I Conditional expectation

Vocabulary (english/*français*): σ-field = *tribu*; nested = *emboité(es)*; $(0, 1) =$ *]0,1[*.

Exercise I.1 (Nested σ-fields). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a real random variable non-negative or integrable. Let $\mathcal{H}$ and $\mathcal{G}$ be two σ-fields such that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$. Prove that:

$$\mathbb{E}\left[\mathbb{E}[X|\mathcal{G}] \,|\, \mathcal{H}\right] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}] \,|\, \mathcal{G}\right] = \mathbb{E}[X|\mathcal{H}].$$

△

Exercise I.2 (Indicators). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B \in \mathcal{F}$ be two events. Describe $\sigma(\mathbf{1}_B)$ and then compute $\mathbb{E}\left[\mathbf{1}_A|\mathbf{1}_B\right]$.

△

Exercise I.3 (Random walk). Let $(X_n, n \in \mathbb{N}^*)$ be identically distributed independent real random variables non-negative or integrable. Let $S_n = \sum_{k=1}^{n} X_k$ for $k \in \mathbb{N}^*$. Compute $\mathbb{E}[X_1|S_2]$ and deduce $\mathbb{E}[X_1|S_n]$ for $n \geq 2$.

△

Exercise I.4 (Symmetric random variable). Let X be a real random variable which is symmetric, that is X and $-X$ have the same distribution. Compute $\mathbb{E}\left[X|X^2\right]$.

△

Exercise I.5 (X conditioned on $|X|$). Let X be an absolutely continuous real random variable with density f. Compute $\mathbb{E}\left[X|\,|X|\right]$ and $\mathbb{E}\left[X|\,X^2\right]$.

△

Exercise I.6 (Geometric distribution). Let $(X_n, n \in \mathbb{N}^*)$ be independent Bernoulli random variables with parameter $p \in (0, 1)$. Let $T = \inf\{n \geq 1;\ X_n = 1\}$ with the convention that $\inf \emptyset = +\infty$. (Notice that T is geometric with parameter p.) Compute $\mathbb{E}[T|\,X_1]$ and deduce $\mathbb{E}[T]$.

△

Exercise I.7 (Variance). Let X be a real random variable such that $\mathbb{E}[X^2] < +\infty$. Let $\mathcal{F}$ be a σ-field. Check that $\mathbb{E}[X|\mathcal{F}]^2$ is integrable and prove that:

$$\mathrm{Var}(\mathbb{E}[X|\mathcal{F}]) \leq \mathrm{Var}(X).$$

$\triangle$

Exercise I.8 (Equality). Let X and Y be two integrable real random variables such that a.s. $\mathbb{E}[X|\,Y] = Y$ and $\mathbb{E}[Y|\,X] = X$.

1. Check that $\mathbb{E}\left[(X-Y)\mathbf{1}_{\{Y \leq a\}}\right] = 0$ for all $a \in \mathbb{R}$.
2. Prove that $\mathbb{E}\left[|X-Y|\mathbf{1}_{\{Y \leq a < X \text{ or } X \leq a < Y\}}\right] = 0$ for all $a \in \mathbb{R}$.
3. Deduce that a.s. $X = Y$.

$\triangle$

Exercise I.9 (Kolmogorov's maximal inequality). Let $(X_n, n \in \mathbb{N}^*)$ be independent real random variables with the same distribution. We assume $\mathbb{E}[X_1^2] < +\infty$ and $\mathbb{E}[X_1] = 0$. Let $x > 0$. We set $S_n = \sum_{k=1}^n X_k$ for $n \in \mathbb{N}^*$ and $T = \inf\{n \in \mathbb{N}^*; |S_n| \geq x\}$ with the convention that $\inf \emptyset = +\infty$.

1. Prove that $\mathbb{P}(T = k) \leq \frac{1}{x^2}\mathbb{E}\left[S_k^2 \mathbf{1}_{\{T=k\}}\right]$ for all $k \in \mathbb{N}^*$.
2. Check that $\sum_{k=1}^n \mathbb{P}(T = k) = \mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\right)$.
3. By noticing that $S_n^2 \geq S_k^2 + 2S_k(S_n - S_k)$, prove Kolmogorov's maximal inequality: for all $x > 0$ and $n \in \mathbb{N}^*$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\right) \leq \frac{\mathbb{E}[S_n^2]}{x^2}.$$

$\triangle$