ENPC Processus Stochastiques
Lundi 30 Janvier 2017 (2h30)

Vocabulary (english/*français*) : distribution =*distribution, loi* ; offspring distribution =*loi de reproduction* ; generating function =*fonction génératrice* ; positive = *strictement positif* ; Brownian bridge = *pont brownien* ; $(0,1]$ = *]0,1]*.

Exercice 1 (Brownian bridge and Wiener integral)**.** Let $(B_t, t \geq 0)$ be a Brownian motion. We recall that $(W_t = B_t - tB_1, t \in [0,1])$ is distributed as the Brownian bridge. We set $X_0 = 0$ and for $t \in (0,1]$:

$$X_t = t \int_t^1 \frac{dB_r}{r}.$$

1. Compute $\mathbb{E}[W_s W_t]$ for $s, t \in [0,1]$.
2. Check the random variables $(X_t, t \in [0,1])$ are well defined with Gaussian distributions.
3. Prove that $(X_t, t \in [0,1])$ is a Brownian bridge.

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Exercice 2 (Galton-Watson process)**.** A Galton-Watson process is an elementary stochastic model for population evolution. We assume that each individual at generation n dies out at generation $n+1$ and produces a random number of children who will compose the population at generation $n+1$. We also assume that the individuals reproduce independently from each other and that the number of its children is random and distributed according to a random variable X whose distribution on $\mathbb{N}$, say $p = (p_k = \mathbb{P}(X = k), k \in \mathbb{N})$, which does not depend neither on the individual nor on its generation, is called offspring distribution. Eventually, we assume there is only one initial individual at generation 0.

More precisely, let $(X_{n,i}, n \in \mathbb{N}^*, i \in \mathbb{N}^*)$ be independent random variables distributed as X. The random variable $X_{n+1,i}$ represents the number of children of individual i living at generation n (if it exists). For $n \in \mathbb{N}$, let Z_n be the size of the population at generation n. We have :

$$Z_0 = 1 \quad \text{and, for } n \in \mathbb{N}, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n+1,i}\,,$$

with the convention that $\sum_\emptyset = 0$. We shall study the extinction time of the population defined by $T = \inf\{n \in \mathbb{N}; Z_n = 0\}$, with the convention that $\inf \emptyset = +\infty$.

We assume the offspring distribution p has its first moment that is $\mu = \mathbb{E}[X] < +\infty$. We consider the generating function of X : $\varphi(x) = \mathbb{E}[x^X]$ for $x \in [0,1]$. We recall that φ is continuous.

I A Markov chain

1. Prove that $Z = (Z_n, n \in \mathbb{N})$ is a Markov chain.
2. Give the distribution of T in the following cases : (i) $p_0 = 0$; (ii) $p_0 + p_1 = 1$ and $p_0 > 0$.

From now on we assume that $0 < p_0 \leq p_0 + p_1 < 1$.

3. Classify the states of the Markov chain and give the invariant probability measures.
4. Prove that Z converges a.s. and identify the possible limits.

II Two martingales

We assume that $0 < p_0 \leq p_0 + p_1 < 1$. We consider the process $M = (M_n, n \in \mathbb{N})$ with $M_n = Z_n/\mu^n$.

1. Prove that M is a martingale which converges to a limit say M_∞.
2. If $\mu \leq 1$, then proves that T is a.s. finite.

From now on we assume that $\mu > 1$. Elementary arguments give that there exists a unique $\rho \in (0,1)$ such that $\varphi(\rho) = \rho$. We consider the process $Q = (Q_n, n \in \mathbb{N})$ with $Q_n = \rho^{Z_n}$.

3. Prove that Q is a martingale which converges to a limit say Q_∞.
4. Prove that $\mathbb{P}(T < +\infty) = \rho$.

III The L^2 case

We assume that $0 < p_0 \leq p_0 + p_1 < 1$ and $\mu > 1$. Let φ_n be the generating function of Z_n : $\varphi_n(x) = \mathbb{E}[x^{Z_n}]$ for $x \in [0,1]$, so that $\varphi_1 = \varphi$ and $\varphi_0(x) = x$ for all $x \in [0,1]$.

1. Prove that $\varphi_n = \varphi_{n-1} \circ \varphi$ for all $n \in \mathbb{N}^*$. Deduce that $\varphi_n = \varphi \circ \varphi_{n-1}$ for all $n \in \mathbb{N}^*$.
2. Prove that $\mathbb{E}\left[\mathrm{e}^{-uM_n}\right] = \varphi\left(\mathbb{E}\left[\mathrm{e}^{-uM_{n-1}/\mu}\right]\right)$ for $u \geq 0$ and $n \in \mathbb{N}^*$. And deduce that for $u \geq 0$:
$$\mathbb{E}\left[\mathrm{e}^{-uM_\infty}\right] = \varphi\left(\mathbb{E}\left[\mathrm{e}^{-uM_\infty/\mu}\right]\right).$$
3. Using the previous question, prove that $\mathbb{P}(M_\infty = 0) = \varphi(\mathbb{P}(M_\infty = 0))$ and deduce that $\mathbb{P}(M_\infty = 0) \in \{\rho, 1\}$.

We assume that the offspring distribution has a second moment, that is $\sigma^2 = \mathrm{Var}(X) < \infty$.

4. Compute $\mathbb{E}\left[(Z_n - \mu Z_{n-1})^2 \middle| Z_{n-1}\right]$ and deduce that :
$$\mathbb{E}\left[(Z_n - \mu^n)^2\right] = \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1}\, \sigma^2.$$
5. Deduce that M is bounded in L^2 and the value of $\mathbb{E}[M_\infty]$. Prove that $\mathbb{P}(M_\infty = 0) = \rho$.
6. Prove that $\mathbb{P}(M_\infty > 0 | T = +\infty) = 1$. Therefore, either the population becomes extinct in finite time or it grows exponentially as fast as μ^n (times a positive random variable).

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