III Markov chains

Vocabulary (english/*français*): countable = *dénombrable*; irreducible = *irréductible*; $\mathbb{N} = \mathbb{N}$ *or* $\mathbb{N}^*$.

Exercise III.1 (Markov chains built from a Markov chain-I). Let $X = (X_n, n \in \mathbb{N})$ be a Markov chain on a finite or countable set E with transition matrix P. Set $Z = (Z_n = X_{2n}, n \in \mathbb{N})$.

1. Compute $\mathbb{P}(X_2 = y | X_0 = x)$ for $x, y \in E$. Prove that Z is a Markov chain and gives its transition matrix.

2. Prove that any invariant probability distribution for X is also invariant for Z. Prove the converse is false in general.

△

Exercise III.2 (Markov chains built from a Markov chain-II). Let $X = (X_n, n \in \mathbb{N})$ be a Markov chain on a finite or countable set E with transition matrix P. Set $Y = (Y_n, n \in \mathbb{N}^*)$ where $Y_n = (X_{n-1}, X_n)$.

1. Prove that Y is a Markov chain on E^2 and give its transition matrix.

2. Give an example with X irreducible and Y nor irreducible. If X is irreducible, change the state space of Y so that it is also irreducible.

3. Let π be an invariant probability distribution of X. Deduce an invariant probability distribution for Y.

△

Exercise III.3 (2 states Markov chain). Let $E = a, b$. The most general stochastic matrix can be written as:

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

1. Compute the invariant probability distribution(s). Give a necessary and sufficient condition for uniqueness of the invariant probability distribution.

2. Let us assume that P is irreducible (*i.e.* $P(a,b)P(b,a) > 0$). Prove that:
$$P^n = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix} + \gamma^n \begin{pmatrix} p & -p \\ -1+p & 1-p \end{pmatrix},$$
for some p and γ which you will compute.

3. Prove that if P is irreducible and $P(a,a)+P(b,b) > 0$, then $\lim_{n\to+\infty} \nu_0 P^n$ exists and does not depend on ν_0.

△

Exercise III.4 (Skeleton Markov chains). Let $X = (X_n, n \in \mathbb{N})$ be a Markov chain on a countable space E with transition matrix P. We use the convention $\inf \emptyset = +\infty$. We define $\tau_1 = \inf\{k \geq 1; X_k \neq X_0\}$.

1. Let $x \in E$. Give the distribution of τ_1 conditionally on $\{X_0 = x\}$. Check that, conditionally on $\{X_0 = x\}$, $\tau_1 = +\infty$ a.s. if x is an absorbing state and otherwise a.s. τ_1 is finite.

2. Conditionally on $\{X_0 = x\}$, if x is not an absorbing state, give the distribution of X_{τ_1}.

We set $S_0 = 0$, $Y_0 = X_0$ and by recurrence for $n \geq 1$: $S_n = S_{n-1} + \tau_n$, and if $S_n < +\infty$: $Y_n = X_{S_n}$ as well as $\tau_{n+1} = \inf\{k \geq 1; X_{k+S_n} \neq X_{S_n}\}$. Let $R = \inf\{n; \tau_n = +\infty\} = \inf\{n; S_n = +\infty\}$.

3. Prove that if X does not have absorbing states, then a.s. $R = +\infty$.

We assume that X does not have absorbing states.

4. Prove that $Y = (Y_n, n \in \mathbb{N})$ is a Markov chain (it is called skeleton of X). Prove that its transition matrix, Q, is given by:
$$Q(x,y) = \frac{P(x,y)}{1-P(x,x)} \mathbf{1}_{\{x \neq y\}} \quad \text{for } x, y \in E.$$

5. Let π be an invariant probability distribution of X. We define a measure ν on E by:
$$\nu(x) = \frac{\pi(x)(1-P(x,x))}{\sum_{y\in E} \pi(y)(1-P(y,y))}, \quad x \in E.$$
Check that ν is an invariant probability measure of Y.

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