IV Ergodic properties of Markov chains

Vocabulary (english/*français*): irreducible = *irréductible*; convergence in distribution = *convergence en loi.*

Exercise IV.1 (Metropolis-Hastings algorithm). Let π be a given positive probability distribution on a discrete space E. The aim of the Metropolis-Hastings[1] algorithm is to sample (asymptotically) according to π.

We consider an irreducible transition matrix Q, called selection matrix, on E such that for all $x, y \in E$, if $Q(x, y) = 0$ then $Q(y, x) = 0$.

For $x, y \in E$ such that $Q(x, y) > 0$, let $(\rho(x, y), \rho(y, x)) \in (0, 1]^2$ be such that:

$$\rho(x, y)\pi(x)Q(x, y) = \rho(y, x)\pi(y)Q(y, x). \tag{IV.1}$$

The function ρ is called the acceptance probability. Notice the following function ρ satisfies IV.1:

$$\rho(x, y) = \gamma\left(\frac{\pi(y)Q(y, x)}{\pi(x)Q(x, y)}\right), \text{ for all } x, y \in E \text{ such that } Q(x, y) > 0, \tag{IV.2}$$

with γ a function taking values in $(0, 1]$ such that $\gamma(u) = u\gamma(1/u)$.

Let X_0 be a random variable on E with probability distribution μ_0. At step n, conditionally on the random variables $X_0, \ldots, X_n$, let Y_{n+1} distributed according to $Q(X_n, \cdot)$ and with probability $\rho(X_n, Y_{n+1})$, we accept the transition and we set $X_{n+1} = Y_{n+1}$. If the transition is rejected, we set $X_{n+1} = X_n$.

1. Prove the functions $\gamma(u) = \min(1, u)$ (which corresponds to the Metropolis algorithm, which is very commonly used) and $\gamma(u) = u/(1 + u)$

[1] W. Hastings: Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57:97–109, 1970.

(which corresponds to the Boltzmann or Barker algorithm) defined on $(0, +\infty)$ are taking values in $(0, 1]$ and satisfies $\gamma(u) = u\gamma(1/u)$.

2. Check that $X = (X_n, n \in \mathbb{N})$ is a Markov chain and compute its transition matrix P.

3. Prove that X is irreducible. And check that X is reversible with respect to π.

4. Prove that unless Q is periodic and reversible with respect to π, then X is aperiodic. Prove that X converge in distribution towards π.

5. Study the convergence of the sequence $(\frac{1}{n}\sum_{k=1}^{n} f(X_k), n \in \mathbb{N}^*)$ for some real function defined on E.

△

Exercise IV.2 (Parameter estimation). Let $X = (X_n, n \in \mathbb{N})$ be an irreducible positive recurrent Markov chain on a countable state space E with transition matrix P and invariant probability π. The aim of this exercise is to give an estimation of the parameter π and P of the Markov chain X.

1. For $x \in E$ and $n \in \mathbb{N}^*$, we set $\hat{\pi}(x; n) = \frac{1}{n} \operatorname{Card} \{1 \leq k \leq n;\ X_k = x\}$. Prove that a.s. for all $x \in E$, $\lim_{n\to+\infty} \hat{\pi}(x; n) = \pi(x)$.

We set $Z = (Z_n, n \in \mathbb{N}^*)$ with $Z_n = (X_{n-1}, X_n)$.

2. Prove that Z is an irreducible Markov chain on $E_2 = \big\{(x, y) \in E^2;\ P(x, y) > 0\big\}$. And compute its transition matrix.

3. Compute the invariant probability distribution of Y and deduce that Y is recurrent positive.

4. For $x, y \in E$ and $n \in \mathbb{N}^*$, we set:

$$\hat{P}(x, y; n) = \frac{\operatorname{Card} \{1 \leq k \leq n;\ Z_k = (x, y)\}}{\operatorname{Card} \{0 \leq k \leq n-1;\ X_k = x\}},$$

with the convention that $\hat{P}(x, y; n) = 0$ if $\operatorname{Card} \{0 \leq k \leq n-1;\ X_k = x\} = 0$. Prove that a.s. for all $(x, y) \in E_2$, $\lim_{n\to+\infty} \hat{P}(x, y; n) = P(x, y)$.

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