Stochastic Process (ENPC)
Monday, 22nd of January 2018 (2h30)

Vocabulary (english/*français*) : distribution =*distribution, loi* ; positive = *strictement positif* ; $[0, 1) =$ *[0,1[*.

We write $\mathbb{N}^* = \mathbb{Z} \cap [1, +\infty[$ and $\mathbb{N} = \mathbb{N}^* \bigcup \{0\}$. We use the convention $\inf \emptyset = \infty$.

Exercice 1 (Distribution of the maximum of a random walk)**.** Let X be a $\mathbb{Z}$-valued random variable and $(X_n,\, n \in \mathbb{N}^*)$ be independent random variables distributed as X. We assume that $\mathbb{P}(X \geq 2) = 0$, $p = \mathbb{P}(X = 1) > 0$, $\mathbb{E}[|X|] < +\infty$ and $m = \mathbb{E}[X] < 0$. We consider the random walk $S = (S_n,\, n \in \mathbb{N})$ defined by $S_0 = 0$ and $S_{n+1} = S_n + X_{n+1}$ for $n \in \mathbb{N}$, and its natural filtration $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N})$. Our aim is to study the distribution of the global maximum of S :

$$M = \sup_{n \in \mathbb{N}} S_n.$$

I Preliminaries

1. Give the limit of $(S_n/n, n \in \mathbb{N}^*)$ as n goes to infinity. Deduce that $\mathbb{P}(M < +\infty) = 1$.
2. Check that S is a Markov chain. Is it irreducible, transient, recurrent ?
3. Check that the first hitting time of the level $k \in \mathbb{N}$ for the random walk S :

$$\tau_k = \inf \{n \geq 0;\, S_n \geq k\}$$

   is a stopping time with respect to filtration $\mathbb{F}$.
4. Is the random time $\eta = \inf\{n \geq 0;\, S_n = M\}$ a stopping time ?

For $\lambda \geq 0$, we set :

$$\varphi(\lambda) = \mathbb{E}\left[e^{\lambda X}\right].$$

5. Prove that φ is of class $\mathcal{C}^\infty$ over $(0, +\infty)$, and check it is strictly convex. Prove there exists a unique $\lambda_0 > 0$ such that $\varphi(\lambda_0) = 1$. Check that $\varphi'(\lambda_0) > 0$.

II An auxiliary Markov chain

We set $q_k = e^{\lambda_0 k}\, \mathbb{P}(X_n = k)$ for $k \in \mathbb{Z}$, so that according to the previous question $q = (q_k, k \in \mathbb{Z})$ is a probability distribution. Let $(Y_n, n \in \mathbb{N}^*)$ be independent $\mathbb{Z}$-valued random variables with probability distribution q. We introduce the random walk $V = (V_n,\, n \in \mathbb{N})$ defined by $V_0 = 0$ and $V_{n+1} = V_n + Y_{n+1}$ for $n \in \mathbb{N}$.

1. Prove that for all measurable bounded or non-negative function f, we have :

$$\mathbb{E}\left[f(Y_1, \ldots, Y_n)\, e^{-\lambda_0 V_n}\right] = \mathbb{E}\left[f(X_1, \ldots, X_n)\right].$$

2. Check that Y_1 is integrable and that $\mathbb{E}[Y_1] > 0$.

Let $\mathbb{G} = (\mathcal{G}_n, n \in \mathbb{N})$ be the natural filtration of V and $\rho_k = \inf \{n \geq 0;\, V_n \geq k\}$ be the first hitting time of the level $k \in \mathbb{N}$ for the random walk V.

3. Give the limit of $(V_n/n, n \in \mathbb{N}^*)$ as n goes to infinity. Deduce that ρ_k is a.s. finite and compute V_{ρ_k}.

4. Prove that $\left(\mathrm{e}^{-\lambda_0 V_n}, n \geq 1\right)$ is a martingale (with respect to the filtration $\mathbb{G}$). Compute :

$$\mathbb{E}\left[\mathbf{1}_{\rho_k \leq n}\, \mathrm{e}^{-\lambda_0 V_n} \mid \mathcal{G}_{\rho_k \wedge n}\right].$$

5. Deduce that $\mathbb{P}(\tau_k \leq n) = \mathrm{e}^{-\lambda_0 k}\, \mathbb{P}(\rho_k \leq n)$, and then compute $\mathbb{P}(\tau_k < \infty)$.
6. Prove that M has a shifted geometric distribution, that is $\mathbb{P}(M = k) = \alpha_0(1-\alpha_0)^k$ for $k \in \mathbb{N}$. Write α_0 using λ_0.

III The simple random walk

We consider the simple random walk with negative drift : $\mathbb{P}(X = -1) = 1-p$ and $p \in (0, 1/2)$.

1. Prove that $\mathbb{P}(S_n < 0;\, \forall n \geq 1) = (1-p)\mathbb{P}(M = 0)$.
2. Compute α_0 and deduce that the probability, $\mathbb{P}(S_n < 0; \forall n \geq 1)$, for the simple random walk S with negative drift to never come back to $\{0\}$ is equal to $1 - 2p$.

△

Exercice 2 (Some Brownian martingale)**.** Let $B = (B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion and $\mathbb{F} = (\mathcal{F}_t, t \in \mathbb{R}_+)$ its natural filtration. For $t \in [0, 1)$, we set :

$$X_t = \frac{1}{\sqrt{1-t}}\, \mathrm{e}^{-B_t^2/2(1-t)}.$$

We recall that the density of the standard Gaussian distribution $\mathcal{N}(0,1)$ is given by :

$$f(x) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}, \quad x \in \mathbb{R},$$

and that if G is a standard Gaussian random variable then, for all $\lambda \in \mathbb{C}$, we have :

$$\mathbb{E}\left[\mathrm{e}^{-\lambda G}\right] = \mathrm{e}^{\lambda^2/2}.$$

1. Let $0 \leq t \leq s + t < 1$. Compute $\mathbb{E}[X_{t+s} | \mathcal{F}_t]$.
2. Deduce that $(X_t, t \in [0, 1))$ is a continuous martingale. And deduce $\mathbb{E}[X_t]$ for all $t \in [0, 1)$.
3. Prove that a.s. $\lim_{t \uparrow 1} X_t = 0$.
4. Deduce that
$$\mathbb{E}\left[\sup_{t \in [0,1)} X_t\right] = +\infty.$$

△

Correction

I Preliminaries

Exercice 1 1. By the law of large number we have that a.s. $\lim_{n\to+\infty} S_n/n = m < 0$. This implies that a.s. $\lim_{n\to+\infty} S_n = -\infty$ and thus that a.s. $M < +\infty$.

2. The process S is a stochastic dynamical system and thus a Markov chain. Since $\mathbb{P}(X = 1) > 0$ and since there exists $k \in \mathbb{N}^*$ such that $\mathbb{P}(X = -k) > 0$ (as $m < 0$), we deduce that S is irreducible. Since $\lim_{n\to+\infty} S_n = -\infty$, we get that S is transient.
3. We have $\{\tau_k > n\} = \{S_j < k,\, 0 \leq j \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and thus τ_k is a stopping time with respect to $\mathbb{F}$.
4. The random time η is not a stopping time because $\{\eta = 0\} = \bigcap_{n\in\mathbb{N}^*}\{S_n < 0\}$ doesn't belong to $\mathcal{F}_0$.
5. Notice that $|X|^n \,\mathrm{e}^{\lambda X} \leq \max(\mathrm{e}^\lambda, \sup_{x\geq 0} x^n \,\mathrm{e}^{-\lambda x})$, so the functions $\varphi_n(\lambda) = \mathbb{E}\left[X^n \,\mathrm{e}^{\lambda X}\right]$, for $\lambda \in (0, +\infty)$, are well defined for $n \in \mathbb{N}$ and locally bounded. Using Fubini, we get that $\int_a^b \varphi_{n+1}(r)\, dr = \varphi_n(b) - \varphi_n(a)$ for all $0 < a \leq b < +\infty$. This implies that φ is of class $\mathcal{C}^\infty$ over $(0, +\infty)$ with n-th derivative φ_n. Since $X^2 \,\mathrm{e}^{\lambda X}$ is non-negative and positive with positive probability, we deduce that $\varphi_2 > 0$ and thus φ is strictly convex on $(0, \infty)$. By dominated convergence, we get that φ and φ' are continuous over $[0, +\infty)$, and we have $\varphi(0) = 1$, $\varphi(+\infty) = +\infty$, $\varphi'(0) = m < 0$. The strict convexity of φ implies the existence of a unique root of $\varphi(\lambda) = 1$ on $(0, +\infty)$, say λ_0. Furthermore, we have that $\varphi'(\lambda_0) > 0$.

II An auxiliary Markov chain

1. We have that :
$$\mathbb{E}\left[\mathbf{1}_{\{Y_1=k_1,\ldots,Y_n=k_n\}} \,\mathrm{e}^{-\lambda_0 V_n}\right] = \left(\prod_{j=1}^n q_{k_j}\right) \mathrm{e}^{-\lambda_0 \sum_{j=1}^n k_j} = \prod_{j=1}^n p_{k_j} = \mathbb{E}\left[\mathbf{1}_{\{X_1=k_1,\ldots,X_n=k_n\}}\right].$$
This proves the result as the random variables X_ℓ and Y_ℓ are discrete.
2. We deduce from the previous question that $\mathbb{E}[|Y_1|] = \mathbb{E}[|X| \,\mathrm{e}^{\lambda_0 X}] \leq \mathrm{e}^{\lambda_0} \mathbb{E}[|X|] < +\infty$. Thus Y_1 is integrable and we have $\mathbb{E}[Y_1] = \mathbb{E}[X \,\mathrm{e}^{\lambda_0 X}] = \varphi'(\lambda_0) > 0$.
3. By the law of large number we have that a.s. $\lim_{n\to+\infty} V_n/n = \mathbb{E}[Y_1] > 0$. This implies that a.s. $\lim_{n\to+\infty} V_n = +\infty$ and thus that a.s. $\rho_k < +\infty$.
4. Set $N_n = \mathrm{e}^{-\lambda_0 V_n}$ for $n \in \mathbb{N}$. Notice that the process $N = (N_n, n \in \mathbb{N})$ is $\mathbb{G}$-adapted and non-negative. We also have :
$$\mathbb{E}\left[N_{n+1}|\mathcal{F}_n\right] = N_n \mathbb{E}\left[\mathrm{e}^{-\lambda_0 Y_{n+1}} |\mathcal{G}_n\right] = N_n \mathbb{E}\left[\mathrm{e}^{-\lambda_0 Y_{n+1}}\right] = N_n,$$
where we used that N_n is $\mathcal{G}_n$-measurable for the first equality, that Y_{n+1} is independent of $\mathcal{G}_n$ for the second and for the third that, by definition of the distribution of Y_{n+1}, we have $\mathbb{E}\left[\mathrm{e}^{-\lambda_0 Y_{n+1}}\right] = \mathbb{E}\left[\mathrm{e}^{-\lambda_0 X_{n+1} + \lambda_0 X_{n+1}}\right] = 1$. We deduce that $\mathbb{E}[N_{n+1}] = \mathbb{E}[N_0] = 1$ and thus that N_{n+1} is integrable. We have proven that N is a non-negative martingale.
Similarly to Question I.3, we get that ρ_k is a stopping time with respect to $\mathbb{G}$. Therefore $\{\rho_k \leq n\}$ belongs to $\mathcal{G}_{\rho_k \wedge n}$. By the stopping time theorem, since $\rho_k \wedge n$ is a bounded stopping time, we get :
$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{\rho_k\leq n\}} \,\mathrm{e}^{-\lambda_0 V_n} \mid \mathcal{G}_{\rho_k\wedge n}\right] &= \mathbf{1}_{\{\rho_k\leq n\}} \mathbb{E}\left[\mathrm{e}^{-\lambda_0 V_n} \mid \mathcal{G}_{\rho_k\wedge n}\right]\\ &= \mathbf{1}_{\{\rho_k\leq n\}} \,\mathrm{e}^{-\lambda_0 V_{\rho_k\wedge n}}\\ &= \mathbf{1}_{\{\rho_k\leq n\}} \,\mathrm{e}^{-\lambda_0 k},\end{aligned}$$

where, for the last equality, we used that $V_{\rho_k \wedge n} = V_{\rho_k} = k$ on $\{\rho_k \leq n\}$.

5. We have thanks to the previous question and Question II.1 that :

$$\mathbb{P}(\tau_k \leq n) = \mathbb{E}\left[\mathbf{1}_{\{\rho_k \leq n\}} \mathrm{e}^{-\lambda_0 V_n}\right] = \mathbb{E}\left[\mathbf{1}_{\{\rho_k \leq n\}} \mathrm{e}^{-\lambda_0 k}\right] = \mathrm{e}^{-\lambda_0 k} \mathbb{P}(\rho_k \leq n).$$

Letting n goes to infinity, and as ρ_k is a.s. finite, we get that $\mathbb{P}(\tau_k < \infty) = \mathrm{e}^{-\lambda_0 k}$.

6. Let $k \in \mathbb{N}$. As $\{M \geq k\} = \{\tau_k < \infty\}$, we deduce that $\mathbb{P}(M \geq k) = (1 - \alpha_0)^k$ with $\alpha_0 = 1 - \mathrm{e}^{-\lambda_0}$. We deduce that $\mathbb{P}(M = k) = \mathbb{P}(M \geq k) - \mathbb{P}(M \geq k+1) = \alpha_0(1 - \alpha_0)^k$.

III The simple random walk

1. Let $M' = \max_{n \in \mathbb{N}^*} S_n - X_1$. Notice that M' is distributed as M and is independent of X_1. Since $\{S_n < 0;\, \forall n \geq 1\} = \{X_1 = -1\} \bigcap \{M' = 0\}$, we deduce that :

$$\mathbb{P}(S_n < 0;\, \forall n \geq 1) = \mathbb{P}(X_1 = -1)\mathbb{P}(M' = 0) = (1-p)\mathbb{P}(M = 0).$$

2. We have $\varphi(\lambda) = p\,\mathrm{e}^{\lambda} + (1-p)\,\mathrm{e}^{-\lambda}$. We get that $\mathrm{e}^{-\lambda_0} \in (0,1)$ is a root of $x \mapsto (1-p)x^2 - x + p = 0$. This gives $\mathrm{e}^{-\lambda_0} = p/(1-p)$. We deduce that $\alpha_0 = 1 - \mathrm{e}^{-\lambda_0} = (1-2p)/(1-p)$. This gives :

$$(1-p)\mathbb{P}(M = 0) = (1-p)\alpha_0 = 1 - 2p.$$

Exercice 2 1. The process $X = (X_t, t \in [0,1))$ is $\mathbb{F}$-adapted. Since X_{t+s} is non-negative, we can compute its conditional expectation. Set $\eta^2 = 1 - t - s$. Using that $B_{t+s} - B_t$ is independent of $\mathcal{F}_t$ and distributed as $\sqrt{s}G$, with G a standard Gaussian random variable, we get that :

$$\mathbb{E}[X_{t+s}|\mathcal{F}_t] = \frac{1}{\eta}\,\mathbb{E}\left[\mathrm{e}^{-(B_{t+s}-B_t+B_t)^2/2\eta^2} \,\middle|\, \mathcal{F}_t\right] = \frac{1}{\eta}\,\mathrm{e}^{-B_t^2/2\eta^2} H(B_t),$$

with

$$H(x) = \mathbb{E}\left[\mathrm{e}^{-[(B_{t+s}-B_t+x)^2 - x^2]/2\eta^2}\right] = \mathbb{E}\left[\mathrm{e}^{-[(\sqrt{s}G+x)^2 - x^2]/2\eta^2}\right].$$

We get with $z = y\sqrt{1-t}/\eta$:

$$\begin{aligned}
H(x) &= \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-[(\sqrt{s}y+x)^2 - x^2]/2\eta^2 - y^2/2}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-y^2(s+\eta^2)/2\eta^2 - \sqrt{s}xy/\eta^2}\, dy \\
&= \frac{\eta}{\sqrt{1-t}} \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-z^2/2 - \sqrt{s}xz/\eta\sqrt{1-t}}\, dz \\
&= \frac{\eta}{\sqrt{1-t}} \mathbb{E}\left[\mathrm{e}^{-\sqrt{s}xG/\eta\sqrt{1-t}}\right] \\
&= \frac{\eta}{\sqrt{1-t}}\, \mathrm{e}^{sx^2/2\eta^2(1-t)}.
\end{aligned}$$

We get :

$$\frac{1}{\eta}\,\mathrm{e}^{-x^2/2\eta^2} H(x) = \frac{1}{\sqrt{1-t}}\,\mathrm{e}^{-x^2/2(1-t)}.$$

2. We deduce from the previous question that $\mathbb{E}[X_{t+s}|\mathcal{F}_t] = X_t$. This gives, with $t = 0$, that $\mathbb{E}[X_s] = 1$ for all $s \in [0,1)$. Thus, as X_t is non-negative, we have that $X_t \in L^1$ for all $t \in [0,1)$. Since the process X is $\mathbb{F}$-adapted, we deduce that X is a martingale. Since B is continuous, we get that X is continuous.

3. As $B_1 \neq 0$ a.s. and B is continuous, we deduce there exists $\varepsilon \in (0,1)$ (random), such that $B_t \neq 0$ for all $t \in (1-\varepsilon, 1]$. Then use that $\lim_{x\to+\infty} x\,\mathrm{e}^{-x^2/2} = 0$ to deduce that a.s. $\lim_{t\uparrow 1} X_t = 0$.
4. Consider $M = (M_n = X_{1-1/(n+1)}, n \in \mathbb{N})$, which is a non-negative martingale with $M_0 = 1$ and a.s. $\lim_{n\to\infty} M_n = 0$. If $\mathbb{E}[\sup_{n\in\mathbb{N}} M_n] < +\infty$, then the martingale would also converge in L^1 by dominated convergence. Since this not the case, we get that $\mathbb{E}[\sup_{n\in\mathbb{N}} M_n] = +\infty$. Then use that $\sup_{t\in[0,1)} X_t \geq \sup_{n\in\mathbb{N}} M_n$ to conclude.