Stochastic Process
(Discrete Markov chains, Martingales, Brownian motion)
2A ENPC, 2016

Vocabulary (english/*français*) : positive = *strictement positif* ; irreducible = *irréductible* ; hitting time = *temps d'atteinte* ; eigen-value = *valeur propre* ; eigen-vector = *vecteur propre*.

Exercice 1 (Q-process)**.** Let E be a finite state space and $E_* \subset E$ such that $2 \leq \text{Card}\,(E_*) < \text{Card}\,(E)$. Let $X = (X_n, n \in \mathbb{N})$ be an irreducible Markov chain on E. We consider the first hitting time of E_*^c :

$$\tau = \inf\{n \geq 0; X_n \notin E_*\}.$$

The aim of this problem is to study the distribution of X conditionally on $\{\tau = +\infty\}$, which will be called the Q-process associated to X and E_*.

I Preliminaries

1. Compute $\mathbb{P}(\tau = +\infty)$. Explain why the distribution of X conditionally on $\{\tau = +\infty\}$ is not well defined.

Let P be the transition matrix of X and π its invariant probability measure. We set $P_* = (P(x, y); x, y \in E_*)$. The notation ${P_*}^n$ corresponds to the usual matrix product of P_* with itself n times. For g a function defined on E_* or E, we define $P_* g$ by :

$$P_* g(x) = \sum_{y \in E_*} P(x, y) g(y), \quad x \in E_*.$$

2. (a) Check that $P_* g(x) = \mathbb{E}_x \left[g(X_1) \mathbf{1}_{\{\tau > 1\}} \right]$ for $x \in E_*$.
   For all $x \in E$ and $n \in \mathbb{N}$, we set :

   $$h_n(x) = \mathbb{P}_x(\tau > n),$$

   so that $h_0(x) = \mathbf{1}_{E_*}$. We set $\mathbf{1}$ the constant function equal to 1.
   (b) Prove that, on E_*, we have $h_{n+1} = P_* h_n$ and thus $h_n = {P_*}^n \mathbf{1}$.
   (c) More generally, prove that for all $x \in E_*$ and g a function defined on E_*, we have :

   $${P_*}^n g(x) = \mathbb{E}_x \left[g(X_n) \mathbf{1}_{\{\tau > n\}} \right].$$

We assume that there exists $n \geq 1$ such that for all $x, y \in E_*$, we have ${P_*}^n(x, y) > 0$. Perron-Frobenius' Theorem asserts that there exists for P_* :

- an eigen-value $\lambda > 0$,
- a function φ (seen as a column vector) defined on E_* positive which is an eigen-vector on the right associated to λ,
- a probability measure ν (see as a lign vector) defined on E_* with $\nu(x) > 0$ for all $x \in E_*$ which is an eigen-vector on the left associated to à λ,

such that $\lim_{n\to+\infty} \lambda^{-n} P_*^n = \varphi\nu$ that is :

$$\lim_{n\to+\infty} \lambda^{-n} {P_*}^n(x,y) = \varphi(x)\nu(y), \quad x, y \in E_*. \tag{1}$$

3. (a) We assume in this question only that the distribution of X_0 is ν, that is $\mathbb{P}(X_0 = x) = \nu(x)$ for $x \in E_*$ and $\mathbb{P}(X_0 = x) = 0$ for $x \notin E_*$. Compute $\mathbb{P}(\tau > n)$ for $n \in \mathbb{N}$.
   (b) Identify the distribution of τ if the distribution of X_0 is ν. Deduce that $\lambda < 1$.

We set $\varphi(x) = 0$ for $x \notin E_*$. We define $M = (M_n, n \in \mathbb{N})$ with :

$$M_n = \lambda^{-n}\varphi(X_n)\mathbf{1}_{\{\tau>n\}}.$$

4. (a) Prove that M converges a.s. and give its limit. Prove that M is a martingale.
   (b) Using (1), prove that $\lim_{n\to+\infty} \lambda^{-n} h_n(x) = \varphi(x)$, for $x \in E_*$.
   (c) We assume that $\mathbb{P}(X_0 \in E_*) > 0$. Let ν_0 denote the distribution of X_0. Let $n \in \mathbb{N}$ be fixed. Prove that, for p_0 large enough, the sequence $(h_p(x)/\mathbb{E}[h_{p+n}(X_0)], p \geq p_0)$ is uniformly bounded in $x \in E$ and that for all $x \in E$:

$$\lim_{p\to+\infty} \frac{h_p(x)}{\mathbb{E}[h_{p+n}(X_0)]} = \lambda^{-n}\frac{\varphi(x)}{\nu_0\varphi}. \tag{2}$$

II Q-process

We denote by ν_0 the distribution of X_0 and we assume that $\nu_0(E_*) = 1$, that is $\mathbb{P}(\tau \geq 1) = 1$. Let $Y = (Y_n, n \in \mathbb{N})$ be a sequence of random variables taking values in E_*, such that for all $A \subset (E_*)^n$, we have :

$$\mathbb{P}\Big((Y_0, \ldots, Y_n) \in A\Big) = \mathbb{E}\left[\frac{M_n}{\mathbb{E}[M_n]}\mathbf{1}_{\{(X_0,\ldots,X_n)\in A\}}\right].$$

1. Using (2), prove that for all $A \in (E_*)^n$, we have :

$$\lim_{p\to+\infty} \mathbb{P}\Big((X_0, \ldots, X_n) \in A | \tau > n+p\Big) = \mathbb{P}\Big((Y_0, \ldots, Y_n) \in A\Big).$$

   We shall say the process Y is the process X conditioned to stay in E_*.

2. (a) Let $n \in \mathbb{N}$. Let f be a function defined on E_* and g a function defined on ${E_*}^{n+1}$. Prove that :

$$\mathbb{E}\left[f(Y_{n+1})g(Y_0, \ldots, Y_n)\right] = \mathbb{E}\left[g(Y_0, \ldots, Y_n)F(Y_n)\right],$$

   with a function F which shall be precised.
   (b) Deduce that Y is a Markov chain with transition matrix Q defined by :

$$Q(x,y) = \frac{\varphi(y)}{\lambda\varphi(x)}P_*(x,y), \quad x, y \in E_*.$$

   (c) Check that Y is irreducible that it has an invariant probability measure ρ, which shall not be computed. Check that Y is aperiodic.

3. (a) Compute Q^2, Q^n and then $\lim_{n\to+\infty} Q^n$. Deduce a formula for ρ and compute $\nu\varphi = \sum_{z\in E_*} \nu(z)\varphi(z)$.

   (b) Prove that if X is reversible with respect to a probability measure π on E, then Y is reversible with respect to a probability measure, say $\hat{\rho}$ on E_*. Determine $\hat{\rho}$ using π and φ.

   (c) If X is reversible with respect to a probability measure say π on E, deduce from the previous question an expression of ν using π and φ. Check that $\sum_{z\in E} \pi(z)\varphi(z) = \sum_{z\in E} \pi(z)\varphi(z)^2$.

4. (a) Compute the following limits for $x \in E_*$ and $A \subset E_*$:

$$\lim_{n\to+\infty} \lim_{p\to+\infty} \mathbb{P}_x(X_n \in A | \tau > n+p),$$
$$\lim_{p\to+\infty} \lim_{n\to+\infty} \mathbb{P}_x(X_n \in A | \tau > n+p).$$

   Check that those two limits are equal.

   (b) Compute for $x \in E_*$ and $A \subset E_*$:

$$\lim_{n\to+\infty} \mathbb{P}_x(X_n \in A | \tau > n).$$

   And check if this limit is equal to the ones of the previous question.

△

Vocabulary (english/*français*) : bounded below = *minoré*.

Solutions

I Preliminaries

Exercice 1 1. As the Markov chain is irreducible on a finite state space, it is recurrent. Thus for any initial condition X_0, we get that a.s. τ is finite. Therefore, the problem is not well posed as the conditionning event $\{\tau = +\infty\}$ is of zero probability.

2. (a) For $x \in E_*$, we have $\tau > 0$ and thus :

$$P_* g(x) = \sum_{y \in E} P(x,y) g(y) \mathbf{1}_{\{y \in E_*\}} = \mathbb{E}_x \left[g(X_1) \mathbf{1}_{\{\tau > 1\}} \right],$$

as under $\mathbb{P}_x$, $\{\tau > 1\} = \{X_1 \in E_*\}$.

(b) We prove the relation by induction. Using the Markov property, we get for $x \in E_*$:

$$P_* h_n(x) = \mathbb{E}_x \left[h_n(X_1) \mathbf{1}_{\{\tau>1\}} \right] = \mathbb{E}_x \left[\mathbb{P}_{X_1}(\tau > n) \mathbf{1}_{\{\tau>1\}} \right] = \mathbb{P}_x(\tau > n+1) = h_{n+1}(x).$$

(c) We prove the relation by induction. We set $g_n(x) = {P_*}^n g(x)$. Question 1 gives that $g_1(x) = \mathbb{E}_x \left[g(X_1) \mathbf{1}_{\{\tau>1\}} \right]$. We assume the relation $g_k(x) = \mathbb{E}_x \left[g(X_k) \mathbf{1}_{\{\tau>k\}} \right]$ is true for $k \leq n$. Using the Markov property at time 1, we get :

$$g_{n+1}(x) = P_* g_n(x) = \mathbb{E}_x \left[\mathbb{E}_{X_1} \left[g(X_n) \mathbf{1}_{\{\tau>n\}} \right] \mathbf{1}_{\{\tau>1\}} \right] = \mathbb{E}_x \left[g(X_{n+1}) \mathbf{1}_{\{\tau>n+1\}} \right].$$

3. (a) We deduce from the previous question with $g = \mathbf{1}$ that, for $n \in \mathbb{N}$, we have :

$$\mathbb{P}(\tau > n) = \sum_{x \in E_*} \nu(x) \mathbb{E}_x \left[\mathbf{1}_{\{\tau>n\}} \right] = \nu {P_*}^n \mathbf{1} = \lambda^n \nu \mathbf{1} = \lambda^n.$$

(b) We get that τ has a geometric distribution with parameter $(1-\lambda)$ if $\lambda < 1$ and that $\mathbb{P}(\tau = +\infty) = 1$ if $\lambda = 1$. According to question 1, τ is a.s. finite for all initial random condition X_0. We deduce that $\lambda < 1$.

4. (a) As τ is finite, we deduce that $M_n = 0$ on $\{n \geq \tau\}$. Thus M converges a.s. towards 0. M is a martingale according to question I.4 as φ is an eigen-vector of P_* associated with the eigen-value λ. The martingale is not uniformly integrable if $\mathbb{P}(X_0 \in E_*) > 0$ as $\mathbb{E}[\varphi(X_0)] > 0 = \mathbb{E}[M_\infty]$. If $\mathbb{P}(X_0 \in E_*) = 0$, then the martingale is constant equal to 0 and is thus uniformly integrable.

(b) We get $P_*^n = \lambda^n (\varphi \nu + R_n)$ with $\lim_{n \to +\infty} R_n(x,y) = 0$ for all $x, y \in E_*$. As ν is a probability measure on E_*, we get :

$$h_n = {P_*}^n \mathbf{1}_{E_*} = \lambda^n (\varphi + r_n),$$

with $r_n = R_n \mathbf{1}$. As E_* is finite, we deduce that $\lim_{n \to +\infty} \sup_{x \in E_*} |r_n(x)| = 0$.

(c) Let ν_0 be the distribution of X_0. As E is finite, we get $\lim_{p \to +\infty} \lambda^{-p} \nu_0 h_p = \nu_0 \varphi > 0$. In particular, for p_0 big enough, the sequence $(\lambda^{-p} \nu_0 h_p, p \geq 0)$ is bounded below by a positive constant. Thus, for all $x \in E$, the sequence $(h_p(x)/\nu_0 h_{n+p}, p \geq p_0)$ converges towards $\lambda^{-n} \varphi(x)/\nu_0 \varphi$. Furthermore, the sequences are uniformly bounded below in x as E is finite.

II Q-process

1. We have :

$$\begin{aligned}\mathbb{P}((X_0,\dots,X_n)\in A|\tau>n+p) &= \frac{\mathbb{E}\left[\mathbf{1}_{\{(X_0,\dots,X_n)\in A\}}\mathbf{1}_{\{\tau>n+p\}}\right]}{\nu_0 h_{n+p}}\\ &= \frac{\mathbb{E}\left[\mathbf{1}_{\{(X_0,\dots,X_n)\in A\}}\mathbf{1}_{\{\tau>n\}}\mathbb{P}_{X_n}(\tau>p)\right]}{\nu_0 h_{n+p}}\\ &= \mathbb{E}\left[\mathbf{1}_{\{(X_0,\dots,X_n)\in A\}}\mathbf{1}_{\{\tau>n\}}\frac{h_p(X_n)}{\nu_0 h_{p+n}}\right].\end{aligned}$$

where we conditioned with respect to $(X_0,\dots,X_n)$ in the second equality, and used the Markov property in the second. We deduce from the previous question that the sequence $(h_p(X_n)/\mathbb{E}[h_{p+n}(X_0)], p\geq p_0)$ is and that it converges $\mathbb{P}$-a.s. towards $\lambda^{-n}\varphi(X_n)/\nu_0\varphi = M_n/\mathbb{E}[M_0]=M_n/\mathbb{E}[M_n]$. The dominated convergence theorem ensures that :

$$\begin{aligned}\lim_{p\to+\infty}\mathbb{E}\left[\mathbf{1}_{\{(X_0,\dots,X_n)\in A\}}\mathbf{1}_{\{\tau>n\}}\frac{h_p(X_n)}{\nu_0 h_{p+n}}\right] &= \mathbb{E}\left[\mathbf{1}_{\{(X_0,\dots,X_n)\in A\}}\mathbf{1}_{\{\tau>n\}}\frac{M_n}{\mathbb{E}[M_n]}\right]\\ &= \mathbb{P}((Y_0,\dots,Y_n)\in A).\end{aligned}$$

2. (a) Let $n\in\mathbb{N}$. Let f be a function defined on E_* and g a function defined on ${E_*}^{n+1}$. We have :

$$\begin{aligned}\mathbb{E}\left[f(Y_{n+1})g(Y_0,\dots,Y_n)\right] &= \mathbb{E}\left[f(X_{n+1})\frac{M_{n+1}}{\mathbb{E}[M_{n+1}]}g(X_0,\dots,X_n)\right]\\ &= \frac{1}{\mathbb{E}[M_0]}\,\mathbb{E}\left[\mathbf{1}_{\{\tau>n\}}g(X_0,\dots,X_n)\lambda^{-n-1}\mathbf{1}_{\{X_{n+1}\in E_*\}}f(X_{n+1})\right]\\ &= \frac{1}{\mathbb{E}[M_0]}\,\mathbb{E}\left[g(X_0,\dots,X_n)M_nF(X_n)\right]\\ &= \mathbb{E}\left[g(Y_0,\dots,Y_n)F(Y_n)\right],\end{aligned}$$

where we used the Markov property for the last but one inequality with

$$F(x)=\frac{1}{\lambda\varphi(x)}\,\mathbb{E}_x\left[\mathbf{1}_{\{X_1\in E_*\}}\varphi(X_1)f(X_1)\right]=\frac{1}{\lambda\varphi(x)}\,P_*(\varphi f)(x).$$

(b) We deduce from the previous question that :

$$\mathbb{E}\left[f(Y_{n+1})|Y_0,\dots,Y_n\right]=F(Y_n).$$

The sequence $(Y_n, n\in\mathbb{N})$ is thus a Markov chain. Its transition matrix Q is given by

$$Q(x,y)=\frac{\varphi(y)}{\lambda\varphi(x)}P_*(x,y),\quad x,y\in E_*.$$

(c) We have $Q(x,y)>0$ if and only if $P_*(x,y)>0$. Since there exists $n>0$ such that $P_*^n(x,y)>0$ for all $x,y\in E_*$, we deduce that $Q^n(x,y)>0$ for all $x,y\in E_*$. Therefore the Markov chain Y is irreducible and aperiodic. Since E_* is finite, we deduce it has a unique invariant probability measure.

3. (a) We have :

$$Q^2(x,y) = \sum_{z\in E_*} \frac{1}{\lambda^2\varphi(x)}\varphi(y)P_*(x,z)P_*(z,y) = \frac{\varphi(y)}{\lambda^2\varphi(x)}{P_*}^2(x,y).$$

By itereation, we obtain :

$$Q^n(x,y) = \frac{\varphi(y)}{\lambda^n\varphi(x)}{P_*}^n(x,y).$$

We get :

$$\lim_{n\to+\infty} Q^n(x,y) = \varphi(y)\nu(y).$$

As Y is aperiodic, we deduce from the ergodic theorem that $\lim_{n\to+\infty} Q^n(x,y) = \rho(y)$ and thus $\rho(y) = \varphi(y)\nu(y)$ for $y \in E_*$.

Since ρ is a probability measure, we notice that $\nu\varphi = 1$.

(b) If P is reversible with respect to π, then for $x, y \in E_*$, we have $\pi(x)P_*(x,y) = \pi(y)P_*(y,x)$. We deduce that :

$$Q(x,y) = \frac{\varphi(y)}{\lambda\varphi(x)}P_*(x,y) = \frac{\pi(y)\varphi(y)}{\lambda\pi(x)\varphi(x)}P_*(y,x) = \frac{\pi(y)\varphi(y)^2}{\pi(x)\varphi(x)^2}Q(y,x).$$

We deduce also that :

$$\pi(x)\varphi(x)^2Q(x,y) = \pi(y)\varphi(y)^2Q(y,x).$$

Therefore, Q is reversible with respect to the probability measure $\hat{\rho}$, with :

$$\hat{\rho}(x) = \frac{\pi(x)\varphi(x)^2}{\sum_{z\in E}\pi(z)\varphi(z)^2}.$$

(c) The probability measure $\hat{\rho}$ is also invariant for Q. By uniqueness, it is equal to ρ. We deduce that :

$$\nu(x) = \frac{\pi(x)\varphi(x)}{\pi(\varphi^2)}, \quad x \in E_*.$$

As ν is a probability measure, we deduce that $\pi\varphi = \pi(\varphi^2)$.

4. We have :

$$\lim_{n\to+\infty}\lim_{p\to+\infty} \mathbb{P}_x(X_n \in A|\tau > n+p) = \lim_{n\to+\infty} \mathbb{P}(Y_n \in A) = \rho(A).$$

With $g_p(x) = \lambda^{-p}\mathbf{1}_A(x)h_p(x)$, we have :

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{P}_x(X_n \in A|\tau > n+p) &= \lim_{n\to+\infty} \frac{1}{h_{n+p}(x)}\mathbb{E}_x\left[\mathbf{1}_A(X_n)h_p(X_n)\mathbf{1}_{\{\tau>n\}}\right] \\
&= \lim_{n\to+\infty} \frac{\lambda^p}{h_{n+p}(x)}P_*^n g_p(x) \\
&= \nu g_p.
\end{aligned}$$

As $\lim_{p\to+\infty} g_p = \mathbf{1}_A\varphi$, we deduce that :

$$\lim_{p\to+\infty}\lim_{n\to+\infty}\mathbb{P}_x(X_n\in A|\tau>n+p)=\nu(\varphi\mathbf{1}_A)=\rho(A).$$

We get :

$$\lim_{n\to+\infty}\mathbb{P}_x(X_n\in A|\tau>n)=\lim_{n\to+\infty}\frac{P_*^n(\mathbf{1}_A)(x)}{P_*^n\mathbf{1}}=\nu(A).$$

The measure ν is called the quasi-stationary distribution of X_n in E_*.
We deduce that, for all $A\subset E_*$:

$$\lim_{n\to+\infty}\lim_{p\to+\infty}\mathbb{P}_x(X_n\in A|\tau>n+p)=\lim_{p\to+\infty}\lim_{n\to+\infty}\mathbb{P}_x(X_n\in A|\tau>n+p).$$

But, unless $\varphi=1$, this quantity is not equal to $\lim_{n\to+\infty}\mathbb{P}_x(X_n\in A|\tau>n)$ for all $A\subset E_*$.