^\infty$ fulfilling the compatibility conditions (2.10), problem (2.1) with the control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ admits a unique solution $(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ with the following estimate,*

$$\|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty} \le C(\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty}), \quad (2.33)$$

where C does not depend on the initial conditions and the source terms.

Proof. Let $(\mathbf{v}, q, \theta_1, \theta_2)$ be the solution to (2.1) with $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$. We now consider $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2) = (\mathbf{v} - \mathbf{v_g}, q, \theta_1, \theta_2)$, it is solution to problem (2.28) where

$$\begin{aligned} \widetilde{\mathbf{v}}_0 &= \mathbf{v}_0 - \mathbf{v_g}(0), \\ \widetilde{\mathbf{f}} &= \mathbf{f} - \partial_t\mathbf{v_g} - (\mathbf{w} \cdot \nabla)\mathbf{v_g} - (\mathbf{v_g} \cdot \nabla)\mathbf{w} + \nu\Delta\mathbf{v_g}, \\ \widetilde{\mathbf{s}} &= \mathbf{s} - \left(\int_{\partial S_s} \nu(\nabla\mathbf{v_g} + \nabla\mathbf{v_g}^T)\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}(0, 0, \gamma_y)\right)_{j=1,2}. \end{aligned}$$

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The initial data $(\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ fulfil (2.29), then all the terms have the good regularity and we have the estimate (2.30). We now use (2.32) and get

$$\|\widetilde{\mathbf{f}}\|_{\mathbb{F}_\delta^\infty} \leq \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + C\|\mathbf{g}\|_{\mathbb{G}_\delta^\infty},$$
$$\|\widetilde{\mathbf{s}}\|_{\mathbb{S}_\delta^\infty} \leq \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty} + C\|\mathbf{g}\|_{\mathbb{G}_\delta^\infty}.$$

All these estimates prove the estimate (2.33). $\qquad\square$

2.5.3 Proof of Proposition 2.1

Proof. In Proposition 2.16, we have proven that the control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ stabilizes the problem (2.1). We now want to prove that the control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ also stabilizes the same problem.

Let $(\mathbf{v}, q, \theta_1, \theta_2)$ be the solution to (2.1) with $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$. According to Proposition 2.16, we have the estimate (2.33).

Let $(\widehat{\mathbf{v}}, \widehat{q}, \widehat{\theta}_1, \widehat{\theta}_2)$ be the solution to (2.1) with $\mathbf{h} = \mathcal{K}_\delta(\widehat{\mathbf{v}}, \widehat{\theta}_1, \widehat{\theta}_2, \dot{\widehat{\theta}}_1, \dot{\widehat{\theta}}_2)$. We now consider

$$(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2) = (\mathbf{v}, q, \theta_1, \theta_2) - (\widehat{\mathbf{v}}, \widehat{q}, \widehat{\theta}_1, \widehat{\theta}_2),$$

it is solution to (2.28) with $\mathbf{f} = 0$, $\mathbf{v}_0 = 0$, $\mathbf{s} = -\mathcal{K}_\delta(\mathbf{v_g}, 0, 0, 0, 0)$, $\theta_{1,0} = 0$, $\theta_{2,0} = 0$, $\omega_{1,0} = 0$, $\omega_{2,0} = 0$ and $\mathbf{h} = \mathcal{K}_\delta(\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2)$. The initial data $(0,0,0,0,0)$ fulfil the compatibility conditions (2.29), hence we have the estimate (2.30). When combining the estimates (2.30) and (2.33), we get

$$\|\widehat{\mathbf{v}}\|_{\mathbb{U}_\delta^\infty} + \|\widehat{q}\|_{\mathbb{P}_\delta^\infty} + \|(\widehat{\theta}_1, \widehat{\theta}_2)\|_{\Theta_\delta^\infty} \leq \|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|\widetilde{\mathbf{v}}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|\widetilde{q}\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty} + \|(\widetilde{\theta}_1, \widetilde{\theta}_2)\|_{\Theta_\delta^\infty}$$
$$\leq C(\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty}).$$

This ends the proof of Proposition 2.1. $\qquad\square$

3 Stabilization of the nonlinear closed loop system

The proof of Theorem 1.6 will be developed in this section. As in Section 2 we consider $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f_s}) \in \mathbf{W}^{1,\infty}(\Omega) \times \mathbf{U}^i \times \mathbb{R}^2$ and a stationary state $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}_\beta^2(\mathscr{F}_s) \times \mathrm{H}_\beta^1(\mathscr{F}_s)$ that fulfil (1.17) (see Remark 1.9).

3.1 The nonlinear problem in a fixed domain

In this section, we are interested in writing the equations fulfilled by the difference between the solution to (1.15) and the stationary state. In order to do so, we consider the change of variables

$$\forall \mathbf{y} \in \mathscr{F}_s, \quad \forall t \in (0, \infty), \quad \begin{cases} \mathbf{v}(t, \mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\boldsymbol{\Phi}}(\theta_1(t), \theta_2(t), \mathbf{y}))^T \mathbf{u}(t, \boldsymbol{\Phi}(\theta_1(t), \theta_2(t), \mathbf{y})) - \mathbf{w}(\mathbf{y}), \\ q(t, \mathbf{y}) = p(t, \boldsymbol{\Phi}(\theta_1(t), \theta_2(t), \mathbf{y})) - p_{\mathbf{w}}(\mathbf{y}), \end{cases} \tag{3.1}$$

where $\boldsymbol{\Phi}$ is the diffeomorphism defined in (1.19) fulfilling (1.21), moreover $\mathcal{J}_{\boldsymbol{\Phi}}(\theta_1, \theta_2, \mathbf{y}) = \nabla_{\mathbf{y}}\boldsymbol{\Phi}(\theta_1, \theta_2, \mathbf{y})$ and $\mathrm{cof}(\mathcal{J}_{\boldsymbol{\Phi}})$ is the cofactor matrix of $\mathcal{J}_{\boldsymbol{\Phi}}$. This change of variables has been chosen in order to have a divergence free velocity $\mathbf{v}$ in the fixed domain.

One can show that under the feedback control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ defined in Proposition 2.1, $(\mathbf{v}, q, \theta_1, \theta_2)$ is solution to the closed loop system

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{w} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{w} - \mathbf{L_F}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, \mathbf{y}) - \nu\Delta\mathbf{v} + \nabla q = \mathbf{f} & \text{in } (0, \infty) \times \mathscr{F}_s, \\ \mathrm{div}\,\mathbf{v} = 0 & \text{in } (0, \infty) \times \mathscr{F}_s, \\ \mathbf{v} = \dot{\theta}_1\partial_{\theta_1}\boldsymbol{\Phi}(0, 0, \gamma_y) + \dot{\theta}_2\partial_{\theta_2}\boldsymbol{\Phi}(0, 0, \gamma_y) + \mathbf{g} & \text{on } (0, \infty) \times \partial S_s, \\ \mathbf{v} = 0 & \text{on } (0, \infty) \times \Gamma_{\mathrm{D}}, \\ \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 & \text{on } (0, \infty) \times \Gamma_{\mathrm{N}}, \\ \mathbf{v}(0, \mathbf{y}) = \mathbf{v}_0(\mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\boldsymbol{\Phi}}(\theta_{1,0}, \theta_{2,0}, \mathbf{y}))^T \mathbf{u}_0(\boldsymbol{\Phi}(\theta_{1,0}, \theta_{2,0}, \mathbf{y})) - \mathbf{w}(\mathbf{y}) & \text{in } \mathscr{F}_s, \\ \mathcal{M}_{0,0}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_1}\boldsymbol{\Phi}(0, 0, \gamma_y) \\ \displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_2}\boldsymbol{\Phi}(0, 0, \gamma_y) \end{pmatrix} \\ \qquad\qquad\qquad + \mathbf{L_S}(\theta_1, \theta_2) + \mathbf{s} + \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) & \text{on } (0, \infty), \\ \theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}, \end{cases} \tag{3.2}$$

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where the linear terms are

$$\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y}) = \mathbf{L}_1(\mathbf{y})\theta_1 + \mathbf{L}_2(\mathbf{y})\theta_2 + \mathbf{L}_3(\mathbf{y})\dot\theta_1 + \mathbf{L}_4(\mathbf{y})\dot\theta_2, \quad \forall \mathbf{y}\in\mathscr{F}_s,$$
$$\mathbf{L_S}(\theta_1,\theta_2) = \mathbf{L}_5\theta_1 + \mathbf{L}_6\theta_2,$$

and the coefficients $\mathbf{L}_1$–$\mathbf{L}_6$ are defined in Appendix A. Moreover $\mathcal{K}_\delta$ is the feedback operator given in (2.25) and the source terms are given by the nonlinear (at least quadratic) terms

$$\begin{cases} \mathbf{f} = \mathbf{f}^{NL}(\theta_1,\theta_2,\mathbf{v},q), \\ \mathbf{g} = \mathbf{g}^{NL}(\theta_1,\theta_2), \\ \mathbf{s} = \mathbf{s}^{NL}(\theta_1,\theta_2,\mathbf{v},q), \end{cases} \tag{3.3}$$

defined below

$$\begin{cases} \mathbf{f}^{NL}(\theta_1,\theta_2,\mathbf{v},q) = \mathbf{F}(\theta_1,\theta_2,\mathbf{w}+\mathbf{v},p_{\mathbf{w}}+q) + (\mathbf{w}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{w} - \mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,.) \\ \hspace{6cm} + \mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}(\theta_1,\theta_2,.)) - \mathbf{f}_{\mathscr{F}}, \\ \mathbf{g}^{NL}(\theta_1,\theta_2) = \mathbf{G}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2), \\ \mathbf{s}^{NL}(\theta_1,\theta_2,\mathbf{v},q) = \mathbf{S}(\theta_1,\theta_2,\mathbf{w}+\mathbf{v},p_{\mathbf{w}}+q) - \mathbf{L_S}(\theta_1,\theta_2), \end{cases} \tag{3.4}$$

where $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{S}$ are defined as follows

$$\mathbf{F}(\theta_1,\theta_2,\mathbf{v},q) = \mathbf{F}^1(\theta_1,\theta_2,\mathbf{v}) + \mathbf{F}^2(\theta_1,\theta_2,\mathbf{v}) + \mathbf{F}^3(\theta_1,\theta_2,\mathbf{v}) + \mathbf{F}^4(\theta_1,\theta_2,\mathbf{v}) + \mathbf{F}^5(\theta_1,\theta_2,q),$$
$$\mathbf{F}^1(\theta_1,\theta_2,\mathbf{v}) = (\mathrm{I} - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^T)\frac{\partial\mathbf{v}}{\partial t},$$
$$\mathbf{F}^2(\theta_1,\theta_2,\mathbf{v}) = -\mathrm{cof}(\dot\theta_1\nabla_{\mathbf{x}}\partial_{\theta_1}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi}) + \dot\theta_2\nabla_{\mathbf{x}}\partial_{\theta_2}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi}))^T\mathbf{v}$$
$$\hspace{2cm} -\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^T(\nabla_{\mathbf{y}}\mathbf{v})\Big(\dot\theta_1\partial_{\theta_1}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi}) + \dot\theta_2\partial_{\theta_2}\mathbf{\Psi}(\theta_1,\theta_2,\mathbf{\Phi})\Big),$$
$$\mathbf{F}^3(\theta_1,\theta_2,\mathbf{v})_i = \nu\sum_{j,k,\ell,m}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial^2 v_k}{\partial y_\ell\partial y_m}\frac{\partial\Psi_\ell}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi})\frac{\partial\Psi_m}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi}) \tag{3.5}$$
$$\hspace{2cm} +2\nu\sum_{j,k,\ell}\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial v_k}{\partial y_\ell}\frac{\partial\Psi_\ell}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi})$$
$$\hspace{2cm} +\nu\sum_{j,k,\ell}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial v_k}{\partial y_\ell}\frac{\partial^2\Psi_\ell}{\partial x_j^2}(\theta_1,\theta_2,\mathbf{\Phi})$$
$$\hspace{2cm} +\nu\sum_{j,k}\mathrm{cof}(\partial_{x_j}^2\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}v_k - \nu\Delta_{\mathbf{y}}v_i(t,\mathbf{y}),$$

$$\mathbf{F}^4(\theta_1,\theta_2,\mathbf{v})_i = -\sum_{j,k,r}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{kj}\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ri}v_k v_r - \sum_{k,r}\det(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^2\frac{\partial\Phi_i}{\partial y_r}\frac{\partial v_r}{\partial y_k}v_k,$$
$$\mathbf{F}^5(\theta_1,\theta_2,q) = (\mathrm{I} - \mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))^T\nabla_{\mathbf{y}}q,$$

$$\mathbf{G}(\theta_1,\theta_2,\omega_1,\omega_2) = \sum_{j=1}^{2}\omega_j\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1,\theta_2,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi}(\theta_1,\theta_2,\mathbf{y}) - \partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\Big),$$
$$\mathbf{S}(\theta_1,\theta_2,\mathbf{v},q) = -(\mathcal{M}_{\theta_1,\theta_2} - \mathcal{M}_{0,0})\begin{pmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{pmatrix} + \mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)$$
$$+ \begin{pmatrix}\displaystyle\int_{\partial S_s}|\mathcal{J}_{\mathbf{\Phi}}(\theta_1,\theta_2,\gamma_y)\mathbf{t}_s|\,[q\mathrm{I} - \nu(\mathcal{G}(\theta_1,\theta_2,\mathbf{v}) + \mathcal{G}(\theta_1,\theta_2,\mathbf{v})^T)]\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi})\cdot\partial_{\theta_1}\mathbf{\Phi}(\theta_1,\theta_2,\gamma_y) \\ \displaystyle\int_{\partial S_s}|\mathcal{J}_{\mathbf{\Phi}}(\theta_1,\theta_2,\gamma_y)\mathbf{t}_s|\,[q\mathrm{I} - \nu(\mathcal{G}(\theta_1,\theta_2,\mathbf{v}) + \mathcal{G}(\theta_1,\theta_2,\mathbf{v})^T)]\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi})\cdot\partial_{\theta_2}\mathbf{\Phi}(\theta_1,\theta_2,\gamma_y)\end{pmatrix} \tag{3.6}$$
$$- \begin{pmatrix}\displaystyle\int_{\partial S_s}[q\mathrm{I} - \nu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)]\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi}(0,0,\gamma_y) \\ \displaystyle\int_{\partial S_s}[q\mathrm{I} - \nu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)]\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi}(0,0,\gamma_y)\end{pmatrix},$$

where $\mathbf{M_I}$ and $\mathcal{M}_{\theta_1,\theta_2}$ are defined in (1.11), (1.12) and

$$\mathcal{G}(\theta_1,\theta_2,\mathbf{v})_{ij} = \sum_k\mathrm{cof}\big[\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,.))\circ\mathbf{\Phi}\big]_{ki}v_k + \sum_{k,\ell}\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}))_{ki}\frac{\partial v_k}{\partial y_\ell}\frac{\partial\Psi_\ell}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi}). \tag{3.7}$$

For the sake of intelligibility, we have used the notation $\mathbf{\Phi} = \mathbf{\Phi}(\theta_1,\theta_2,.)$.

3.2 Proof of the stabilization result in the fixed domain

In this section, we develop the fixed point argument used to prove the stabilization result of the nonlinear problem (1.15) in the fixed domain $\mathscr{F}_s$ (i.e. the stabilization result for (3.2)–(3.3)).

Proposition 3.1. *Let $\delta > 0$ and let $(\mathcal{H})_\delta$ be fulfilled. There exists $\varepsilon > 0$, such that for every $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ satisfying the compatibility conditions*

$$\begin{cases} \operatorname{div} \mathbf{v}_0 = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v}_0 = \displaystyle\sum_{j=1}^{2} \omega_{j,0} \operatorname{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_{1,0}, \theta_{2,0}, .))^T \partial_{\theta_j} \mathbf{\Phi}(\theta_{1,0}, \theta_{2,0}, .) & \text{on } \partial S_s, \\ \mathbf{v}_0 = 0 & \text{on } \Gamma_{\mathrm{D}}, \end{cases} \tag{3.8}$$

and

$$\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \leq \varepsilon, \tag{3.9}$$

problem (3.2)–(3.3) admits a solution $(\mathbf{v}, q, \theta_1, \theta_2)$ that tends to zero with exponential decay rate δ. For all $t > 0$, we have

$$\|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\omega_1(t)| + |\omega_2(t)| \leq C e^{-\delta t}. \tag{3.10}$$

Proof. Let $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ in $\mathbf{H}^1(\mathscr{F}_s) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ fulfilling the compatibility conditions (3.8) and (3.9) for some $\varepsilon > 0$. We consider the space

$$\mathbb{N}_\delta^\infty = \left\{ \begin{array}{ll} (\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty \text{ with } & (\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)(0) = (\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}), \\ & \text{for every } t \text{ in } (0, \infty),\ (\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta \end{array} \right\}, \tag{3.11}$$

equipped with the natural norm of $\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$

$$\|(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} = \|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty},$$

where all the spaces used are defined in (2.4)–(2.9).

We define the application Λ^∞ on $\mathbb{N}_\delta^\infty$ as follows. For $(\overline{\mathbf{v}}, \overline{q}, \overline\theta_1, \overline\theta_2) \in \mathbb{N}_\delta^\infty$, $(\mathbf{v}, q, \theta_1, \theta_2) = \Lambda^\infty(\overline{\mathbf{v}}, \overline{q}, \overline\theta_1, \overline\theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ is the solution to problem (3.2) where the source terms are $\mathbf{f} = \mathbf{f}^{NL}(\overline\theta_1, \overline\theta_2, \overline{\mathbf{v}}, \overline{q})$, $\mathbf{g} = \mathbf{g}^{NL}(\overline\theta_1, \overline\theta_2)$ and $\mathbf{s} = \mathbf{s}^{NL}(\overline\theta_1, \overline\theta_2, \overline{\mathbf{v}}, \overline{q})$ (defined in (3.4)). Note that Λ^∞ depends on the initial data.

The compatibility conditions (3.8) correspond to (2.10) with $\mathbf{g}(0) = \mathbf{G}(\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ and the conditions in (3.11) impose that $\mathbf{g}^{NL}(\overline\theta_1, \overline\theta_2, \overline{\mathbf{v}}, \overline{q})(0) = \mathbf{G}(\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$. Moreover, we prove in (3.12) that, for every $(\overline\theta_1, \overline\theta_2, \overline{\mathbf{v}}, \overline{q}) \in \mathbb{N}_\delta^\infty$, we have $\mathbf{f}^{NL}(\overline\theta_1, \overline\theta_2, \overline{\mathbf{v}}, \overline{q}) \in \mathbb{F}_\delta^\infty$, $\mathbf{g}^{NL}(\overline\theta_1, \overline\theta_2) \in \mathbb{G}_\delta^\infty$ and $\mathbf{s}^{NL}(\overline\theta_1, \overline\theta_2, \overline{\mathbf{v}}, \overline{q}) \in \mathbb{S}_\delta^\infty$. Hence, according to Proposition 2.1, Λ^∞ is well defined.

The domain $\mathbb{D}_\Theta$ is open and $(0, 0) \in \mathbb{D}_\Theta$, then there exists $R_0 > 0$ such that $B((0, 0), R_0) \subset \mathbb{D}_\Theta$. The application Λ^∞ will be studied on a ball of radius $R \leq R_0$

$$\mathbb{B}_R^\infty = \{(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{N}_\delta^\infty \text{ with } \|(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq R\}.$$

We also consider the space

$$\widetilde{\mathbb{B}_R^\infty} = \{(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty \text{ with } \|(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq R, \quad \forall t \in (0, \infty), (\theta_1, \theta_2)(t) \in \mathbb{D}_\Theta\},$$

in which the initial value for $(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ is free. The following lemma will be used.

Lemma 3.2. *There exists a constant $C' = C'(R_0)$, such that for every $R \in (0, R_0)$, every $(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a)$ and $(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b) \in \widetilde{\mathbb{B}_R^\infty}$, the following estimates hold*

$$\|\mathbf{f}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{f}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)\|_{\mathbb{F}_\delta^\infty} \leq C'R(\|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a - q^b\|_{\mathbb{P}_\delta^\infty} + \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}),$$

$$\|\mathbf{g}^{NL}(\theta_1^a, \theta_2^a) - \mathbf{g}^{NL}(\theta_1^b, \theta_2^b)\|_{\mathbb{G}_\delta^\infty} \leq C'R\|\theta^a - \theta^b\|_{\Theta_\delta^\infty},$$

$$\|\mathbf{s}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{s}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)\|_{\mathbb{S}_\delta^\infty} \leq C'R(\|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a - q^b\|_{\mathbb{P}_\delta^\infty} + \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}).$$

This lemma is proven in Appendix C.2. We use the spaces $\mathbf{H}_\beta^2(\mathscr{F}_s)$ and $\mathrm{H}_\beta^1(\mathscr{F}_s)$ in this proof.

We now denote C the constant in (2.11) and C' the one in Lemma 3.2.

At this point, we use Lemma 3.2 with $(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b) = (0, 0, 0, 0)$, we thus get for every $R > 0$ and for every $(\mathbf{v}, q, \theta_1, \theta_2)$ in $\mathbb{B}_R^\infty$,

$$\|\mathbf{f}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}^{NL}(\theta_1, \theta_2)\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{S}_\delta^\infty} \leq 3C'R^2. \tag{3.12}$$

Let $R = \min(R_0, 1/(6CC'))$ and $\varepsilon = R/(2C)$, hence $3CC'R^2 \leq R/2$.

We can build (θ_1, θ_2) as the solution to

$$\begin{cases} \ddot{\theta}_j(t) + 2\delta\dot{\theta}_j(t) + 2\delta^2\theta_j(t) = 0, & t > 0, \\ \theta_j(0) = \theta_{j,0}, & \dot{\theta}_j(0) = \omega_{j,0}. \end{cases}$$

For ε small enough and $|\omega_{j,0}| + |\theta_{j,0}| \leq \varepsilon$, $(0, 0, \theta_1, \theta_2)$ belongs to $\mathbb{B}_R^\infty$. Thus, taking $\varepsilon > 0$ smaller if necessary, we can assume that $\mathbb{B}_R^\infty \neq \emptyset$. In the sequel, we choose such a ε.

According to Proposition 2.1, we have for every $(\mathbf{v}, q, \theta_1, \theta_2)$ in $\mathbb{B}_R^\infty$,

$$\|\Lambda^\infty(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathrm{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq C(\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{F}_\delta^\infty}$$
$$+ \|\mathbf{g}^{NL}(\theta_1, \theta_2)\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{S}_\delta^\infty}),$$

we combine this result with (3.12). Then, by using (3.9), $\varepsilon = R/(2C)$ and $3CC'R^2 \leq R/2$, we have

$$\|\Lambda^\infty(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathrm{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq R.$$

Hence, if we write $(\tilde{\mathbf{v}}, \tilde{q}, \tilde{\theta}_1, \tilde{\theta}_2) = \Lambda^\infty(\theta_1, \theta_2, \mathbf{v}, q)$, then we have $\|(\tilde{\theta}_1, \tilde{\theta}_2)\|_{\mathbf{L}^\infty(0,\infty)} \leq \|\Lambda^\infty(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{N}_\delta^\infty} \leq R \leq R_0$. Then, $(\tilde{\theta}_1, \tilde{\theta}_2)$ belongs to $\mathbb{D}_\Theta$. This proves that $\Lambda^\infty : \mathbb{B}_R^\infty \to \mathbb{B}_R^\infty$.

For $(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a)$ and $(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b)$ in $\mathbb{B}_R^\infty$, $\Lambda^\infty(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a) - \Lambda^\infty(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b)$ solves problem (3.2) where the source terms are $\mathbf{f}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{f}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)$, $\mathbf{g}^{NL}(\theta_1^a, \theta_2^a) - \mathbf{g}^{NL}(\theta_1^b, \theta_2^b)$, $\mathbf{s}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{s}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)$ and the initial data are null. Then, according to Proposition 2.1 and Lemma 3.2, we have

$$\|\Lambda^\infty(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a) - \Lambda^\infty(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b)\|_{\mathrm{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq 3CC'R(\|\theta^a - \theta^b\|_{\Theta_\delta^\infty} + \|\mathbf{v}^a - \mathbf{v}^b\|_{\mathrm{U}_\delta^\infty} + \|q^a - q^b\|_{\mathbb{P}_\delta^\infty}). \quad (3.13)$$

As $R \leq 1/(6CC')$, the estimate (3.13) yields that Λ^∞ is a contraction on $\mathbb{B}_R^\infty$. Hence, according to the Picard fixed point theorem, there exists a unique fixed point $(\mathbf{v}, q, \theta_1, \theta_2)$ to Λ^∞ in $\mathbb{B}_R^\infty$. This fixed point solves the closed–loop nonlinear problem (3.2)–(3.3).

Estimate (3.10) is then a consequence of the fact that $(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{B}_R^\infty$:

$$\|\mathbf{v}e^{\delta t}\|_{\mathscr{C}^0([0,\infty);\mathbf{H}^1(\mathscr{F}_s))} + \|\theta e^{\delta t}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{D}_\Theta)} + \|\dot{\theta}e^{\delta t}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^2)} \leq R \leq R_0.$$

$\square$

3.3 Proof of Theorem 1.6

Proof. The last step towards proving Theorem 1.6 is to use the change of variables (3.1) to prove the result on $(\mathbf{u}, p)$ as a consequence of Proposition 3.1 that states properties of $(\mathbf{v}, q)$. The only difficulty is to handle the nonlinear term $\mathrm{cof}(\mathcal{J}_\Phi)^T$ that is present in this change of variables. This nonlinearity creates some difficulties for using the smallness assumption and compatibility conditions on the initial data.

Let $\delta > 0$, we consider that $(\mathcal{H})_\delta$ is fulfilled and that the initial data satisfy the compatibility conditions (1.24). Moreover, we consider that for some $\varepsilon_1 > 0$ we have

$$\|\mathbf{u}_0(\boldsymbol{\Phi}(\theta_{1,0}, \theta_{2,0}, .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \leq \varepsilon_1.$$

Since $\mathbf{v}_0(\mathbf{y}) = \mathrm{cof}(\mathcal{J}_\Phi(\theta_{1,0}, \theta_{2,0}, \mathbf{y}))^T \mathbf{u}_0 \circ \boldsymbol{\Phi}(\theta_{1,0}, \theta_{2,0}, \mathbf{y}) - \mathbf{w}(\mathbf{y})$, we have (3.8) and

$$\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} \leq \|\mathrm{cof}(\mathcal{J}_\Psi(\theta_{1,0}, \theta_{2,0}, \boldsymbol{\Phi}))^T(\mathbf{u}_0 \circ \boldsymbol{\Phi} - \mathbf{w})\|_{\mathbf{H}^1(\mathscr{F}_s)} + \|\mathrm{cof}(\mathcal{J}_\Psi(\theta_{1,0}, \theta_{2,0}, \boldsymbol{\Phi}) - \mathrm{I})^T\mathbf{w}\|_{\mathbf{H}^1(\mathscr{F}_s)}$$
$$\leq K\varepsilon_1 + K\|\mathbf{w}\|_{\mathbf{H}_\beta^2(\mathscr{F}_s)}\varepsilon_1,$$

where K is the constant in (C.18). In order to fulfil (3.9), we choose

$$\varepsilon_1 = \frac{\varepsilon}{K + K\|\mathbf{w}\|_{\mathbf{H}_\beta^2(\mathscr{F}_s)}},$$

where $\varepsilon > 0$ is the bound in Proposition 3.1, we have (3.9). Then, according to Proposition 3.1, the feedback $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ stabilizes problem (3.2)–(3.3) at decay rate δ. We denote $(\mathbf{v}, q, \theta_1, \theta_2)$ the solution to (3.2)–(3.3).

We use the identity

$$\forall t \in (0, \infty), \forall \mathbf{x} \in \mathscr{F}(\theta_1(t), \theta_2(t)),$$
$$\begin{cases} \mathbf{u}(t, \mathbf{x}) = \mathrm{cof}(\mathcal{J}_\Psi(\theta_1(t), \theta_2(t), \mathbf{x}))^T \Big(\mathbf{v}(t, \boldsymbol{\Psi}(\theta_1(t), \theta_2(t), \mathbf{x})) + \mathbf{w}(\boldsymbol{\Psi}(\theta_1(t), \theta_2(t), \mathbf{x})) \Big), \\ p(t, \mathbf{x}) = q(t, \boldsymbol{\Psi}(\theta_1(t), \theta_2(t), \mathbf{x})) + p_\mathbf{w} \circ \boldsymbol{\Psi}(\theta_1(t), \theta_2(t), \mathbf{x}). \end{cases}$$

The quadruplet $(\mathbf{u}, p, \theta_1, \theta_2)$ is solution to (1.15) where

$$\mathbf{h} = \mathcal{K}_\delta\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}(\theta_1(t), \theta_2(t), .))^T \mathbf{u}(t, \mathbf{\Phi}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.), \theta_1(t), \theta_2(t), \dot{\theta}_1(t), \dot{\theta}_2(t)\Big).$$

Moreover, for ε small enough and for every $t > 0$, we have the estimate

$$\|\mathbf{u}(t, \mathbf{\Phi}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}\|_{\mathbf{H}^1(\mathscr{F}_s)} \leq \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1(t), \theta_2(t), .)) - \mathrm{I}\|_{\mathbf{H}^2(\mathscr{F}_s)}\|\mathbf{v}(t) + \mathbf{w}\|_{\mathbf{H}^1(\mathscr{F}_s)} + \|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)}$$
$$\leq K(|(\theta_1(t), \theta_2(t))| + \|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)}).$$

Hence the estimate (3.10) implies that

$$\forall t > 0, \quad \|\mathbf{u}(t, \mathbf{\Phi}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot{\theta}_1(t)| + |\dot{\theta}_2(t)| \leq C e^{-\delta t}.$$

This concludes the proof of Theorem 1.6. $\qquad\square$

Appendix

A   The linearized terms

In what follows we give the explicit expression of the functions $\mathbf{L}_1$–$\mathbf{L}_4$ and of the constants $\mathbf{L}_5$–$\mathbf{L}_6$. We denote $\mathbf{t}_s = \mathbf{n}_s^\perp = (-(\mathbf{n}_s)_2, (\mathbf{n}_s)_1)$ a unit tangent vector to ∂S_s. We have

$$\mathbf{L}_1(\mathbf{y})_i = \big(\nu \mathbf{L}_{\mathbf{F}^3}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^4}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^5}(\mathbf{y})\big)_{i1} + (\nabla \mathbf{f}_{\mathscr{F}}(\mathbf{y})\partial_{\theta_1}\mathbf{\Phi}(0, 0, \mathbf{y}))_i, \tag{A.1}$$

$$\mathbf{L}_2(\mathbf{y})_i = \big(\nu \mathbf{L}_{\mathbf{F}^3}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^4}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^5}(\mathbf{y})\big)_{i2} + (\nabla \mathbf{f}_{\mathscr{F}}(\mathbf{y})\partial_{\theta_2}\mathbf{\Phi}(0, 0, \mathbf{y}))_i, \tag{A.2}$$

$$\mathbf{L}_3(\mathbf{y})_i = \big(\mathbf{L}_{\mathbf{F}^2}(\mathbf{y})\big)_{i1}, \tag{A.3}$$

$$\mathbf{L}_4(\mathbf{y})_i = \big(\mathbf{L}_{\mathbf{F}^2}(\mathbf{y})\big)_{i2}, \tag{A.4}$$

$$(\mathbf{L}_5)_i = \int_{\partial S_s} (-\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s) \cdot \partial_{\theta_1 \theta_i}\mathbf{\Phi}(0, 0, \gamma_y) - \sum_{k,\ell} \sigma_F(\mathbf{w}, p_{\mathbf{w}})_{\ell k}(\mathbf{L}_{\mathbf{n}_{\theta_1, \theta_2}})_{k1}\partial_{\theta_i}\Phi_\ell(0, 0, \gamma_y)$$
$$-(\nabla_{\mathbf{y}}\partial_{\theta_1}\mathbf{\Phi}(0, 0, \gamma_y)\mathbf{t}_s \cdot \mathbf{t}_s)\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_i}\mathbf{\Phi}(0, 0, \gamma_y) \tag{A.5}$$
$$-\nu \sum_{k,\ell} ((\mathbf{L}_{\mathcal{G}})_{k\ell 1} + (\mathbf{L}_{\mathcal{G}})_{\ell k1})(\mathbf{n}_s)_k \partial_{\theta_i}\Phi_\ell(0, 0, \gamma_y) \, d\gamma_y,$$

$$(\mathbf{L}_6)_i = \int_{\partial S_s} (-\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s) \cdot \partial_{\theta_2 \theta_i}\mathbf{\Phi}(0, 0, \gamma_y) - \sum_{k,\ell} \sigma_F(\mathbf{w}, p_{\mathbf{w}})_{\ell k}(\mathbf{L}_{\mathbf{n}_{\theta_1, \theta_2}})_{k2}\partial_{\theta_i}\Phi_\ell(0, 0, \gamma_y)$$
$$-(\nabla_{\mathbf{y}}\partial_{\theta_2}\mathbf{\Phi}(0, 0, \gamma_y)\mathbf{t}_s \cdot \mathbf{t}_s)\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_i}\mathbf{\Phi}(0, 0, \gamma_y) \tag{A.6}$$
$$-\nu \sum_{k,\ell} ((\mathbf{L}_{\mathcal{G}})_{k\ell 2} + (\mathbf{L}_{\mathcal{G}})_{\ell k2})(\mathbf{n}_s)_k \partial_{\theta_i}\Phi_\ell(0, 0, \gamma_y) \, d\gamma_y,$$

where the terms used are defined in (A.7)–(A.12). In the previous paragraph, we denoted by $\big(\mathbf{L}_{\mathbf{F}^k}\big)_{ij}$ for $3 \leq k \leq 5$, the linearization of $(\mathbf{F}^k)_i$ (defined in (3.6)) around the stationary state $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$ solution of (1.17) with respect to the parameter θ_j and $\big(\mathbf{L}_{\mathbf{F}^2}\big)_{ij}$ is the linearization of $(\mathbf{F}^2)_i$ with respect to $\dot{\theta}_j$. These terms are given below

$$(\mathbf{L}_{\mathbf{F}^2}(\mathbf{y}))_{ij} = \Big((\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}) \cdot \nabla_{\mathbf{y}})\mathbf{w} + \mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}))^T \mathbf{w}\Big)_i, \tag{A.7}$$

$$(\mathbf{L}_{\mathbf{F}^3}(\mathbf{y}))_{ij} = -2\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}) : \nabla^2 w_i(\mathbf{y}) - (\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}))^T \Delta\mathbf{w}(\mathbf{y}))_i$$
$$+2 \sum_{k,\ell} \mathrm{cof}\left(\frac{\partial}{\partial x_\ell}\big(\nabla_{\mathbf{x}}\partial_{\theta_j}\mathbf{\Psi}(0, 0, \mathbf{y})\big)\right)_{ki} \frac{\partial w_k}{\partial y_\ell}(\mathbf{y})$$
$$+\sum_{\ell,m} \frac{\partial w_i}{\partial y_\ell}(\mathbf{y}) \frac{\partial^2}{\partial x_m^2}(\partial_{\theta_j}\mathbf{\Psi}(0, 0, \mathbf{y}))_\ell \tag{A.8}$$
$$+\sum_{k,m} \mathrm{cof}\left(\frac{\partial^2}{\partial x_m^2}\big(\nabla_{\mathbf{x}}\partial_{\theta_j}\mathbf{\Psi}(0, 0, \mathbf{y})\big)\right)_{ki} w_k(\mathbf{y}),$$

$$(\mathbf{L}_{\mathbf{F}^4}(\mathbf{y}))_{ij} = -\sum_{k,\ell} \mathrm{cof}\left(\frac{\partial}{\partial x_k}(\nabla_{\mathbf{x}}\partial_{\theta_j}\mathbf{\Psi}(0, 0, \mathbf{y}))\right)_{\ell i} w_k(\mathbf{y})w_\ell(\mathbf{y})$$
$$+\sum_{k,\ell} \Big(2\mathrm{Tr}(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}))\delta_{i\ell} - (\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}))_{i\ell}\Big)w_k(\mathbf{y})\frac{\partial w_\ell}{\partial y_k}(\mathbf{y}), \tag{A.9}$$

and

$$(\mathbf{L}_{\mathbf{F}^5}(\mathbf{y}))_{ij} = \Big((\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0, 0, \mathbf{y}))^T \nabla_{\mathbf{y}}p_{\mathbf{w}}(\mathbf{y})\Big)_i. \tag{A.10}$$

We also define the linearization of $\mathbf{n}_{\theta_1,\theta_2}$ (the unit outward normal to $\mathscr{F}(\theta_1,\theta_2)$) and $\mathcal{G}$ (defined in (3.7)) with respect to θ_j by

$$(\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}}(\gamma_y))_{j,n} = (\text{cof}(\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\gamma_y))\mathbf{n}_s)_j - (\text{cof}(\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\gamma_y))\mathbf{n}_s \cdot \mathbf{n}_s)(\mathbf{n}_s)_j, \tag{A.11}$$

and

$$\begin{aligned}
(\mathbf{L}_{\mathcal{G}}(\gamma_y))_{i,j,n} = &\sum_k w_k(\gamma_y)\text{cof}\left(\frac{\partial}{\partial x_j}(\nabla_{\mathbf{x}}\partial_{\theta_n}\boldsymbol{\Psi}(0,0,\gamma_y))\right)_{ki} \\
&- \sum_k \text{cof}(\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\gamma_y))_{ki}\frac{\partial w_k}{\partial y_j}(\gamma_y) + \sum_k \nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\gamma_y)_{kj}\frac{\partial w_i}{\partial y_k}(\gamma_y).
\end{aligned} \tag{A.12}$$

B Independence of the hypothesis $(\mathcal{H})_\delta$ with respect to the diffeomorphism $\boldsymbol{\Phi}$

Let $\boldsymbol{\Phi}^a(\theta_1,\theta_2,\mathbf{y})$ and $\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y})$ be two diffeomorphisms from $\mathscr{F}_s$ to $\mathscr{F}(\theta_1,\theta_2)$ that are extensions of $\mathbf{X}(\theta_1,\theta_2,\mathbf{y})$ into the fluid domain. We also assume that $\boldsymbol{\Phi}^a(0,0,\mathbf{y}) = \boldsymbol{\Phi}^b(0,0,\mathbf{y}) = \mathbf{y}$, $\forall \mathbf{y} \in \mathscr{F}_s$. Let us denote $\boldsymbol{\Psi}^a(\theta_1,\theta_2,.)$ and $\boldsymbol{\Psi}^b(\theta_1,\theta_2,.)$ the inverse diffeomorphisms of, respectively, $\boldsymbol{\Phi}^a(\theta_1,\theta_2,.)$ and $\boldsymbol{\Phi}^b(\theta_1,\theta_2,.)$. We also denote $\mathcal{J}_{\boldsymbol{\Phi}}^a(\theta_1,\theta_2,\mathbf{y})$, $\mathcal{J}_{\boldsymbol{\Phi}}^b(\theta_1,\theta_2,\mathbf{y})$, $\mathcal{J}_{\boldsymbol{\Psi}}^a(\theta_1,\theta_2,\mathbf{x})$ and $\mathcal{J}_{\boldsymbol{\Psi}}^b(\theta_1,\theta_2,\mathbf{x})$ the corresponding Jacobian matrices.

We define the difference between the velocity at time t in $\mathscr{F}_s$ and the stationary velocity for each of these diffeomorphisms

$$\forall \mathbf{y} \in \mathscr{F}_s, \quad \forall t \in (0,\infty), \quad \begin{cases}
\mathbf{v}^a(t,\mathbf{y}) = \text{cof}(\mathcal{J}_{\boldsymbol{\Phi}}^a(\theta_1(t),\theta_2(t),\mathbf{y}))^T\mathbf{u}(t,\boldsymbol{\Phi}^a(\theta_1(t),\theta_2(t),\mathbf{y})) - \mathbf{w}(\mathbf{y}), \\
q^a(t,\mathbf{y}) = p(t,\boldsymbol{\Phi}^a(\theta_1,\theta_2,\mathbf{y})) - p_{\mathbf{w}}(\mathbf{y}), \\
\mathbf{v}^b(t,\mathbf{y}) = \text{cof}(\mathcal{J}_{\boldsymbol{\Phi}}^b(\theta_1(t),\theta_2(t),\mathbf{y}))^T\mathbf{u}(t,\boldsymbol{\Phi}^b(\theta_1(t),\theta_2(t),\mathbf{y})) - \mathbf{w}(\mathbf{y}), \\
q^b(t,\mathbf{y}) = p(t,\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y})) - p_{\mathbf{w}}(\mathbf{y}).
\end{cases}$$

We have the relations

$$\begin{cases}
\mathbf{v}^b(t,\mathbf{y}) = \text{cof}(\mathcal{J}_{\boldsymbol{\Phi}}^b(\theta_1,\theta_2,\mathbf{y}))^T\text{cof}(\mathcal{J}_{\boldsymbol{\Psi}}^a(\theta_1,\theta_2,\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y})))^T\Big(\mathbf{v}^a\big(t,\boldsymbol{\Psi}^a(\theta_1,\theta_2,\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))\big) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{w}\big(\boldsymbol{\Psi}^a(\theta_1,\theta_2,\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))\big)\Big) - \mathbf{w}(\mathbf{y}), \\
q^b(t,\mathbf{y}) = q^a(t,\boldsymbol{\Psi}^a(\theta_1,\theta_2,\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))) + p_{\mathbf{w}}(\boldsymbol{\Psi}^a(\theta_1,\theta_2,\boldsymbol{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))) - p_{\mathbf{w}}(\mathbf{y}).
\end{cases}$$

Of course, $(\mathbf{v}^a,q^a,\theta_1,\theta_2)$ satisfies the nonlinear system (3.2)–(3.3) corresponding to the diffeomorphisms $\boldsymbol{\Phi}^a(\theta_1,\theta_2,.)$ and $\boldsymbol{\Psi}^a(\theta_1,\theta_2,.)$. Similarly, $(\mathbf{v}^b,q^b,\theta_1,\theta_2)$ fulfils (3.2)–(3.3) given by $\boldsymbol{\Phi}^b(\theta_1,\theta_2,.)$ and $\boldsymbol{\Psi}^b(\theta_1,\theta_2,.)$.

We are interested in the following system

$$\begin{cases}
\dfrac{\partial \mathbf{v}_L^\alpha}{\partial t} + (\mathbf{w}\cdot\nabla)\mathbf{v}_L^\alpha + (\mathbf{v}_L^\alpha \cdot \nabla)\mathbf{w} - \mathbf{L}_{\mathbf{F}}^\alpha(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2,\mathbf{y}) - \nu\Delta\mathbf{v}_L^\alpha + \nabla q_L^\alpha = 0 & \text{in } (0,\infty)\times\mathscr{F}_s, \\
\text{div } \mathbf{v}_L^\alpha = 0 & \text{in } (0,\infty)\times\mathscr{F}_s, \\
\mathbf{v}_L^\alpha = \dot{\theta}_1\partial_{\theta_1}\boldsymbol{\Phi}^\alpha(0,0,.) + \dot{\theta}_2\partial_{\theta_2}\boldsymbol{\Phi}^\alpha(0,0,.) & \text{on } (0,\infty)\times\partial S_s, \\
\mathbf{v}_L^\alpha = 0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{D}}, \\
\sigma_F(\mathbf{v}_L^\alpha,q_L^\alpha)\mathbf{n} = 0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{N}}, \\
\mathcal{M}_{0,0}\begin{pmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{pmatrix} = \left(\displaystyle\int_{\partial S_s}\Big(q_L^\alpha\mathbf{I} - \nu(\nabla\mathbf{v}_L^\alpha + (\nabla\mathbf{v}_L^\alpha)^T)\Big)\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}^\alpha(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{L}_{\mathbf{S}}^\alpha(\theta_1,\theta_2) & \text{on } (0,\infty),
\end{cases} \tag{B.1}$$

where $\alpha = a$ or b, and $\mathbf{L}_{\mathbf{F}}^\alpha$ and $\mathbf{L}_{\mathbf{S}}^\alpha$ are given respectively by (2.2) and (2.3) corresponding to the choice of diffeomorphism $\boldsymbol{\Phi}(\theta_1,\theta_2,.) = \boldsymbol{\Phi}^\alpha(\theta_1,\theta_2,.)$.

We can then show that $(\mathbf{v}_L^a,q_L^a)$ fulfils the linear system (B.1) with $\alpha = a$ if and only if

$$\begin{cases}
\mathbf{v}_L^b(t,\mathbf{y}) = \mathbf{v}_L^a(t,\mathbf{y}) + \displaystyle\sum_j \theta_j\Big(\text{cof}(\partial_{\theta_j}\mathcal{J}_{\boldsymbol{\Phi}}^b(0,0,\mathbf{y}))^T\mathbf{w}(\mathbf{y}) + \text{cof}(\partial_{\theta_j}\mathcal{J}_{\boldsymbol{\Psi}}^a(0,0,\mathbf{y}))^T\mathbf{w}(\mathbf{y}) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \nabla\mathbf{w}\times(\partial_{\theta_j}\boldsymbol{\Psi}^a(0,0,\mathbf{y}) + \partial_{\theta_j}\boldsymbol{\Phi}^b(0,0,\mathbf{y}))\Big), \\
q_L^b(t,\mathbf{y}) = q_L^a(t,\mathbf{y}) + \displaystyle\sum_j \theta_j\nabla p_{\mathbf{w}}\cdot(\partial_{\theta_j}\boldsymbol{\Phi}^a(0,0,\mathbf{y}) - \partial_{\theta_j}\boldsymbol{\Phi}^b(0,0,\mathbf{y})),
\end{cases} \tag{B.2}$$

fulfils the linear system (B.1) with $\alpha = b$.

The proof is a direct computation, for instance

$$\frac{\partial \mathbf{v}_L^b}{\partial t} - \mathbf{L}_3^b \dot{\theta}_1 - \mathbf{L}_4^b \dot{\theta}_2 = \frac{\partial \mathbf{v}_L^a}{\partial t} + \sum_j \dot{\theta}_j \Big(\operatorname{cof}\big(\partial_{\theta_j} \mathcal{J}_{\boldsymbol{\Phi}}^b(0,0,\mathbf{y}) - \partial_{\theta_j} \mathcal{J}_{\boldsymbol{\Phi}}^a(0,0,\mathbf{y})\big)^T \mathbf{w} + \nabla \mathbf{w} \times \big(\partial_{\theta_j} \boldsymbol{\Phi}^b(0,0,\mathbf{y}) - \partial_{\theta_j} \boldsymbol{\Phi}^a(0,0,\mathbf{y})\big) \Big)$$

$$- \mathbf{L}_3^b \dot{\theta}_1 - \mathbf{L}_4^b \dot{\theta}_2$$

$$= \frac{\partial \mathbf{v}_L^a}{\partial t} - \mathbf{L}_3^a \dot{\theta}_1 - \mathbf{L}_4^a \dot{\theta}_2.$$

Moreover by using div $\mathbf{w} = 0$, $\operatorname{cof}(\partial_{\theta_k} \mathcal{J}_{\boldsymbol{\Phi}}(0,0,\mathbf{y}))^T : \nabla \mathbf{w} = \partial_{\theta_k} \mathcal{J}_{\boldsymbol{\Psi}}(0,0,\mathbf{y}) : \nabla \mathbf{w} = -\partial_{\theta_k} \mathcal{J}_{\boldsymbol{\Phi}}(0,0,\mathbf{y}) : \nabla \mathbf{w}$ and the Piola identity, we get

$$\operatorname{div} \mathbf{v}_L^b = \operatorname{div} \mathbf{v}_L^a = 0.$$

This implies that the solution $(\mathbf{v}_L^b, q_L^b, \theta_1, \theta_2)$ to the system (B.1) written with the diffeomorphism $\boldsymbol{\Phi}^b$ can be derived from the solution to the system (B.1) written with the diffeomorphism $\boldsymbol{\Phi}^a$ via the relation (B.2). Then the stabilizability of (2.1) is independent with respect to the choice of the diffeomorphism $\boldsymbol{\Phi}$. As the Hautus test $(\mathcal{H})_\delta$ is equivalent to the stabilizability of (2.1), the hypothesis $(\mathcal{H})_\delta$ is independent from the choice of $\boldsymbol{\Phi}$.

C   Proof of Lemma 3.2

We first prove some technical lemmas that are used later to decompose the intricate terms of Lemma 3.2.

C.1   Technical lemmas

Lemma C.1. *Let $R_0 > 0$ be small enough, then there exists a constant K, such that for every $R \leq R_0$ and every $(.,.,\theta_1^j, \theta_2^j)$ in $\widetilde{\mathbb{B}_R^\infty}$, the following estimates hold*

$$\left\| e^{\delta t} \Big(\boldsymbol{\Phi}(\theta_1^a, \theta_2^a) - \boldsymbol{\Phi}(\theta_1^b, \theta_2^b) - \sum_n (\theta_n^a - \theta_n^b) \partial_{\theta_n} \boldsymbol{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\Omega))} \leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.1}$$

$$\left\| e^{\delta t} \Big(\mathcal{J}_{\boldsymbol{\Phi}}(\theta_1^a, \theta_2^a) - \mathcal{J}_{\boldsymbol{\Phi}}(\theta_1^b, \theta_2^b) - \sum_n (\theta_n^a - \theta_n^b) \nabla_{\mathbf{y}} \partial_{\theta_n} \boldsymbol{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.2}$$

$$\left\| e^{\delta t} \Big(\mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^a, \theta_2^a) \circ \boldsymbol{\Phi}(\theta_1^a, \theta_2^a) - \mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^b, \theta_2^b) \circ \boldsymbol{\Phi}(\theta_1^b, \theta_2^b) + \sum_j (\theta_j^a - \theta_j^b) \nabla_{\mathbf{y}} \partial_{\theta_j} \boldsymbol{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.3}$$

$$\left\| e^{\delta t} \Big(\partial_{x_j} \mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^a, \theta_2^a) \circ \boldsymbol{\Phi}(\theta_1^a, \theta_2^a) - \partial_{x_j} \mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^b, \theta_2^b) \circ \boldsymbol{\Phi}(\theta_1^b, \theta_2^b) - \sum_n (\theta_n^a - \theta_n^b) \partial_{x_j} \nabla_{\mathbf{x}} \partial_{\theta_n} \boldsymbol{\Psi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\Omega))} \leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.4}$$

$$\left\| e^{\delta t} \Big(\partial_{x_j}^2 \mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^a, \theta_2^a) \circ \boldsymbol{\Phi}(\theta_1^a, \theta_2^a) - \partial_{x_j}^2 \mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^b, \theta_2^b) \circ \boldsymbol{\Phi}(\theta_1^b, \theta_2^b) - \sum_n (\theta_n^a - \theta_n^b) \partial_{x_j}^2 \nabla_{\mathbf{x}} \partial_{\theta_n} \boldsymbol{\Psi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\Omega))} \leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.5}$$

$$\left\| e^{\delta t} \Big(\mathbf{n}_{\theta_1^a, \theta_2^a}(\boldsymbol{\Phi}(\theta_1^a, \theta_2^a))_j - \mathbf{n}_{\theta_1^b, \theta_2^b}(\boldsymbol{\Phi}(\theta_1^b, \theta_2^b))_j - \sum_n (\theta_n^a - \theta_n^b)(\mathbf{L}_{\mathbf{n}_{\theta_1, \theta_2}})_{j,n} \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathrm{L}^\infty(\partial S_s))}$$
$$\leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.6}$$

$$\left\| e^{\delta t} \Big(\det(\mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^a, \theta_2^a, \boldsymbol{\Phi}^a)) - \det(\mathcal{J}_{\boldsymbol{\Psi}}(\theta_1^b, \theta_2^b, \boldsymbol{\Phi}^b)) + \sum_n (\theta_n^a - \theta_n^b) \operatorname{Tr}(\nabla_{\mathbf{y}} \partial_{\theta_n} \boldsymbol{\Phi}(0,0,.)) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.7}$$

$$\left\| e^{\delta t} \Big(|\mathcal{J}_{\boldsymbol{\Phi}}(\theta_1^a, \theta_2^a) \mathbf{t}_s| - |\mathcal{J}_{\boldsymbol{\Phi}}(\theta_1^b, \theta_2^b) \mathbf{t}_s| - \sum_k (\theta_k^a - \theta_k^b) \nabla_{\mathbf{y}} \partial_{\theta_k} \boldsymbol{\Phi}(0,0,.) \mathbf{t}_s \cdot \mathbf{t}_s \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\partial S_s))}$$
$$\leq KR \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.8}$$

where $\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}}$ is defined in (A.11) and

$$\left\| e^{\delta t}\Big((\partial_t \mathbf{\Psi}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi}(\theta_1^a, \theta_2^a) - (\partial_t \mathbf{\Psi}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi}(\theta_1^b, \theta_2^b) \right.$$
$$\left. + \sum_j (\dot{\theta}_j^a - \dot{\theta}_j^b)\partial_{\theta_j}\mathbf{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.9}$$

$$\left\| e^{\delta t}\Big(\partial_t(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi}(\theta_1^a, \theta_2^a) - \partial_t(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi}(\theta_1^b, \theta_2^b) \right.$$
$$\left. + \sum_j (\dot{\theta}_j^a - \dot{\theta}_j^b)\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.10}$$

and

$$\left\| e^{\delta t}\Big(\mathcal{M}_{\theta_1^a,\theta_2^a} - \mathcal{M}_{\theta_1^b,\theta_2^b} \Big) \right\|_{\mathrm{L}^\infty(0,\infty)} \leq K\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.11}$$

$$\left\| e^{\delta t}\Big(\partial_{\theta_j}\mathbf{\Phi}(\theta_1^a, \theta_2^a, .) - \partial_{\theta_j}\mathbf{\Phi}(\theta_1^b, \theta_2^b, .) - \sum_k (\theta_k^a - \theta_k^b)\partial_{\theta_j\theta_k}\mathbf{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq K\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.12}$$

$$\left\| e^{\delta t}\Big(\partial_{\theta_j\theta_k}\mathbf{\Phi}(\theta_1^a, \theta_2^a, .) - \partial_{\theta_j\theta_k}\mathbf{\Phi}(\theta_1^b, \theta_2^b, .) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq K\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.13}$$

$$\left\| e^{\delta t}\Big(\partial_t \mathcal{J}_{\mathbf{\Phi}}(\theta_1^a, \theta_2^a) - \partial_t \mathcal{J}_{\mathbf{\Phi}}(\theta_1^b, \theta_2^b) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq K\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}. \tag{C.14}$$

Moreover for every $(\mathbf{v}^j, ., \theta_1^j, \theta_2^j)$ in $\widetilde{\mathbb{B}_R^\infty}$, we have

$$\left\| e^{\delta t}\Big(\nabla\mathbf{v}^a - \mathcal{G}(\theta_1^a, \theta_2^a, \mathbf{v}^a) - \nabla\mathbf{v}^b + \mathcal{G}(\theta_1^b, \theta_2^b, \mathbf{v}^b) \Big) \right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\partial S_s))} \leq KR(\|\theta^a - \theta^b\|_{\Theta_\delta^\infty} + \|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_\delta^\infty}), \tag{C.15}$$

$$\left\| e^{\delta t}\Big(\mathcal{G}(\theta_1^a, \theta_2^a, \mathbf{w})_{ij} - \mathcal{G}(\theta_1^b, \theta_2^b, \mathbf{w})_{ij} - \sum_n (\theta_n^a - \theta_n^b)(\mathbf{L}_{\mathcal{G}})_{i,j,n} \Big) \right\|_{\mathrm{L}^2(0,+\infty;\mathbf{L}^2(\partial S_s))} \leq KR\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}, \tag{C.16}$$

where $\mathbf{L}_{\mathcal{G}}$ is defined in (A.12) and $\mathcal{G}$ is defined in (3.7).

A direct application of Lemma C.1 obtained by taking $(\theta_1^b, \theta_2^b) = (0,0)$ is the following lemma.

Lemma C.2. Let $R_0 > 0$ be small enough, then there exists $K > 0$ such that for every $R \leq R_0$ and for every $(., ., \theta_1, \theta_2) \in \widetilde{\mathbb{B}_R^\infty}$, the following estimates hold

$$\left\| e^{\delta t}\Big(\mathcal{J}_{\mathbf{\Phi}}(\theta_1, \theta_2) - \mathrm{I} \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR, \tag{C.17}$$

$$\left\| e^{\delta t}\Big(\mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2, \mathbf{\Phi}(\theta_1, \theta_2)) - \mathrm{I} \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR, \tag{C.18}$$

$$\left\| e^{\delta t}\Big(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2) \circ \mathbf{\Phi}(\theta_1, \theta_2) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\Omega))} \leq KR, \tag{C.19}$$

$$\left\| e^{\delta t}\Big(\partial_{x_j}^2 \mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2) \circ \mathbf{\Phi}(\theta_1, \theta_2) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\Omega))} \leq KR, \tag{C.20}$$

$$\left\| e^{\delta t}\Big(\mathcal{M}_{\theta_1,\theta_2} - \mathcal{M}_{0,0} \Big) \right\|_{\mathrm{L}^\infty(0,\infty)} \leq KR, \tag{C.21}$$

$$\left\| e^{\delta t}\Big(\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi}(\theta_1, \theta_2)) - \mathbf{n}_s \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\partial S_s))} \leq KR, \tag{C.22}$$

$$\left\| e^{\delta t}\Big(|\mathcal{J}_{\mathbf{\Phi}}(\theta_1, \theta_2)\mathbf{t}_s| - 1 \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\partial S_s))} \leq KR, \tag{C.23}$$

$$\left\| e^{\delta t}\Big(\partial_{\theta_j}\mathbf{\Phi}(\theta_1, \theta_2, .) - \partial_{\theta_j}\mathbf{\Phi}(0,0,.) \Big) \right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\Omega))} \leq KR, \tag{C.24}$$

and

$$\| e^{\delta t}\partial_t \mathcal{J}_{\mathbf{\Phi}}(\theta_1, \theta_2) \|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR, \tag{C.25}$$

$$\| e^{\delta t}(\partial_t \mathbf{\Psi}(\theta_1, \theta_2)) \circ \mathbf{\Phi}(\theta_1, \theta_2) \|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR, \tag{C.26}$$

$$\| e^{\delta t}(\partial_t \mathcal{J}_{\mathbf{\Psi}}(\theta_1, \theta_2)) \circ \mathbf{\Phi}(\theta_1, \theta_2) \|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR, \tag{C.27}$$

and

$$\left\| e^{\delta t}\Big(\nabla\mathbf{w} - \mathcal{G}(\theta_1, \theta_2, \mathbf{w}) \Big) \right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\partial S_s))} \leq KR, \tag{C.28}$$

where $\mathcal{G}$ is defined in (3.7).

Proof of Lemma C.1. Let $\mathbb{X}$ be some Banach space. All functions in Lemma C.1 can be written as $\alpha = \alpha(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ valued in $\mathbb{X}$. The proof of estimates (C.11)–(C.14) uses the mean value theorem,

$$
\begin{aligned}
&\left\| e^{\delta t}\Big(\alpha(\theta_1^a, \theta_2^a, \dot\theta_1^a, \dot\theta_2^a) - \alpha(\theta_1^b, \theta_2^b, \dot\theta_1^b, \dot\theta_2^b) \Big) \right\|_{L^\infty(0,\infty;\mathbb{X})} \\
&\qquad \leq \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta \\ |\omega_1|+|\omega_2|\leq R}} \|\nabla_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{L^\infty(0,\infty;\mathbb{X})} \Big(\|e^{\delta t}(\theta^a-\theta^b)\|_{L^\infty(0,\infty)} + \|e^{\delta t}(\dot\theta^a-\dot\theta^b)\|_{L^\infty(0,\infty)} \Big) \\
&\qquad \leq 2C \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta \\ |\omega_1|+|\omega_2|\leq R}} \|\nabla_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{L^\infty(0,\infty;\mathbb{X})} \|\theta^a-\theta^b\|_{\Theta_\delta^\infty}.
\end{aligned}
$$

To prove the other estimates (C.1)–(C.10) and (C.16), we do a Taylor expansion of order 2 and use the mean value theorem. We get

$$
\begin{aligned}
&\left\| e^{\delta t}\Big(\alpha(\theta_1^a, \theta_2^a, \dot\theta_1^a, \dot\theta_2^a) - \alpha(\theta_1^b, \theta_2^b, \dot\theta_1^b, \dot\theta_2^b) - \sum_{j=1}^2 \partial_{\theta_j}\alpha(0,0,0,0)(\theta_j^a-\theta_j^b) - \sum_{j=1}^2 \partial_{\omega_j}\alpha(0,0,0,0)(\dot\theta_j^a-\dot\theta_j^b) \Big) \right\|_{L^\infty(0,\infty;\mathbb{X})} \\
&\qquad \leq \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta \\ |\omega_1|+|\omega_2|\leq R}} \|\nabla_{\theta,\omega}^2\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{L^\infty(0,\infty;\mathbb{X})} R (\|e^{\delta t}(\theta^a-\theta^b)\|_{L^\infty(0,\infty)} + \|e^{\delta t}(\dot\theta^a-\dot\theta^b)\|_{L^\infty(0,\infty)}).
\end{aligned}
$$

The estimate (C.15) is proven by using the previous estimates. $\qquad\square$

C.2 Detailed proof of Lemma 3.2

Proof. The weights in time $e^{\delta t}$ of the nonlinear terms can be easily handled. All the difficulties then come from space regularity issues, which can be handled as in [11, Appendix B] for the terms $\mathbf{F}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)$, $\mathbf{G}(\theta_1^a, \theta_2^a, \dot\theta_1^a, \dot\theta_2^a) - \mathbf{G}(\theta_1^b, \theta_2^b, \dot\theta_1^b, \dot\theta_2^b)$ and $\mathbf{S}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)$ using the estimates of Lammas C.1 and C.2:

$$
\|\mathbf{F}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)\|_{\mathbb{F}_\delta^\infty} \leq KR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a-q^b\|_{\mathbb{P}_\delta^\infty} + \|\theta^a-\theta^b\|_{\Theta_\delta^\infty}),
$$

$$
\|\mathbf{G}(\theta_1^a, \theta_2^a, \dot\theta_1^a, \dot\theta_2^a) - \mathbf{G}(\theta_1^b, \theta_2^b, \dot\theta_1^b, \dot\theta_2^b)\|_{\mathbb{G}_\delta^\infty} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty},
$$

and

$$
\|\mathbf{S}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)\|_{\mathbb{S}_\delta^\infty} \leq KR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a-q^b\|_{\mathbb{P}_\delta^\infty} + \|\theta^a-\theta^b\|_{\Theta_\delta^\infty}),
$$

where the terms $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{S}$ are defined in (3.5)–(3.6).

Now, it remains to prove

$$
\begin{aligned}
&\|\mathbf{F}(\theta_1^a, \theta_2^a, \mathbf{w}+\mathbf{v}^a, p_\mathbf{w}+q^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \mathbf{w}+\mathbf{v}^b, p_\mathbf{w}+q^b) - \mathbf{F}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) + \mathbf{F}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b) \\
&+ (\mathbf{w}\cdot\nabla)(\mathbf{v}^a-\mathbf{v}^b) + ((\mathbf{v}^a-\mathbf{v}^b)\cdot\nabla)\mathbf{w} - \mathbf{L_F}(\theta_1^a-\theta_1^b, \theta_2^a-\theta_2^b, \dot\theta_1^a-\dot\theta_1^b, \dot\theta_2^a-\dot\theta_2^b) + \mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}(\theta_1^a,\theta_2^a)) - \mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}(\theta_1^b,\theta_2^b))\|_{\mathbb{F}_\delta^\infty} \\
&\qquad\qquad \leq CR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|\theta^a-\theta^b\|_{\Theta_\delta^\infty}),
\end{aligned} \tag{C.29}
$$

and

$$
\begin{aligned}
&\|\mathbf{S}(\theta_1^a, \theta_2^a, \mathbf{w}+\mathbf{v}^a, p_\mathbf{w}+q^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \mathbf{w}+\mathbf{v}^b, p_\mathbf{w}+q^b) - \mathbf{S}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) + \mathbf{S}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b) \\
&\qquad\qquad - \mathbf{L_S}(\theta_1^a-\theta_1^b, \theta_2^a-\theta_2^b)\|_{\mathbb{S}_T} \leq CR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty},
\end{aligned} \tag{C.30}
$$

and then the proof will be complete.

The estimate (C.29) is a consequence of the following relations

$$
\mathbf{F}^1(\theta_1^a, \theta_2^a, \mathbf{w}+\mathbf{v}^a) - \mathbf{F}^1(\theta_1^b, \theta_2^b, \mathbf{w}+\mathbf{v}^b) - \mathbf{F}^1(\theta_1^a, \theta_2^a, \mathbf{v}^a) + \mathbf{F}^1(\theta_1^b, \theta_2^b, \mathbf{v}^b) = 0, \tag{C.31}
$$

$$
\left\| \mathbf{F}^2(\theta_1^a, \theta_2^a, \mathbf{w}+\mathbf{v}^a) - \mathbf{F}^2(\theta_1^b, \theta_2^b, \mathbf{w}+\mathbf{v}^b) - \mathbf{F}^2(\theta_1^a, \theta_2^a, \mathbf{v}^a) + \mathbf{F}^2(\theta_1^b, \theta_2^b, \mathbf{v}^b) \right. \\
\left. - \sum_j (\dot\theta_j^a-\dot\theta_j^b)\Big((\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\cdot\nabla_\mathbf{y})\mathbf{w} + \mathrm{cof}(\nabla_\mathbf{y}\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y}))^T\mathbf{w} \Big) \right\|_{\mathbb{F}_\delta^\infty} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{C.32}
$$

$$
\left\| (\mathbf{F}^3(\theta_1^a, \theta_2^a, \mathbf{w}+\mathbf{v}^a) - \mathbf{F}^3(\theta_1^b, \theta_2^b, \mathbf{w}+\mathbf{v}^b) - \mathbf{F}^3(\theta_1^a, \theta_2^a, \mathbf{v}^a) + \mathbf{F}^3(\theta_1^b, \theta_2^b, \mathbf{v}^b))_i - \nu \sum_j (\theta_j^a-\theta_j^b)(\mathbf{L_{F^3}})_{ij} \right\|_{\mathbb{F}_\delta^\infty} \\
\leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{C.33}
$$

$$\left\|\left(\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)+\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)+((\mathbf{v}^a-\mathbf{v}^b)\cdot\nabla)\mathbf{w}\right.\right.$$
$$\left.\left.+(\mathbf{w}\cdot\nabla)(\mathbf{v}^a-\mathbf{v}^b)\right)_i-\sum_n(\theta_n^a-\theta_n^b)(\mathbf{L}_{\mathbf{F}^4})_{i,n}\right\|_{\mathbb{F}_\delta^\infty}\leq KR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_\delta^\infty}+\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}),\quad\text{(C.34)}$$

$$\left\|\mathbf{F}^5(\theta_1^a,\theta_2^a,p_{\mathbf{w}}+q^a)-\mathbf{F}^5(\theta_1^a,\theta_2^a,q^a)-\mathbf{F}^5(\theta_1^b,\theta_2^b,p_{\mathbf{w}}+q^b)+\mathbf{F}^5(\theta_1^b,\theta_2^b,q^b)\right.$$
$$\left.-\sum_j(\theta_j^a-\theta_j^b)(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y}))^T\nabla_{\mathbf{y}}p_{\mathbf{w}}\right\|_{\mathbb{F}_\delta^\infty}\leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty},\quad\text{(C.35)}$$

and

$$\left\|\mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}(\theta_1^a,\theta_2^a))-\mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}(\theta_1^b,\theta_2^b))-\sum_k(\theta_k^a-\theta_k^b)\nabla_{\mathbf{y}}\mathbf{f}_{\mathscr{F}}\partial_{\theta_k}\mathbf{\Phi}(0,0,.)\right\|_{\mathbb{F}_\delta^\infty}\leq K\|\mathbf{f}_{\mathscr{F}}\|_{\mathbf{W}^{1,\infty}(\Omega)}R\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}.\quad\text{(C.36)}$$

In the sequel, we use the compact notations $\mathcal{J}_{\mathbf{\Psi}}^a=\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a,\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y}))$ and $\mathcal{J}_{\mathbf{\Psi}}^b=\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b,\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y}))$, and similarly for other functions. We now prove the estimates (C.30)–(C.36). We keep (C.30) for the end.
- **Identity** (C.31): the proof is immediate.
- **Estimate** (C.32): we use the decomposition

$$\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{v}^b)$$
$$-\sum_j(\dot\theta_j^a-\dot\theta_j^b)\Big((\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\cdot\nabla_{\mathbf{y}})\mathbf{w}+\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y}))^T\mathbf{w}\Big)$$
$$=-\mathrm{cof}\Big(\partial_t(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a,\theta_2^a))\circ\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})-\partial_t(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b,\theta_2^b))\circ\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y})+\sum_j(\dot\theta_j^a-\dot\theta_j^b)\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\Big)^T\mathbf{w}$$
$$-\mathrm{cof}\Big(\mathcal{J}_{\mathbf{\Psi}}^a-\mathcal{J}_{\mathbf{\Psi}}^b\Big)^T\nabla_{\mathbf{y}}\mathbf{w}(\partial_t\mathbf{\Psi}(\theta_1^a,\theta_2^a,.))\circ\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})$$
$$-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^b)^T\nabla_{\mathbf{y}}\mathbf{w}\Big(\partial_t(\mathbf{\Psi}(\theta_1^a,\theta_2^a,.))\circ\mathbf{\Phi}(\theta_1^a,\theta_2^a,\mathbf{y})-\partial_t(\mathbf{\Psi}(\theta_1^b,\theta_2^b,.))\circ\mathbf{\Phi}(\theta_1^b,\theta_2^b,\mathbf{y})+\sum_j(\dot\theta_j^a-\dot\theta_j^b)\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y})\Big)$$
$$+\sum_j(\dot\theta_j^a-\dot\theta_j^b)\mathrm{cof}\Big(\mathcal{J}_{\mathbf{\Psi}}^b-\mathrm{I}\Big)^T\nabla_{\mathbf{y}}\mathbf{w}\partial_{\theta_j}\mathbf{\Phi}(0,0,\mathbf{y}),$$

and the estimates (C.18), (C.26), (C.3), (C.9) and (C.10) yield (C.32).
- **Estimate** (C.33): we use the decomposition

$$\Big(\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\Big)_i-\nu\sum_j(\theta_j^a-\theta_j^b)(\mathbf{L}_{\mathbf{F}^3})_{ij}$$
$$=\Big(\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{w})-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{w})\Big)_i-\nu\sum_j(\theta_j^a-\theta_j^b)(\mathbf{L}_{\mathbf{F}^3})_{ij}$$
$$=A_{1,i}+A_{2,i}+A_{3,i}+A_{4,i},$$

where

$$A_{1,i}=\nu\sum_{j,k,\ell,m}\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^a)_{ki}\frac{\partial\Psi_\ell^a}{\partial x_j}\frac{\partial\Psi_m^a}{\partial x_j}-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^b)_{ki}\frac{\partial\Psi_\ell^b}{\partial x_j}\frac{\partial\Psi_m^b}{\partial x_j}$$
$$+\sum_n(\theta_n^a-\theta_n^b)\Big(\partial_{y_j}\partial_{\theta_n}\Phi_\ell(0,0,\mathbf{y})\delta_{ki}\delta_{mj}+\partial_{y_j}\partial_{\theta_n}\Phi_m(0,0,\mathbf{y})\delta_{ki}\delta_{\ell j}+\mathrm{cof}(\partial_{\theta_n}\mathcal{J}_{\mathbf{\Phi}}(0,0,\mathbf{y}))_{ki}\delta_{mj}\delta_{\ell j}\Big)\Big)\frac{\partial^2 w_k}{\partial y_m\partial y_\ell},$$
$$A_{2,i}=2\nu\sum_{j,k,\ell}\Big(\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}^a)_{ki}\frac{\partial\Psi_\ell^a}{\partial x_j}-\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}^b)_{ki}\frac{\partial\Psi_\ell^b}{\partial x_j}-\sum_n(\theta_n^a-\theta_n^b)\mathrm{cof}(\partial_{x_j}\partial_{\theta_n}\mathcal{J}_{\mathbf{\Psi}}(0,0,\mathbf{y}))_{ki}\delta_{\ell j}\Big)\frac{\partial w_k}{\partial y_\ell},$$
$$A_{3,i}=\nu\sum_{j,k,\ell}\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^a)_{ki}\frac{\partial^2\Psi_\ell^a}{\partial x_j^2}-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^b)_{ki}\frac{\partial^2\Psi_\ell^b}{\partial x_j^2}-\sum_n(\theta_n^a-\theta_n^b)\delta_{ki}\frac{\partial^2}{\partial x_j^2}\partial_{\theta_n}\Psi_\ell(0,0,\mathbf{y})\Big)\frac{\partial w_k}{\partial y_\ell},$$

and

$$A_{4,i}=\nu\sum_{j,k}\left(\mathrm{cof}\left(\frac{\partial^2}{\partial x_j^2}\mathcal{J}_{\mathbf{\Psi}}^a\right)_{ki}-\mathrm{cof}\left(\frac{\partial^2}{\partial x_j^2}\mathcal{J}_{\mathbf{\Psi}}^b\right)_{ki}-\sum_n(\theta_n^a-\theta_n^b)\frac{\partial^2}{\partial x_j^2}\partial_{\theta_n}\mathcal{J}_{\mathbf{\Psi}}(0,0,\mathbf{y})_{ki}\right)w_k.$$

The term $A_{1,i}$ is of the form $\displaystyle\sum_{j,k,\ell,m}a_{ijk\ell m}(\mathbf{y})\times\frac{\partial^2 w_k}{\partial y_m\partial y_\ell}$, where

$$a_{ijk\ell m}(\mathbf{y})=\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^a)_{ki}\frac{\partial\Psi_\ell^a}{\partial x_j}\frac{\partial\Psi_m^a}{\partial x_j}-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}^b)_{ki}\frac{\partial\Psi_\ell^b}{\partial x_j}\frac{\partial\Psi_m^b}{\partial x_j}$$
$$+\sum_n(\theta_n^a-\theta_n^b)\Big(\partial_{y_j}\partial_{\theta_n}\Phi_\ell(0,0,\mathbf{y})\delta_{ki}\delta_{mj}+\partial_{y_j}\partial_{\theta_n}\Phi_m(0,0,\mathbf{y})\delta_{ki}\delta_{\ell j}+\mathrm{cof}(\partial_{\theta_n}\mathcal{J}_{\mathbf{\Phi}}(0,0,\mathbf{y}))_{ki}\delta_{mj}\delta_{\ell j}\Big)\Big).$$

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Moreover, according to (1.21), $\boldsymbol{\Psi}(\theta_1, \theta_2, \mathbf{y}) = \mathbf{y}$ in $\Omega \backslash \Omega_\varepsilon$, then $a_{ijk\ell m} = 0$ in $\Omega \backslash \Omega_\varepsilon$ (Ω_ε is defined in Lemma 1.4). We then estimate $A_{1,i}$ as

$$\left\| a_{ijk\ell m} \times \frac{\partial^2 w_k}{\partial y_m \partial y_\ell} \right\|_{\mathbf{L}^2(\mathscr{F}_s)} \leq \left\| \frac{a_{ijk\ell m}}{\prod_{j \in \mathscr{J}_{d,n}} r_j^\beta} \right\|_{\mathbf{L}^\infty(\mathscr{F}_s)} \left\| \frac{\partial^2 w_k}{\partial y_m \partial y_\ell} \prod_{j \in \mathscr{J}_{d,n}} r_j^\beta \right\|_{\mathbf{L}^2(\mathscr{F}_s)} \leq \|a_{ijk\ell m}\|_{\mathbf{L}^\infty(\Omega)} \left\| \frac{1}{\prod_{j \in \mathscr{J}_{d,n}} r_j^\beta} \right\|_{\mathbf{L}^\infty(\Omega \backslash \Omega_\varepsilon)} \|\mathbf{w}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)}.$$

Moreover, we get the estimate $\|a_{ijk\ell m}\|_{\mathbf{L}^\infty(\Omega)} \leq KR\|\theta^a - \theta^b\|_{\Theta^\infty_\delta}$ by using three times the estimate (C.3). We then use estimates (C.3), (C.4) and (C.19) for $A_{2,i}$ and $A_{3,i}$ and estimate (C.5) for $A_{4,i}$. We obtain (C.33).

- **Estimate** (C.34)**:** we have the decomposition

$$\Big(\mathbf{F}^4(\theta_1^a, \theta_2^a, \mathbf{w} + \mathbf{v}^a) - \mathbf{F}^4(\theta_1^a, \theta_2^a, \mathbf{v}^a) - \mathbf{F}^4(\theta_1^b, \theta_2^b, \mathbf{w} + \mathbf{v}^b) + \mathbf{F}^4(\theta_1^b, \theta_2^b, \mathbf{v}^b) + ((\mathbf{v}^a - \mathbf{v}^b) \cdot \nabla)\mathbf{w}$$
$$+ (\mathbf{w} \cdot \nabla)(\mathbf{v}^a - \mathbf{v}^b) \Big)_i - \sum_n (\theta_n^a - \theta_n^b)(\mathbf{L}_{\mathbf{F}^4})_{i,n}$$
$$= B_{1,i} + B_{2,i} + B_{3,i} + B_{4,i} + B_{5,i} + B_{6,i},$$

where

$$B_{1,i} = -\sum_{j,k,\ell} \Big(\text{cof}(\mathcal{J}_{\boldsymbol{\Psi}}^a)_{kj}\text{cof}(\partial_{x_j}\mathcal{J}_{\boldsymbol{\Psi}}^a)_{\ell i} - \text{cof}(\mathcal{J}_{\boldsymbol{\Psi}}^b)_{kj}\text{cof}(\partial_{x_j}\mathcal{J}_{\boldsymbol{\Psi}}^b)_{\ell i} - \sum_n (\theta_n^a - \theta_n^b)\text{cof}(\partial_{x_j}\partial_{\theta_n}\mathcal{J}_{\boldsymbol{\Psi}}(0,0,\mathbf{y}))_{\ell i}\delta_{kj} \Big) w_k w_\ell,$$

$$B_{2,i} = -\sum_{j,k,\ell} \Big(\text{cof}(\mathcal{J}_{\boldsymbol{\Psi}}^a)_{kj}\text{cof}(\partial_{x_j}\mathcal{J}_{\boldsymbol{\Psi}}^a)_{\ell i} - \text{cof}(\mathcal{J}_{\boldsymbol{\Psi}}^b)_{kj}\text{cof}(\partial_{x_j}\mathcal{J}_{\boldsymbol{\Psi}}^b)_{\ell i} \Big)(w_k v_\ell^a + v_k^a w_\ell),$$

$$B_{3,i} = -\sum_{j,k,\ell} \text{cof}(\mathcal{J}_{\boldsymbol{\Psi}}^b)_{kj}\text{cof}(\partial_{x_j}\mathcal{J}_{\boldsymbol{\Psi}}^b)_{\ell i} \big(w_k(v_\ell^a - v_\ell^b) + (v_k^a - v_k^b)w_\ell \big),$$

$$B_{4,i} = -\sum_{k,\ell} \Big(\det(\mathcal{J}_{\boldsymbol{\Psi}}^a)^2 \frac{\partial \Phi_i^a}{\partial y_\ell} - \det(\mathcal{J}_{\boldsymbol{\Psi}}^b)^2 \frac{\partial \Phi_i^b}{\partial y_\ell} + \sum_n (\theta_n^a - \theta_n^b)\big(2\text{Tr}(\partial_{\theta_n}\mathcal{J}_{\boldsymbol{\Phi}}(0,0,\mathbf{y}))\delta_{i\ell} - \partial_{\theta_n}\mathcal{J}_{\boldsymbol{\Phi}}(0,0,\mathbf{y})_{i\ell} \big) \Big) w_k \frac{\partial w_\ell}{\partial y_k},$$

$$B_{5,i} = -\sum_{k,\ell} \Big(\det(\mathcal{J}_{\boldsymbol{\Psi}}^a)^2 \frac{\partial \Phi_i^a}{\partial y_\ell} - \det(\mathcal{J}_{\boldsymbol{\Psi}}^b)^2 \frac{\partial \Phi_i^b}{\partial y_\ell} \Big)\Big(\frac{\partial w_\ell}{\partial y_\ell} v_k^a + \frac{\partial v_\ell^a}{\partial y_\ell} w_k \Big),$$

and

$$B_{6,i} = -\sum_{k,\ell} \Big(\det(\mathcal{J}_{\boldsymbol{\Psi}}^b)^2 \frac{\partial \Phi_i^b}{\partial y_\ell} - \delta_{i\ell} \Big)\Big(\frac{\partial w_\ell}{\partial y_k}(v_k^a - v_k^b) + (\frac{\partial v_\ell^a}{\partial y_k} - \frac{\partial v_\ell^b}{\partial y_k})w_k \Big).$$

Now, we use

- estimates (C.3), (C.4), (C.18) and (C.19) for $B_{1,i}$,

- estimates (C.3), (C.4) for $B_{2,i}$,

- estimates (C.18) and (C.19) for $B_{3,i}$,

- estimates (C.2), (C.7), (C.17) and (C.18) for $B_{4,i}$ and $B_{5,i}$,

- estimates (C.17) and (C.18) for $B_{6,i}$.

We get (C.34).
- **Estimate** (C.35)**:** we have

$$\mathbf{F}^5(\theta_1^a, \theta_2^a, p_\mathbf{w} + q^a) - \mathbf{F}^5(\theta_1^a, \theta_2^a, q^a) - \mathbf{F}^5(\theta_1^b, \theta_2^b, p_\mathbf{w} + q^b) + \mathbf{F}^5(\theta_1^b, \theta_2^b, q^b) - \sum_n (\theta_n^a - \theta_n^b)(\nabla_\mathbf{y}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\mathbf{y}))^T \nabla_\mathbf{y} p_\mathbf{w}$$
$$= \mathbf{F}^5(\theta_1^a, \theta_2^a, p_\mathbf{w}) - \mathbf{F}^5(\theta_1^b, \theta_2^b, p_\mathbf{w}) - \sum_n (\theta_n^a - \theta_n^b)(\nabla_\mathbf{y}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\mathbf{y}))^T \nabla_\mathbf{y} p_\mathbf{w}$$
$$= -\Big(\mathcal{J}_{\boldsymbol{\Psi}}^a - \mathcal{J}_{\boldsymbol{\Psi}}^b + \sum_n (\theta_n^a - \theta_n^b)\nabla_\mathbf{y}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\mathbf{y}) \Big)^T \nabla_\mathbf{y} p_\mathbf{w},$$

and estimate (C.3) gives (C.35).
- **Estimate** (C.36)**:** it is a consequence of (C.1).
- **Estimate** (C.30)**:** we have

$$\Big(\mathbf{S}(\theta_1^a, \theta_2^a, \mathbf{w} + \mathbf{v}^a, p_\mathbf{w} + q^a) - \mathbf{S}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \mathbf{w} + \mathbf{v}^b, p_\mathbf{w} + q^b) + \mathbf{S}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b) - \mathbf{L}_\mathbf{S}(\theta_1^a - \theta_1^b, \theta_2^a - \theta_2^b) \Big)_j$$
$$= D_{1,j} + D_{2,j} + D_{3,j} + D_{4,j},$$

where

$$D_{1,j} = \int_{\partial S_s} (|\mathcal{J}_{\boldsymbol{\Phi}}^a \mathbf{t}_s| - |\mathcal{J}_{\boldsymbol{\Phi}}^b \mathbf{t}_s|)(p_{\mathbf{w}}\mathrm{I} - \nu(\mathcal{G}^a + (\mathcal{G}^a)^T))(\mathbf{n}_{\theta_1,\theta_2}^a \circ \boldsymbol{\Phi}^a) \cdot \partial_{\theta_j}\boldsymbol{\Phi}^a$$
$$+ \sum_n (\theta_n^a - \theta_n^b) \int_{\partial S_s} ((\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}(0,0,\gamma_y))\mathbf{t}_s \cdot \mathbf{t}_s)\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_j}\boldsymbol{\Phi}(0,0,\gamma_y),$$

$$D_{2,j} = -\nu \int_{\partial S_s} |\mathcal{J}_{\boldsymbol{\Phi}}^b \mathbf{t}_s|(\mathcal{G}^a + (\mathcal{G}^a)^T - \mathcal{G}^b - (\mathcal{G}^b)^T)(\mathbf{n}_{\theta_1,\theta_2}^a \circ \boldsymbol{\Phi}^a) \cdot \partial_{\theta_j}\boldsymbol{\Phi}^a$$
$$+ \nu \sum_{k,\ell,n} (\theta_n^a - \theta_n^b) \int_{\partial S_s} ((\mathbf{L}_{\mathcal{G}})_{k\ell n} + (\mathbf{L}_{\mathcal{G}})_{\ell kn})(\mathbf{n}_s)_k \partial_{\theta_j}\Phi_\ell(0,0,\gamma_y),$$

$$D_{3,j} = \int_{\partial S_s} |\mathcal{J}_{\boldsymbol{\Phi}}^b \mathbf{t}_s|(p_{\mathbf{w}}\mathrm{I} - \nu(\mathcal{G}^b + (\mathcal{G}^b)^T))(\mathbf{n}_{\theta_1,\theta_2}^a \circ \boldsymbol{\Phi}^a - \mathbf{n}_{\theta_1,\theta_2}^b \circ \boldsymbol{\Phi}^b) \cdot \partial_{\theta_j}\boldsymbol{\Phi}^a$$
$$+ \sum_{k,\ell,n} (\theta_n^a - \theta_n^b) \int_{\partial S_s} \sigma_F(\mathbf{w}, p_{\mathbf{w}})_{\ell k}(\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{kn} \partial_{\theta_j}\Phi_\ell(0,0,\gamma_y),$$

and

$$D_{4,j} = \int_{\partial S_s} |\mathcal{J}_{\boldsymbol{\Phi}}^b \mathbf{t}_s|(p_{\mathbf{w}}\mathrm{I} - \nu(\mathcal{G}^b + (\mathcal{G}^b)^T))(\mathbf{n}_{\theta_1,\theta_2}^b \circ \boldsymbol{\Phi}^b) \cdot (\partial_{\theta_j}\boldsymbol{\Phi}^a - \partial_{\theta_j}\boldsymbol{\Phi}^b)$$
$$+ \sum_n (\theta_n^a - \theta_n^b) \int_{\partial S_s} \sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_n\theta_j}\boldsymbol{\Phi}(0,0,\gamma_y).$$

We use

- estimates (C.8), (C.28), (C.22) and (C.24) for $D_{1,j}$,

- estimates (C.23), (C.16), (C.22) and (C.24) for $D_{2,j}$,

- estimates (C.23), (C.28), (C.6) and (C.24) for $D_{3,j}$,

- estimates (C.23), (C.28), (C.22) and (C.12) for $D_{4,j}$.

Combining these estimates yields (C.30). $\qquad\square$

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