Chapter III

Optimal stopping

III.1 Introduction

The goal of this chapter is to determine the best time (if any) at which one has to stop a stochastic process in order to maximize a given criterion. The following two examples are typical of the problems which will be solved. Their solution are given respectively in Examples III.8 and III.18.

Example III.1 (Marriage of a princess: the setting). In a faraway old age, a princess had to choose a prince for a marriage among ζ candidates. At step $1 \leq n < \zeta$, she interviews the n-th candidate and at the end of the interview she either accepts to marry the candidate or refuses. In the former case the process stop and she get married with the n-th candidate; in the latter case the rebuked candidate leaves forever and the princess moves on to step $n+1$. If $n = \zeta$, she has no more choice but to marry the last candidate. What is the best strategy for the princess if she wants to maximize the probability to marry the best prince?

Example III.2 (Castle to sell). A princess want to sell her castle, let X_n be the n-th price offer. However, preparing the castle for a visit of a potential buyer has a cost, say $c > 0$ per visit. So the gain of the selling at step $n \geq 1$ will be $G_n = X_n - nc$ or $G_n = \max_{1\leq k\leq n} X_k - nc$ if the princess can recall a previous interested buyer. In this infinite time horizon setting, what is the best strategy of the princess to maximize her gain.

We consider a game over the discrete time interval $[\![0, \zeta]\!]$ with horizon $\zeta \in \bar{\mathbb{N}} = \mathbb{N} \bigcup \{\infty\}$, where at step $n \leq \zeta$ we can either stop and receive the gain or reward G_n or continue to step $n+1$ if $n+1 \leq \zeta$. Eventually in the infinite horizon case, $\zeta = \infty$, if we never stop, we receive the gain G_∞. We assume the gains $G = (G_n, n \in [\![0, \zeta]\!])$ form a sequence of random variables on a probability space $(\Omega, \mathbb{P}, \mathcal{G})$ taking values in $[-\infty, +\infty)$.

We assume the information available is given by a filtration $\mathcal{F} = (\mathcal{F}_n, n \in [\![0, \zeta]\!])$ with $\mathcal{F}_n \subset \mathcal{G}$, and a strategy corresponds to a stopping time. Let $\mathbb{T}(\zeta)$ be the set of all stopping times with respect to the filtration $\mathcal{F}$ taking values in $[\![0, \zeta]\!]$. We shall assume that $\mathbb{E}[G_\tau^+] < +\infty$ for all $\tau \in \mathbb{T}(\zeta)$, where $x^+ = \max(0, x)$. In particular, the expectation $\mathbb{E}[G_\tau]$ is well defined and belongs to $[-\infty, +\infty)$. Thus, the maximal gain of the game G is:

$$V_* = \sup_{\tau\in\mathbb{T}(\zeta)} \mathbb{E}[G_\tau].$$

A stopping time $\tau \in \mathbb{T}(\zeta)$ will be said optimal for the game G if $V_* = \mathbb{E}[G_\tau]$.

The next theorem, which is a direct consequences of Corollaries III.7 and III.20, is the main result of this Chapter. For a real sequence $(a_n, n \in \bar{\mathbb{N}})$, we set $\limsup a_n = \lim_{n \nearrow \infty} \sup_{\infty > k \geq n} a_k$.

Theorem III.3. *Let $\zeta \in \bar{\mathbb{N}}$, $G = (G_n, n \in [\![0, \zeta]\!])$ be a sequence of random variables taking values in $[-\infty, +\infty)$ and $\mathcal{F} = (\mathcal{F}_n, n \in \bar{\mathbb{N}})$ be a filtration. Assume the integrability condition:*

$$\mathbb{E}\Big[\sup_{n \in [\![0,\zeta]\!]} G_n^+\Big] < \infty. \tag{III.1}$$

If $\zeta \in \mathbb{N}$ or if $\zeta = \infty$ and

$$\limsup G_n \leq G_\infty \quad a.s., \tag{III.2}$$

then, there exists an optimal stopping time.

Notice that (III.1) implies that $\mathbb{E}[G_\tau^+] < +\infty$ for all $\tau \in \mathbb{T}(\zeta)$. When the horizon ζ is finite, then condition (III.1) is equivalent to

$$\mathbb{E}[G_n^+] < +\infty \quad \text{for all } n \in [\![0, \zeta]\!]. \tag{III.3}$$

When the sequence G is adapted to the filtration $\mathcal{F}$, we shall also consider a particular solution $S = (S_n, n \in [\![0, \zeta]\!])$ to the so called optimal (or Bellman) equations:

$$S_n = \max\left(G_n, \mathbb{E}[S_{n+1}|\mathcal{F}_n]\right) \qquad \text{for } 0 \leq n < \zeta, \tag{III.4}$$

as well as the stopping time $\tau_* \in \mathbb{T}(\zeta)$:

$$\tau_* = \inf\{n \in [\![0, \zeta[\![;\, S_n = G_n\}, \tag{III.5}$$

with the convention $\inf \emptyset = \zeta$. In this setting, we shall prove that τ_* is the minimal optimal stopping time provided that $V_* > -\infty$. We shall also prove that the stopping time:

$$\tau_{**} = \inf\{n \in [\![0, \zeta]\!];\, S_n > \mathbb{E}[S_{n+1}|\mathcal{F}_n]\}.$$

is the maximal optimal stopping time provided that $V_* > -\infty$, see Exercises III.1 and III.4.

The finite horizon case ($\zeta < \infty$) is presented in Section III.2, and the infinite horizon case ($\zeta = \infty$), which is much more delicate (in particular for the definition of S), is presented in Section III.3. We consider the approximation of the infinite horizon case by finite horizon cases in Section III.4. Eventually, Section III.5 is devoted to the Markov chain setting.

The presentation of this Chapter follows [2] also inspired by [4], see also [1, 3] and the references therein. Concerning the infinite horizon case, we consider stopping time taking values in $\bar{\mathbb{N}}$ (instead of $\mathbb{N}$ in most text books). Since in some standard applications, the gain of not stopping in finite time is $G_\infty = -\infty$ (which *de facto* implies the optimal stopping time is finite unless $V_* = -\infty$), we shall consider rewards G_n taking values in $[-\infty, +\infty)$ (instead of assuming that $\mathbb{E}[|G_n|] < +\infty$ in most text books). In this setting, the results are general and easy to present, see Theorem III.3. The drawback of this setting is that we shall not rely on the martingale approach which is the corner stone of the Snell envelope approach, see Remark III.6 and Exercise III.1.

III.2 Finite horizon case

We assume in this section that $\zeta \in \mathbb{N}$. Example III.4 on the marriage of a princess stresses out that the process G may not be adapted to the filtration $\mathcal{F}$. We shall first consider in Section III.2.1 the adapted case, and then deduce in Section III.2.2 the general case.

Example III.4 (Marriage of a princess: the mathematical framework). We continue Example III.1. The princess wants to maximize the probability to marry the best prince. This corresponds to the gain $G_n = \mathbf{1}_{\{\Sigma_n=1\}}$, with Σ_n the (random) rank of the n-th candidate among the ζ candidates. The observation at step n is the relative rank R_n of the n-th candidate. So the available information at step n is given by the σ-field $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$. Notice in particular that Σ_n is unobserved at step n and thus not $\mathcal{F}_n$-measurable (unless $n = \zeta$). Therefore the sequence $G = (G_n, n \in [\![1, \zeta]\!])$ is not adapted to the filtration $\mathcal{F} = (\mathcal{F}_n, n \in [\![1, \zeta]\!])$. (Notice that to stick to the presentation of this section, we could set $G_0 = -\infty$ and $\mathcal{F}_0$ the trivial σ-field.)

III.2.1 The adapted case

We assume $\zeta \in \mathbb{N}$, the sequence $G = (G_n, n \in [\![0, \zeta]\!])$ is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n, n \in [\![0, \zeta]\!])$, and that the integrability condition (III.3) (or equivalently (III.1)) holds. Recall $\mathbb{T}(\zeta)$ is the set of stopping times with respect to the filtration $\mathcal{F}$ taking values in $[\![0, \zeta]\!]$. Notice that (III.3) implies that $\mathbb{E}[G_\tau^+] < +\infty$ for all $\tau \in \mathbb{T}(\zeta)$.

We define the sequence $S = (S_n, n \in [\![0, \zeta]\!])$ recursively by $S_\zeta = G_\zeta$ and the optimal equations (III.4). The following Proposition gives a solution to the optimal stopping time in the setting of this section.

Proposition III.5. *Let $\zeta \in \mathbb{N}$ and $G = (G_n, n \in [\![0, \zeta]\!])$ be an adapted sequence such that $\mathbb{E}[G_n^+] < +\infty$ for all $n \in [\![0, \zeta]\!]$. The stopping time τ_* given by (III.5) (with $(S_n, n \in [\![0, \zeta]\!])$ defined by $S_\zeta = G_\zeta$ and (III.4)) is optimal and $V_* = \mathbb{E}[S_0]$.*

Proof. For $n \in [\![0, \zeta]\!]$, we define $\mathbb{T}_n$ as the set of all stopping times with respect to the filtration $\mathcal{F}$ taking values in $[\![n, \zeta]\!]$, as well as the stopping time $\tau_n = \inf\{k \in [\![n, \zeta]\!];\, S_k = G_k\}$. Notice that $n \leq \tau_n \leq \zeta$. We first prove by downward induction that:

$$S_n \geq \mathbb{E}[G_\tau | \mathcal{F}_n] \quad \text{a.s. for all } \tau \in \mathbb{T}_n, \tag{III.6}$$

$$S_n = \mathbb{E}[G_{\tau_n} | \mathcal{F}_n] \quad \text{a.s.}. \tag{III.7}$$

Notice that (III.6) and (III.7) are clear for $n = \zeta$.

Let $n \in [\![0, \zeta - 1]\!]$. We assume (III.6) and (III.7) hold for $n+1$ and prove them for n. Let $\tau \in \mathbb{T}_n$ and consider the stopping time $\tau' = \max(\tau, n+1) \in \mathbb{T}_{n+1}$. We have:

$$\mathbb{E}[G_\tau | \mathcal{F}_n] = G_n \mathbf{1}_{\{\tau = n\}} + \mathbb{E}[G_{\tau'} | \mathcal{F}_n] \mathbf{1}_{\{\tau > n\}}, \tag{III.8}$$

as $\tau = \tau'$ on $\{\tau > n\}$. We get that a.s.:

$$\mathbb{E}[G_{\tau'} | \mathcal{F}_n] = \mathbb{E}\left[\mathbb{E}[G_{\tau'} | \mathcal{F}_{n+1}] \mathcal{F}_n\right] \leq \mathbb{E}[S_{n+1} | \mathcal{F}_n] \tag{III.9}$$

where we used (III.6) (with $n+1$ and τ') for the first inequality. Using the optimal equations (III.4), we get:

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] \leq S_n. \tag{III.10}$$

Since (III.4) gives also $G_n \leq S_n$, we get using (III.8) that a.s.

$$\mathbb{E}[G_\tau|\mathcal{F}_n] \leq S_n. \tag{III.11}$$

This gives (III.6).

Consider τ_n instead of τ in (III.8). Then notice that on $\{\tau_n > n\}$, we have $\max(\tau_n, n+1) = \tau_{n+1}$. Then the inequality in (III.9) (with $\tau' = \tau_{n+1}$) is in fact an equality thanks to (III.7) (with $n+1$). The inequality in (III.10) is also an equality on $\{\tau_n > n\}$ by definition of τ_n. Then use that $G_n = S_n$ on $\{\tau_n = n\}$, so that (III.11), with τ_n instead of τ, is also an equality. This gives (III.7). We then deduce that (III.6) and (III.7) hold for all $n \in [\![0, \zeta]\!]$.

Notice that $\tau_* = \tau_0$ by definition. We deduce from (III.6), with $n = 0$, that $\mathbb{E}[S_0] \geq \mathbb{E}[G_\tau]$ for all $\tau \in \mathbb{T}(\zeta)$, and from (III.7), that $\mathbb{E}[S_0] = \mathbb{E}[G_{\tau_*}]$. This gives $V_* = \mathbb{E}[S_0]$ and τ_* is optimal. □

Remark III.6 (Snell envelope). Assume that $\mathbb{E}[|G_n|] < \infty$ for all $n \in [\![0, \zeta]\!]$. Notice from (III.4) that S is a super-martingale and that S dominates G. It is left to the reader to check that S is in fact the smallest super-martingale which dominates G. It is called the Snell enveloppe of G. For $n \in [\![0, \zeta]\!]$, using that $S_n = \mathbb{E}[S_{n+1}|\mathcal{F}_n]$ on $\{\tau_* > n\}$, we have:

$$S_{n\wedge\tau_*} = S_{\tau_*}\mathbf{1}_{\{\tau_*\leq n\}} + S_n\mathbf{1}_{\{\tau_*>n\}} = S_{\tau_*}\mathbf{1}_{\{\tau_*\leq n\}} + \mathbb{E}[S_{n+1}\mathbf{1}_{\{\tau_*>n\}}|\mathcal{F}_n] = \mathbb{E}\left[S_{(n+1)\wedge\tau_*}|\mathcal{F}_n\right]. \tag{III.12}$$

This gives that $(S_{n\wedge\tau_*}, n \in [\![0, \zeta]\!])$ is a martingale.

Exercise III.1. Assume that $\mathbb{E}[|G_n|] < \infty$ for all $n \in [\![0, \zeta]\!]$. Prove that τ is an optimal stopping time if and only if $S_\tau = G_\tau$ a.s. and $(S_{n\wedge\tau}, n \in [\![0, \zeta]\!])$ is a martingale. Deduce that τ_* is the minimal optimal stopping time (that is: if τ is optimal, then a.s. $\tau \geq \tau_*$). Using the Doob decomposition, see Remark IV.1, of the super-martingale S, prove that the stopping time:

$$\tau_{**} = \inf\{n \in [\![0, \zeta[\![;\ S_n > \mathbb{E}[S_{n+1}|\mathcal{F}_n]\},$$

with the convention $\inf\emptyset = \zeta$, is the maximal optimal stopping time. △

III.2.2 The general case

If the sequence $G = (G_n, n \in [\![0, \zeta]\!])$ is not adapted to the filtration $\mathcal{F}$, then we shall consider the corresponding adapted sequence $G' = (G'_n, n \in [\![0, \zeta]\!])$ given by:

$$G'_n = \mathbb{E}[G_n|\mathcal{F}_n].$$

Thanks to Jensen inequality, we have $\mathbb{E}[(G'_n)^+] \leq \mathbb{E}[G_n^+] < +\infty$ for all $n \in [\![0, \zeta]\!]$. Recall $\mathbb{T}(\zeta)$ is the set of all stopping time with respect to the filtration $\mathcal{F}$ taking values in $[\![0, \zeta]\!]$. Thanks to Fubini, we get that for $\tau \in \mathbb{T}(\zeta)$:

$$\mathbb{E}[G_\tau] = \sum_{n=0}^{\zeta} \mathbb{E}[G_n \mathbf{1}_{\{\tau=n\}}] = \sum_{n=0}^{\zeta} \mathbb{E}[G'_n \mathbf{1}_{\{\tau=n\}}] = \mathbb{E}[G'_\tau].$$

We thus deduce the maximal gain for the game G is also the maximal gain for the game G'. The following Corollary is then an immediate consequence of Proposition III.5.

Corollary III.7. *Let $\zeta \in \mathbb{N}$ and $G = (G_n, n \in [\![0, \zeta]\!])$ be such that $\mathbb{E}[G_n^+] < +\infty$ for all $n \in [\![0, \zeta]\!]$. Set $S_\zeta = \mathbb{E}[G_\zeta|\mathcal{F}_\zeta]$ and $S_n = \max(\mathbb{E}[G_n|\mathcal{F}_n], \mathbb{E}[S_{n+1}|\mathcal{F}_n])$ for $0 \leq n < \zeta$. Then the stopping time $\tau_* = \inf\{n \in [\![0, \zeta]\!];\ S_n = \mathbb{E}[G_n|\mathcal{F}_n]\}$ is optimal and $V_* = \mathbb{E}[S_0]$.*

Example III.8 (Marriage of a princess: the solution). We continue Example III.4. Recall Σ_n is the rank of the n-th candidate among the ζ candidates, and $G_n = \mathbf{1}_{\{\Sigma_n=1\}}$ is the gain for the princess if she chooses the n-th candidate. We assume the random permutation $\Sigma = (\Sigma_n, n \in [\![1, \zeta]\!])$ is uniformly distributed on the set $\mathcal{S}_\zeta$ of permutation on $[\![1, \zeta]\!]$.

For a permutation $\sigma = (\sigma_1, \ldots, \sigma_\zeta)$, we define the sequence of partial ranks $r(\sigma) = (r_1, \ldots, r_\zeta)$ such that r_n is the partial rank of σ_n in $(\sigma_1, \ldots, \sigma_n)$. In particular, we have $r_1 = 1$ and $r_\zeta = \sigma_\zeta$. Set $E = \prod_{n=1}^{\zeta} [\![1, n]\!]$ the state space of $r(\sigma)$. It is easy to get that r is one-to-one from $\mathcal{S}_\zeta$ to E. Set $r(\Sigma) = (R_1, \ldots, R_n)$, so that R_n is the observed partial rank of the n-th candidate. The filtration $\mathcal{F} = (\mathcal{F}_n, n \in [\![1, \zeta]\!])$ generated by the observations is thus given by $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$.

Since r is one-to-one, we deduce that $r(\Sigma)$ is uniform on E. Since E has a product form, we get that the random variables $R_1, \ldots, R_n$ are independent and R_n is uniform on $[\![1, n]\!]$. The event $\{\Sigma_n = 1\}$ is equal to $\{R_n = 1\} \bigcap_{k=n+1}^{\zeta} \{R_k > 1\}$. Using the independence of $(R_{n+1}, \ldots, R_\zeta)$ with $\mathcal{F}_n$, we deduce that:

$$\mathbb{E}[G_n|\mathcal{F}_n] = \mathbb{E}[\mathbf{1}_{\{\Sigma_n=1\}}|\mathcal{F}_n] = \mathbf{1}_{\{R_n=1\}} \prod_{k=n+1}^{\zeta} \mathbb{P}(R_k > 1) = \frac{n}{\zeta} \mathbf{1}_{\{R_n=1\}}.$$

By a direct induction, we get from the definition of S_n given in Corollary III.7 that S_n is a function of R_n and more precisely $S_n = \max\left(\frac{n}{\zeta} \mathbf{1}_{\{R_n=1\}}, s_{n+1}\right)$, with $s_{n+1} = \mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_{n+1}]$ as S_{n+1}, which is a function of R_{n+1}, is independent of $\mathcal{F}_n$. The sequence $(s_n, n \in [\![1, \zeta]\!])$ is non-increasing as $(S_n, n \in [\![1, \zeta]\!])$ is a super-martingale. We deduce that $\tau_* = \gamma_{n_*}$ for some n_*, where for $n \in [\![1, \zeta]\!]$, the strategy γ_n corresponds to first observe $n-1$ candidate and then choose the next one who is better than those who have been observed (or the last if there is none): $\gamma_n = \inf\{k \in [\![n, \zeta]\!];\ R_k = 1\}$, with the convention that $\inf \emptyset = \zeta$. We set $\Gamma_n = \mathbb{E}[G_{\gamma_n}]$ the gain corresponding to the strategy γ_n. We have $\Gamma_1 = 1/\zeta$ and for $n \in [\![2, \zeta]\!]$:

$$\Gamma_n = \sum_{k=n}^{\zeta} \mathbb{P}(\gamma_n = k, \Sigma_k = 1) = \sum_{k=n}^{\zeta} \mathbb{P}(R_n > 1, \ldots, R_k = 1, \ldots, R_\zeta > 1) = \frac{n-1}{\zeta} \sum_{k=n}^{\zeta} \frac{1}{k-1} \cdot$$

Notice that $\zeta\Gamma_1 = \zeta\Gamma_\zeta = 1$. For $n \in [\![1, \zeta - 1]\!]$, we have $\zeta(\Gamma_n - \Gamma_{n+1}) = 1 - \sum_{j=n}^{\zeta-1} 1/j$. We deduce that Γ_n is maximal for $n_* = \inf\{n \geq 1;\ \sum_{j=n}^{\zeta-1} 1/j \leq 1\}$. We also have $V_* = \Gamma_{n_*}$.

For ζ large, we get $n_* \sim \zeta/\mathrm{e}$, so the optimal strategy is to observe a fraction $1/\mathrm{e} \simeq 37\%$ of the candidates, and then choose the next best one (or the last if there is none); the probability to get the best prince is then $V_* = \Gamma_{n_*} \simeq n_*/\zeta \simeq 1/\mathrm{e} \simeq 37\%$.

III.3 Infinite horizon case

We assume in this section that $\zeta = \infty$. Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration. For simplicity, we write $\mathbb{T} = \mathbb{T}(\infty)$ for the set of stopping times taking values in $\bar{\mathbb{N}}$. Notice the definition of stopping time, and thus of the set $\mathbb{T}$, does not depend on the choice of $\mathcal{F}_\infty$ as long as this σ-field contains $\mathcal{F}_n$ for all $n \in \mathbb{N}$. For this reason, we shall take for $\mathcal{F}_\infty$ the smallest possible σ-field whose existence is given by the next lemma.

Lemma III.9. *There exists a smallest σ-field containing $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$.*

The smallest σ-field containing $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is denoted by $\bigvee_{n\in\mathbb{N}} \mathcal{F}_n$.

Proof. Let $\mathbb{F}$ be the set of all σ-fields containing $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Notice $\mathbb{F}$ is non-empty as it contains $\mathcal{G}$. Since the intersection of any family of σ-fields is a σ-field, we deduce that $\bigcap_{\mathcal{F}'\in\mathbb{F}} \mathcal{F}'$ is the smallest σ-field containing $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. □

We use the following convention. The limit operator $\lim_{n\to\infty}$ will be understood as $\lim_{n\to\infty;\, n<\infty}$, and for a real sequence $(a_n, n \in \bar{\mathbb{N}})$, we set $\limsup a_n = \lim_{n\to\infty} \sup_{\infty>k\geq n} a_k$. as well as $\liminf a_n = \lim_{n\to\infty} \inf_{\infty>k\geq n} a_k$.

The next two Examples illustrates the hypothesis on the gain process $G = (G_n, n \in \bar{\mathbb{N}})$ which we shall use to get the existence of an optimal stopping time.

Example III.10. We consider the gain process G given by $G_n = 1 - 1/n$ for $n \in \mathbb{N}$ and $G_\infty = 0$. Clearly we have $V_* = 1$ and there is no optimal stopping time.

The absence of optimal stopping time in Example III.10 is due to the "bad" value of G_∞. For this reason, we will consider the continuity condition (III.2).

Example III.11. Let $(X_n, n \in \mathbb{N})$ be independent Bernoulli random variables such that $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = 0) = 1/2$. We consider the gain process $G = (G_n, n \in \bar{\mathbb{N}})$ given by $G_0 = 0$, $G_n = (2^n - 1)\prod_{k=1}^n X_k$ for $n \in \mathbb{N}^*$ and a.s. $G_\infty = \lim_{n\to\infty} G_n = 0$. Let $\mathcal{F}$ be the natural filtration of the process G. We have $\mathbb{E}[G_n] = 1 - 2^{-n}$ so that $V_* \geq 1$. Notice G is a non-negative sub-martingale as:

$$\mathbb{E}[G_{n+1}|\mathcal{F}_n] = \frac{2^{n+1}-1}{2^{n+1}-2} G_n \geq G_n.$$

Thus, for any $\tau \in \mathbb{T}$, we have $\mathbb{E}[G_{\tau\wedge n}] \leq \mathbb{E}[G_n] \leq 1$. And by Fatou Lemma, we get $\mathbb{E}[G_\tau] \leq 1$. Thus, we deduce that $V_* = 1$.

Since $\mathbb{E}[G_{n+1}|\mathcal{F}_n] > G_n$ on $\{G_n \neq 0\}$ and $G_{n+1} = G_n$ on $\{G_n = 0\}$, we get at step n that the expected future gain at step $n+1$ is better than the gain G_n. Therefore it is more interesting to continue than to stop at step n. However this strategy will provide the gain $G_\infty = 0$, and is thus not optimal. We deduce there is no optimal stopping time.

The absence of optimal stopping time in Example III.11 is due to the "bad" integrability condition as $\mathbb{E}[\lim_{n\to\infty} G_n] < \lim_{n\to\infty} \mathbb{E}[G_n]$. For this reason, we will consider the integrability condition (III.1).

The main result of this section is that if (III.1) and (III.2) hold, then there exists an optimal stopping time $\tau_* \in \mathbb{T}$, see Corollary III.20. The main idea of the infinite horizon case, inspired by the finite horizon case, is to consider a process $S = (S_n, n \in [\![0, \zeta]\!])$ satisfying the optimal equations (III.4). But since the initialization of S given in the finite horizon case is now useless, we shall rely on a definition given by (III.6) and (III.7). However, we need to consider a measurable version of the supremum of $\mathbb{E}[G_\tau|\mathcal{F}_n]$, where τ is any stopping time such that $\tau \geq n$. This is developed in Section III.3.1. Then, as in Section III.2, we will consider separately the adapted case in Sections III.3.2 and III.3.3, and then the general case in Section III.3.4.

III.3.1 Essential supremum

The following proposition asserts the existence of a minimal random variable dominating a family (which might be uncountable) of random variables in the sense of a.s. inequality. We set $\bar{\mathbb{R}} = [-\infty, +\infty]$.

Proposition III.12. *Let $(X_t, t \in T)$ be a family of random variables taking values in $\bar{\mathbb{R}}$. There exists a unique (up to the a.s. equivalence) random variable X_* taking values in $\bar{\mathbb{R}}$ such that:*

(i) For all $t \in T$, $\mathbb{P}(X_* \geq X_t) = 1$.

(ii) If there exists a random variable Y such that for all $t \in T$, $\mathbb{P}(Y \geq X_t) = 1$, then a.s. $Y \geq X_*$.

The random variable X_* of the previous proposition is called the essential supremum of $(X_t, t \in T)$ and is denoted by:

$$X_* = \operatorname*{ess\,sup}_{t \in T} X_t.$$

Proof. Since we are only considering inequalities between real random variables, by mapping $\bar{\mathbb{R}}$ onto $[0, 1]$ with an increasing one-to-one function, we can assume that X_t takes values in $[0, 1]$ for all $t \in T$.

Let $\mathcal{I}$ be the family of all countable sub-families of T. For each $I \in \mathcal{I}$, consider the (well defined) random variable $X_I = \sup_{t \in I} X_t$ and define $\alpha = \sup_{I \in \mathcal{I}} \mathbb{E}[X_I]$. There exists a sequence $(I_n, n \in \mathbb{N})$ such that $\lim_{n\to+\infty} \mathbb{E}[X_{I_n}] = \alpha$. The set $I_* = \bigcup_{n \in \mathbb{N}} I_n$ is countable and thus $I_* \in \mathcal{I}$. Set $X_* = X_{I_*}$. Since $\mathbb{E}[X_{I_n}] \leq \mathbb{E}[X_*] \leq \alpha$ for all $n \in \mathbb{N}$, we get $\mathbb{E}[X_*] = \alpha$.

For any $t \in T$, consider $J = I_* \bigcup \{t\}$, which belongs to $\mathcal{I}$, and notice that $X_J = \max(X_t, X_*)$. Since $\alpha = \mathbb{E}[X_*] \leq \mathbb{E}[X_J] \leq \alpha$, we deduce that $\mathbb{E}[X_*] = \mathbb{E}[X_J]$ and thus a.s. $X_J = X_*$, that is $\mathbb{P}(X_* \geq X_t) = 1$. This gives (i).

Let Y be as in (ii). Since I_* is countable, we get that a.s. $Y \geq X_*$. This gives (ii). □

III.3.2 The adapted case: regular stopping times

We assume in this section that the sequence $G = (G_n, n \in \bar{\mathbb{N}})$ is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n, n \in \bar{\mathbb{N}})$, with $\mathcal{F}_\infty = \bigvee_{n \in \mathbb{N}} \mathcal{F}_n$. We shall consider the following hypothesis which is slightly weaker than (III.1):

(H) There exists a non-negative integrable random variable M such that for all $n \in \bar{\mathbb{N}}$, we have a.s. $G_n \leq \mathbb{E}[M|\mathcal{F}_n]$.

Condition (H) implies that for all $\tau \in \mathbb{T}$, we have a.s. $G_\tau^+ \leq \mathbb{E}[M|\mathcal{F}_\tau]$. Notice that if (III.1) holds then (H) holds with $M = \sup_{k \in \bar{\mathbb{N}}} G_k^+$.

For $n \in \mathbb{N}$, let $\mathbb{T}_n = \{\tau \in \mathbb{T};\, \tau \geq n\}$ be the set of stopping times larger than or equal to n. We define the subset of $\mathbb{T}_n$ of regular stopping times, $\mathbb{T}'_n$, such that for all finite $k \geq n$:

$$\mathbb{E}[G_\tau|\mathcal{F}_k] > G_k \quad \text{a.s. on } \{\tau > k\}.$$

Notice that $\mathbb{T}'_n$ is non-empty as it contains n and that the definition of regular stopping time depends on the gain sequence G.

Lemma III.13. *Assume that G is adapted and a.s. $\mathbb{E}[G_\tau^+] < +\infty$ for all $\tau \in \mathbb{T}$. Let $n \in \mathbb{N}$.*

(i) If $\tau \in \mathbb{T}_n$, then there exists a regular stopping time $\tau' \in \mathbb{T}'_n$ such that $\tau' \leq \tau$ and a.s. $\mathbb{E}[G_{\tau'}|\mathcal{F}_n] \geq \mathbb{E}[G_\tau|\mathcal{F}_n]$.

(ii) If $\tau', \tau'' \in \mathbb{T}'_n$ are regular, then the stopping time $\tau = \max(\tau', \tau'') \in \mathbb{T}'_n$ is regular and a.s. $\mathbb{E}[G_\tau|\mathcal{F}_n] \geq \max\left(\mathbb{E}[G_{\tau'}|\mathcal{F}_n], \mathbb{E}[G_{\tau''}|\mathcal{F}_n]\right)$.

Proof. Let $\tau \in \mathbb{T}_n$ and set $\tau' = \inf\{k \geq n;\, \mathbb{E}[G_\tau|\mathcal{F}_k] \leq G_k\}$ with the convention that $\inf \emptyset = \infty$. Notice that τ' is a stopping time and that a.s. $n \leq \tau' \leq \tau$. On $\{\tau' = \infty\}$, we have $\tau = \infty$ and a.s. $G_{\tau'} = G_\infty = G_\tau$. For $\infty > m \geq n$, we have, on $\{\tau' = m\}$, that a.s. $\mathbb{E}[G_{\tau'}|\mathcal{F}_m] = G_m \geq \mathbb{E}[G_\tau|\mathcal{F}_m]$. We deduce that for all finite $k \geq n$ a.s. on $\{\tau' \geq k\}$:

$$\mathbb{E}[G_{\tau'}|\mathcal{F}_k] = \sum_{m \in [\![k,\infty]\!]} \mathbb{E}\left[\mathbb{E}[G_{\tau'}|\mathcal{F}_m]\mathbf{1}_{\{\tau'=m\}}|\mathcal{F}_k\right] \geq \sum_{m \in [\![k,\infty]\!]} \mathbb{E}\left[\mathbb{E}[G_\tau|\mathcal{F}_m]\mathbf{1}_{\{\tau'=m\}}|\mathcal{F}_k\right].$$

And thus, for all finite $k \geq n$:

$$\mathbb{E}[G_{\tau'}|\mathcal{F}_k]\mathbf{1}_{\{\tau' \geq k\}} \geq \mathbb{E}[G_\tau|\mathcal{F}_k]\mathbf{1}_{\{\tau' \geq k\}}. \tag{III.13}$$

We have on $\{\tau' > k\}$, $\mathbb{E}[G_\tau|\mathcal{F}_k] > G_k$. Then use (III.13) to get that τ' is regular. Take $k = n$ in (III.13) and use that $\tau' \geq n$ a.s. to get the last part of (i).

Let $\tau', \tau'' \in \mathbb{T}'_n$ and $\tau = \max(\tau', \tau'')$. By construction τ is a stopping time. We have for all $m \geq k \geq n$ and k finite:

$$\mathbb{E}[G_\tau \mathbf{1}_{\{\tau'=m\}}|\mathcal{F}_k] = \mathbb{E}[G_{\tau'} \mathbf{1}_{\{m=\tau'\geq\tau''\}}|\mathcal{F}_k] + \mathbb{E}[G_{\tau''} \mathbf{1}_{\{\tau''>\tau'=m\}}|\mathcal{F}_k].$$

Using that $\tau'' \in \mathbb{T}'_n$, we get for all finite $m \geq k \geq n$:

$$\mathbb{E}[G_{\tau''} \mathbf{1}_{\{\tau''>\tau'=m\}}|\mathcal{F}_k] = \mathbb{E}\left[\mathbb{E}[G_{\tau''}|\mathcal{F}_m]\mathbf{1}_{\{\tau''>m\}}\mathbf{1}_{\{\tau'=m\}}|\mathcal{F}_k\right] \geq \mathbb{E}[G_m \mathbf{1}_{\{\tau''>\tau'=m\}}|\mathcal{F}_k].$$

We deduce that for all $m \geq k \geq n$ and k finite:

$$\mathbb{E}[G_\tau \mathbf{1}_{\{\tau'=m\}}|\mathcal{F}_k] \geq \mathbb{E}[G_{\tau'} \mathbf{1}_{\{\tau'=m\}}|\mathcal{F}_k]. \tag{III.14}$$

By summing (III.14) over m with $m > k$ and using that $\tau' \in \mathbb{T}'_n$, we get:

$$\mathbb{E}[G_\tau|\mathcal{F}_k]\mathbf{1}_{\{\tau'>k\}} \geq \mathbb{E}[G_{\tau'}|\mathcal{F}_k]\mathbf{1}_{\{\tau'>k\}} > G_k \mathbf{1}_{\{\tau'>k\}}.$$

By symmetry, we also get $\mathbb{E}[G_\tau|\mathcal{F}_k]\mathbf{1}_{\{\tau''>k\}} > G_k \mathbf{1}_{\{\tau''>k\}}$. Since $\{\tau > k\} = \{\tau' > k\} \bigcup \{\tau'' > k\}$, this implies that $\mathbb{E}[G_\tau|\mathcal{F}_k] > G_k$ a.s. on $\{\tau > k\}$. Thus, τ is regular.

By summing (III.14) over m with $m \geq k = n$, and using that $\tau' \geq n$ a.s., we get $\mathbb{E}[G_\tau|\mathcal{F}_n] \geq \mathbb{E}[G_{\tau'}|\mathcal{F}_n]$. By symmetry, we also have $\mathbb{E}[G_\tau|\mathcal{F}_n] \geq \mathbb{E}[G_{\tau''}|\mathcal{F}_n]$. We deduce the last part of (ii). □

The next lemma is the main result of this section.

Lemma III.14. *We assume that G is adapted and hypothesis (H) and (III.2) hold. Then, for all $n \in \mathbb{N}$, there exists $\tau_n^\circ \in \mathbb{T}_n$ such that a.s.* $\operatorname{ess\,sup}_{\tau\in\mathbb{T}_n} \mathbb{E}[G_\tau|\mathcal{F}_n] = \mathbb{E}[G_{\tau_n^\circ}|\mathcal{F}_n]$.

Proof. We set $X_* = \operatorname{ess\,sup}_{\tau\in\mathbb{T}_n} \mathbb{E}[G_\tau|\mathcal{F}_n]$. According to the proof of Proposition III.12, there exists a sequence $(\tau_k, k \in \mathbb{N})$ of elements of $\mathbb{T}_n$ such that $X_* = \sup_{k\in\mathbb{N}} \mathbb{E}[G_{\tau_k}|\mathcal{F}_n]$. Thanks to (i) of Lemma III.13, there exists a sequence $(\tau'_k, k \in \mathbb{N})$ of regular stopping times, elements of $\mathbb{T}'_n$, such that $\mathbb{E}[G_{\tau'_k}|\mathcal{F}_n] \geq \mathbb{E}[G_{\tau_k}|\mathcal{F}_n]$. According to (ii) of Lemma III.13, for all $k \in \mathbb{N}$, the stopping time $\tau''_k = \max_{0\leq j\leq k} \tau'_j$ belongs to $\mathbb{T}'_n$, the sequence $(\mathbb{E}[G_{\tau''_k}|\mathcal{F}_n], k \in \mathbb{N})$ is non-decreasing and $\mathbb{E}[G_{\tau''_k}|\mathcal{F}_n] \geq \mathbb{E}[G_{\tau'_k}|\mathcal{F}_n] \geq \mathbb{E}[G_{\tau_k}|\mathcal{F}_n]$. In particular, we get $X_* = \sup_{k\in\mathbb{N}} \mathbb{E}[G_{\tau_k}|\mathcal{F}_n] \leq \sup_{k\in\mathbb{N}} \mathbb{E}[G_{\tau''_k}|\mathcal{F}_n] \leq X_*$, so that a.s. $X_* = \limsup_{k\to\infty} \mathbb{E}[G_{\tau''_k}|\mathcal{F}_n]$.

Let $\tau_n^\circ \in \mathbb{T}_n$ be the limit of the non-decreasing sequence $(\tau''_k, k \in \mathbb{N})$. Set $Y_k = \mathbb{E}[M|\mathcal{F}_{\tau''_k}]$. Thanks to Corollary IV.3, we have that the sequence $(Y_k, k \in \mathbb{N})$ converges a.s. and in L^1 towards $Y_\infty = \mathbb{E}[M|\mathcal{F}_{\tau_n^\circ}]$. Then, we use Lemma IV.2 with $X_k = G_{\tau''_k}$ to get that $X_* \leq \mathbb{E}[\limsup_{k\to\infty} G_{\tau''_k}|\mathcal{F}_n]$. Thanks to (III.2), we have a.s. $\limsup_{k\to\infty} G_{\tau''_k} \leq G_{\tau_n^\circ}$. So we get that a.s. $X_* \leq \mathbb{E}[G_{\tau_n^\circ}|\mathcal{F}_n]$. To conclude use that by definition of X_*, we have $\mathbb{E}[G_{\tau_n^\circ}|\mathcal{F}_n] \leq X_*$ and thus $X_* = \mathbb{E}[G_{\tau_n^\circ}|\mathcal{F}_n]$. □

We have the following Corollary.

Corollary III.15. *We assume that G is adapted and hypothesis (H) and (III.2) hold. Then, we have that τ_0° is optimal that is $V_* = \mathbb{E}[G_{\tau_0^\circ}]$.*

Proof. Lemma III.14 gives that $\mathbb{E}[G_\tau] \leq \mathbb{E}[G_{\tau_0^\circ}]$ for all $\tau \in \mathbb{T}$. Thus τ_0° is optimal. □

Exercise III.2. Assume that hypothesis (H) and (III.2) hold. Let $n \in \mathbb{N}$. Prove that the limit of a non-decreasing sequence of regular stopping times, elements of $\mathbb{T}'_n$, is regular. Deduce that τ_n° in Lemma III.14 is regular, that is τ_n° belongs to $\mathbb{T}'_n$. △

III.3.3 The adapted case: optimal equations

We assume in this section that the sequence $G = (G_n, n \in \bar{\mathbb{N}})$ is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n, n \in \bar{\mathbb{N}})$, with $\mathcal{F}_\infty = \bigvee_{n\in\mathbb{N}} \mathcal{F}_n$. Recall that $\mathbb{T}_n = \{\tau \in \mathbb{T};\ \tau \geq n\}$ for $n \in \mathbb{N}$. We assume (H) holds. We set for $n \in \mathbb{N}$:

$$S_n = \operatorname*{ess\,sup}_{\tau\in\mathbb{T}_n} \mathbb{E}[G_\tau |\, \mathcal{F}_n]. \tag{III.15}$$

The next proposition is the main result of this section.

Proposition III.16. *We assume that G is adapted and hypothesis (H) and (III.2) hold. Then, for all $n \in \mathbb{N}$, we have $\mathbb{E}[S_n^+] < +\infty$. The sequence $(S_n, n \in \mathbb{N})$ satisfies the optimal equations (III.4). We also have $V_* = \mathbb{E}[S_0]$.*

Proof. Recall that (H) implies $\mathbb{E}[G_\tau^+] < +\infty$ for all $\tau \in \mathbb{T}_n$. Then use Lemma III.14 to deduce that $\mathbb{E}[S_n^+] = \mathbb{E}[G_{\tau_n^\circ}^+] < +\infty$. For $\tau \in \mathbb{T}_n$, we have (III.8) and (III.9) by definition of the essential supremum for S_{n+1}. We deduce that a.s. $\mathbb{E}[G_\tau |\, \mathcal{F}_n] \leq \max(G_n, \mathbb{E}[S_{n+1} |\, \mathcal{F}_n])$. This implies, see (ii) of Proposition III.12, that a.s. $S_n \leq \max(G_n, \mathbb{E}[S_{n+1} |\, \mathcal{F}_n])$.

According to Lemma III.14, there exists $\tau_{n+1}^\circ \in \mathbb{T}_{n+1}$ such that a.s. $S_{n+1} = \mathbb{E}[G_{\tau_{n+1}^\circ} | \mathcal{F}_{n+1}]$. Since τ_{n+1}° (resp. n) belongs also to $\mathbb{T}_n$, we have $S_n \geq \mathbb{E}[G_{\tau_{n+1}^\circ} | \mathcal{F}_n] = \mathbb{E}[S_{n+1} | \mathcal{F}_n]$ (resp. $S_n \geq G_n$). This implies that $S_n \geq \max(G_n, \mathbb{E}[S_{n+1} |\, \mathcal{F}_n])$. And thus $(S_n, n \in \mathbb{N})$ satisfies the optimal equations.

Use Corollary III.15 and Lemma III.14 to get $V_* = \mathbb{E}[S_0]$. □

We conclude this section by giving an explicit optimal stopping time.

Proposition III.17. *We assume that G is adapted and hypothesis (H) and (III.2) hold. Then τ_* defined by (III.5), with $(S_n, n \in \mathbb{N})$ given by (III.15), is optimal.*

Proof. If $V_* = -\infty$ then nothing has to be proven. So, we assume $V_* > -\infty$. According to Corollary III.15, there exists an optimal stopping time τ.

In a first step, we check that $\tau' = \min(\tau, \tau_*)$ is also optimal. Since $\mathbb{E}[G_\tau^+] < +\infty$, by Fubini and the definition of S_n, we have:

$$\mathbb{E}\left[G_\tau \mathbf{1}_{\{\tau>\tau_*\}}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[G_\tau \mathbf{1}_{\{\tau>\tau_*=n\}}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[\mathbb{E}[G_\tau|\mathcal{F}_n] \mathbf{1}_{\{\tau>\tau_*=n\}}\right] \leq \sum_{n\in\mathbb{N}} \mathbb{E}\left[S_n \mathbf{1}_{\{\tau>\tau_*=n\}}\right].$$

Since $S_n = G_n$ on $\{\tau_* = n\}$ for $n \in \mathbb{N}$, we deduce that:

$$\mathbb{E}\left[G_\tau \mathbf{1}_{\{\tau>\tau_*\}}\right] \leq \sum_{n\in\mathbb{N}} \mathbb{E}\left[G_n \mathbf{1}_{\{\tau>\tau_*=n\}}\right] = \mathbb{E}\left[G_{\tau_*} \mathbf{1}_{\{\tau>\tau_*\}}\right].$$

This implies that:

$$\mathbb{E}\left[G_\tau\right] = \mathbb{E}\left[G_\tau \mathbf{1}_{\{\tau>\tau_*\}}\right] + \mathbb{E}\left[G_\tau \mathbf{1}_{\{\tau\leq\tau_*\}}\right] \leq \mathbb{E}\left[G_{\tau_*} \mathbf{1}_{\{\tau>\tau_*\}}\right] + \mathbb{E}\left[G_\tau \mathbf{1}_{\{\tau\leq\tau_*\}}\right] = \mathbb{E}\left[G_{\tau'}\right].$$

And thus τ' is optimal.

In a second step we check that a.s. $\tau' = \tau_*$. Let us assume that $\mathbb{P}(\tau' < \tau_*) > 0$. Recall τ_n° defined in Lemma III.14. We define the stopping time τ'' by $\tau'' = \tau_*$ on $\{\tau' = \tau_*\}$ and $\tau'' = \tau_n^\circ$ on $\{n = \tau' < \tau_*\}$ for $n \in \mathbb{N}$. Since $\mathbb{E}[G_{\tau''}^+] < +\infty$, by Fubini and the definition of S_n, we have:

$$\mathbb{E}\left[G_{\tau''}\mathbf{1}_{\{\tau'<\tau_*\}}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[G_{\tau_n^\circ}\mathbf{1}_{\{n=\tau'<\tau_*\}}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[\mathbb{E}[G_{\tau_n^\circ}|\mathcal{F}_n]\mathbf{1}_{\{n=\tau'<\tau_*\}}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[S_n\mathbf{1}_{\{n=\tau'<\tau_*\}}\right].$$

Since $S_n > G_n$ on $\{\tau_* > n\}$ for $n \in \mathbb{N}$, we deduce that:

$$\mathbb{E}\left[G_{\tau''}\mathbf{1}_{\{\tau'<\tau_*\}}\right] > \sum_{n\in\mathbb{N}} \mathbb{E}\left[G_n\mathbf{1}_{\{n=\tau'<\tau_*\}}\right] = \mathbb{E}\left[G_{\tau'}\mathbf{1}_{\{\tau'<\tau_*\}}\right]$$

unless $\mathbb{E}\left[G_{\tau''}\mathbf{1}_{\{\tau'<\tau_*\}}\right] = \mathbb{E}\left[G_{\tau'}\mathbf{1}_{\{\tau'<\tau_*\}}\right] = -\infty$. The latter case is not possible since $\mathbb{E}\left[G_{\tau'}\right] = V_* > -\infty$. Thus, we deduce that $\mathbb{E}\left[G_{\tau''}\mathbf{1}_{\{\tau'<\tau_*\}}\right] > \mathbb{E}\left[G_{\tau'}\mathbf{1}_{\{\tau'<\tau_*\}}\right]$. This implies (using again that $\mathbb{E}[G_{\tau'}] > -\infty$) that:

$$\mathbb{E}\left[G_{\tau''}\right] = \mathbb{E}\left[G_{\tau'}\mathbf{1}_{\{\tau'=\tau_*\}}\right] + \mathbb{E}\left[G_{\tau''}\mathbf{1}_{\{\tau'<\tau_*\}}\right] > \mathbb{E}\left[G_{\tau'}\mathbf{1}_{\{\tau'=\tau_*\}}\right] + \mathbb{E}\left[G_{\tau'}\mathbf{1}_{\{\tau'<\tau_*\}}\right] = \mathbb{E}\left[G_{\tau'}\right].$$

This is impossible as τ' is optimal. Thus, we have a.s. $\tau' = \tau_*$ and τ_* is optimal. □

Exercise III.3. Assume that G is adapted and hypothesis (H) and (III.2) hold and $V_* > -\infty$.

1. Deduce from the proof of Proposition III.17, that τ_* is the minimal optimal stopping time.
2. Deduce that if $G_\infty = -\infty$ a.s., then a.s. τ_* is finite.

△

Exercise III.4. We assume that G is adapted and hypothesis (H) and (III.2) hold. We set:

$$W_n = \operatorname*{ess\,sup}_{\tau\in\mathbb{T}_{n+1}} \mathbb{E}[G_\tau|\,\mathcal{F}_n] \quad \text{and} \quad \tau_{**} = \inf\{n \in \mathbb{N};\, G_n > W_n\}, \tag{III.16}$$

with the convention that $\inf\emptyset = +\infty$.

1. Prove that $W_n = \mathbb{E}[S_{n+1}|\mathcal{F}_n]$ and that τ_{**} is optimal.
2. Assume that $V_* > -\infty$. Prove that if τ is optimal, then a.s. $\tau_* \leq \tau \leq \tau_{**}$.

△

Exercise III.5. Assume that G is adapted and hypothesis (H) and (III.2) hold, as well as $V_* > -\infty$. Prove that τ_* is regular. △

Example III.18 (Castle to sell: setting and solution). Continuation of Example III.2. Let X be a random variable taking values in $[-\infty, +\infty)$ such that $\mathbb{E}[(X^+)^2] < +\infty$ and $\mathbb{P}(X > -\infty) > 0$. Let $(X_n, n \in \mathbb{N}^*)$ be a sequence of independent random variables distributed as X. Let $c > 0$. We assume we can call back a previous buyer so that the gain at step $n \in \mathbb{N}^*$ is given by $G_n = M_n - nc$, with $M_n = \max_{1\leq k\leq n} X_k$ and X_k is the proposal of the k-th buyer of the castle. We set $G_\infty = -\infty$. We consider the σ-field $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \in \mathbb{N}^*$ and $\mathcal{F}_\infty = \bigvee_{n\in\mathbb{N}^*} \mathcal{F}_n$. (Notice that to stick to the presentation of this section, we could set $G_0 = -\infty$ and $\mathcal{F}_0$ the trivial σ-field.)

We first assume that X is bounded below, that is there exists $a \in \mathbb{R}$ such that a.s. $X \geq a$. Notice that $\max(x, y) = (x-y)^+ + y$ for $x, y \in \mathbb{R}$. In particular, if Y is a real random variable independent of X, we get:

$$\mathbb{E}[\max(X, Y)|Y] = f(Y) + Y \quad \text{with} \quad f(x) = \mathbb{E}[(X - x)^+].$$

We deduce that:

$$\mathbb{E}[G_{n+1}|\mathcal{F}_n] = \mathbb{E}[\max(X_{n+1}, M_n)|M_n] - (n+1)c = f(M_n) - c + G_n.$$

Since $\mathbb{E}[X^+]$ is finite, we get that the function $f(x) = \mathbb{E}[(X - x)^+]$ is continuous strictly decreasing on $(-\infty, x_0)$, with $x_0 = \sup\{x \in \mathbb{R}; \mathbb{P}(X \geq x) > 0\}$, and such that $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to x_0} f(x) = 0$. Since a.s. $\lim_{n\to\infty} M_n = x_0$, we get that a.s. $\lim_{n\to\infty} f(M_n) = 0$. Thus the stopping time $\tau = \inf\{n \in \mathbb{N}^*,\ f(M_n) \leq c\}$ is a.s. finite. From the properties of f, we deduce there exists a unique $c_* \in \mathbb{R}$ such that $f(c_*) = c$. Using that $(f(M_n), n \in \mathbb{N}^*)$ is non increasing and that it jumps at record times of the sequence $(X_n, n \in \mathbb{N}^*)$, we get the representation:

$$\tau = \inf\{n \in \mathbb{N}^*,\ X_n \geq c_*\}.$$

Furthermore, for $n \in \mathbb{N}^*$, we have a.s. that:

$$\mathbb{E}[G_{n+1}|\mathcal{F}_n] > G_n \quad \text{on} \quad \{n < \tau\}, \tag{III.17}$$

$$\mathbb{E}[G_{n+1}|\mathcal{F}_n] \leq G_n \quad \text{on} \quad \{n \geq \tau\}. \tag{III.18}$$

According to Lemma III.19, we have that (III.1) and (III.2) hold. According to Proposition III.17, τ_* given by (III.5) is optimal. We deduce from (III.17) that a.s. $\tau_* \geq \tau$. Since a.s. $X \geq a$ and (III.1) holds, we get that $\mathbb{E}[|G_n|] < +\infty$ for all $n \in \mathbb{N}^*$. Mimicking the proof of the stopping theorem for super-martingale, using that $\tau_* \geq \tau$, we deduce from (III.18) that a.s. on $\{\tau = n\}$ for all finite $k \geq n$:

$$\mathbb{E}[G_{\tau_*\wedge k}|\mathcal{F}_n] \leq G_n.$$

Letting k goes to infinity, we deduce from (III.2) and Fatou lemma that $\mathbb{E}[G_{\tau_*}|\mathcal{F}_n] \leq G_n$ a.s. on $\{\tau = n\}$. Since τ is bounded, this gives $\mathbb{E}[G_{\tau_*}] \leq \mathbb{E}[G_\tau]$. Since τ_* is optimal, we deduce that τ is also optimal (and in fact equal to τ_* as τ_* is the minimal optimal stopping time according to Exercise III.3). We have:

$$V_* = \mathbb{E}[G_\tau] = \mathbb{E}[X_\tau] - c\mathbb{E}[\tau] = \frac{\mathbb{E}[X\mathbf{1}_{\{X\geq c_*\}}]}{\mathbb{P}(X \geq c_*)} - \frac{c}{\mathbb{P}(X \geq c_*)} = \frac{\mathbb{E}[(X - c_*)^+] - c}{\mathbb{P}(X \geq c_*)} + c_* = c_*,$$

where we used that τ is geometric with parameter $\mathbb{P}(X \geq c_*)$ for the third equality and $\mathbb{E}[(X - c_*)^+] = f(c_*) = c$ for the last.

If X is not bounded below by a constant a, we set $X^a = \max(a, X)$, $M_n^a = \max(a, M_n)$ and $G_n^a = M_n^a - nc$ for $n \in \mathbb{N}^*$. Let $V_*^a = \sup_{\tau \in \mathbb{T}} \mathbb{E}[G_\tau^a]$, with $\mathbb{T}$ the set of stopping times with respect to the filtration $\mathcal{F}$. Since $G_n \leq G_n^a$ for all $n \in \mathbb{N}^*$, we deduce that $V_* \leq V_*^a$. According to the first part, we get that $V_*^a = \mathbb{E}[G_{\tau^a}^a] = c_*^a$ with $\tau^a = \inf\{n \in \mathbb{N}^*,\ X_n^a \geq c_*^a\}$ and c_*^a the unique root of $\mathbb{E}[(\max(X, a) - x)^+] = c$. Let c_* be the unique root of $\mathbb{E}[(X - x)^+] = c$ and set $\tau = \inf\{n \in \mathbb{N}^*,\ X_n \geq c_*\}$. Notice that for $a < c_*$ we have $c_*^a = c_*$, and thus $\tau^a = \tau$ as well as $G_{\tau^a}^a = G_\tau$. We deduce that $V_* \geq \mathbb{E}[G_\tau] = V_*^a$, and thus $V_* = \mathbb{E}[G_\tau] = c_*$ and τ is optimal.

If one can not call back a previous buyer, then the gain is $G_n'' = X_n - nc$. Let V_*'' be the corresponding maximal gain. On the one hand, since $G_n'' \leq G_n$ for all $n \in \mathbb{N}$, we deduce that $V_*'' \leq V_*$. On the other hand, we have $G_\tau'' = G_\tau$. This implies that $V_*'' \geq \mathbb{E}[G_\tau''] = \mathbb{E}[G_\tau] = V_*$. We deduce that $V_*'' = c_*$ and τ is also optimal in this case.

Lemma III.19. *Let X be a random variable taking values in $[-\infty, +\infty)$. Let $(X_n, n \in \mathbb{N}^*)$ be a sequence of random variables distributed as X. Let $c \in]0, +\infty[$. Set $G_n = \max_{1 \leq k \leq n} X_k - nc$ for $n \in \mathbb{N}^*$. If $\mathbb{E}[(X^+)^2] < +\infty$, then $\mathbb{E}[\sup_{n \in \mathbb{N}^*} G_n^+] < +\infty$ and $\limsup G_n = -\infty$.*

With the notation of Lemma III.19, one can prove that if the random variables $(X_n, n \in \mathbb{N}^*)$ are independent then $\mathbb{E}[\sup_{n \in \mathbb{N}^*} (G_n'')^+] < +\infty$ implies that $\mathbb{E}[(X^+)^2] < +\infty$.

Proof. Assume that $\mathbb{E}[(X^+)^2] < +\infty$. Since $X_n - nc \leq G_n \leq \max_{1 \leq k \leq n}(X_k - kc)$ for all $n \in \mathbb{N}^*$, we deduce that $\sup_{n \in \mathbb{N}^*} G_n = \sup_{n \in \mathbb{N}^*} (X_n - nc)$. This gives:

$$\mathbb{E}\Big[\sup_{n \in \mathbb{N}^*} G_n^+\Big] = \mathbb{E}\Big[\sup_{n \in \mathbb{N}^*} (X_n - nc)^+\Big] \leq \mathbb{E}\Big[\sum_{n \in \mathbb{N}^*} (X_n - nc)^+\Big] = \mathbb{E}\Big[\sum_{n \in \mathbb{N}^*} (X - nc)^+\Big],$$

where we used Fubini (twice) and that X_n is distributed as X in the last equality. Then use that for $x \in \mathbb{R}$:

$$\sum_{n \in \mathbb{N}^*} (x - n)^+ \leq \sum_{n \in \mathbb{N}^*} x^+ \mathbf{1}_{\{n < x^+\}} \leq (x^+)^2,$$

to get $\mathbb{E}\Big[\sum_{n \in \mathbb{N}^*} (X - nc)^+\Big] \leq \mathbb{E}[(X^+)^2]/c < +\infty$. So we obtain $\mathbb{E}\Big[\sup_{n \in \mathbb{N}^*} G_n^+\Big] < +\infty$.

Set $G_n' = \max_{1 \leq k \leq n} X_k - nc/2$. Using the previous result (with c replaced by $c/2$), we deduce that $\sup_{n \in \mathbb{N}^*} (G_n')^+$ is integrable and thus a.s. $\limsup G_n' < +\infty$. Since $G_n = G_n' - nc/2$, we get that a.s. $\limsup G_n \leq \limsup G_n' - \lim nc/2 = -\infty$. □

III.3.4 The general case

We state the main result of this section. Let $\mathbb{T}$ denote the set of stopping times (taking values in $\bar{\mathbb{N}}$) with respect to the filtration $(\mathcal{F}_n, n \in \mathbb{N})$.

Corollary III.20. *Let $G = (G_n, n \in \bar{\mathbb{N}})$ be a sequence of random variables such that (III.1) and (III.2) hold. Then there exists an optimal stopping time.*

Proof. According to the first paragraph of Section III.3, without loss of generality, we can assume that $\mathcal{F}_\infty = \bigvee_{n \in \mathbb{N}} \mathcal{F}_n$. If G is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n, n \in \bar{\mathbb{N}})$ then use $M = \sup_{n \in \bar{\mathbb{N}}} G_n^+$, so that (H) holds, and Corollary III.15 to conclude.

If the sequence G is not adapted to the filtration $\mathcal{F}$, then we shall consider the corresponding adapted sequence $G' = (G'_n, n \in \bar{\mathbb{N}})$ given by $G'_n = \mathbb{E}[G_n|\mathcal{F}_n]$ for $n \in \bar{\mathbb{N}}$. Notice G' is well defined thanks to (III.1). Thanks to (III.1), we can use Fubini lemma to get for $\tau \in \mathbb{T}$:

$$\mathbb{E}[G_\tau] = \sum_{n\in\bar{\mathbb{N}}} \mathbb{E}[G_n \mathbf{1}_{\{\tau=n\}}] = \sum_{n\in\bar{\mathbb{N}}} \mathbb{E}[G'_n \mathbf{1}_{\{\tau=n\}}] = \mathbb{E}[G'_\tau].$$

We thus deduce the maximal gain for the game G is also the maximal gain for the game G'.

Let $M = \mathbb{E}\big[\sup_{n\in\bar{\mathbb{N}}} G_n^+|\mathcal{F}_\infty\big]$. Notice then that (H) holds with G replaced by G'. Then, to conclude using Corollary III.15, it is enough to check that (III.2) holds with G replaced by G'.

For $a \in \mathbb{R}$ and $n \geq k$ finite, using Jensen inequality, we get:

$$\max(a, G'_n) \leq \mathbb{E}[\max(a, G_n)|\mathcal{F}_n] \leq \mathbb{E}\big[\max\big(a, \sup_{\ell\in[\![k,\infty]\!]} G_\ell\big)\big|\,\mathcal{F}_n\big].$$

Since $\mathbb{E}\big[|\max\big(a, \sup_{\ell\in[\![k,\infty]\!]} G_\ell\big)|\big]$ is finite thanks to (III.1), we deduce from Corollary IV.3 that:

$$\lim_{n\to+\infty} \mathbb{E}\big[\max\big(a, \sup_{\ell\in[\![k,\infty]\!]} G_\ell\big)\big|\,\mathcal{F}_n\big] = \mathbb{E}\big[\max\big(a, \sup_{\ell\in[\![k,\infty]\!]} G_\ell\big)\big|\,\mathcal{F}_\infty\big].$$

Thus we get $\max(a, \limsup G'_n) \leq \mathbb{E}\big[\max\big(a, \sup_{\ell\in[\![k,\infty]\!]} G_\ell\big)\big|\,\mathcal{F}_\infty\big]$. Letting k goes to infinity, we get by dominated convergence and using (III.2) that:

$$\max(a, \limsup G'_n) \leq \mathbb{E}\big[\max\big(a, \limsup G_n\big)\big|\,\mathcal{F}_\infty\big] \leq \mathbb{E}[\max(a, G_\infty)|\,\mathcal{F}_\infty].$$

This gives $\limsup G'_n \leq \mathbb{E}[\max(a, G_\infty)|\,\mathcal{F}_\infty]$. Letting a goes to $-\infty$, we get by monotone convergence that $\limsup G'_n \leq \mathbb{E}[G_\infty|\,\mathcal{F}_\infty] = G'_\infty$. Thus (III.2) holds with G replaced by G'. This finishes the proof. □

Exercise III.6. Let $G = (G_n, n \in \bar{\mathbb{N}})$ be a sequence of random variables such that (III.1) and (III.2) hold. Let $\tau_* = \inf\{n \in \mathbb{N};\, \operatorname{ess\,sup}_{\tau\in\mathbb{T}_n} \mathbb{E}[G_\tau|\mathcal{F}_n] = \mathbb{E}[G_n|\mathcal{F}_n]\}$ with $\inf\emptyset = \infty$. Prove that τ_* is optimal. △

III.4 From finite horizon to infinite horizon

In the finite horizon case, the random variable S_n is defined recursively, and thus defined in a constructive way. There is no such constructive way in the infinite horizon case. Thus, is it a natural to ask if the infinite horizon case is the limit of finite horizon cases, when the horizon ζ goes to infinity. We shall give sufficient condition for this to hold.

We assume in this section that the sequence $G = (G_n, n \in \bar{\mathbb{N}})$ is adapted to the filtration $\mathcal{F} = (\mathcal{F}_n, n \in \bar{\mathbb{N}})$ (with $\mathcal{F}_\infty = \bigvee_{n\in\mathbb{N}} \mathcal{F}_n$), and that (III.1) holds. We shall also consider the following assumption which is stronger than (III.2):

$$\limsup G_n = G_\infty \quad \text{a.s..} \tag{III.19}$$

Recall that $\mathbb{T}_n = \{\tau \in \mathbb{T};\ \tau \geq n\}$ for $n \in \mathbb{N}$ and set $\mathbb{T}^\zeta = \{\tau \in \mathbb{T};\ \tau \leq \zeta\}$ for $\zeta \in \mathbb{N}$. For $\zeta \in \mathbb{N}$ and $n \in [\![0, \zeta]\!]$ we define $\mathbb{T}_n^\zeta = \mathbb{T}_n \bigcap \mathbb{T}^\zeta$ as well as:

$$S_n^\zeta = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}_n^\zeta} \mathbb{E}[G_\tau |\, \mathcal{F}_n]. \tag{III.20}$$

From Sections III.2.1 and III.3.3, we get that $S_\zeta^\zeta = G_\zeta$ and $S^\zeta = (S_n^\zeta, n \in [\![0, \zeta]\!])$ satisfies the optimal equations (III.4). For $n \in \mathbb{N}$, the sequence $(S_n^\zeta, \zeta \in [\![n, \infty[\![)$ is non-decreasing and denote by S_n^* its limit. Notice that $(S_n^*, n \in \mathbb{N})$ satisfies the optimal equations (III.4) with $\zeta = \infty$. In fact $(S_n^*, n \in \mathbb{N})$ is the smallest sequence satisfying the optimal equations (III.4) with $\zeta = \infty$. For $n \in \mathbb{N}$, we have a.s. $S_n^* = \operatorname{ess\,sup}_{\tau \in \mathbb{T}_n^{(\mathrm{b})}} \mathbb{E}[G_\tau |\, \mathcal{F}_n]$, where $\mathbb{T}_n^{(\mathrm{b})} = \mathbb{T}_n \bigcap \mathbb{T}^{(\mathrm{b})}$ and $\mathbb{T}^{(\mathrm{b})} \subset \mathbb{T}$ is the subset of bounded stopping times.

By construction of S_n, we have for all $n \in \mathbb{N}$:

$$S_n^* \leq S_n, \tag{III.21}$$

The sequence $(\tau_*^\zeta, \zeta \in \mathbb{N})$, with $\tau_*^\zeta = \inf\{n \in [\![0, \zeta]\!];\ S_n^\zeta = G_n\}$, is non-decreasing and thus converge to a limit, say $\tau_*^* \leq \tau_*$ with $\tau_*^* \in \bar{\mathbb{N}}$ and

$$\tau_*^* = \inf\{n \in \mathbb{N};\ S_n^* = G_n\}. \tag{III.22}$$

We set $V_*^\zeta = \mathbb{E}[S_0^\zeta] = \sup_{\tau \in \mathbb{T}^\zeta} \mathbb{E}[G_\tau]$ and $V_* = \mathbb{E}[S_0] = \sup_{\tau \in \mathbb{T}} \mathbb{E}[G_\tau]$. Let V_*^* be the non-decreasing limit of the sequence $(V_*^\zeta, \zeta \in \mathbb{N})$, so that $V_*^* \leq V_*$. We shall say the infinite horizon case is the limit of the finite horizon cases if $V_*^* = V_*$. Notice, we don't have $V_*^* = V_*$ in all cases, see Example III.21 taken from [3].

Example III.21. Let $(X_n, n \in \mathbb{N}^*)$ be independent random variables such that $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$ for all $n \in \mathbb{N}$. Let $c = (c_n, n \in \mathbb{N}^*)$ be a strictly increasing sequence such that $0 < c_n < 1$ for all $n \in \mathbb{N}^*$ and $\lim_{n \to \infty} c_n = 1$. We define $G_0 = 0$, $G_\infty = 0$, and for $n \in \mathbb{N}^*$:

$$G_n = \min\big(1, W_n\big) - c_n,$$

with $W_n = \sum_{k=1}^n X_k$. Notice that $G_n \leq 1$ and a.s. $\limsup G_n = G_\infty$ so that (III.1) and (III.19) hold. (Notice also that $\mathbb{E}[|G_n|]$ for all $n \in \bar{\mathbb{N}}$.) Since $\mathbb{E}[W_{n+1}|\mathcal{F}_n] = W_n$, we deduce from Jensen inequality that a.s. $\mathbb{E}[\min(1, W_{n+1})|\mathcal{F}_n] \geq \min(1, W_n)$. Then use that the sequence c is strictly increasing to get that for all $n \in \mathbb{N}$ a.s. $G_n > \mathbb{E}[G_{n+1}|\mathcal{F}_n]$. The optimal equations imply that $S_n^\zeta = G_n$ for all $n \in [\![0, \zeta]\!]$ and $\zeta \in \mathbb{N}$ and thus $\tau_*^\zeta = 0$. We deduce that $S_n^* = G_n$ for all $n \in \mathbb{N}$, $\tau_*^* = 0$ and $V_*^* = 0$.

Since (III.1) and (III.2) hold, we deduce there exists an optimal stopping time for the infinite horizon case. The stopping time $\tau = \inf\{n \in \mathbb{N}^*;\ W_n = 1\}$ is a.s. strictly positive and finite. On $\{\tau = n\}$, we have that $G_n = 1 - c_n$ as well as $G_m \leq 0 < G_n$ for all $m \in [\![0, n-1]\!]$ and $G_m \leq 1 - c_m < G_n$ for all $m \in]\!]n, \infty]\!]$. We deduce that $G_\tau = \sup_{\tau' \in \mathbb{T}} G_{\tau'}$, that is $\tau = \tau_*$ is optimal. Notice that $V_* > V_*^* = 0$ and a.s. $\tau_* > \tau_*^* = 0$. Thus, the infinite horizon case is not the limit of the finite horizon cases.

We end this Section by giving sufficient conditions so that the infinite horizon case is the limit of the finite horizon cases. In particular, we shall consider the following condition (which is stronger than (III.2)):

$$\lim_{n\nearrow\infty} G_n = G_\infty \quad \text{a.s..} \tag{III.23}$$

Proposition III.22. *Let $(G_n, n \in \mathbb{N})$ be an adapted sequence of random variables taking values in $\mathbb{R}$ and define G_∞ by (III.19). Assume that (III.1) holds and that the sequence $(T_n, n \in \mathbb{N})$, with $T_n = \sup_{k\geq n} G_k - G_n$, is uniformly integrable. If there exists an a.s. finite optimal stopping time or if (III.23) holds, then the infinite horizon case is the limit of the finite horizon cases.*

Proof. If $V_* = -\infty$, nothing has to be proven. Let us assume that $V_* > -\infty$. According to Proposition III.17 (and since (III.1) implies condition (H)), there exists an optimal stopping time, say τ. Since $\mathbb{E}[G_{\min(\tau,n)}] \leq V_*^n$, we get:

$$\begin{aligned} 0 \leq V_* - V_*^n \leq \mathbb{E}[G_\tau - G_{\min(\tau,n)}] &= \mathbb{E}\left[\mathbf{1}_{\{n<\tau<\infty\}}(G_\tau - G_n)\right] + \mathbb{E}\left[\mathbf{1}_{\{\tau=\infty\}}(G_\infty - G_n)\right] \\ &\leq \mathbb{E}\left[\mathbf{1}_{\{n<\tau<\infty\}}T_n\right] + \mathbb{E}\left[\mathbf{1}_{\{\tau=\infty\}}(G_\infty - G_n)^+\right]. \end{aligned}$$

Recall $(T_n, n \in \mathbb{N})$ is uniformly integrable. Since a.s. $\lim_{n\to+\infty} \mathbf{1}_{\{n<\tau<\infty\}} = 0$, we deduce from Corollary IV.5 (with $Y_n = |T_n|$) that $\lim_{n\to+\infty} \mathbb{E}\left[\mathbf{1}_{\{n<\tau<\infty\}}T_n\right] = 0$.

If τ is a.s. finite, then we have $\mathbb{E}\left[\mathbf{1}_{\{\tau=\infty\}}(G_\infty - G_n)^+\right] = 0$. If (III.23) holds, then we have that a.s. $\lim_{n\to+\infty}(G_\infty - G_n)^+ = 0$. Since $\mathbf{1}_{\{\tau=\infty\}}(G_\infty - G_n)^+ \leq |T_n|$, we deduce from Corollary IV.5 (with $Y_n = |T_n|$) that $\lim_{n\to+\infty} \mathbb{E}\left[\mathbf{1}_{\{\tau=\infty\}}(G_\infty - G_n)^+\right] = 0$. In both cases, we deduce that $\lim_{n\to\infty} V_* - V_*^n = 0$. This gives the result. □

We give an immediate Corollary of this result.

Corollary III.23. *Let $(G_n, n \in \mathbb{N})$ be an adapted sequence of random variables taking values in $\mathbb{R}$ and define G_∞ by (III.19). Assume that for $n \in \mathbb{N}$ we have $G_n = Z_n - W_n$, with $(Z_n, n \in \mathbb{N})$ adapted, $\mathbb{E}[\sup_{n\in\mathbb{N}} |Z_n|] < +\infty$ and $(W_n, n \in \mathbb{N})$ an adapted non-decreasing sequence of non-negative random variables. If there exists an a.s. finite optimal stopping time or if (III.23) holds, then the infinite horizon case is the limit of the finite horizon cases.*

Proof. For $k \geq n$, we have $G_k - G_n \leq Z_k - Z_n \leq 2\sup_{\ell\in\mathbb{N}} |Z_\ell|$. This gives that the sequence $(T_n = \sup_{k\geq n} G_k - G_n, n \in \mathbb{N})$ is non-negative and bounded by $2\sup_{\ell\in\mathbb{N}} |Z_\ell|$, hence it is uniformly integrable. We conclude using Proposition III.22. □

Using super-martingale theory, we can prove directly the following result (which is not a direct consequence of the previous Corollary with $W_n = 0$).

Proposition III.24. *Let $(G_n, n \in \mathbb{N})$ be an adapted sequence of random variables taking values in $\mathbb{R}$ and define G_∞ by (III.19). Assume that $\mathbb{E}[\sup_{n\in\mathbb{N}} |G_n|] < +\infty$. Then the infinite horizon case is the limit of the finite horizon cases. Further more, we have that a.s. $\tau_* = \tau_*^*$, where τ_* is defined by (III.5) with $(S_n, n \in \mathbb{N})$ given by (III.15), and τ_*^* is defined by (III.22) with $(S_n^*, n \in \mathbb{N})$ given by (III.20).*

Proof. Recall that $(S_n^*, n \in \mathbb{N})$ satisfies the optimal equations (III.4) with $\zeta = \infty$. Since it is bounded by $\sup_{n\in\mathbb{N}} |G_n|$ which is integrable, it is a super-martingale and it converges a.s. to a limit say S_∞^*. We have $S_n^* \geq G_n$ for all $n \in \mathbb{N}$, which implies thanks to (III.19) that $S_\infty^* \geq G_\infty$. Since $(S_n^*, n \in \mathbb{N})$ is a uniformly integrable super-martingale, we deduce from the stopping theorem, see Corollary IV.4, that for $n \in \mathbb{N}$ fixed, we have $S_n^* \geq \mathbb{E}[S_\tau^*|\mathcal{F}_n] \geq \mathbb{E}[G_\tau|\mathcal{F}_n]$ a.s. for all stopping time $\tau \geq n$. This implies that $S_n^* \geq S_n$ a.s. according to Proposition III.12. Thanks to (III.21), we get that a.s. $S_n^* \leq S_n$ and thus a.s. $S_n^* = S_n$ for all $n \in \mathbb{N}$. In particular, we have $V_* = V_*^*$. Thus, the infinite horizon case is the limit of the finite horizon cases. Using (III.22), we get that a.s. $\tau_* = \tau_*^*$. □

Example III.25 (Castle to sell: from finite to infinite horizon). We keep notations and hypothesis from Example III.18 and consider the gain $G_n = M_n - nc$ with $M_n = \max_{1\leq k\leq n} X_k$ and $c > 0$. Recall the random variables $(X_k, k \in \mathbb{N}^*$ are independent and distributed as X. We assume furthermore that $\mathbb{E}[|X|] < +\infty$. Notice hypothesis of Corollary III.23 are not fulfilled (with $Z_n = M_n$ and $W_n = nc$) unless X is bounded. However, we can follow the proof of Proposition III.22 to check the infinite horizon case is the limit of the finite horizon cases. Since the optimal stopping time $\tau_* = \inf\{n \in \mathbb{N}^*, X_n \geq c_*\}$ is a.s. finite, see Example III.18, we have, with the notation of the proof of Proposition III.22 that:

$$0 \leq V_* - V_*^n \leq \mathbb{E}[G_{\tau_*} - G_{\min(\tau_*,n)}] = \mathbb{E}\left[\mathbf{1}_{\{n<\tau_*<\infty\}}(G_{\tau_*} - G_n)\right] \leq \mathbb{E}\left[\mathbf{1}_{\{n<\tau_*<\infty\}}(X_{\tau_*} - X_1)\right],$$

where we used $G_{\tau_*} - G_n = X_{\tau_*} - \tau_* c - M_n + nc \leq X_{\tau_*} - X_1$ for the last equation. Since X_{τ_*} and X_1 are integrable, we get $\lim_{n\to+\infty} \mathbb{E}\left[\mathbf{1}_{\{n<\tau_*<\infty\}}(X_{\tau_*} - X_1)\right] = 0$ by dominated convergence. That is $\lim_{n\to+\infty} V_*^n = V_*$ and thus the infinite horizon case is the limit of the finite horizon cases. (Notice that if $1 > \mathbb{P}(X = -\infty) > 0$, then the infinite horizon case is no more the limit of the finite horizon cases as $V_*^n = -\infty$ for all $n \in \mathbb{N}^*$.)

If one can not call back a previous buyer, then the gain is $G_n'' = X_n - nc$. Let V_* (resp. $(V_*'')^n$) denote the maximal gain when the horizon is infinite (resp. equal to n). Arguing as above, we get:

$$0 \leq V_*'' - (V_*'')^n \leq \mathbb{E}\left[\mathbf{1}_{\{n<\tau_*<\infty\}}(X_{\tau_*} - X_n)\right] = \mathbb{E}\left[\mathbf{1}_{\{n<\tau_*<\infty\}}(X_{\tau_*} - X_1)\right],$$

as conditionally on $\{n < \tau_* < \infty\}$, (X_{τ_*}, X_n) and (X_{τ_*}, X_1) have the same distribution. And we deduce the infinite horizon case is the limit of the finite horizon cases.

Example III.26. Extend Proposition III.24 to the non adapted case.

III.5 The Markovian case

We assume in this section that $\zeta = \infty$. Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a filtration and we write $\mathbb{T}^{(b)} \subset \mathbb{T}$ for the subset of bounded stopping times. Let $(X_n, n \in \mathbb{N})$ be a Markov chain with state space E (at most countable) and transition kernel P. Let φ be a non-negative function defined on E. We shall consider the optimal stopping problem for the game with gain $G_n = \varphi(X_n)$ for $n \in \mathbb{N}$ and $G_\infty = \limsup G_n$.

We set:

$$\varphi_0 = \varphi \quad \text{and, for } n \in \mathbb{N}, \quad \varphi_{n+1} = \max(\varphi, P\varphi_n).$$

Lemma III.27. *The sequence of functions* $(\varphi_n, n \in \mathbb{N})$ *is non-decreasing and converges to a limit say* φ_* *such that* $\varphi_* = \max(\varphi, P\varphi_*)$. *For any non-negative function* g *such that* $g \geq \max(\varphi, Pg)$, *we have that* $g \geq \varphi_*$.

Proof. By an elementary induction argument, we get that the sequence $(\varphi_n, n \in N)$ is non-decreasing. Let φ_* be its limit. By monotone convergence, we get that $\varphi_* = \max(\varphi, P\varphi_*)$. Let g be a non-negative function g such that $g \geq \max(\varphi, Pg)$, we have by induction that $g \geq \varphi_n$ and thus $g \geq \varphi_*$. □

We now give the main result of this section.

Proposition III.28. *Let* $x \in E$ *and* φ *a non-negative function defined on* E. *Assume that* $\mathbb{E}_x[\sup_{n\in\mathbb{N}} \varphi(X_n)] < +\infty$. *Then, we have:*

$$\varphi_*(x) = \sup_{\tau\in\mathbb{T}^{(b)}} \mathbb{E}_x[\varphi(X_\tau)] = \sup_{\tau\in\mathbb{T}} \mathbb{E}_x[\varphi(X_\tau)] = \mathbb{E}[\varphi(X_{\tau_*})],$$

$$\textit{with} \quad \tau_* = \inf\{n \in \mathbb{N};\, X_n \in \{\varphi = \varphi_*\}\},$$

and the conventions $\inf \emptyset = +\infty$ *and* $\varphi(X_\infty) = \limsup \varphi(X_n)$.

Proof. We keep notations from Section III.4. Recall definition (III.20) of S_n^ζ for the finite horizon $\zeta \in \mathbb{N}$. We deduce from (III.4) and $S_\zeta^\zeta = G_\zeta$, that $S_n^\zeta = \varphi_{\zeta-n}(X_n)$ for all $0 \leq n \leq \zeta < \infty$ and $\tau_*^\zeta = \inf\{n \in [\![0, \zeta]\!];\, \varphi_{\zeta-n}(X_n) = \varphi(X_n)\}$. Lemma III.27 implies that $S_n^* = \lim_{\zeta\nearrow\infty} S_n^\zeta = \varphi_*(X_n)$ and thus

$$\tau_*^* = \lim_{\zeta\nearrow\infty} \tau_*^\zeta = \inf\{n \in \mathbb{N};\, X_n \in \{\varphi = \varphi_*\}\}, \tag{III.24}$$

with the convention that $\inf \emptyset = \infty$. According to Proposition III.24, the infinite horizon case is the limit of the finite horizon cases and the optimal stopping time τ_* given by (III.5) is a.s. equal to τ_*^*. We deduce that $V_* = \mathbb{E}_x[S_0^*] = \varphi_*(x)$. □

Chapter IV

Divers

Let $\mathbb{T}$ be the set of stopping times taking values in $\bar{\mathbb{N}} = \mathbb{N} \bigcup \{+\infty\}$.

Remark IV.1 (Doob decomposition of a super-martingale).

We shall use the following result.

Lemma IV.2. *Let $\mathcal{G}' \subset \mathcal{G}$ be a σ-field. Let $X = (X_n, n \in \mathbb{N})$, resp. $Y = (Y_n, n \in \mathbb{N})$, be a sequence of random variables taking values in $[-\infty, +\infty)$, resp. in $[0, +\infty)$, such that a.s. for all $n \in \mathbb{N}$, $X_n^+ \leq Y_n$ and Y converges a.s. towards a limits, say Y_∞ and $\lim_{n\to+\infty} \mathbb{E}[Y_n] = \mathbb{E}[Y_\infty]$. Then, we have $\limsup \mathbb{E}[X_n|\mathcal{G}'] \leq \mathbb{E}[\limsup X_n|\mathcal{G}']$.*

Proof. Recall $x^- = \max(0, -x)$ and $x = x^+ - x^-$. By Fatou lemma, we have that a.s.:

$$\begin{aligned}\mathbb{E}[Y_\infty|\mathcal{G}'] - \limsup \mathbb{E}[X_n^+|\mathcal{G}'] &= \liminf \mathbb{E}[Y_n - X_n^+|\mathcal{G}'] \\ &\geq \mathbb{E}[\liminf (Y_n - X_n^+)|\mathcal{G}'] = \mathbb{E}[Y_\infty|\mathcal{G}'] - \mathbb{E}[\limsup X_n^+|\mathcal{G}'].\end{aligned}$$

Thus, we have that a.s. $\limsup \mathbb{E}[X_n^+|\mathcal{G}'] \leq \mathbb{E}[\limsup X_n^+|\mathcal{G}']$. By Fatou lemma, we have also that a.s. $\liminf \mathbb{E}[X_n^-|\mathcal{G}'] \geq \mathbb{E}[\liminf X_n^-|\mathcal{G}']$, that is $\limsup \mathbb{E}[-X_n^-|\mathcal{G}'] \leq \mathbb{E}[\limsup -X_n^-|\mathcal{G}']$. By summing, we deduce that a.s. $\limsup \mathbb{E}[X_n|\mathcal{G}'] \leq \mathbb{E}[\limsup X_n|\mathcal{G}']$. □

Corollary IV.3. *Let M be an integrable random variables. Let $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be a filtration. Set $\mathcal{F}_\infty = \bigvee_{n\in\mathbb{N}} \mathcal{F}_n$. Set $M_\tau = \mathbb{E}[M|\mathcal{F}_\tau]$ for any stopping time τ, taking values in $\bar{\mathbb{N}}$, with respect to the filtration $\mathcal{F}$. Then, for any converging sequence of stopping times $(\tau_n, n \in \mathbb{N})$, with $\tau = \lim_{n\to+\infty} \tau_n$, we have that the sequence $(M_{\tau_n}, n \in \mathbb{N})$ converges a.s. and in L^1 towards M_τ.*

Corollary IV.4. *Let $(M_n, n \in \mathbb{N})$ be a super-martingale converging to M_∞. Assume that $(M_n, n \in \mathbb{N})$ is uniformly integrable. Then for all stopping times $\tau \leq \tau'$ taking values in $\bar{\mathbb{N}}$, we have a.s. $M_\tau \geq \mathbb{E}[M_{\tau'}|\mathcal{F}_\tau]$.*

Corollary IV.5. *Let $(X_n, n \in \mathbb{N})$ a sequence of random variables taking values in $\mathbb{R}$ which converges a.s. to a finite random variable and such that $|X_n| \leq Y_n$, where the sequence of random variables $(Y_n, n \in \mathbb{N})$ is uniformly integrable. Then the sequence $(X_n, n \in \mathbb{N})$ converges in L^1.*

Proof. Set $X_\infty = \lim_{n\to+\infty} X_n$ and $W = \sup_{n\in\mathbb{N}} |X_n|$. We have:

$$\mathbb{E}[|X_\infty|] \leq \liminf_{n\to+\infty} \mathbb{E}[|X_n|] \leq \sup_{n\in\mathbb{N}} \mathbb{E}[Y_n] < +\infty,$$

where we used Fatou lemma for the first inequality and that $(Y_n, n \in \mathbb{N})$ is uniformly integrable for the last. We have:

$$\mathbb{E}[|X_n - X_\infty|] \leq \mathbb{E}\left[\mathbf{1}_{\{W\leq c\}}|X_n - X_\infty|\right] + \mathbb{E}\left[\mathbf{1}_{\{W>c\}}(Y_n + |X_\infty|)\right].$$

Let $\varepsilon > 0$. Since W is finite a.s. as X_∞ is finite, using that $(Y_n, n \in \mathbb{N})$ and X_∞ are uniformly integrable, we have that for c large enough $\sup_{n\in\mathbb{N}} \mathbb{E}\left[\mathbf{1}_{\{W>c\}}(Y_n + |X_\infty|)\right] \leq \varepsilon$. by dominated convergence, we get that $\lim_{n\to+\infty} \mathbb{E}\left[\mathbf{1}_{\{W\leq c\}}|X_n - X_\infty|\right] = 0$. This gives that $\limsup_{n\to+\infty} \mathbb{E}[|X_n - X_\infty|] \leq \varepsilon$. Then use that $\varepsilon > 0$ is arbitrary to conclude. $\square$

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