V Martingales

Vocabulary (english/*français*): sub-martingale = *sous-martingale*, super-martingale = *sur-martingale*.

Exercise V.1 (Martingale not converging in L^1). Let $(X_n, n \geq 1)$ be a sequence of independent Bernoulli random variables of parameter $\mathbb{E}[X_n] = (1+\mathrm{e})^{-1}$. We define $M_0 = 1$ and for $n \in \mathbb{N}^*$:

$$M_n = \mathrm{e}^{-n+2\sum_{i=1}^n X_i}.$$

1. Prove that $(M_n, n \geq 1)$ is a martingale and that a.s. $\lim_{n\to\infty} M_n = 0$.
2. Compute $\mathbb{E}[M_n]$ and check that $(M_n, n \in \mathbb{N}^*)$ doesn't converge in L^1.

△

Exercise V.2 (Martingale not converging a.s.). Let $(Z_n, n \geq 1)$ be independent random variable such that $\mathbb{P}(Z_n = 1) = \mathbb{P}(Z_n = -1) = 1/(2n)$ and $\mathbb{P}(Z_n = 0) = 1 - n^{-1}$. We set $X_1 = Z_1$ and for $n \geq 2$:

$$X_n = Z_n \mathbf{1}_{\{X_{n-1}=0\}} + nX_{n-1}|Z_n|\mathbf{1}_{\{X_{n-1}\neq 0\}}.$$

1. Check that $|X_n| \leq n!$ and that $X = (X_n, n \geq 1)$ is a martingale.
2. Prove directly that X converge in probability towards 0.
3. Using Borel-Cantelli's lemma, prove that $\mathbb{P}(Z_n \neq 0 \text{ infinitely often}) = 1$. Deduce that $\mathbb{P}(\lim_{n\to\infty} X_n \text{ existe}) = 0$. In particular, the martingale does not converge a.s. towards 0.

△

Exercise V.3 (Wright-Fisher model). We consider a population of constant size N. We assume that the reproduction is random: this corresponds in the end to each individual choosing his parent independently in the previous generation. The Wright-Fisher model study the evolution of the number of individuals carrying one of the two alleles A and a. For $k \geq 0$, let X_n denote the number of alleles A at generation n in the population. We assume that $X_0 = i \in \{0, \ldots, N\}$ is given.

1. Prove that $X = (X_n, n \geq 0)$ is a martingale (specify the filtration).
2. Prove that X converges to a limit, say X_∞, and give the type of convergence.
3. Prove that $M = (M_n = (\frac{N}{N-1})^n X_n(N - X_n), n \geq 0)$ is a martingale and compute $\mathbb{E}[X_\infty(N - X_\infty)]$.
4. Prove that one of the allele disappears a.s in finite time. Compute the probability that allele A disappears.
5. Compute $\lim_{n\to\infty} M_n$ and deduce that M doesn't converge in L^1.

△

Exercise V.4 (Waiting time of a given sequence). Let $(X_n, n \geq 1)$ be a sequence of independent Bernoulli random variable with parameter $p \in (0,1)$: $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0) = p$. Let $\tau_{ijk} = \inf\{n \geq 3; (X_{n-2}, X_{n-1}, X_n) = (i,j,k)\}$ be the waiting time of the sequence $(i,j,k) \in \{0,1\}^3$.

1. Prove that τ_{ijk} is a stopping time a.s. finite.
2. We set $S_0 = 0$ and $S_n = (S_{n-1} + 1)\dfrac{X_n}{p}$ for $n \geq 1$. Prove that $(S_n - n, n \geq 0)$ is a martingale. Deduce $\mathbb{E}[\tau_{111}]$.
3. Compute $\mathbb{P}(\tau_{111} > \tau_{110})$.
4. By considering $T_2 = \dfrac{X_1 X_2}{p^2} + \dfrac{X_2}{p}$ and $T_n = T_{n-1}\dfrac{1 - X_n}{1-p} + \dfrac{X_{n-1}X_n}{p^2} - \dfrac{X_{n-1}(1 - X_n)}{p(1-p)} + \dfrac{X_n}{p}$ for $n \geq 3$, compute $\mathbb{E}[\tau_{110}]$.
5. Using similar arguments, compute $\mathbb{E}[\tau_{100}]$ and $\mathbb{E}[\tau_{101}]$.

It can be proved that for any $(i,j,k) \in \{0,1\}$, there exists $(i',j',k') \in \{0,1\}$ such that $\mathbb{P}(\tau_{i'j'k'} < \tau_{ijk}) > 1/2$. △

Exercise V.5 (When daes an insurance companies goes bankrupt?). We consider the evolution of the capital of an insurance companies. Let $S_0 = x > 0$ be the initial capital, $c > 0$ the fixed income per year and $X_n \geq 0$ the (random) cost of the damage for the year n. The capital at the end of year $n \geq 1$ is thus $S_n = x + nc - \sum_{k=1}^n X_k$. Bankruptcy happens if the capital becomes negative that is $\tau = \inf\{k; S_k < 0\}$ is finite. The goal of this exercise is to find an upper bound of the bankruptcy probability $\mathbb{P}(\tau < \infty)$.

We assume the real random variables $(X_k, k \geq 1)$ are independent, identically distributed, a.s. non constant, and have all its exponential moments (i.e. $\mathbb{E}[\mathrm{e}^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$).

1. Check that $\mathbb{E}[X_1] \geq c$ implies $\mathbb{P}(\tau < \infty) = 1$, and that $\mathbb{P}(X_1 > c) = 0$ implies $\mathbb{P}(\tau < \infty) = 0$.

We assume that $\mathbb{E}[X_1] < c$ and $\mathbb{P}(X_1 > c) > 0$.

2. Check that if $\mathbb{E}[\mathrm{e}^{\lambda X_k}] \geq \mathrm{e}^{\lambda c}$, then $(V_n = \mathrm{e}^{-\lambda S_n + \lambda x}, n \geq 0)$ is a non-negative sub-martingale.

3. Let $N \geq 1$. Prove that $\{\tau \leq N\}$ is the disjoint union of the events $F_k = \{S_r \geq 0 \text{ for } r < k, S_k < 0\} = \{\tau = k\}$ for $k \in \{1, \ldots, N\}$. Deduce that:

$$\mathbb{E}[V_N \mathbf{1}_{\{\tau \leq N\}}] \geq \sum_{k=1}^N \mathbb{E}[V_k \mathbf{1}_{\{\tau = k\}}] \geq \mathrm{e}^{\lambda x}\, \mathbb{P}(\tau \leq N).$$

   (You can check, we recover the maximal inequality for the positive suub-martingale.)

4. Deduce that $\mathbb{P}(\tau < \infty) \leq \mathrm{e}^{-\lambda_0 x}$, where $\lambda_0 \in (0, \infty)$ is the unique root of $\mathbb{E}[\mathrm{e}^{\lambda X_1}] = \mathrm{e}^{\lambda c}$.

△

Exercise V.6 (A.s. convergence and convergence in distribution). Let $(X_n, n \geq 1)$ be a sequence of independent real random variables. We set $S_n = \sum_{k=1}^n X_k$ for $n \geq 1$. The goal of this exercise is to prove that if the sequence $(S_n, n \geq 1)$ converges in distribution, then it converges a.s. also.

For $t \in \mathbb{R}$, we set $\psi_n(t) = \mathbb{E}[\mathrm{e}^{itX_n}]$ and $M_n(t) = \dfrac{\mathrm{e}^{itS_n}}{\prod_{k=1}^n \psi_k(t)}$ for $n \geq 1$ if $\prod_{k=1}^n \psi_k(t) \neq 0$.

1. Let $t \in \mathbb{R}$ be such that $\prod_{k=1}^{n} \psi_k(t) \neq 0$. Prove that $(M_k(t), 1 \leq k \leq n)$ is a martingale.

We assume that $(S_n, n \geq 1)$ converges in distribution towards S.

2. Prove there exists $\varepsilon > 0$ such that for all $t \in [-\varepsilon, \varepsilon]$, a.s. the sequence $(\mathrm{e}^{itS_n}, n \geq 1)$ converges.

3. We recall that if there exists $\varepsilon > 0$ s.t., for almost all $t \in [-\varepsilon, \varepsilon]$, the sequence $(\mathrm{e}^{its_n}, n \geq 1)$ converges, then the sequence $(s_n, n \geq 1)$ converges. Prove that $(S_n, n \geq 1)$ converges a.s. towards a random variable distributed as S.

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