ASYMPTOTIC PROPERTIES OF EXPANSIVE GALTON-WATSON TREES

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Abstract. We consider a super-critical Galton-Watson tree whose non-degenerate offspring distribution has finite mean. We consider the random trees τ_n distributed as τ conditioned on the n-th generation, Z_n, to be of size $a_n \in \mathbb{N}$. We identify the possible local limits of τ_n as n goes to infinity according to the growth rate of a_n. In the low regime, the local limit τ^0 is the Kesten tree, in the moderate regime the family of local limits, τ^θ for $\theta \in (0, +\infty)$, is distributed as τ conditionally on $\{W = \theta\}$, where W is the (non-trivial) limit of the renormalization of Z_n. In the high regime, we prove the local convergence towards τ^∞ in the Harris case (finite support of the offspring distribution) and we give a conjecture for the possible limit when the offspring distribution has some exponential moments. When the offspring distribution has a fat tail, the problem is open. The proof relies on the strong ratio theorem for Galton-Watson processes. Those latter results are new in the low regime and high regime, and they can be used to complete the description of the (space-time) Martin boundary of Galton-Watson processes. Eventually, we consider the continuity in distribution of the local limits $(\tau^\theta, \theta \in [0, \infty])$.

1. Introduction

The study of Galton-Watson (GW) processes and more generally GW trees conditioned to be non extinct goes back to Kesten [24], see Lemma 1.14 therein. In the sub-critical and non-degenerate critical case the extinction event $\mathcal{E}$ being of probability one, there are many non equivalent limiting procedure to define a GW tree conditioned on the non-extinction event. Those so-called local limits of GW trees have received a renewed interest recently because of the possibility of condensation phenomenon: a node in the limiting tree has an infinite degree. This appears when conditioning sub-critical GW trees to be large, see Jonsson and Stefánson [22] and Janson [21] when conditioning on large total population and Abraham and Delmas [2], when conditioning on large sub-population or [4] for a survey from the same authors. The other typical behavior for the local limit of GW trees is to exhibit an infinite spine on which are grafted independent finite GW sub-trees, such as in [24]. Various conditionings lead to such local limit, which we call Kesten tree, for critical or subcritical GW tree, see Abraham and Delmas [3] and references therein for a general study and [4] for other recent references in this direction also. Intuitively, the local limit is the Kesten tree when the events approximating the non-extinction event decrease in probability at polynomial rate. One of the motivations of the current work is to present local limits of sub-critical GW trees with different behavior (that is other than an infinite spine or a node of infinite degree), see the partial results from Section 9, where we present a family of local limits with an infinite backbone not reduced to a spine.

Recently, with Bouaziz, we considered in [1] the local limits of GW trees τ with geometric offspring distribution (see Section 1.4 for a precise definition) conditioned on the size Z_n of the

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population at generation n being equal to $a_n \in \mathbb{N}$. Because the distribution of Z_n is explicit for the geometric offspring distribution, it is possible to compute all the possible local limits (if any) for the sub-critical, critical and super-critical cases and for all the possible sequences $(a_n, n \in \mathbb{N}^*)$. The local limit is a random tree which depends on the rate of convergence of $(a_n, n \in \mathbb{N}^*)$ towards infinity. When this sequence is positive bounded or grows slowly to infinity, the limit is still the Kesten tree. This result already appears in the critical case in [3], see Section 6. When the growth to infinity is moderate, then the local limit can be described as an infinite random backbone on which are grafted independent finite GW trees. Surprisingly the backbone does not enjoy the branching property as the numbers of children of individuals at generation n on the backbone are not independent and depend also on the size of the backbone at generation n. If the growth to infinity is high, then the local limit exhibits the condensation phenomenon: the root, and only the root, of the local limit has an infinite number of children. The aim of the present work is to extend those results mainly to general super-critical offspring distribution and marginally to sub-critical offspring distribution.

1.1. **The main results.** Let $p = (p(k), k \in \mathbb{N})$ be a non-degenerate offspring distribution with finite mean $\mu = \sum_{k\in\mathbb{N}} kp(k)$. Let f denote the corresponding generating function so that $f'(1) = \mu$, and let $R_c \geq 1$ be its radius of convergence. We shall mainly consider the super-critical case $\mu \in (1, +\infty)$, but in Section 9 where we consider a particular sub-critical offspring distribution (that is $\mu \in (0, 1)$).

We recall the local convergence of random ordered rooted tree. The ordered rooted trees, defined in Section 2.1, are subsets of the set of finite sequences of positive integers $\mathcal{U} = \bigcup_{n\geq 0} (\mathbb{N}^*)^n$ with the convention $(\mathbb{N}^*)^0 = \{\partial\}$, and ∂ being the root of the tree. For a tree $\mathbf{t}$ and $u \in \mathcal{U}$, we denote by $k_u(\mathbf{t}) \in \bar{\mathbb{N}} = \mathbb{N} \bigcup \{\infty\}$ the out-degree of a node $u \in \mathbf{t}$ or equivalently the number of children of u in $\mathbf{t}$, with the convention that $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$. We denote by $z_h(\mathbf{t})$ the size of $\mathbf{t}$ at generation $h \in \mathbb{N}$. A sequence of trees $\mathbf{t}_n$ converges locally to a tree $\mathbf{t}$ if $k_u(\mathbf{t}_n)$ converges to $k_u(\mathbf{t})$ for all $u \in \mathcal{U}$. And we say that a sequence of random trees T_n converges locally in distribution to a random tree T if $(k_u(T_n), u \in \mathcal{U})$ converges in distribution to $(k_u(T), u \in \mathcal{U})$ for the finite dimensional marginals. See Section 2.2 for a precise setting.

We consider the random tree τ defined as the GW tree with super-critical non-degenerate offspring distribution p and finite mean μ, and we define $Z = (Z_n = z_n(\tau), n \in \mathbb{N})$ the corresponding GW process, with Z_n being the size of τ at generation n, strating at $Z_0 = 1$. Let $\mathfrak{a} \in \mathbb{N}$ and $\mathfrak{b} \in \bar{\mathbb{N}}$ be respectively the lower and upper bound of the support of p. We have $\mathfrak{a} < \mathfrak{b}$ as p is non-degenerate. Let $\mathfrak{c} = \mathbb{P}(\mathcal{E})$ be the probability of the extinction event. We recall that $\mathfrak{c} \in [0, 1)$ is the only root of $f(r) = r$ on $[0, 1)$. Notice that $\mathfrak{c} = 0$ if and only if $\mathfrak{a} \geq 1$. When $\mathbb{P}(Z_n = a_n) > 0$, we denote by τ_n a random tree distributed as τ conditioned on $\{Z_n = a_n\}$. We study the local convergence in distribution of $(\tau_n, n \in \mathbb{N}^*)$ according to the rate of growth of the sequence $(a_n, n \in \mathbb{N}^*)$. According to Seneta [32] or Asmussen and Hering [6], we shall consider the Seneta-Heyde norming $(c_n, n \in \mathbb{N})$ which is a sequence such that Z_n/c_n converges a.s. to a limit W and $\mathbb{P}(W = 0) = \mathfrak{c}$, see its definition in Section 4. When $\mu = +\infty$, then such a normalization does not exists and when the $L\log(L)$ condition holds, that is $\sum_{k\in\mathbb{N}^*} p_k \log(p_k) < +$, then c_n is equivalent to μ^n up to an arbitrary positive multiplicative constant, see Seneta [33]. However, we stress out that the $L\log(L)$ condition is not assumed in this paper and that we only consider the case μ finite. It is well known that the distribution of W, restricted to $(0, +\infty)$, has a continuous positive density w with respect to the Lebesgue measure, see the seminal work of Harris [20] and the general result from Dubuc [12]. However, w is explicitly known in only two cases: the geometric offspring distribution, see Section 1.4 below and the example developed by Hambly [19].

We now introduce the possible local limiting trees.

Definition 1.1. *Let τ be a GW tree with non-degenerate super-critical offspring distribution p with finite mean.*

- *If $\mathfrak{c} > 0$, we denote by $\tau^{0,0}$ a random tree distributed as τ conditionally on the extinction event $\mathcal{E}$.*
- *If $\mathfrak{c} > 0$, we denote by τ^0 the corresponding Kesten tree, see Definition 3.3. If $\mathfrak{c} = 0$ (that is $\mathfrak{a} \geq 1$), we denote by τ^0 the deterministic regular $\mathfrak{a}$-ary tree.*
- *For $\theta \in (0, +\infty)$, we denote by τ^θ a random tree distributed as τ conditioned on $\{W = \theta\}$.*
- *If $\mathfrak{b} < \infty$, we denote by τ^∞ the deterministic regular $\mathfrak{b}$-ary tree. If $\mathfrak{b} = +\infty$ and $R_c > 1$, we denote by τ^∞ the random tree $T^{(\lambda_c)}$ given in Section 8.*

According to Remark 5.3, the distribution of τ^θ, which is defined in Section 5, is a regular version of the distribution of τ conditioned on $\{W = \theta\}$ for $\theta \in (0, +\infty)$. The tree τ^θ can be intuitively described as a random (non-homogeneous in time) infinite backbone on which, if $\mathfrak{c} > 0$, are grafted independent GW trees distributed as $\tau^{0,0}$. The description of the backbone and of its offspring distribution is one of the main contribution of this paper. The infinite backbone does not enjoy the branching property, and the offspring distribution $\rho_{\theta,r}$ of the individuals of the current generation depends on the size r of the current generation, see Definition (20). The probability distribution $\rho_{\theta,r}$ is a function of the density w. The distribution of τ^θ is in a sense a generalization of the Kesten tree distribution.

The tree τ^∞ appears as a natural local limit of non-homogeneous GW trees $T^{(\lambda)}$ introduced in Section 8.1 and with a nice representation given in Section 8.2 using two-type GW trees.

The condensation holds for τ^∞, defined in Section 8, at least at the root if $\mathfrak{b} = +\infty$ and $f(R_c) = +\infty$. Furthermore, the tree τ^∞ is not homogeneous in general.

We can now give the first main result on the local convergence in distribution of τ_n according to the growth rate of $(a_n, n \in \mathbb{N}^*)$.

Theorem 1.2. *Let τ be a GW tree with non-degenerate super-critical offspring distribution p with finite mean. We assume that the sequence $(a_n, n \in \mathbb{N}^*)$ is such that τ_n is well defined for all $n \in \mathbb{N}^*$.*

- ***Extinction case:*** *$a_n = 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}^*$. If $\mathfrak{c} = 0$, then τ_n is not defined. If $\mathfrak{c} > 0$, then τ_n is well defined and we have:*
$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^{0,0}.$$
- ***Low regime:*** *$\lim_{n\to\infty} a_n/c_n = 0$ and $a_n > 0$ for all $n \in \mathbb{N}^*$. Then, we have:*
$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^{0}.$$
- ***Moderate regime:*** *$\lim_{n\to\infty} a_n/c_n = \theta \in (0, +\infty)$. Then, we have:*
$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^{\theta}.$$
- ***High regime:*** *$\lim_{n\to\infty} a_n/c_n = +\infty$. (Partial results.) If $\mathfrak{b} < \infty$ (Harris case) or if p is geometric, then we have:*
$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^{\infty}.$$

The local convergence is well known in the extinction case, see Proposition 6.4. For the low regime, it is stated in Proposition 6.5. For the moderate regime, it is stated in Proposition 6.2. For the high regime, it is stated in [1] for p geometric and in Proposition 6.3 for the Harris case. All the proofs rely on the strong ratio theorem, see Section 1.2 below. We conjecture that the local convergence of τ_n towards τ^∞ in the high regime holds if $R_c > 1$ (or equivalently W has some positive exponential moments, see the first part of Section 8.1). The existence of local limits in the high regime when $R_c = 1$ is open.

Remark 1.3. We recall from Dubuc [13] some sufficient conditions on $x \in \mathbb{N}$ such that $\mathbb{P}_k(Z_n = x) > 0$, where $\mathbb{P}_k$ denote, for $k \in \mathbb{N}^*$, the distribution of the GW process Z started from $Z_0 = k$. Notice first that if $x = 0$, then $\mathbb{P}_k(Z_n = x) > 0$ if and only if $\mathfrak{c} > 0$, that is $\mathfrak{a} = 0$.

We now consider the case $x > 0$. The offspring distribution p is said to be of type (L_0, r_0), if L_0 is the period of p, that is the greatest common divisor of $\{n-\ell;\, n > \ell \text{ and } p(n)p(\ell) \neq 0\}$, and r_0 is the residue (mod L_0) of any n such that $p(n) \neq 0$. It is clear that $\mathbb{P}_k(Z_n = x) > 0$ implies $x = k\, r_0^n (\text{mod } L_0)$. According to [13], for any $b > \mathfrak{a}$ such that $p(b) > 0$ (take $b = \mathfrak{b}$ if $\mathfrak{b} < \infty$), there exists $d \in \mathbb{N}$ such that for all $k \in \mathbb{N}^*$ and $x \in [\![k\mathfrak{a}^n + d, kb^n - d]\!]$ with $x = kr_0^n (\text{mod } L_0)$, we have $\mathbb{P}_k(Z_n = x) > 0$.

Taking $k = 1$ and $x = a_n$, this provides sufficient conditions for τ_n to be well defined. In particular, notice that there exist sequences $(a_n, n \in \mathbb{N}^*)$ in all the regime such that τ_n is well defined.

Moreover, we have the following continuity result in distribution for the family of limiting trees.

Theorem 1.4. *Let p be a non-degenerate super-critical offspring distribution with finite mean. The family $(\tau^\theta, \theta \in [0, +\infty))$ is continuous for the local convergence in distribution. Furthermore, if $\mathfrak{b} < \infty$ or if p is geometric, then we have:*

$$\tau^\theta \xrightarrow[\theta \to +\infty]{(d)} \tau^\infty.$$

The continuity of $(\tau^\theta, \theta \in [0, +\infty))$ is proven in Section 7 and more precisely in Corollary 7.1 for the continuity at 0. The continuity at 0 allows to explain and extend Corollary 3 from Berestycki, Gantert and Mörters [9] on the convergence in distribution of $\tau_{(\varepsilon)}$ (distributed as τ conditionally on $\{0 < W \leq \varepsilon\}$) towards τ^0 as ε goes down to 0, see Corollary 7.2.

The continuity at infinity is proven in [1] for the geometric case and in Proposition 8.10 for the Harris case. If $R_c > 1$, we also conjecture that the local convergence in distribution of τ^θ towards τ^∞ as θ goes to infinity. We get a partial result in this direction in Section 8.3, as if $R_c > 1$ and if τ^θ converges locally in distribution as θ goes to infinity, then the limit is indeed τ^∞, see Corollary 8.8. If $R_c = 1$, then we have not hint concerning the existence or non-existence of possible limits for τ^θ as θ goes to infinity. Notice that it is not clear that τ^θ is stochastically non-decreasing with θ.

Remark 1.5. Partial results concerning the sub-critical case are presented in Section 9, under the assumption that $R_c > 1$ and the equation $f(r) = r$ has a finite root in $(1, R_c]$. This assumption is equivalent to assume that the sub-critical GW tree is distributed as a super-critical GW tree conditioned on the extinction event. In this case, we can use the previous results in the super-critical case to get results in the sub-critical case.

1.2. **Strong ratio theorem for super-critical GW process.** We set for $k, h \in \mathbb{N}^*$:

$$H_n(h,k) = \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)}, \tag{1}$$

where Z is under $\mathbb{P}_k$ a GW process starting from $Z_0 = k$. The proofs of Theorem 1.2, when there is non condensation, rely on the elementary identity (12) which states that $\mathbb{P}(r_h(\tau_n) = \mathbf{t}) = H_n(h, z_h(\mathbf{t}))\mathbb{P}(r_h(\tau) = \mathbf{t})$, where $r_h(\mathbf{s})$ denotes the restriction of the tree $\mathbf{s}$ up to generation $h \in \mathbb{N}^*$, and $\mathbf{t}$ is a tree with height h (that is $z_h(\mathbf{t}) > 0$ and $z_{h+1}(\mathbf{t}) = 0$). Since the local convergence in distribution of τ_n towards a tree with finite nodes is equivalent to the convergence of $\mathbb{P}(r_h(\tau_n) = \mathbf{t})$ for all $h \in \mathbb{N}^*$ and all tree $\mathbf{t}$ of height h, up to the identification of the limit, the local convergence can be deduced from the convergence as n goes to infinity of $H_n(h, k)$ for all $h, k \in \mathbb{N}^*$. The result is in the same spirit as the strong ratio theorem for random walks. Notice that all the regimes described in the following theorem are valid thanks to Remark 1.3.

Theorem 1.6. *Let p be a non-degenerate super-critical offspring distribution with μ finite. We assume that the sequence $(a_n, n \in \mathbb{N}^*)$ is such that $\mathbb{P}(Z_n = a_n) > 0$ for all $n \in \mathbb{N}^*$.*

- ***Extinction case:*** *$a_n = 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}^*$. If $\mathfrak{c} = 0$, then $\mathbb{P}(Z_n = 0) = 0$ for all $n \in \mathbb{N}$, and thus H_n is not defined. If $\mathfrak{c} > 0$, then we have:*

$$H^{0,0}(h, k) := \lim_{n\to\infty} H_n(h, k) = \mathfrak{c}^{k-1}. \tag{2}$$

- ***Low regime:*** *$\lim_{n\to\infty} a_n/c_n = 0$ and $a_n > 0$ for all $n \in \mathbb{N}^*$. We have[1]:*

$$H^{0}(h, k) := \lim_{n\to\infty} H_n(h, k) = \begin{cases} k\mathfrak{c}^{k-1} f'(\mathfrak{c})^{-h} & \text{if } \mathfrak{a} = 0, \\ f'(\mathfrak{c})^{-h} \mathbf{1}_{\{k=1\}} & \text{if } \mathfrak{a} = 1, \\ p(\mathfrak{a})^{-(\mathfrak{a}^h-1)/(\mathfrak{a}-1)} \mathbf{1}_{\{k=\mathfrak{a}^h\}} & \text{if } \mathfrak{a} \geq 2. \end{cases} \tag{3}$$

- ***Moderate regime:*** *$\lim_{n\to\infty} a_n/c_n = \theta \in (0, +\infty)$. We have, with the notation $w_k(\theta) = \sum_{i=1}^k \binom{k}{i} \mathfrak{c}^{k-i} w^{*i}(\theta)$:*

$$H^{\theta}(h, k) := \lim_{n\to\infty} H_n(h, k) = \mu^h \, \frac{w_k\left(\mu^h \theta\right)}{w(\theta)} \mathbf{1}_{\{k=r_0^h \,(\text{mod } L_0)\}}, \tag{4}$$

  where (L_0, r_0) is the type of p.
- ***High regime:*** *$\lim_{n\to\infty} a_n/c_n = +\infty$. (Partial results.) We have:*

$$H^{\infty}(h, k) := \lim_{n\to\infty} H_n(h, k) = \begin{cases} p(\mathfrak{b})^{-(\mathfrak{b}^h-1)/(\mathfrak{b}-1)} \mathbf{1}_{\{k=\mathfrak{b}^h\}} & \text{if } \mathfrak{b} < \infty, \\ 0 & \text{if } p \text{ is geometric.} \end{cases} \tag{5}$$

Contrary to the short proof of the strong ratio theorem for random walks given by Neveu [27], the proof presented here for the strong ratio theorem rely on explicit equivalent of $\mathbb{P}_k(Z_{n-h} = a_n)$ for n large. The well known extinction case is given in Remark 3.2.

The result for the low regime is much more delicate. We shall distinguish between the Schröder case $f'(\mathfrak{c}) > 0$ and the Böttcher case $f'(\mathfrak{c}) = 0$, and in those two cases consider the sequence $(a_n, n \in \mathbb{N}^*)$ bounded or unbounded. The case a_n bounded and $\mathfrak{a} = 0$ can be found in Papangelou [30]. The case a_n bounded, $\mathfrak{a} = 1$ is an easy extension of [30], see Case I in the proof of Proposition 6.5 in the Schröder case. The case a_n unbounded and $\mathfrak{a} \leq 1$ (Schröder case) can be derived, see Lemma 6.6, from the precise asymptotics of $\mathbb{P}_\ell(Z_n = a_n)$ given by Fleischmann and Wachtel [17]. The case $\mathfrak{a} \geq 2$ (Böttcher case) is given in Lemma 12.4 and Lemma 12.5. The former lemma relies on a precise approximation of $\mathbb{P}_\ell(Z_n = a_n)$ given in Lemma 12.3 for a_n unbounded, which is an extension of the precise asymptotics given by Fleischmann and Wachtel [18].

[1] Notice that $\mathfrak{a} = 0$, resp. $\mathfrak{a} = 1$, resp. $\mathfrak{a} \geq 2$, is equivalent to $\mathfrak{c} > 0$, resp. $\mathfrak{c} = 0$ and $f'(\mathfrak{c}) > 0$, resp. $f'(\mathfrak{c}) = 0$.

The moderate regime is a direct consequence of the local limit theorem in Dubuc and Seneta [14], see Lemma 6.1 here.

The high regime in the Harris case when $\limsup_{n\to\infty} a_n/\mathfrak{b}^n < 1$ is detailed in Lemma 11.6 with $\ell = 1$. It relies on techniques similar to those developed in [18] or in Flajolet and Odlyzko [16] to get an equivalent to $\mathbb{P}_k(Z_n = a_n)$, see Lemma 11.5. The proof is however given in details because the adaptation is not straightforward. The high regime for the geometric offspring distribution is given in [1].

If $\mathfrak{b} = \infty$ and $f(R_c) = +\infty$, we conjecture that τ_n converges locally in distribution towards a limit τ^∞ whose root has an infinite number of children. Using the elementary identity (12), we deduce the following conjecture that if $\mathfrak{b} = \infty$ and $f(R_c) = +\infty$, then:

$$H^\infty(h,k) := \lim_{n\to\infty} H_n(h,k) = 0. \tag{6}$$

If $\mathfrak{b} = +\infty$ and $f(R_c) < +\infty$, then τ^∞ has no condensation and thus $H^\infty(n,k)$ might exists and be given by $f_{-h+1}(R_c)^k/f(R_c)$, where, for $n \in \mathbb{N}^*$, f_n denotes the n-th iterate of f and f_{-n} its inverse (which is well defined because f_n is increasing). See the martingale term in the right hand side of (39) with $\lambda = \lambda_c$.

If $R_c = 1$, the possible existence of a limit for H_n is an open question. See Wachtel, Denisov and Korshunov [34] for a first step in the study of this so-called heavy-tailed case.

1.3. **Link with the Martin boundary of super-critical GW process.** Recall that Z is a super-critical GW process with non-degenerate offspring distribution p with finite mean μ. The Martin boundary $\mathcal{M}$ of the non-negative space-time GW process corresponds to all extremal non-negative space-time harmonic functions H defined on $\mathbb{N}^2$, and is related to the set of all extremal non-negative martingales $N = (N_n = H(n, Z_n), n \in \mathbb{N})$. Considering only the case $Z_0 = 1$, then Remark 1.3 implies that the functions H are only defined for (n,k) such that $k = r_0^n (\text{mod } L_0)$, where (L_0, r_0) is the type of p. Let $\mathcal{H}$ denote the set of non-negative space-time function H such that there exists a sequence $(a_n, n \in \mathbb{N}^*)$ with $H(h,k) = \lim_{n\to\infty} \mathbb{P}_k(Z_{n-h} = a_n)/\mathbb{P}(Z_n = a_n)$ for all $h,k \in \mathbb{N}$. According to Kemeny, Snell and Knapp [23] Chapter 10, we have $\mathcal{M} \subset \mathcal{H}$.

Consider the collection $\mathcal{H}^* = \{H^\theta, \theta \in [0,\infty)\}$. We deduce from Section 1.2 that $\mathcal{H}^* \subset \mathcal{H}$. This appears already in Athreya and Ney [7], see also Section II.9 from Athreya and Ney [8]. We also deduce from Section 1.2 that $H^{0,0} \in \mathcal{H}$ if and only if $\mathfrak{a} = 0$. We get a complete description of $\mathcal{H}$ and $\mathcal{M}$ in the Harris case and geometric case. To our knowledge, the results for the Harris case in the present work and for the geometric case in [1] are the first complete descriptions of the Martin boundary for super-critical GW process. This (partially) answers a question raised in [7], on the identification of $\mathcal{H}\backslash\mathcal{H}^*$.

Theorem 1.7. *Let p be a non-degenerate super-critical offspring distribution with μ finite. If $\mathfrak{b} < \infty$, then we have:*

$$\mathcal{M} = \mathcal{H} = \begin{cases} \mathcal{H}^* \cup \{H^{0,0}, H^\infty\} & \text{if } \mathfrak{a} = 0, \\ \mathcal{H}^* \cup \{H^\infty\} & \text{if } \mathfrak{a} \geq 1. \end{cases}$$

If p is geometric, then we have $\mathcal{M} = \mathcal{H} = \mathcal{H}^* \cup \{H^{0,0}\}$ if $\mathfrak{a} = 0$ and $\mathcal{M} = \mathcal{H} = \mathcal{H}^*$ if $\mathfrak{a} > 0$.

In the previous theorem, the description of $\mathcal{H}$ is a consequence of Theorem 1.6; the equality $\mathcal{M} = \mathcal{H}$ follows directly from the fact that for all $\theta \in (0, +\infty)$ a.s. $\lim_{n\to\infty} z_n(\tau^\theta)/c_n = \theta$, see Remark 5.3 or Lootgieter [26], Corollary 2.3.II c) which states that all the functions in $\mathcal{H}^*$ are extremal under the $L\log(L)$ condition and the aperiodic condition, that is $L_0 = 1$. In the same spirit, Overbeck [29] has given an explicit description of the Martin boundary for some time-continuous branching processes, see for example Theorem 2 therein.

We conjecture that $\mathcal{H} = \mathcal{H}^*$ or $\mathcal{H} = \mathcal{H}^* \cup \{H^{0,0}\}$ as soon as $\mathfrak{b} = +\infty$ and $f(R_c) = +\infty$, keeping $H^{0,0}$ if and only if $\mathfrak{a} = 0$. Otherwise, existence of a limit function H when $\lim_{n\to\infty} a_n/c_n = +\infty$ is still open in the general case.

We end this section with some works related to Martin boundary for GW process. We refer to Dynkin [15] or to [23] for a presentation of the Martin boundary. For the extremal non-negative harmonic functions (space only) of GW process, we refer to Theorem 3 in Cohn [11], which is stated under the $L\log(L)$ condition and an aperiodic condition. (Notice that the $L\log(L)$ and aperiodic conditions are indeed required in the proof of Theorem 3 in [11] as it relies on Corollary 2.3.II a) from [26].) For the Martin entrance boundary of GW process, see Alsmeyer and Rösler [5].

1.4. **The geometric offspring distribution case.** We consider the geometric super-critical offspring distribution. We collect results developed in [1] and in this paper.

Let $0 < q < \eta \leq 1$ and define the $\mathcal{G}(\eta, q)$ geometric offspring distribution by

$$\begin{cases} p(0) = 1 - \eta, \\ p(k) = \eta q (1-q)^{k-1} \quad \text{for } \in \mathbb{N}^*. \end{cases}$$

We have $\mathfrak{a} = 0$ if $\eta < 1$ and $\mathfrak{a} = 1$ if $\eta = 1$. Moreover, we have $\mathfrak{b} = +\infty$, $(L_0, r_0) = (1, 0)$, $\mu = \eta/q \in (1, +\infty)$. It is easy to compute

$$f(s) = \frac{(1-\eta) - s(1-q-\eta)}{1 - s(1-q)},$$

and deduce that $R_c = 1/(1-q)$, $f(R_c) = +\infty$, $\mathfrak{c} = (1-\eta)/(1-q) \in [0,1)$ and $f'(\mathfrak{c}) = q/\eta$. It is also easy to check that

$$w(\theta) = (1-\mathfrak{c})^2 \, \mathrm{e}^{-(1-\mathfrak{c})\theta}$$

for $\theta > 0$ and thus $\lambda_c = \sup\{\lambda \in \mathbb{R};\, \mathbb{E}[\exp(\lambda W)] < +\infty\} = 1 - \mathfrak{c} > 0$. If $\mathfrak{c} > 0$ or equivalently $\eta < 1$, then $\tau^{0,0}$ has geometric offspring distribution $\mathcal{G}(q, \eta)$. We have for $\theta \in (0, +\infty)$, $r \in \mathbb{N}^*$:

$$\rho_{\theta,r}(s) = \frac{(r-1)!}{(|s|_1 - 1)!} \Big(\theta(1-\mathfrak{c})(\mu-1)\Big)^{|s|_1 - r} \mathrm{e}^{-\theta(1-\mathfrak{c})(\mu-1)}, \quad s \in (\mathbb{N}^*)^r,$$

with $|s|_1 = \sum_{i=1}^r s_i$ for $s = (s_1, \ldots, s_r)$; and

$$H^\theta(h,k) = \mu^h \, \mathrm{e}^{-\theta(1-\mathfrak{c})(\mu^h - 1)} \sum_{i=1}^k \binom{k}{i} \mathfrak{c}^{k-i} \frac{\Big(\theta(1-\mathfrak{c})^2 \mu^h\Big)^{i-1}}{(i-1)!}.$$

Notice that the definition of H^θ is similar to the extremal space-time harmonic functions given in Theorem 2 from [29] for binary splitting in continuous time.

We have that $(\tau_n, n \in \mathbb{N}^*)$ converges locally in distribution towards τ^θ if $\lim_{n\to\infty} a_n/\mu^n = \theta \in [0, +\infty]$ and $a_n > 0$ for all $n \in \mathbb{N}^*$. The family $(\tau^\theta, \theta \in [0, +\infty])$ is continuous in distribution for the local convergence. The random tree τ^∞ has only one node of infinite degree which happens to be the root. The space-time Martin boundary is $\mathcal{M} = \mathcal{H} = \mathcal{H}^*$ if $\mathfrak{c} = 0$ and $\mathcal{M} = \mathcal{H} = \mathcal{H}^* \cup \{H^{0,0}\}$ if $\mathfrak{c} > 0$.

1.5. **Organization of the paper.** We recall the definition of trees, the local convergence and the distribution of the Galton-Watson tree τ in Section 2. Section 3 is devoted to the Kesten tree associated with τ. We introduce in Section 4 a probability distribution $\rho_{\theta,r}$ in (20) which plays a crucial role to describe the local limits in the moderate regime. We present the local limits in the moderate regime in Section 5. The statements of the local convergence are in Section 6. The continuity of the local limits is studied in Section 7 and the partial results on the continuity at $\theta = +\infty$ are presented in Section 8. Section 9 is devoted to the sub-critical case (when it is seen as the super-critical case conditioned to the extinction event). After some ancillary results given in Section 10, we give detailed proofs in the technical Section 11 for the Harris case and state the results for the Böttcher case in Section 12.

2. Notations

We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of non-negative integers, by $\mathbb{N}^* = \{1, 2, \dots\}$ the set of positive integers and $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. For any finite set E, we denote by $\sharp E$ its cardinal.

We say that a function g defined on $(0, +\infty)$ is multiplicatively periodic with period $c > 0$ if $g(cx) = g(x)$ for all $x > 0$. Notice that g is also multiplicatively periodic with period $1/c$.

2.1. **The set of discrete trees.** We recall Neveu's formalism [28] for ordered rooted trees. Let $\mathcal{U} = \bigcup_{n\geq 0} (\mathbb{N}^*)^n$ be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\partial\}$. We also set $\mathcal{U}^* = \bigcup_{n\geq 1} (\mathbb{N}^*)^n = \mathcal{U}\backslash\{\partial\}$.

For $u \in \mathcal{U}$, let $|u|$ be the length or the generation of u defined as the integer n such that $u \in (\mathbb{N}^*)^n$. If u and v are two sequences of $\mathcal{U}$, we denote by uv the concatenation of two sequences, with the convention that $uv = vu = u$ if $v = \partial$. The set of strict ancestors of $u \in \mathcal{U}^*$ is defined by

$$\mathrm{Anc}(u) = \{v \in \mathcal{U},\ \exists w \in \mathcal{U}^*,\ u = vw\},$$

and for $\mathscr{S} \subset \mathcal{U}^*$, being non-empty, we set $\mathrm{Anc}(\mathscr{S}) = \bigcup_{u\in\mathscr{S}} \mathrm{Anc}(u)$.

A tree $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies :

- $\partial \in \mathbf{t}$.
- If $u \in \mathbf{t}\backslash\{\partial\}$, then $\mathrm{Anc}(u) \subset \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists $k_u(\mathbf{t}) \in \bar{\mathbb{N}}$ such that, for every $i \in \mathbb{N}^*$, $ui \in \mathbf{t} \iff 1 \leq i \leq k_u(\mathbf{t})$.

We denote by $\mathbb{T}_\infty$ the set of trees. For $r \in \bar{\mathbb{N}}$, $r \geq 1$, we denote by $\mathbf{t}_r$ the regular r-ary tree, defined by $k_u(\mathbf{t}_r) = r$ for all $u \in \mathbf{t}_r$. Let $\mathbf{t} \in \mathbb{T}_\infty$ be a tree. The vertex ∂ is called the root of the tree $\mathbf{t}$ and we denote by $\mathbf{t}^* = \mathbf{t}\backslash\{\partial\}$ the tree without its root. For a vertex $u \in \mathbf{t}$, the integer $k_u(\mathbf{t})$ represents the number of offsprings (also called the out-degree) of the vertex $u \in \mathbf{t}$. By convention, we shall write $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$. The height $H(\mathbf{t})$ of the tree $\mathbf{t}$ is defined by:

$$H(\mathbf{t}) = \sup\{|u|,\ u \in \mathbf{t}\} \in \bar{\mathbb{N}}.$$

For $n \in \mathbb{N}$, the size of the n-th generation of $\mathbf{t}$ is defined by:

$$z_n(\mathbf{t}) = \sharp\{u \in \mathbf{t}, |u| = n\}.$$

We denote by $\mathbb{T}_\mathrm{f}^*$ the subset of trees with finite out-degrees except the root's:

$$\mathbb{T}_\mathrm{f}^* = \{\mathbf{t} \in \mathbb{T}_\infty;\ \forall u \in \mathbf{t}^*,\ k_u(\mathbf{t}) < +\infty\}$$

and by $\mathbb{T}_\mathrm{f} = \{\mathbf{t} \in \mathbb{T}_\mathrm{f}^*;\ k_\partial(\mathbf{t}) < +\infty\}$ the subset of trees with finite out-degrees.

Let $h, k \in \mathbb{N}^*$. We define $\mathbb{T}_{\mathrm{f}}^{(h)}$ the subset of finite trees with height h:

$$\mathbb{T}_{\mathrm{f}}^{(h)} = \{\mathbf{t} \in \mathbb{T}_{\mathrm{f}};\ H(\mathbf{t}) = h\}$$

and $\mathbb{T}_k^{(h)} = \{\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)};\ k_\partial(\mathbf{t}) = k\}$ the subset of finite trees with height h and out-degree of the root equal to k. The restriction operators r_h and $r_{h,k}$ are defined, for every $\mathbf{t} \in \mathbb{T}_\infty$, by:

$$r_h(\mathbf{t}) = \{u \in \mathbf{t};\ |u| \leq h\} \quad \text{and} \quad r_{h,k}(\mathbf{t}) = \{\partial\} \cup \{u \in r_h(\mathbf{t})^*;\ \mathrm{Anc}(u) \cap \{1, \ldots, k\} \neq \emptyset\},$$

so that, for $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}$, if $H(\mathbf{t}) \geq h$, then $r_h(\mathbf{t}) \in \mathbb{T}_{\mathrm{f}}^{(h)}$; and for $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^*$, if $H(\mathbf{t}) \geq h$ and $k_\partial(\mathbf{t}) \geq k$, then $r_{h,k}(\mathbf{t}) \in \mathbb{T}_k^{(h)}$.

2.2. **Convergence of trees.** Set $\mathbb{N}_1 = \{-1\} \cup \bar{\mathbb{N}}$, endowed with the usual topology of the one-point compactification of the discrete space $\{-1\} \cup \mathbb{N}$. For a tree $\mathbf{t} \in \mathbb{T}_\infty$, recall that by convention the out-degree $k_u(\mathbf{t})$ of u is set to -1 if u does not belong to $\mathbf{t}$. Thus a tree $\mathbf{t} \in \mathbb{T}_\infty$ is uniquely determined by the $\mathbb{N}_1$-valued sequence $(k_u(\mathbf{t}), u \in \mathcal{U})$ and then $\mathbb{T}_\infty$ is a subset of $\mathbb{N}_1^{\mathcal{U}}$. By Tychonoff theorem, the set $\mathbb{N}_1^{\mathcal{U}}$ endowed with the product topology is compact. Since $\mathbb{T}_\infty$ is closed it is thus compact. In fact, the set $\mathbb{T}_\infty$ is a Polish space (but we don't need any precise metric at this point). The local convergence of sequences of trees is then characterized as follows. Let $(\mathbf{t}_n, n \in \mathbb{N})$ and $\mathbf{t}$ be trees in $\mathbb{T}_\infty$. We say that $\lim_{n\to\infty} \mathbf{t}_n = \mathbf{t}$ if and only if $\lim_{n\to\infty} k_u(\mathbf{t}_n) = k_u(\mathbf{t})$ for all $u \in \mathcal{U}$. It is easy to see that:

- If $(\mathbf{t}_n, n \in \mathbb{N})$ and $\mathbf{t}$ are trees in $\mathbb{T}_{\mathrm{f}}$, then we have $\lim_{n\to\infty} \mathbf{t}_n = \mathbf{t}$ if and only if $\lim_{n\to\infty} r_h(\mathbf{t}_n) = r_h(\mathbf{t})$ for all $h \in \mathbb{N}^*$.
- If $(\mathbf{t}_n, n \in \mathbb{N})$ and $\mathbf{t}$ are trees in $\mathbb{T}_{\mathrm{f}}^*$, then we have $\lim_{n\to\infty} \mathbf{t}_n = \mathbf{t}$ if and only if $\lim_{n\to\infty} r_{h,k}(\mathbf{t}_n) = r_{h,k}(\mathbf{t})$ for all $h, k \in \mathbb{N}^*$.

If T is a $\mathbb{T}_{\mathrm{f}}$-valued (resp. $\mathbb{T}_{\mathrm{f}}^*$-valued) random variable, then its distribution is characterized by $\left(\mathbb{P}(r_h(T) = \mathbf{t});\ h \in \mathbb{N}^*,\ \mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}\right)$ (resp. $\left(\mathbb{P}(r_{h,k}(T) = \mathbf{t});\ h, k \in \mathbb{N}^*,\ \mathbf{t} \in \mathbb{T}_k^{(h)}\right)$). Using the Portmanteau theorem, we deduce the following characterization of the convergence in distribution:

- Let $(T_n, n \in \mathbb{N})$ and T be $\mathbb{T}_{\mathrm{f}}$-valued random variables. Then, if a.s. $H(T) = +\infty$, we have:

$$T_n \xrightarrow[n\to\infty]{\textbf{(d)}} T \iff \lim_{n\to\infty} \mathbb{P}(r_h(T_n) = \mathbf{t}) = \mathbb{P}(r_h(T) = \mathbf{t}) \quad \text{for all } h \in \mathbb{N}^*,\ \mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}. \tag{7}$$

- Let $(T_n, n \in \mathbb{N})$ and T be $\mathbb{T}_{\mathrm{f}}^*$-valued random variables. Then, if a.s. $H(T) = +\infty$ and $k_\partial(T) = +\infty$, we have:

$$T_n \xrightarrow[n\to\infty]{\textbf{(d)}} T \iff \lim_{n\to\infty} \mathbb{P}(r_{h,k}(T_n) = \mathbf{t}) = \mathbb{P}(r_{h,k}(T) = \mathbf{t}) \quad \text{for all } h, k \in \mathbb{N}^*,\ \mathbf{t} \in \mathbb{T}_k^{(h)}. \tag{8}$$

2.3. **Galton-Watson trees.** Let $p = (p(n), n \in \mathbb{N})$ be a probability distribution on $\mathbb{N}$. A $\mathbb{T}_{\mathrm{f}}$-valued random variable τ is called a GW tree with offspring distribution p if for all $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}$ with $H(\mathbf{t}) \leq h$:

$$\mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t})).$$

The generation size process defined by $(Z_n = z_n(\tau),\ n \in \mathbb{N})$ is the so-called GW process. We refer to [8] and [6] for a general study of GW processes.

We recall here the classical result on the extinction probability of the GW tree and introduce some notations. We denote by $\mathcal{E} = \{H(\tau) < +\infty\} = \bigcup_{n\in\mathbb{N}}\{Z_n = 0\}$ the extinction event and denote by $\mathfrak{c}$ the extinction probability:

$$\mathfrak{c} = \mathbb{P}(\mathcal{E}). \tag{9}$$

Then, if f denotes the generating function of p, $\mathfrak{c}$ is the smallest non-negative root of $f(s) = s$. We denote by μ the mean of p i.e. $\mu = f'(1)$. We recall the three following cases:

- The sub-critical case ($\mu < 1$): $\mathfrak{c} = 1$.
- The critical case ($\mu = 1$): $\mathfrak{c} = 1$ (unless $p(1) = 1$ and then $\mathfrak{c} = 0$).
- The super-critical case ($\mu > 1$): $\mathfrak{c} \in [0, 1)$, the process has a positive probability of non-extinction. Notice that $\mathfrak{c} = 0$ if and only if $\mathfrak{a} \geq 1$.

We consider the lower and upper bounds of the support of p:

$$\mathfrak{a} = \inf\{n \in \mathbb{N};\, p(n) > 0\} \quad \text{and} \quad \mathfrak{b} = \sup\{k;\, p(k) > 0\} \in \bar{\mathbb{N}}. \tag{10}$$

We say that p is non-degenerate if $\mathfrak{a} < \mathfrak{b}$. We define f_n the n-th iterate of f, which is the generating function of Z_n. We recall that $\lim_{n\to\infty} f_n(0) = \mathfrak{c}$. We also introduce in the supercritical case ($\mu > 1$) the Schröder constant α defined by:

$$f'(\mathfrak{c}) = \mu^{-\alpha}, \quad \alpha \in (0, +\infty]. \tag{11}$$

We set $\mathbb{P}_k$ the probability under which the GW process $(Z_n, n \geq 0)$ starts with $Z_0 = k$ individuals and write $\mathbb{P}$ for $\mathbb{P}_1$ so that:

$$\mathbb{P}_k(Z_n = a) = \mathbb{P}(Z_n^{(1)} + \cdots + Z_n^{(k)} = a),$$

where the $(Z^{(i)}, 1 \leq i \leq k)$ are independent random variables distributed as Z under $\mathbb{P}$.

We consider a sequence $(a_n, n \in \mathbb{N}^*)$ of elements of $\mathbb{N}$ and, when $\mathbb{P}(Z_n = a_n) > 0$, τ_n a random tree distributed as the GW tree τ conditioned on $\{Z_n = a_n\}$. Let $n \geq h \geq 1$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$. We have, with $k = z_h(\mathbf{t})$:

$$\mathbb{P}(r_h(\tau_n) = \mathbf{t}) = \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)}\cdot \tag{12}$$

3. The Kesten tree

In this section, we consider a GW tree τ with offspring distribution $p = (p(n), n \in \mathbb{N})$ having mean $\mu \in (0, +\infty)$. Recall that $\mathfrak{c} \in [0, 1]$ denotes the extinction probability of τ. We define an associated probability distribution $\mathfrak{p}$ on $\mathbb{N}$ as follows:

Definition 3.1. (i) *If $\mathfrak{c} = 0$, we define $\mathfrak{p}$ as the Dirac mass at point $\mathfrak{a}$.*
(ii) *If $\mathfrak{c} > 0$, we define the probability distribution $\mathfrak{p} = (\mathfrak{p}(n), n \in \mathbb{N})$ by:*

$$\mathfrak{p}(n) = \mathfrak{c}^{n-1} p(n) \quad \textit{for } n \in \mathbb{N}. \tag{13}$$

We denote by $\mathfrak{m}$ the mean of $\mathfrak{p}$. If $\mu \leq 1$ and $p(1) \neq 1$, as $\mathfrak{c} = 1$, we have $\mathfrak{p} = p$ and $\mathfrak{m} = \mu$. If $\mathfrak{c} > 0$, we have $\mathfrak{m} = f'(\mathfrak{c}) \in (0, 1]$.

Remark 3.2. If $\mathfrak{c} > 0$, let $\tau^{0,0}$ be a GW tree with offspring distribution $\mathfrak{p}$ defined in (13). It is well known that the GW tree τ conditioned on the extinction event $\mathcal{E}$ is distributed as $\tau^{0,0}$. Indeed, we have using the branching property that, for $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$, and setting $k = z_h(\mathbf{t})$:

$$\mathbb{P}(r_h(\tau) = \mathbf{t} \,|\, \mathcal{E}) = \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{\mathbb{P}_k(\mathcal{E})}{\mathbb{P}(\mathcal{E})} = \mathfrak{c}^{k-1} \mathbb{P}(r_h(\tau) = \mathbf{t}) = \mathbb{P}(r_h(\tau^{0,0}) = \mathbf{t}).$$

Let $k \in \mathbb{N}^*$. If $f^{(k)}(1) \in (0, +\infty)$, that is p has finite moment of order k and the support of p is not a subset of $\{0, \ldots, k-1\}$, then we define the k-th order size-biased probability distribution of p as $p_{[k]} = (p_{[k]}(n), n \in \mathbb{N})$ with:

$$p_{[k]}(n) = \mathbf{1}_{\{n \geq k\}} \binom{n}{k} \frac{k!}{f^{(k)}(1)}\, p(n). \tag{14}$$

The generating function of $p_{[k]}$ is $f_{[k]}(s) = s^k f^{(k)}(s)/f^{(k)}(1)$. The probability distribution $p_{[1]}$ is the so-called size-biased probability distribution of p.

We now define the so-called Kesten tree $\hat{\tau}^0$ associated with the offspring distribution p.

Definition 3.3 (Kesten tree)**.** (i) *If $\mathfrak{c} > 0$, the Kesten tree $\hat{\tau}^0$ is a two-type GW tree where the vertices are either of type* s *(for survivor) or of type* e *(for extinction). Its distribution is characterized as follows.*
- *The root is of type* s*.*
- *The number of offsprings of a vertex depends, conditionally on the vertices of lower or same height, only on its own type (branching property).*
- *A vertex of type* e *produces only vertices of type* e *with offspring distribution $\mathfrak{p}$.*
- *The random number of children of a vertex of type* s *has the size-biased distribution of $\mathfrak{p}$ that is $\mathfrak{p}_{[1]}$ defined by (14) with $k = 1$. (Notice that $\mathfrak{p}_{[1]}$ is well defined as $\mathfrak{c} > 0$.) Furthermore, all of the children are of type* e *but one, uniformly chosen at random which is of type* s*.*

(ii) *If $\mathfrak{c} = 0$, the (degenerate) Kesten tree $\hat{\tau}^0$ is given by $\mathbf{t}_{\mathfrak{a}}$ the regular $\mathfrak{a}$-ary tree, with $\mathfrak{a} \geq 1$ defined by (10). It can be seen as a GW tree with degenerate offspring distribution the Dirac mass at point $\mathfrak{a}$. In this case all the individuals have type* s*.*

Informally, when $\mathfrak{c} > 0$, the individuals of type s in $\hat{\tau}^0$ form an infinite spine on which are grafted independent GW trees distributed (see Remark 3.2) as τ conditionally on the extinction event $\mathcal{E}$.

We define $\tau^0 = \mathrm{Ske}(\hat{\tau}^0)$ as the tree $\hat{\tau}^0$ when one forgets the types of the vertices. If $\mathfrak{c} = 0$, then τ^0 is the regular $\mathfrak{a}$-ary tree. If $\mathfrak{c} > 0$, the distribution of τ^0 is given in the following classical result.

Lemma 3.4. *Let p be an offspring distribution with finite positive mean such that $\mathfrak{c} > 0$. The distribution of τ^0 is characterized by: for all $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$ with $k = z_h(\mathbf{t})$:*

$$\mathbb{P}(r_h(\tau^0) = \mathbf{t}) = k\mathfrak{c}^{k-1}\mathfrak{m}^{-h}\, \mathbb{P}(r_h(\tau) = \mathbf{t}). \tag{15}$$

If $\mu \leq 1$, this is the usual link between Kesten tree and the size-biased GW tree. If $\mu > 1$, the lemma just means that the Kesten tree is the sized biased tree associated with the tree conditioned on extinction (which is the subcritical GW tree with offspring distribution $\mathfrak{p}$). We give a short proof of this well-known result.

Proof. According to Section 2.2, the distribution of τ^0 is characterized by (15) for all $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$ with $k = z_h(\mathbf{t})$.

Let $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$ and $v \in \mathbf{t}$ such that $|v| = h$. Let V be the vertex of type s at level h in $\hat{\tau}^0$. We have, with $k = z_h(\mathbf{t})$:

$$\begin{aligned}\mathbb{P}(r_h(\tau^0) = \mathbf{t}, V = v) &= \prod_{u \in r_{h-1}(\mathbf{t}) \setminus \mathrm{Anc}(\{v\})} \mathfrak{p}(k_u(\mathbf{t})) \prod_{u \in \mathrm{Anc}(\{v\})} \frac{1}{k_u(\mathbf{t})}\, \mathfrak{p}_{[1]}(k_u(\mathbf{t}))\\ &= \mathfrak{m}^{-h} \mathfrak{c}^{\sum_{u \in r_{h-1}(\mathbf{t})}(k_u(\mathbf{t})-1)} \prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t}))\\ &= \mathfrak{m}^{-h} \mathfrak{c}^{k-1}\, \mathbb{P}(r_h(\tau) = \mathbf{t}),\end{aligned}$$

where we used (14) (with $k = 1$, $n = k_u(\mathbf{t})$ and p replaced by $\mathfrak{p}$) and (13) (with $n = k_u(\mathbf{t})$) for the second equality and that $\sum_{u \in r_{h-1}(\mathbf{t})}(k_u(\mathbf{t}) - 1) = k - 1$ for the last one. Summing over all $v \in \mathbf{t}$ such that $|v| = h$ gives the result. $\square$

4. A probability distribution associated with super-critical GW trees

In this section, we consider a super-critical GW tree τ with non-degenerate offspring distribution $p = (p(n), n \in \mathbb{N})$ with finite mean $\mu \in (1, +\infty)$. We recall that f denotes the generating function of p and $\mathfrak{c}$ is the smallest root in $[0, 1)$ of $f(s) = s$. Notice that $\mathfrak{a} = 0$ is equivalent to $\mathfrak{c} > 0$. Following [32] or [6], we consider the Seneta-Heyde norming: $(c_n, n \in \mathbb{N})$ is a sequence such that $\left(\mathrm{e}^{-Z_n/c_n}, n \in \mathbb{N}\right)$ is a martingale and $c_0 \in (-1/\log(\mathfrak{c}), +\infty)$. This sequence is increasing positive and unbounded. Furthermore, we have that $\mathfrak{a} < c_{n+1}/c_n < \mu$ for all $n \in \mathbb{N}$ and that the sequence $(c_{n+1}/c_n, n \in \mathbb{N})$ is increasing[2] and converges towards μ. We also have that $(Z_n/c_n, n \in \mathbb{N})$ converges a.s. towards a non-negative random variable W with Laplace transform $\varphi(\lambda) = \mathbb{E}\left[\mathrm{e}^{-\lambda W}\right]$ such that $\varphi(+\infty) = \mathbb{P}(W = 0) = \mathfrak{c}$ and for all $\lambda \geq 0$:

$$f(\varphi(\lambda/\mu)) = \varphi(\lambda). \tag{16}$$

The probability distribution of W, up to a multiplicative constant, is the unique probability distribution solution of (16).

Remark 4.1. If one assumes that p satisfies $\mathbb{E}[Z_1 \log(Z_1)] < +\infty$, then Kesten and Stigum results asserts that $(\mu^{-n} Z_n, n \in \mathbb{N})$ converges a.s. towards W up to a scaling factor and that $\lim_{n\to\infty} \mu^{-n} c_n$ exists and belongs to $(0, +\infty)$.

Remark 4.2. Let $R_c = \sup\{r \geq 1;\, f(r) < +\infty\} \geq 1$ be the convergence radius of the generating function f of p. Set

$$\mathcal{K} = \{\lambda \in \mathbb{R};\, \mathbb{E}[\mathrm{e}^{\lambda W}] < +\infty\}, \tag{17}$$

and $\lambda_c = \sup \mathcal{K} \geq 0$. According to Theorem 8.1 in [25] (see also [31]), we have that $\lambda_c > 0$ if and only if $R_c > 1$. We then deduce that (16) holds for $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \in \mathcal{K}$. We get that $f(R_c) = \varphi(-\lambda_c) \in [1, +\infty]$ and thus that:

$$R_c = \varphi(-\lambda_c/\mu). \tag{18}$$

[2]We provide a short proof of the fact that the sequence $(c_{n+1}/c_n, n \in \mathbb{N})$ is increasing, as we didn't find a reference. Define $g_1(\lambda) = \log(f(\mathrm{e}^{-\lambda}))/\lambda$ so that $g_1(1/c_{n+1}) = -c_{n+1}/c_n$. So to prove that the sequence $(c_{n+1}/c_n, n \in \mathbb{N})$ is increasing, it is enough to check that g_1 is increasing, or more generally that the function $g_2(\lambda) = \log(\mathbb{E}[\mathrm{e}^{-\lambda X}])/\lambda$ defined for $\lambda > 0$ is increasing, where X is a non constant real-valued random variable with finite Laplace transform. Indeed, we have $g_2'(\lambda) > 0$ as $\mathbb{E}[Y\,\mathrm{e}^{-Y}] + \mathbb{E}[\mathrm{e}^{-Y}] \log(\mathbb{E}[\mathrm{e}^{-Y}]) < 0$ for any random variable Y such that $Y\,\mathrm{e}^{-Y}$ is integrable, thanks to Jensen inequality with the strictly concave function $-x\log(x)$ applied to e^{-Y}.

According to [14] and references therein, the distribution of W is $\mathfrak{c}\delta_0(dt) + w(t)\mathbf{1}_{\{t>0\}}dt$, where w is a positive continuous function defined on $(0, +\infty)$. Let $(W_\ell, \ell \in \mathbb{N}^*)$ be independent random variables distributed as W. The distribution of $\sum_{\ell=1}^k W_\ell$ is $\mathfrak{c}^k\delta_0(dt) + w_k(t)dt$, where (by decomposing according to the number $k-i$ of random variables W_ℓ which are equal to 0):

$$w_k(\theta) = \sum_{i=1}^{k} \binom{k}{i} \mathfrak{c}^{k-i} w^{*i}(\theta) \quad \text{for } \theta > 0, \tag{19}$$

and w^{*i} denotes the i-fold convolution of the function w. We now define a new probability distribution related to the function w. For $r \in \mathbb{N}^*$, $s = (s_1, \ldots, s_r) \in (\mathbb{N}^*)^r$ and $\theta \in (0, +\infty)$, we set $|s|_1 = \sum_{i=1}^r s_i$ and:

$$\rho_{\theta,r}(s) = \mu \frac{w^{*|s|_1}(\mu\theta)}{w^{*r}(\theta)} \prod_{i=1}^{r} \frac{f^{(s_i)}(\mathfrak{c})}{s_i!}. \tag{20}$$

Lemma 4.3. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Let $\theta \in (0, +\infty)$, $r \in \mathbb{N}^*$. Then $\rho_{\theta,r} = (\rho_{\theta,r}(s),\ s \in (\mathbb{N}^*)^r)$ defines a probability distribution on $(\mathbb{N}^*)^r$.*

Proof. For convenience, we shall prove that $\rho_{\theta/\mu,r}$ is a probability distribution. Let $\hat{w}$ denote the Laplace transform of w: $\hat{w}(\lambda) = \int_0^\infty w(t)\, \mathrm{e}^{-\lambda t}\, dt$ for $\lambda \geq 0$. We deduce from (16) that $f(\mathfrak{c} + \hat{w}(\lambda)) = \mathfrak{c} + \hat{w}(\mu\lambda) = f(\mathfrak{c}) + \hat{w}(\mu\lambda)$. We deduce that for $r \in \mathbb{N}^*$:

$$\begin{aligned}
\hat{w}(\mu\lambda)^r &= \left(f(\mathfrak{c} + \hat{w}(\lambda)) - f(\mathfrak{c})\right)^r \\
&= \sum_{k_1,\ldots,k_r \in \mathbb{N}^*} \prod_{i=1}^{r} p(k_i) \left((\mathfrak{c} + \hat{w}(\lambda))^{k_i} - \mathfrak{c}^{k_i} \right) \\
&= \sum_{k_1,\ldots,k_r \in \mathbb{N}^*} \prod_{i=1}^{r} p(k_i) \sum_{s_i=1}^{k_i} \binom{k_i}{s_i} \mathfrak{c}^{k_i - s_i} \hat{w}(\lambda)^{s_i} \\
&= \sum_{s=(s_1,\ldots,s_r) \in (\mathbb{N}^*)^r} \hat{w}(\lambda)^{|s|_1} \prod_{i=1}^{r} \sum_{k_i = s_i}^{+\infty} \binom{k_i}{s_i} \mathfrak{c}^{k_i - s_i} p(k_i) \\
&= \sum_{s=(s_1,\ldots,s_r) \in (\mathbb{N}^*)^r} \hat{w}(\lambda)^{|s|_1} \prod_{i=1}^{r} \frac{f^{(s_i)}(\mathfrak{c})}{s_i!},
\end{aligned}$$

where we used for the last equality that for $s \in \mathbb{N}^*$, $x \in [0,1]$:

$$f^{(s)}(x) = \sum_{k=s}^{+\infty} \frac{k!}{(k-s)!} x^{k-s} p(k) = s! \sum_{k=s}^{+\infty} \binom{k}{s} x^{k-s} p(k).$$

Since $\hat{w}(\mu\lambda)^r$ is the Laplace transform of $w^{*r}(t/\mu)/\mu$, by uniqueness of the Laplace transform and the continuity of w (and thus of w^{*i}), we get using the definition (20) of $\rho_{\theta,r}$ that for all $\theta \in (0, +\infty)$:

$$\frac{1}{\mu}\, w^{*r}(\theta/\mu) = \sum_{s=(s_1,\ldots,s_r) \in (\mathbb{N}^*)^r} w(\theta)^{*|s|_1} \prod_{i=1}^{r} \frac{f^{(s_i)}(\mathfrak{c})}{s_i!} = \frac{1}{\mu}\, w^{*r}(\theta/\mu) \sum_{s \in (\mathbb{N}^*)^r} \rho_{\theta/\mu,r}(s). \tag{21}$$

Since w is non-zero, we get that $\sum_{s \in (\mathbb{N}^*)^r} \rho_{\theta/\mu,r}(s) = 1$ and thus $\rho_{\theta/\mu,r}$ is a probability distribution as $\rho_{\theta/\mu,r}(s)$ is non-negative. $\square$

We end this section with the limit of $\rho_{\theta,r}$ as θ goes to 0 and in a particular case to $+\infty$. Recall Definitions (10) and (11). One has to distinguish two cases when θ goes to 0: the so-called Schröder case $\mathfrak{a} \leq 1$ (equivalently $p(0)+p(1) \neq 0$, $f'(\mathfrak{c}) > 0$ or $\alpha < +\infty$) and the so-called Böttcher case $\mathfrak{a} \geq 2$ (equivalently $p(0)+p(1) = 0$, $f'(\mathfrak{c}) = 0$ or $\alpha = +\infty$). When θ goes to infinity we consider the particular so-called Harris case where p has a finite support (equivalently $\mathfrak{b}$ is finite).

Lemma 4.4. *Let p be a non-degenerate super-critical offspring distribution with finite mean.*

(i) *In the Schröder case ($\mathfrak{a} \leq 1$), we get that $\rho_{\theta,1}$ converges to the Dirac mass at point 1 as θ goes down to 0.*

(ii) *In the Böttcher case ($\mathfrak{a} \geq 2$), we get that, for all $r \in \mathbb{N}^*$, $\rho_{\theta,r}$ converges to the Dirac mass at $(\mathfrak{a}, \dots, \mathfrak{a}) \in \mathbb{N}^r$ as θ goes down to 0.*

(iii) *In the Harris case ($\mathfrak{b} < \infty$), we get that, for all $r \in \mathbb{N}^*$, $\rho_{\theta,r}$ converges to the Dirac mass at $(\mathfrak{b}, \dots, \mathfrak{b}) \in \mathbb{N}^r$ as θ goes to infinity.*

Proof. We give the proof of (i). The technical proofs of (ii) and (iii) are postponed respectively to Sections 12.3 and 11.3.

According to [10], there exists a positive continuous multiplicatively periodic function V defined on $(0, +\infty)$ with period μ such that for all $x > 0$:

$$x^{1-\alpha} w(x) = V(x) + o(1) \quad \text{as } x \searrow 0. \tag{22}$$

We have for $\theta > 0$ as θ goes down to 0:

$$\rho_{\theta,1}(1) = \mu f'(\mathfrak{c}) \frac{w(\mu\theta)}{w(\theta)} = \frac{V(\mu\theta) + o(1)}{V(\theta) + o(1)} = 1 + o(1),$$

where we used Definition (11) of the Schröder constant for the first equality and that V has multiplicative period μ for the last one. This implies that $\lim_{\theta \to 0} \rho_{\theta,1}(1) = 1$ and thus $\rho_{\theta,1}$ converges to the Dirac mass at 1 as θ goes down to 0. □

5. Extremal GW trees

We are in the setting of Section 4. If $\mathfrak{c} > 0$, we define the sub-critical offspring distribution $\mathfrak{p}$ by (13) and, see (14), the corresponding size-biased distribution $\mathfrak{p}_{[\ell]}$ of order $\ell \in \mathbb{N}^*$. For $\ell \in \mathbb{N}^*$ such that $f^{(\ell)}(\mathfrak{c}) > 0$, we have:

$$\mathfrak{p}_{[\ell]}(k) = \binom{k}{\ell} \frac{\ell!}{f^{(\ell)}(\mathfrak{c})} \, \mathfrak{c}^{k-\ell} p(k), \quad k \geq \ell. \tag{23}$$

If $\mathfrak{c} = 0$ but $p(\ell) > 0$ (or equivalently $f^{(\ell)}(\mathfrak{c}) > 0$), then we define $\mathfrak{p}_{[\ell]}$ as the Dirac mass at point ℓ, so that Definition (23) is consistent for $\mathfrak{c} \geq 0$. Recall Definition (10) of $\mathfrak{a}$ and note that $\mathfrak{p} = \mathfrak{p}_{[\mathfrak{a}]}$ if $\mathfrak{c} = 0$.

Let $\theta \in (0, +\infty)$. We define a two-type random tree $\hat{\tau}^\theta$ and shall consider the corresponding tree $\tau^\theta = \mathrm{Ske}(\hat{\tau}^\theta)$ when one forgets the types of the vertices of $\hat{\tau}^\theta$.

Definition 5.1 (Extremal tree)**.** *Let p be a non-degenerate super-critical offspring distribution with finite mean. The labeled random tree $\hat{\tau}^\theta$ is a two-type random tree where the vertices are either of type* s *(for survivor) or of type* e *(for extinction) and $\tau^\theta = \mathrm{Ske}(\hat{\tau}^\theta)$ denotes the corresponding random $\mathbb{T}_\mathrm{f}$-valued tree when one forgets the labels (or types). The distribution of $\hat{\tau}^\theta$ is characterized as follows:*

- *The root is of type* s.

- *The number of offsprings of a vertex of type* e *does not depend on the vertices of lower or same height (branching property for vertices of type* e*).*
- *A vertex of type* e *produces only vertices of type* e *with offspring distribution* $\mathfrak{p}$ *(as in the Kesten tree).*
- *For every* $h \geq 0$*, we set*

$$\mathscr{S}_h = \{u \in \tau^\theta;\ |u| = h \text{ and the vertex } u \text{ has type s in } \hat{\tau}^\theta\}.$$

For a vertex u *of type* s*, we denote by* $\kappa^{\mathrm{s}}(u)$ *the number of children of* u *with type* s *and by* $\kappa^{\mathrm{e}}(u)$ *the number of children of* u *with type* e*. Conditionally given* $r_h(\tau^\theta)$ *and* $(\mathscr{S}_\ell, 0 \leq \ell \leq h)$*, we have:*
 (i) $(\kappa^{\mathrm{s}}(u), u \in \mathscr{S}_h)$ *has distribution* $\rho_{\mu^h\theta, \sharp\mathscr{S}_h}$.
 (ii) *For every* $u \in \mathscr{S}_h$*, conditionally on* $\{\kappa^{\mathrm{s}}(v) = s_v \geq 1, v \in \mathscr{S}_h\}$*,* $\kappa^{\mathrm{e}}(u)$ *is such that* $k_u(\tau^\theta) = \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u)$ *has distribution* $\mathfrak{p}_{[s_u]}$ *and the* s_u *vertices of type* s *are chosen uniformly at random among the* $k_u(\tau^\theta)$ *children.*

Notice that Property (i) in the above definition breaks down the branching property. If $\mathfrak{c} = 0$, then a.s. $\kappa^{\mathrm{e}}(u) = 0$, so that there are no individuals of type e. We stress out, and shall use later on, that $\hat{\tau}^\theta$ truncated at level h can be recovered from $r_h(\tau^\theta)$ and $\mathscr{S}_h$ as all the ancestors of a vertex of type s are of type s and a vertex of type s has at least one children of type s.

Since all the vertices of type s have at least one offspring of type s, we get $\sharp\mathscr{S}_{h+1} \geq \sharp\mathscr{S}_h$. The offspring distribution of vertices of type s can also be described as follows. For every $h \geq 0$, conditionally given $r_h(\tau^\theta)$ and $\mathscr{S}_h$, we compute the probability that

- we have $\sharp\mathscr{S}_{h+1} - \sharp\mathscr{S}_h = n$ for some $n \geq 0$ i.e. n new vertices of type s appear at generation $h+1$,
- every node u of $\mathscr{S}_h$ has k_u offspring, s_u of them being of type s, where the integers $((s_u, k_u),\ u \in \mathscr{S}_h)$ satisfy $1 \leq s_u \leq k_u$ and $\sum_{u \in \mathscr{S}_h} s_u = n + \sharp\mathscr{S}_h$,
- for every $u \in \mathscr{S}_h$ and every subset $A_u \subset \{1, \ldots, k_u\}$ such that $\sharp A_u = s_u$, the positions of the offspring of u of type s among all the offspring of u, are given by A_u i.e. $\mathscr{S}_{h+1} \cap \{u1, \ldots, uk_u\} = uA_u$ where we recall that uv denotes the concatenation of the two sequences u and v.

We have:

$$\begin{aligned}
(24)\quad \mathbb{P}\Big(\forall u \in \mathscr{S}_h,\ \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u) = k_u \text{ and } \mathscr{S}_{h+1} \cap \{u1, \ldots, uk_u\} = uA_u \,|\, r_h(\tau^\theta), \mathscr{S}_h\Big) \\
= \rho_{\mu^h\theta, \sharp\mathscr{S}_h}((s_u, u \in \mathscr{S}_h)) \prod_{u \in \mathscr{S}_h} \frac{1}{\binom{k_u}{s_u}} \mathfrak{p}_{[s_u]}(k_u) \\
= \mu \frac{w^{*(\sharp\mathscr{S}_h + n)}(\mu^{h+1}\theta)}{w^{*\sharp\mathscr{S}_h}(\mu^h\theta)} \prod_{u \in \mathscr{S}_h} \mathfrak{c}^{k_u - s_u} p(k_u),
\end{aligned}$$

where we used (20) and (23) for the last equality.

By construction, a.s. individuals of type s have a progeny which does not suffer extinction whereas individuals of type e (if any) have a progeny which suffers extinction. Since the individuals of type s do not satisfy the branching property, the random tree $\hat{\tau}^\theta$ is not a two-type inhomogeneous GW tree.

Using this definition, it is easy to get that the distribution of the tree $r_h(\tau^\theta)$ is absolutely continuous with respect to those of the original GW tree $r_h(\tau)$.

Lemma 5.2. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Let $\theta \in (0, +\infty)$. Let $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$. We have, with $k = z_h(\mathbf{t})$:*

$$\mathbb{P}(r_h(\tau^\theta) = \mathbf{t}) = \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mu^h \frac{w_k(\mu^h \theta)}{w(\theta)}.$$

Proof. Let $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$ and $S_h \subset \{u \in \mathbf{t};\, |u| = h\}$ be non empty. Set $k = z_h(\mathbf{t})$. In order to shorten the notations, we set $\mathcal{A} = S_h \bigcup \mathrm{Anc}(S_h)$. We set, for $\ell \in \{0, \dots, h-1\}$, $S_\ell = \{u \in \mathcal{A},\, |u| = \ell\}$ the vertices at level ℓ which have at least one descendant in S_h. For $u \in r_{h-1}(\mathbf{t})$, we set $s_u(\mathbf{t}) = \sharp(\mathcal{A} \bigcap u\mathbb{N}^*)$, the number of children of u having descendants in S_h. We recall that $\hat{\tau}^\theta$ truncated at level h can be recovered from $r_h(\tau^\theta)$ and $\mathscr{S}_h$. We compute $\mathcal{C}_{S_h} = \mathbb{P}(r_h(\tau^\theta) = \mathbf{t},\, \mathscr{S}_h = S_h)$. We have, using (24):

$$\begin{aligned}
\mathcal{C}_{S_h} &= \left[\prod_{u \in r_{h-1}(\mathbf{t}),\, u \notin \mathcal{A}} \mathfrak{p}(k_u(\mathbf{t}))\right] \prod_{\ell=0}^{h-1} \mu \frac{w^{*(\sharp S_{\ell+1})}(\mu^{\ell+1}\theta)}{w^{*(\sharp S_h)}(\mu^\ell \theta)} \prod_{u \in S_\ell} \mathfrak{c}^{k_u(\mathbf{t}) - s_u(\mathbf{t})} p(k_u(\mathbf{t}))\\
&= \left[\prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t}))\right] \left[\prod_{u \in r_{h-1}(\mathbf{t})} \mathfrak{c}^{k_u(\mathbf{t})-1}\right] \left[\prod_{u \in \mathcal{A}} \mathfrak{c}^{-(s_u(\mathbf{t})-1)}\right] \mu^h \frac{w^{*(\sharp S_h)}(\mu^h\theta)}{w(\theta)}\\
&= \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k - \sharp S_h} \mu^h\, \frac{w^{*(\sharp S_h)}(\mu^h \theta)}{w(\theta)}, \qquad (25)
\end{aligned}$$

where we used that for a tree $\mathbf{s}$, we have $\sum_{u \in r_{h-1}(\mathbf{s})} k_u(\mathbf{s}) - 1 = z_h(\mathbf{s}) - 1$ and that $\mathbf{s} = \mathcal{A}$ is tree-like with $z_h(\mathbf{s}) = \sharp S_h$. Remark that $\mathcal{C}_{S_h}$ depends only of $\sharp S_h$. Since $\sharp \mathscr{S}_h \geq 1$ as the root is of type s, we obtain:

$$\begin{aligned}
\mathbb{P}(r_h(\tau^\theta) = \mathbf{t}) &= \sum_{i=1}^k \sum_{S_h \subset \{u \in \mathbf{t};\, |u| = h\}} \mathbf{1}_{\{\sharp S_h = i\}}\, \mathcal{C}_{S_h}\\
&= \sum_{i=1}^k \binom{k}{i} \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k-i} \mu^h\, \frac{w^{*i}(\mu^h\theta)}{w(\theta)} = \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mu^h \frac{w_k(\mu^h\theta)}{w(\theta)},
\end{aligned}$$

where we used (19) for the last equality. □

Remark 5.3. Let $\mathcal{E}^c = \{W > 0\}$ denote the non-extinction event. Using Lemma 5.2, we get for $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$, and g a non-negative measurable function defined on $\mathbb{R}_+$, that:

$$\int_0^{+\infty} g(\theta) \mathbb{P}(r_h(\tau^\theta) = \mathbf{t})\, w(\theta) d\theta = \mathbb{E}\left[g(W) \mathbf{1}_{\{r_h(\tau) = \mathbf{t}, \mathcal{E}^c\}}\right]$$

This implies that for every non-negative measurable function G defined on $\mathbb{T}_\infty \times \mathbb{R}_+$, we have:

$$\int_0^{+\infty} \mathbb{E}[G(\tau^\theta, \theta)]\, w(\theta) d\theta = \mathbb{E}\left[G(\tau, W) \mathbf{1}_{\{\mathcal{E}^c\}}\right].$$

Thus, the distribution probability of τ^θ is a regular version of the distribution of τ conditionally on $\{W = \theta\}$. From Lemma 5.2, we get that this version is continuous on $\mathbb{T}_\mathrm{f}^{(h)}$ for all $h \in \mathbb{N}^*$. In particular, we deduce that for a.e. $\theta \in (0, +\infty)$, a.s. $\lim_{n \to \infty} z_n(\tau^\theta)/c_n = \theta$ (see also Theorem 2.II in [26] for an a.s. convergence for all $\theta \in (0, +\infty)$ under stronger hypothesis). The distribution of τ conditionally on $\mathcal{E}^c$ can be written as a mixture of distributions of τ^θ as for every Borel set A of $\mathbb{T}_\infty$,

$$\int_0^{+\infty} \mathbb{P}(\tau^\theta \in A)\, w(\theta) d\theta = \mathbb{P}(\{\tau \in A\} \cap \mathcal{E}^c).$$

6. Convergence of conditioned super-critical GW trees

We are in the setting of Section 4, with τ a GW tree with super-critical non-degenerate offspring distribution p with finite mean μ. We consider a deterministic $\mathbb{N}$-valued sequence $(a_n, n \in \mathbb{N}^*)$ such that $\mathbb{P}(Z_n = a_n) > 0$ for every $n > 0$. See Remark 1.3 for conditions on the existence of such sequences. We denote by τ_n a random tree distributed as the GW tree τ conditioned on $\{Z_n = a_n\}$. We study the limit in distribution of τ_n as n goes to infinity and we consider different regimes according to the growth speed of the sequence $(a_n, n \in \mathbb{N}^*)$. Recall that Z_n is under $\mathbb{P}_k$ distributed as a GW process with offspring distribution p starting at $Z_0 = k$.

We say that the offspring distribution p is of type (L_0, r_0), when L_0 is the period of p, that is the greatest common divisor of $\{n - \ell;\, n > \ell$ and $p(n)p(\ell) \neq 0\}$, and r_0 is the residue $(\text{mod } L_0)$ of any n such that $p(n) \neq 0$. See Remark 1.3 on sufficient conditions to get $\mathbb{P}_k(Z_n = a) > 0$.

6.1. **The intermediate regime:** $\lim_{n\to\infty} a_n/c_n \in (0, +\infty)$.

We first state a strong ratio limit which is a direct consequence of the local limit theorem in [14].

Lemma 6.1. *Let p be a non-degenerate super-critical offspring distribution with finite mean and type (L_0, r_0). Let $\theta \in (0, +\infty)$. Assume that $\lim_{n\to\infty} a_n/c_n = \theta$ and that $a_n = r_0^n (\text{mod } L_0)$ for all $n \in \mathbb{N}^*$. For all $h, k \in \mathbb{N}^*$, we have:*

$$\lim_{n\to\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = \mu^h \, \frac{w_k\left(\mu^h \theta\right)}{w(\theta)} \mathbf{1}_{\{k = r_0^h (\text{mod } L_0)\}}.$$

Proof. The local limit theorem in [14] states that for all $k \in \mathbb{N}^*$, $\theta \in (0, +\infty)$ and $(a_n, n \in \mathbb{N})$ a sequence of elements of $\mathbb{N}^*$ such that $\lim_{n\to\infty} a_n/c_n = \theta$, we have:

$$\lim_{n\to\infty} \left[c_n \mathbb{P}_k(Z_n = a_n) - L_0 \mathbf{1}_{\{a_n = k r_0^n (\text{mod } L_0)\}} w_k(\theta) \right] = 0. \tag{26}$$

We now assume that $a_n = k r_0^n (\text{mod } L_0)$ and $\lim_{n\to\infty} a_n/c_n = \theta \in (0, +\infty)$. Using Remark 1.3, we deduce that $\mathbb{P}_k(Z_{n-h} = a_n) > 0$ if and only if $a_n = k r_0^{n-h} (\text{mod } L_0)$ that is $k = r_0^h (\text{mod } L_0)$. In this case, noticing that $\lim_{n\to\infty} a_n/c_{n-h} = \mu^h \theta$ as $\lim_{n\to\infty} c_n/c_{n-h} = \mu^h$, using (26), we get that:

$$\lim_{n\to\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = \lim_{n\to\infty} \frac{c_n}{c_{n-h}} \, \frac{w_k\left(\mu^h \theta\right)}{w(\theta)} = \mu^h \, \frac{w_k\left(\mu^h \theta\right)}{w(\theta)}.$$

□

We deduce the following local convergence.

Proposition 6.2. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Let $\theta \in (0, +\infty)$. Assume that $\lim_{n\to\infty} a_n/c_n = \theta$ and that τ_n is well defined for all n. Then, we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^\theta.$$

Proof. Assume that p is of type (L_0, r_0), so that τ_n is well defined for n large if and only if $a_n = r_0^n (\text{mod } L_0)$. Using that a.s. $H(\tau^\theta) = +\infty$, the characterization (7) of the convergence in $\mathbb{T}_\mathrm{f}$, (12) with $k = r_0^h (\text{mod } L_0)$, and Lemmas 5.2 and 6.1, we directly get the result. $\square$

6.2. **The high regime in the Harris case:** $\lim_{n\to\infty} a_n/c_n = +\infty$**.** Let p be a non-degenerate super-critical offspring distribution with finite mean. Recall $\mathfrak{b}$ (the supremum of the support of p) defined in (10). Notice that $\mathfrak{b}$ finite (Harris case) implies that p has finite mean. When $\mathfrak{b} < \infty$, we define τ^∞ as $\mathbf{t}_\mathfrak{b}$, the deterministic regular $\mathfrak{b}$-ary tree.

Proposition 6.3. *Let p be a non-degenerate super-critical offspring distribution with $\mathfrak{b} < \infty$. Assume that $a_n \leq \mathfrak{b}^n$ for all $n \in \mathbb{N}^*$, $\lim_{n\to\infty} a_n/c_n = \infty$ and that τ_n is well defined for all n. Then, we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^\infty.$$

Proof. We assume that τ_n is well defined, that is $\mathbb{P}(Z_n = a_n) > 0$. For $h \in \mathbb{N}^*$, we have $\mathbb{P}(r_h(\tau) = r_h(\mathbf{t}_\mathfrak{b})) = p(\mathfrak{b})^{(\mathfrak{b}^h-1)/(\mathfrak{b}-1)}$. We deduce from (12) and (7), using that $\mathbf{t}_\mathfrak{b}$ has a.s. an infinite height, that the proof of Proposition 6.3 is complete as soon as we prove that for all $k \leq \mathfrak{b}^h$:

$$\lim_{n\to\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = p(\mathfrak{b})^{-(\mathfrak{b}^h-1)/(\mathfrak{b}-1)} \mathbf{1}_{\{k=\mathfrak{b}^h\}}. \tag{27}$$

In fact, it is enough to prove (27) for $k = \mathfrak{b}^h$ as $\mathbb{P}(Z_h = \mathfrak{b}^h) = p(\mathfrak{b})^{-(\mathfrak{b}^h-1)/(\mathfrak{b}-1)}$ and:

$$\mathbb{P}(Z_n = a_n) = \mathbb{P}(Z_h = \mathfrak{b}^h)\mathbb{P}_{\mathfrak{b}^h}(Z_{n-h} = a_n) + \sum_{k\leq \mathfrak{b}^h-1} \mathbb{P}(Z_h = k)\mathbb{P}_k(Z_{n-h} = a_n). \tag{28}$$

It is also enough to consider the two cases: $\lim_{n\to\infty} a_n/\mathfrak{b}^n = 1$ or $\limsup_{n\to\infty} a_n/\mathfrak{b}^n < 1$ with $\lim_{n\to\infty} a_n/c_n = +\infty$.

We first consider the case $\lim_{n\to\infty} a_n/\mathfrak{b}^n = 1$. Notice that $\mathbb{P}_k(Z_{n-h} = a_n) = 0$ for $k\mathfrak{b}^{n-h} < a_n$ as each individual produces at most $\mathfrak{b}$ children. For $k \leq \mathfrak{b}^h - 1$, we have $k\mathfrak{b}^{n-h} \leq \mathfrak{b}^n - \mathfrak{b}^{n-h}$. Since $\lim_{n\to\infty} a_n/\mathfrak{b}^n = 1$, we deduce that for $h, k \in \mathbb{N}^*$, if $k \leq \mathfrak{b}^h - 1$, then $k\mathfrak{b}^{n-h} < a_n$ for n large enough. Using (28), we deduce that for n large enough, $\mathbb{P}(Z_n = a_n) = \mathbb{P}(Z_h = \mathfrak{b}^h)\mathbb{P}_{\mathfrak{b}^h}(Z_{n-h} = a_n)$ as soon as $\mathbb{P}(Z_n = a_n) > 0$. This gives (27).

The case $\limsup_{n\to\infty} a_n/\mathfrak{b}^n < 1$ and $\lim_{n\to\infty} a_n/c_n = +\infty$ is proven in Section 11.4, see Lemma 11.6 with $\ell = 1$. $\square$

6.3. **The low regime:** $\lim_{n\to\infty} a_n/c_n = 0$**.** Let p be a non-degenerate super-critical offspring distribution with finite mean. If $\mathfrak{c} > 0$ (and thus $\mathfrak{a} = 0$), we recall that $\tau^{0,0}$ denote the distribution of the GW tree τ with offspring distribution $\mathfrak{p}$ given in (13). According to Remark 3.2, we have the following result for the extinction regime.

Proposition 6.4. *Let p be a non-degenerate super-critical offspring distribution with finite mean such that $\mathfrak{c} > 0$. Assume that $a_n = 0$ for n large enough so that τ_n is well defined for n large enough. Then, we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^{0,0}.$$

Recall the Kesten tree τ^0 from Definition 3.3. Recall that $\mathfrak{a} \geq 1$ implies that a.s. $\tau^0 = \mathbf{t}_\mathfrak{a}$, the deterministic regular $\mathfrak{a}$-ary tree.

Proposition 6.5. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Assume that $a_n \geq 1 \vee \mathfrak{a}^n$ for all $n \in \mathbb{N}^*$, $\lim_{n\to\infty} a_n/c_n = 0$ and that τ_n is well defined for all n. Then, we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^0.$$

Proof. We give the proof in the Schröder case ($\mathfrak{a} \leq 1$). The Böttcher case ($\mathfrak{a} \geq 2$) is more technical and its proof is postponed to Section 12.5. We suppose throughout the proof that p is of type (L_0, r_0).

Case I: the sequence $(a_n, n \in \mathbb{N}^*)$ is bounded. We first consider the case $\mathfrak{a} = 0$. The ratio theorem, see (4) in [30] (or [8] Theorem A.7.4), implies, that for all $\ell, k, h \in \mathbb{N}^*$, if $\mathbb{P}(Z_n = k) > 0$ for n large enough, then:

$$\lim_{n\to\infty} \frac{\mathbb{P}_\ell(Z_{n-h} = k)}{\mathbb{P}(Z_n = k)} = \ell \mathfrak{c}^{\ell-1} f'(\mathfrak{c})^{-h}.$$

We deduce from (12) and (15), as $\mathfrak{m} = f'(\mathfrak{c})$, that for $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$, we have $\lim_{n\to\infty} \mathbb{P}(r_h(\tau_n) = \mathbf{t}) = \mathbb{P}(r_h(\tau^0) = \mathbf{t})$. Since τ^0 has a.s. an infinite height, we get that τ_n converges in distribution towards τ^0 using the convergence characterization (7).

We consider now the case $\mathfrak{a} = 1$. Recall that $\mathbf{t}_\mathfrak{a}$ is the regular $\mathfrak{a}$-ary tree. According to Remark 1.3, for k large enough, we get that $\mathbb{P}(Z_n = k) > 0$ and $\mathbb{P}(Z_{n-h} = k) > 0$ for n large enough. It is easy to check that for $h \in \mathbb{N}$, $k \in \mathbb{N}^*$:

$$\mathbb{P}(r_h(\tau) = r_h(\mathbf{t}_\mathfrak{a}) |\, Z_n = k) = p(1)^h \frac{\mathbb{P}(Z_{n-h} = k)}{\mathbb{P}(Z_n = k)}.$$

For $k = 1$, the left hand side member is equal to one. For $k > 1$, it is not difficult to get, by considering the lowest vertex of τ with out-degree larger than one, that the sequence $(\mathbb{P}(Z_n = k)/\mathbb{P}(Z_n = 1), n \in \mathbb{N}^*)$ is bounded. Then arguing as in [30], one gets that $\lim_{n\to\infty} \frac{\mathbb{P}(Z_{n-h}=k)}{\mathbb{P}(Z_n=k)} = p(1)^{-h}$. This gives that $\lim_{n\to\infty} \mathbb{P}(r_h(\tau) = r_h(\mathbf{t}_\mathfrak{a}) |\, Z_n = k) = 1$. This implies that τ_n converges in distribution towards $\tau^0 = \mathbf{t}_\mathfrak{a}$ using the convergence characterization (7).

Case II: $\lim_{n\to\infty} a_n = +\infty$. We first consider the case $\mathfrak{a} = 0$. Then we have $f_n(0) > 0$ for all $n \in \mathbb{N}^*$. Since $\{\sum_{i=1}^\ell Z_{n-h}^{(i)} = a_n\}$ contains $\bigcup_{j=1}^\ell \left(\{Z_{n-h}^{(j)} = a_n\} \bigcap_{i\neq j} \{Z_{n-h}^{(i)} = 0\}\right)$, we deduce that $\mathbb{P}_\ell(Z_{n-h} = a_n) \geq \ell f_{n-h}(0)^{\ell-1} \mathbb{P}(Z_{n-h} = a_n)$. Using that $\lim_{n\to\infty} f_{n-h}(0) = \mathfrak{c}$, we deduce from Lemma 6.6, stated below, that:

$$\liminf_{n\to\infty} \frac{\mathbb{P}_\ell(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \geq \ell \mathfrak{c}^{\ell-1} f'(\mathfrak{c})^{-h}.$$

As $f'(\mathfrak{c}) = \mathfrak{m}$, we deduce from (12) and (15) that

$$\liminf_{n\to\infty} \mathbb{P}(r_h(\tau_n) = \mathbf{t}) \geq \mathbb{P}(r_h(\tau^0) = \mathbf{t}). \tag{29}$$

Since τ^0 has a.s. an infinite height, we deduce that (29) holds for all $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h')}$ with $0 \leq h' \leq h$. Since singletons are open subsets of the closed discrete set $\bigcup_{0\leq h'\leq h} \mathbb{T}_\mathrm{f}^{(h')}$, we deduce from the Portmanteau theorem that $(r_h(\tau_n), n \in \mathbb{N})$ converges in distribution towards $r_h(\tau^0)$. Since this holds for all $h \in \mathbb{N}^*$, and since τ^0 has a.s. an infinite height, we conclude using the convergence characterization (7).

We now consider the case $\mathfrak{a} = 1$. Then we have a.s. $\tau^0 = \mathbf{t}_\mathfrak{a}$. We deduce, as $f'(\mathfrak{c}) = p(1)$, that $\mathbb{P}(r_h(\tau) = r_h(\mathbf{t}_\mathfrak{a})) = f'(\mathfrak{c})^h$ and thus, using (12) and Lemma 6.6:

$$\mathbb{P}(r_h(\tau_n) = r_h(\mathbf{t}_\mathfrak{a})) = \mathbb{P}(r_h(\tau) = r_h(\mathbf{t}_\mathfrak{a})) \frac{\mathbb{P}(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \xrightarrow[n\to\infty]{} 1.$$

Since this holds for all $h \in \mathbb{N}^*$, and since $\mathbf{t}_\mathfrak{a}$ has a.s. an infinite height, we conclude using the convergence characterization (7). □

The proof of the previous proposition in the Schröder case is based on the following strong ratio limit.

Lemma 6.6. *Let p be a non-degenerate super-critical offspring distribution with finite mean in the Schröder case ($\mathfrak{a} \leq 1$). Assume that $\lim_{n\to+\infty} a_n = +\infty$, $\lim_{n\to+\infty} a_n/c_n = 0$ and $\mathbb{P}(Z_n = a_n) > 0$ for every $n \in \mathbb{N}^*$. Then we have for all $h \in \mathbb{N}^*$:*

$$\lim_{n\to+\infty} \frac{\mathbb{P}(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = f'(\mathfrak{c})^{-h}. \tag{30}$$

Notice that according to Remark 1.3, the condition $\mathbb{P}(Z_n = a_n) > 0$ in Lemma 6.6 is satisfied as soon as $a_n = r_0^n (\text{mod } L_0)$ as $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to+\infty} a_n/c_n = 0$.

Proof. Since $\mathfrak{a} \leq 1$, we have $r_0 \in \{0, 1\}$. We deduce from Corollary 5 in [17], that for $k_n \leq c_n$ and $\lim_{n\to\infty} k_n = +\infty$:

$$\lim_{n\to\infty} \sup_{k\in[k_n, c_n],\, k=r_0 \,(\text{mod } L_0),} \left| \frac{\mu^{n-\rho_k} c_{\rho_k}}{L_0 w(k/\mu^{n-\rho_k} c_{\rho_k})} \mathbb{P}(Z_n = k) - 1 \right| = 0, \tag{31}$$

where $\rho_k = \min\{\ell \geq 1;\, c_\ell \geq k\}$. Recall that $\lim_{n\to\infty} c_{n+1}/c_n = \mu$. The hypothesis on a_n imply thus that $\lim_{n\to\infty} \rho_{a_n} = +\infty$. Set $\rho = \rho_{a_n}$ for simplicity. Assume that $a_n = r_0^n (\text{mod } L_0)$, so that $\mathbb{P}(Z_n = a_n) > 0$ for n large enough. For n large enough, we have:

$$\frac{\mathbb{P}(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} \sim \mu^h \frac{w(a_n/\mu^{n-h-\rho} c_\rho)}{w(a_n/\mu^{n-\rho} c_\rho)} \sim \mu^{\alpha h} \frac{V(a/c_\rho)}{V(a/c_\rho)} = f'(\mathfrak{c})^{-h},$$

where we used (31) for the first approximation, the representation (22) of w in the Schröder case and that V is multiplicatively periodic with period μ for the second one. □

7. Continuity in law of the extremal GW trees at $\theta = 0$

We are in the setting of Section 4. Recall the definition of $\hat{\tau}^\theta$ given in Section 5 for $\theta > 0$ and in Section 3 for $\theta = 0$. Since the function w is continuous, we get that the distribution of $\hat{\tau}^\theta$ and thus of τ^θ, as a function of $\theta \in (0, +\infty)$ is continuous. From the convergence of the offspring distribution of the individuals of type s which is a consequence of Lemma 4.4, we deduce the continuity in distribution of $\hat{\tau}^\theta$ for $\theta \in [0, +\infty)$. This directly gives the continuity in distribution of τ^θ for $\theta \in [0, +\infty)$. We stress in the next corollary that only the convergence at 0 is non-trivial.

Corollary 7.1. *Let p be a non-degenerate super-critical offspring distribution with finite mean. We have the following convergence in distribution:*

$$\tau^\theta \xrightarrow[\theta\to 0]{(d)} \tau^0.$$

As a consequence of Corollary 7.1, we recover directly Corollary 3 from [9], which is stated only in the Böttcher case ($p(0) + p(1) = 0$) and extend it to the Schröder case, see next corollary. Recall that in the Böttcher case, the random tree τ^0 is in fact the (deterministic) regular $\mathfrak{a}$-ary tree. For $\varepsilon \in (0,1)$, let $\tau_{(\varepsilon)}$ be distributed as τ conditionally on $\{0 < W \leq \varepsilon\}$. Notice that if $\mathfrak{c} = 0$, then conditioning on $\{0 < W \leq \varepsilon\}$ is the same as conditioning on $\{0 \leq W \leq \varepsilon\}$.

Corollary 7.2. *Let p be a non-degenerate super-critical offspring distribution with finite mean. We have the following convergence in distribution:*

$$\tau_{(\varepsilon)} \xrightarrow[\varepsilon\to 0]{(d)} \tau^0.$$

Proof. Let $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$ and set $k = z_h(\mathbf{t})$. We deduce from Lemma 5.2 that for all $\theta \in (0, +\infty)$:

$$\mathbb{P}(r_h(\tau) = \mathbf{t})\, \mu^h\, w_k(\mu^h\theta) = \mathbb{P}(r_h(\tau^\theta) = \mathbf{t})\, w(\theta).$$

Integrating with respect to $\theta \in (0, \varepsilon]$ for some $\varepsilon > 0$, we get:

$$\mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathbb{P}_k(0 < W \leq \varepsilon\mu^h) = \int_0^\varepsilon \mathbb{P}(r_h(\tau^\theta) = \mathbf{t})\, w(\theta)\, d\theta,$$

where W under $\mathbb{P}_k$ is distributed as $\sum_{\ell=1}^k W_\ell$, where $(W_\ell, \ell \in \mathbb{N}^*)$ are independent random variables distributed as W under $\mathbb{P}$. Using Corollary 7.1, we get that:

$$\lim_{\varepsilon\to 0} \frac{\int_0^\varepsilon \mathbb{P}(r_h(\tau^\theta) = \mathbf{t})\, w(\theta)\, d\theta}{\mathbb{P}(0 < W \leq \varepsilon)} = \mathbb{P}(r_h(\tau^0) = \mathbf{t}).$$

This implies that:

$$\lim_{\varepsilon\to 0} \mathbb{P}(r_h(\tau) = \mathbf{t})\, \frac{\mathbb{P}_k(0 < W \leq \varepsilon\mu^h)}{\mathbb{P}(0 < W \leq \varepsilon)} = \mathbb{P}(r_h(\tau^0) = \mathbf{t}). \tag{32}$$

On the other hand, we have:

$$\begin{aligned}
\mathbb{P}(r_h(\tau) = \mathbf{t}, 0 < W \leq \varepsilon) &= \mathbb{P}\left(r_h(\tau) = \mathbf{t}, 0 < \lim_{n\to\infty} \frac{Z_n}{c_n} \leq \varepsilon\right) \\
&= \mathbb{P}(r_h(\tau) = \mathbf{t})\mathbb{P}_k\left(0 < \lim_{n\to\infty} \frac{Z_n}{c_{n+h}} \leq \varepsilon\right) \\
&= \mathbb{P}(r_h(\tau) = \mathbf{t})\mathbb{P}_k\left(0 < W \leq \varepsilon\mu^h\right),
\end{aligned}$$

where we used that $\lim_{n\to\infty} c_n/c_{n+h} = \mu^{-h}$ for the last equality. We deduce that:

$$\mathbb{P}(r_h(\tau) = \mathbf{t} \,|\, 0 < W \leq \varepsilon) = \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{\mathbb{P}_k\left(0 < W \leq \varepsilon\mu^h\right)}{\mathbb{P}(0 < W \leq \varepsilon)}.$$

Then use (32) and the characterization (7) of the convergence in $\mathbb{T}_\mathrm{f}$ to conclude. □

8. Weak continuity in law of the extremal GW trees at $\theta = +\infty$

We are in the setting of Section 4. The continuity of $(\hat\tau^\theta, \theta \in [0, +\infty))$ or of $(\tau^\theta, \theta \in [0, +\infty))$ at infinity is more involved. And, but for the geometric offspring distribution, see [1], and the Harris case, see Proposition 8.10, we have a less precise result.

We first introduce a family of inhomogeneous GW trees (whose offspring distribution depends on the height of the vertex) which converges in distribution toward a random tree τ^∞. These trees are first constructed by absolute continuity with respect to the distribution of

$r_h(\tau)$ and can also be seen as two-type GW trees generalizing the Kesten tree (see Subsection 8.2). The tree τ^∞ will be a good candidate for the limit in distribution of τ^θ as $\theta \to +\infty$ (as for the geometric case of [1]) and the limit in distribution of τ_n when $a_n \gg c_n$. We prove only a weak limit in Subsection 8.3.

8.1. **A family of inhomogeneous GW trees.** We set $\tilde{\varphi}(\lambda) = \varphi(-\lambda) = \mathbb{E}[\exp(\lambda W)]$ for $\lambda \in \mathbb{R}$. Recall from Remark 4.2 that: $\lambda_c = \sup\{\lambda \in \mathbb{R};\, \tilde{\varphi}(\lambda) < +\infty\} \geq 0$, $R_c = \tilde{\varphi}(\lambda_c/\mu) \geq 1$ is the convergence radius of the generating function f of p, see (18), and $f(R_c) = +\infty$ if and only if $\tilde{\varphi}(\lambda_c) = +\infty$. For $\lambda \in [-\infty, \lambda_c]$ and $h \in \mathbb{N}$, we set:

$$\zeta_h(\lambda) = \tilde{\varphi}(\lambda\mu^{-h}) = \mathbb{E}[\mathrm{e}^{\lambda\mu^{-h}W}] \in [\mathfrak{c}, +\infty]. \tag{33}$$

We have for $h, \ell \in \mathbb{N}$ (with an obvious convention when $\zeta_{h+\ell}(\lambda) = +\infty$) that:

$$f_h\left(\zeta_{h+\ell}(\lambda)\right) = \zeta_\ell(\lambda). \tag{34}$$

The sequence $(\zeta_h(\lambda), h \in \mathbb{N})$ is bounded from below by $\mathfrak{c}$ and from above by 1 if $\lambda \leq 0$ and from below by 1 and from above by $\zeta_0(\lambda)$ if $\lambda \geq 0$. Notice that if $\lambda_c = +\infty$, then we have $\zeta_h(\lambda_c) = +\infty$ for all $h \in \mathbb{N}$. Notice that $\zeta_h(-\infty) = \mathfrak{c}$ and thus $\zeta_h(-\infty) = 0$ for all $h \in \mathbb{N}$ if $\mathfrak{a} \geq 1$; and $\zeta_h(-\infty) > 0$ for all $h \in \mathbb{N}$ if $\mathfrak{a} = 0$. We deduce that:

(i) $\zeta_h(\lambda) \in (0, +\infty)$ if and only if $\lambda \in (-\infty, \lambda_c)$, or $\lambda = -\infty$ and $\mathfrak{a} = 0$, or $\lambda = \lambda_c$ and $\zeta_0(\lambda_c) < +\infty$ (the latter condition being equivalent to $f(R_c) < +\infty$).
(ii) $\zeta_h(\lambda) = +\infty$ if and only if $\lambda = \lambda_c = +\infty$, or $\lambda = \lambda_c$, $h = 0$ and $\zeta_0(\lambda_c) = +\infty$ (the latter condition being equivalent to $f(R_c) = +\infty$).
(iii) $\zeta_h(\lambda) = 0$ if and only if $\lambda = -\infty$ and $\mathfrak{a} > 0$.

For $h \in \mathbb{N}$ and $\lambda \in [-\infty, \lambda_c]$, we define the probability $\tilde{p}_h^{(\lambda)} = \left(\tilde{p}_h^{(\lambda)}(k), k \in \bar{\mathbb{N}}\right)$ as follows. Recall $\mathfrak{a} \in \mathbb{N}$ and $\mathfrak{b} \in \bar{\mathbb{N}}$ defined in (10).

(i) If $\zeta_h(\lambda) \in (0, +\infty)$, we set for $k \in \mathbb{N}$:

$$\tilde{p}_h^{(\lambda)}(k) = \frac{\zeta_{h+1}(\lambda)^k}{\zeta_h(\lambda)} p(k). \tag{35}$$

Thanks to (34), we get $\sum_{k\in\mathbb{N}} \tilde{p}_h^{(\lambda)}(k) = f(\zeta_{h+1}(\lambda))/\zeta_h(\lambda) = 1$, so that $\tilde{p}_h^{(\lambda)}$ defined by (35) is a probability distribution on $\mathbb{N}$.
(ii) If $\zeta_h(\lambda) = +\infty$ (which implies $\lambda = \lambda_c$), we set $\tilde{p}_h^{(\lambda)}$ the Dirac mass at $\mathfrak{b}$.
(iii) If $\zeta_h(\lambda) = 0$ (which implies $\lambda = -\infty$ and $\mathfrak{a} > 0$), we set $\tilde{p}_h^{(\lambda)}$ the Dirac mass at $\mathfrak{a} \in \mathbb{N}^*$.

For simplicity, we shall write $\tilde{p}_h$ for $\tilde{p}_h^{(\lambda)}$, and specify the value of λ only if needed.

We define $T^{(\lambda)}$ as a GW tree with offspring distribution $\tilde{p}_h$ at generation $h \in \mathbb{N}$. Since the case $\lambda = \lambda_c$ will appears later, we will particularize it and write

$$\tau^\infty = T^{(\lambda_c)}. \tag{36}$$

If $\lambda_c = +\infty$, then the tree τ^∞ is the regular $\mathfrak{b}$-ary tree $\mathbf{t}_\mathfrak{b}$, where $\mathfrak{b} \in \bar{\mathbb{N}}$. Notice that the root of τ^∞ has an infinite number of children if and only if $\zeta_0(\lambda_c) = +\infty$ and $\mathfrak{b} = \infty$, whereas all the other individuals have an infinite number of children if and only if $\lambda_c = \mathfrak{b} = \infty$.

Notice that $T^{(\lambda)}$, for $\lambda = 0$, is distributed as τ. Since $\lambda \mapsto \tilde{p}_h^{(\lambda)}$ is continuous on the set of probability distributions over $\bar{\mathbb{N}}$, for $\lambda \in (-\infty, \lambda_c)$, we get that the distribution of $T^{(\lambda)}$ is continuous for the local convergence in distribution as a function of λ over $(-\infty, \lambda_c)$. It is easy to check that the tree-valued random variable $T^{(-\infty)}$ is in fact distributed as τ conditionally on the extinction event $\mathcal{E} = \{H(\tau) < +\infty\}$, that is $\tau^{0,0}$, if $\mathfrak{a} = 0$ or as the regular $\mathfrak{a}$-ary tree

$\mathbf{t}_{\mathfrak{a}}$ if $\mathfrak{a} \geq 1$. In the latter case, $T^{(-\infty)}$ is thus defined as the Kesten tree τ^0 defined in Section 3. Taking particular care of the cases $\mathfrak{a} \geq 1$ (when λ goes to $-\infty$), $0 < \lambda_c < +\infty$ (when λ goes to λ_c), and $\lambda_c = +\infty$ with either $\mathfrak{b}$ finite or not (when λ goes to λ_c), it is not difficult to check that the probability distributions $\tilde{p}_h^{(\lambda)}$ over $\bar{\mathbb{N}}$ converge towards $\tilde{p}_h^{(-\infty)}$ as λ goes to $-\infty$, and to $\tilde{p}_h^{(\lambda_c)}$ as λ goes to λ_c. This implies the following result.

Lemma 8.1. *Let p be a non-degenerate super-critical offspring distribution with finite mean. We have that the family $(T^{(\lambda)}, \lambda \in [-\infty, \lambda_c])$ is continuous in distribution.*

Let $\lambda \in [-\infty, \lambda_c]$. If $\zeta_0(\lambda) \in (0, +\infty)$, then for $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$, we have:

$$\mathbb{P}(r_h(T^{(\lambda)}) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})} \tilde{p}_{|u|}(k_u(\mathbf{t})). \tag{37}$$

If $\lambda_c < +\infty$ and $\zeta_0(\lambda_c) = +\infty$, then for $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$, we have:

$$\mathbb{P}(r_{h,k_0}(\tau^\infty) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})^*} \tilde{p}_{|u|}(k_u(\mathbf{t})), \tag{38}$$

where we recall that for a tree $\mathbf{s}$ we set $\mathbf{s}^* = \mathbf{s} \setminus \{\partial\}$. Remark that a.s. $T^{(\lambda)} \in \mathbb{T}_{\mathrm{f}}$ if and only if $\zeta_0(\lambda)$ or $\mathfrak{b}$ is finite, and that a.s. $\tau^\infty \in \mathbb{T}_{\mathrm{f}}^*$ if and only if $\zeta_1(\lambda)$ or $\mathfrak{b}$ is finite.

We give a representation of the distribution of $T^{(\lambda)}$ as the distribution of τ with a martingale weight. The proof of the following lemma is elementary and thus left to the reader.

Lemma 8.2. *Let p be a non-degenerate super-critical offspring distribution with finite mean. For $\lambda \in [-\infty, \lambda_c]$ such that $\zeta_0(\lambda) \in (0, +\infty)$, $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$, we have with $k = z_h(\mathbf{t})$:*

$$\mathbb{P}\left(r_h(T^{(\lambda)}) = \mathbf{t}\right) = \frac{\zeta_h(\lambda)^k}{\zeta_0(\lambda)} \, \mathbb{P}\left(r_h(\tau) = \mathbf{t}\right). \tag{39}$$

For $\lambda_c < +\infty$ and $\zeta_0(\lambda_c) = +\infty$, $h \in \mathbb{N}^*$, $k_0 \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$, we have with $k = z_h(\mathbf{t})$:

$$\mathbb{P}\left(r_{h,k_0}(T^{(\lambda_c)}) = \mathbf{t}\right) = \frac{\zeta_h(\lambda_c)^k}{\zeta_1(\lambda_c)^{k_0}} \, \frac{\mathbb{P}\left(r_h(\tau) = \mathbf{t}\right)}{p(k_0)}. \tag{40}$$

Notice that $\lambda_c < +\infty$ and $\zeta_0(\lambda_c) = +\infty$ occurs in the case of the geometric offspring distribution studied in [1].

8.2. **A family of two-type GW trees.** We keep notations from Section 8.1. For $\lambda \in (-\infty, \lambda_c]$, we give a description of $T^{(\lambda)}$ using a two-type GW tree $\hat{T}^{(\lambda),\mathrm{e}}$.

For $h \in \mathbb{N}$ and $\lambda \in (-\infty, \lambda_c]$ such that $\zeta_h(\lambda)$ is finite, we define the probability distribution $\hat{p}_h^{(\lambda),\mathrm{e}} = (\hat{p}_h^{(\lambda),\mathrm{e}}(\ell), \ell \in \mathbb{N}^*)$ by:

$$\hat{p}_h^{(\lambda),\mathrm{e}}(\ell) = \frac{(\zeta_{h+1}(\lambda) - \mathfrak{c})^\ell}{(\zeta_h(\lambda) - \mathfrak{c})} \frac{f^{(\ell)}(\mathfrak{c})}{\ell!}. \tag{41}$$

Notice that $\hat{p}_h^{(\lambda),\mathrm{e}}$ is indeed a probability as by the Taylor-Lagrange expansion at $\mathfrak{c}$ of f, we have, using (34) that $\sum_{\ell \geq 1} \hat{p}_h^{(\lambda),\mathrm{e}}(\ell) = (f(\zeta_{h+1}(\lambda)) - \mathfrak{c})/(\zeta_h(\lambda) - \mathfrak{c}) = 1$. For $\ell \in \mathbb{N}^*$ such that $f^{(\ell)}(\mathfrak{c}) > 0$, we also recall the ℓth-size biased probability distribution $\mathfrak{p}_{[\ell]}$, see Definition (23), with the convention that $\mathfrak{p}_{[\ell]}$ is the Dirac mass at ℓ if $\mathfrak{c} = 0$.

We define a two type random tree $\hat{T}^{(\lambda),\mathrm{e}}$ in the next definition and write $T^{(\lambda),\mathrm{e}} = \mathrm{Ske}(\hat{T}^{(\lambda),\mathrm{e}})$ for the tree $\hat{T}^{(\lambda),\mathrm{e}}$ when one forgets the types of the vertices of $\hat{T}^{(\lambda),\mathrm{e}}$.

Definition 8.3. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Let $\lambda \in (-\infty, \lambda_c]$. The labeled tree $\hat{T}^{(\lambda),\mathrm{e}}$ is a two-type random tree whose vertices are either of type* s *(for survivor) or of type* e *(for extinct).*

(i) *If $\zeta_0(\lambda) = \zeta_1(\lambda) = +\infty$, then $\hat{T}^{(\lambda),\mathrm{e}}$ is the regular $\mathfrak{b}$-ary tree and all its vertices are of type* s *(and thus there is no vertex of type* e*).*

(ii) *If $\zeta_1(\lambda) < +\infty$, the random tree $\hat{T}^{(\lambda),\mathrm{e}}$ is defined as follows:*
   - *For a vertex, the number of offsprings of each type and their positions depend, conditionally on the vertices of lower or same height, only on its own type (branching property).*
   - *The root is of type* s *with probability $(\zeta_0(\lambda) - \mathfrak{c})/\zeta_0(\lambda)$. This probability is set to 1 if $\zeta_0(\lambda) = +\infty$.*
   - *A vertex of type* e *produces only vertices of type* e *with sub-critical offspring distribution $\mathfrak{p}$.*
   - *Recall that only $\zeta_0(\lambda)$ might be infinite. Let $h \in \mathbb{N}$ such that $\zeta_h(\lambda)$ is finite. A vertex $u \in \hat{T}^{(\lambda),\mathrm{e}}$ at level h of type* s *produces $\kappa^{\mathrm{s}}(u)$ vertices of type* s *with probability distribution $\hat{p}_h^{(\lambda),\mathrm{e}}$ and $\kappa^{\mathrm{e}}(u)$ vertices of type* e *such that conditionally on $\kappa^{\mathrm{s}}(u) = s_u \geq 1$, $k_u(T^{(\lambda),\mathrm{e}}) = \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u)$ has distribution $\mathfrak{p}_{[s_u]}$, defined in (23), and the s_u individuals of type* s *are chosen uniformly at random among the $k_u(T^{(\lambda),\mathrm{e}})$ children. More precisely, as for Definition 5.1, we denote by $\mathscr{S}_h = \{u \in T^{(\lambda),\mathrm{e}};\, |u| = h$ and u is of type s$\}$ the set of vertices of $\hat{T}^{(\lambda),\mathrm{e}}$ with type* s *at level $h \in \mathbb{N}$, and we have for $u \in \mathscr{S}_h$: for all $k_u \in \mathbb{N}^*$, $s_u \in \{1, \ldots, k_u\}$, and $A_u \subset \{1, \ldots, k_u\}$ such that $\sharp A_u = s_u$,*

$$\mathbb{P}\left(\kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{e}}(u) = k_u \text{ and } \mathscr{S}_{h+1} \cap \{u1, \ldots, uk_u\} = uA_u \,|\, r_h(T^{(\lambda),\mathrm{e}}), \mathscr{S}_h\right) \\ = \hat{p}_h^{(\lambda),\mathrm{e}}(s_u)\, \frac{1}{\binom{k_u}{s_u}}\, \mathfrak{p}_{[s_u]}(k_u) = \frac{(\zeta_{h+1}(\lambda) - \mathfrak{c})^{s_u}}{\zeta_h(\lambda) - \mathfrak{c}}\, \mathfrak{c}^{k_u - s_u}\, p(k_u).$$

   - *If $\zeta_0(\lambda) = +\infty$, then the root, which is of type* s *a.s., has infinitely many children of types* s *and* e*, each children being, independently from the other, of type* s *with probability $(\zeta_1(\lambda) - \mathfrak{c})/\zeta_1(\lambda)$. That is for $k_0 \in \mathbb{N}^*$ and $S_1 \subset \{1, \ldots, k_0\}$:*

$$\mathbb{P}\left(\mathscr{S}_1 \cap \{1, \ldots, k_0\} = S_1\right) = \left(\frac{\zeta_1(\lambda) - \mathfrak{c}}{\zeta_1(\lambda)}\right)^{\sharp S_1} \left(\frac{\mathfrak{c}}{\zeta_1(\lambda)}\right)^{k_0 - \sharp S_1}.$$

Unless $\mathfrak{a} \geq 1$ or $\zeta_0(\lambda) = \zeta_1(\lambda) = +\infty$, conditionally on the fact that the root is of type s, a.s. there exists an infinite number of vertices of type s and of type e. By construction individuals of type s have a progeny which does not suffer extinction, whereas individuals of type e have a finite progeny. Informally the individuals of type s in $\hat{T}^{(\lambda),\mathrm{e}}$, if any, form a backbone, on which are grafted, if $\mathfrak{a} = 0$, independent GW trees distributed as τ conditionally on the extinction event $\mathcal{E}$. This is in a sense a generalization of the Kesten tree, where the backbone is reduced to an infinite spine in the case $\mathfrak{a} \leq 1$. We stress out that $\hat{T}^{(\lambda),\mathrm{e}}$, truncated at level h can be recovered from $r_h(T^{(\lambda),\mathrm{e}})$ and $\mathscr{S}_h$ as all the ancestors of a vertex of type s is also of a type s and a vertex of type s has at least one children of type s.

The following result states that the random tree $T^{(\lambda)}$ can be seen as the skeleton of a two-type GW tree.

Lemma 8.4. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Then, for $\lambda \in (-\infty, \lambda_c]$, the tree $T^{(\lambda),\mathrm{e}}$ is distributed as $T^{(\lambda)}$.*

Proof. Let $\lambda \in (-\infty, \lambda_c]$. We first consider the case $\zeta_0(\lambda)$ finite. We assume $\mathfrak{c} > 0$ (or equivalently $\mathfrak{a} = 0$). Let $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$ and $S_h \subset \{u \in \mathbf{t};\, |u| = h\}$. Set $k = z_h(\mathbf{t}) = \sharp\{u \in \mathbf{t};\, |u| = h\}$. In order to shorten the notations, we set $\mathcal{A} = S_h \bigcup \mathrm{Anc}(S_h)$. We set, for $\ell \in \{0, \ldots, h-1\}$, $S_\ell = \{u \in \mathcal{A},\, |u| = \ell\}$ the vertices at level ℓ which have at least one descendant in S_h. For $u \in r_{h-1}(\mathbf{t})$, we set $s_u(\mathbf{t}) = \sharp(\mathcal{A} \bigcap u\mathbb{N}^*)$, the number of children of u having descendants in S_h. We recall that $\hat{T}^{(\lambda),\mathrm{e}}$ truncated at level h can be recovered from $r_h(T^{(\theta),\mathrm{e}})$ and $\mathscr{S}_h$. We compute $\mathcal{C}_{S_h} = \mathbb{P}(r_h(T^{(\lambda),\mathrm{e}}) = \mathbf{t},\, \mathscr{S}_h = S_h)$. We have by construction if $\sharp S_h > 0$:

$$\mathcal{C}_{S_h} = \frac{\zeta_0(\lambda) - \mathfrak{c}}{\zeta_0(\lambda)} \left[\prod_{u \in r_{h-1}(\mathbf{t}),\, u \notin \mathcal{A}} \mathfrak{p}(k_u(\mathbf{t})) \right] \left[\prod_{u \in \mathcal{A}} \frac{\left(\zeta_{|u|+1}(\lambda) - \mathfrak{c}\right)^{s_u(\mathbf{t})}}{\zeta_{|u|}(\lambda) - \mathfrak{c}} \mathfrak{c}^{k_u(\mathbf{t}) - s_u(\mathbf{t})} p(k_u(\mathbf{t})) \right]$$

$$= \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k - \sharp S_h} \frac{\left(\zeta_h(\lambda) - \mathfrak{c}\right)^{\sharp S_h}}{\zeta_0(\lambda)}, \tag{42}$$

where we used that for a tree $\mathbf{s}$, we have $\sum_{u \in r_{h-1}(\mathbf{s})} k_u(\mathbf{s}) - 1 = z_h(\mathbf{s}) - 1$ and that $\mathbf{s} = \mathcal{A}$ is tree-like with $z_h(\mathbf{s}) = \sharp S_h$. It is elementary to check that Formula (42) is also true when S_h is empty, and the root is thus of type e. Since $\mathcal{C}_{S_h}$ depends only of $\sharp S_h$, we shall write $\mathcal{C}_{\sharp S_h}$ for $\mathcal{C}_{S_h}$. We get:

$$\mathbb{P}(r_h(T^{(\lambda),\mathrm{e}}) = \mathbf{t}) = \sum_{i=0}^{k} \binom{k}{i} \mathcal{C}_i = \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{\zeta_h(\lambda)^k}{\zeta_0(\lambda)}.$$

We deduce from (39) that $T^{(\lambda),\mathrm{e}}$ and $T^{(\lambda)}$ have the same distribution.

The case $\zeta_0(\lambda)$ finite and $\mathfrak{c} = 0$ (i.e. $\mathfrak{a} > 0$) is clear, as there is no vertex of type e in $\hat{T}^{(\lambda),\mathrm{e}}$ and the offspring distribution of individuals of type s at level h in $\hat{T}^{(\lambda),\mathrm{e}}$ given by (41), that is:

$$\hat{p}_h^{(\lambda),\mathrm{e}}(\ell) = \frac{\left(\zeta_{h+1}(\lambda) - \mathfrak{c}\right)^\ell}{\left(\zeta_h(\lambda) - \mathfrak{c}\right)} \frac{f^{(\ell)}(\mathfrak{c})}{\ell!} = \frac{\zeta_{h+1}(\lambda)^\ell}{\zeta_h(\lambda)} p(\ell),$$

coincides with the offspring distribution $\tilde{p}_h^{(\lambda)}(k)$ given in (35) of individuals at level h in $T^{(\lambda)}$.

We consider the case $\zeta_0(\lambda) = +\infty$, $\zeta_1(\lambda)$ finite and $\mathfrak{c} > 0$. Let $k_0, h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_{k_0}^{(h)}$ and $S_h \subset \{u \in \mathbf{t};\, |u| = h\}$. Set $k = z_h(\mathbf{t}) = \sharp\{u \in \mathbf{t};\, |u| = h\}$. Arguing as in the case $\zeta_0(\lambda)$ finite, we get if $\mathfrak{c} > 0$:

$$\mathcal{C}_{S_h} = \mathbb{P}(r_{h,k_0}(T^{(\lambda),\mathrm{e}}) = \mathbf{t},\, \mathscr{S}_h = S_h) = \frac{\mathbb{P}(r_h(\tau) = \mathbf{t})}{p(k_0)} \mathfrak{c}^{k - \sharp S_h} \frac{\left(\zeta_h(\lambda) - \mathfrak{c}\right)^{\sharp S_h}}{\zeta_1(\lambda)^{k_0}},$$

and thus, writing $\mathcal{C}_{\sharp S_h}$ for $\mathcal{C}_{S_h}$ as the latter quantity depends only on $\sharp S_h$:

$$\mathbb{P}(r_{h,k_0}(T^{(\lambda),\mathrm{e}}) = \mathbf{t}) = \sum_{i=0}^{k} \binom{k}{i} \mathcal{C}_i = \frac{\mathbb{P}(r_h(\tau) = \mathbf{t})}{p(k_0)} \frac{\zeta_h(\lambda)^k}{\zeta_1(\lambda)^{k_0}}.$$

Then use (40) to conclude. The sub-case $\mathfrak{c} = 0$ is handled in the same way as when $\zeta_0(\lambda)$ is finite.

Eventually, we consider the case $\zeta_1 = +\infty$. In this case $T^{(\lambda),\mathrm{e}}$ and $T^{(\lambda)}$ are by definition regular $\mathfrak{b}$-ary trees, and they are thus a.s. equal. $\square$

For $\lambda > -\infty$, we denote by $T^{(\lambda),*}$ the tree-valued random variable distributed as $T^{(\lambda)}$ conditionally on the non extinction event (which is distributed as the skeleton of $\hat{T}^{(\lambda),\mathrm{e}}$ conditionally on the root being of type s). Recall the Kesten tree τ^0 defined in Section 3.

Lemma 8.5. *Let p be a non-degenerate super-critical offspring distribution with finite mean. We have the following convergence in distribution:*

$$T^{(\lambda),*} \xrightarrow[\lambda\searrow-\infty]{(d)} \tau^0.$$

Proof. Considering the cases $\mathfrak{a} = 0$ and $\mathfrak{a} \geq 1$, it is easy to check that the distributions $\hat{p}_h^{(\lambda),\mathrm{e}}$ over $\bar{\mathbb{N}}$ defined in (41) converge as λ goes to $-\infty$ towards the Dirac mass at $\max(1, \mathfrak{a})$. This implies the convergence in distribution as λ goes to $-\infty$ of $\hat{T}^{(\lambda),\mathrm{e}}$ conditionally on the root being of type s towards $\hat{\tau}^0$. Using that the extinction event of $T^{(\lambda),\mathrm{e}}$ corresponds to the root of $\hat{T}^{(\lambda),\mathrm{e}}$ being of type s, we obtain the convergence of the lemma. □

Remark 8.6. In the proof of Lemma 8.5, we proved in fact the convergence of the two-type random trees $\hat{T}^{(\lambda),\mathrm{e}}$ conditionally on the root being of type s towards $\hat{\tau}^0$ as λ goes to $-\infty$, using the convergence in distribution of the probability distribution $\hat{p}^{(\lambda),\mathrm{e}}$ as λ goes to $-\infty$.

Similarly, considering carefully the three cases $\zeta_0(\lambda_c)$ finite; $\zeta_0(\lambda_c) = +\infty$ and $\zeta_1(\lambda_c)$ finite; $\zeta_0(\lambda_c) = \zeta_1(\lambda_c) = +\infty$, it is not very difficult to check that $\hat{T}^{(\lambda),\mathrm{e}}$ converges in distribution towards $\hat{T}^{(\lambda_c),\mathrm{e}}$ as λ goes up towards λ_c. Then, by considering only the skeleton, this allows to recover the convergence in distribution of $T^{(\lambda)}$ towards τ^∞ as λ goes up to λ_c, thus recovering the continuity at λ_c in Lemma 8.1. Notice that when $\zeta_0(\lambda_c) = +\infty$, then the root of $\hat{T}^{(\lambda_c),\mathrm{e}}$ is a.s. of type s and has infinitely many children.

8.3. **Continuity in law of the extremal GW trees at $\theta = +\infty$.**

Recall that $T^{(\lambda),*}$ is distributed as $T^{(\lambda)}$ conditionally on the non extinction event (which is distributed as the skeleton of $\hat{T}^{(\lambda),\mathrm{e}}$ conditionally on the root being of type s).

Recall that $\zeta_0(\lambda) > \mathfrak{c}$ for $\lambda > -\infty$. For $\lambda \in (-\infty, \lambda_c]$, such that $\zeta_0(\lambda)$ is finite, we consider the function g_λ defined by:

$$g_\lambda(\theta) = \frac{1}{\zeta_0(\lambda) - \mathfrak{c}}\, w(\theta)\, \mathrm{e}^{\lambda\theta}\, \mathbf{1}_{(0,+\infty)}(\theta).$$

Since, by definition, $\int g_\lambda = 1$, we deduce that g_λ is a probability density. Let Θ_λ be a random variable with density g_λ. We consider the random tree τ^{Θ_λ} and the random two-type tree $\hat{\tau}^{\Theta_\lambda}$, which conditionally on $\{\Theta_\lambda = \theta\}$ are distributed respectively as τ^θ and $\hat{\tau}^\theta$. We have the following representation.

Proposition 8.7. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Then, for $\lambda \in (-\infty, \lambda_c]$ such that $\mathbb{E}[\mathrm{e}^{\lambda W}]$ is finite, we have that τ^{Θ_λ} (resp. $\hat{\tau}^{\Theta_\lambda}$) is distributed as $T^{(\lambda),*}$ (resp. as $\hat{T}^{(\lambda),\mathrm{e}}$ conditionally on the root being of type* s*).*

Proof. Let $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$ and $S_h \subset \{u \in \mathbf{t};\ |u| = h\}$ with S_h non empty. We recall that the distribution of $\hat{\tau}^\theta$ up to generation h is completely characterized by $r_h(\tau^\theta)$ its skeleton up to level h and by the set $\mathscr{S}_h$ of vertices at generation h which are of type s. We still denote by $\mathscr{S}_h$ the vertices of τ^{Θ_λ} at generation h which are of type s. We have with $k = z_h(\mathbf{t})$ and

$\ell = \sharp S_h$:

$$\begin{aligned}
\mathbb{P}(r_h(\tau^{\Theta_\lambda}) = \mathbf{t}, \mathscr{S}_h = S_h) &= \int \mathbb{P}(r_h(\tau^\theta) = \mathbf{t}, \mathscr{S}_h = S_h)\, g_\lambda(\theta)\, d\theta \\
&= \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k-\ell}\, \frac{1}{\zeta_0(\lambda) - \mathfrak{c}} \int w^{*\ell}(\mu^h\theta)\, \mathrm{e}^{\lambda\theta}\, \mu^h d\theta \\
&= \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k-\ell}\, \frac{1}{\zeta_0(\lambda) - \mathfrak{c}}\, \mathbb{E}\left[\mathrm{e}^{\lambda\mu^{-h}\sum_{i=1}^{\ell} W_i} \prod_{i=1}^{\ell} \mathbf{1}_{\{W_i>0\}}\right] \\
&= \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k-\ell}\, \frac{(\zeta_h(\lambda) - \mathfrak{c})^\ell}{\zeta_0(\lambda) - \mathfrak{c}}, \qquad (43)
\end{aligned}$$

where we used (25) for the second equality, that $(W_i, i \in \mathbb{N}^*)$ are independent random variables distributed as W for the third one and the definition of ζ_h given in (33). Then use (42) and that the root of $\hat{T}^{(\lambda),\mathrm{e}}$ is of type s with probability $(\zeta_0(\lambda) - \mathfrak{c})/\zeta_0(\lambda)$ to get that:

$$\mathbb{P}(r_h(\tau^{\Theta_\lambda}) = \mathbf{t}, \mathscr{S}_h = S_h) = \mathbb{P}(r_h(T^{(\lambda),\mathrm{e}}) = \mathbf{t}, \mathscr{S}_h = S_h |\text{ type of } \partial \text{ is s}).$$

Since $\hat{\tau}^{\Theta_\lambda}$ up to level h is characterized by τ^{Θ_λ} and $\mathscr{S}_h$, and similarly for $\hat{T}^{(\lambda),\mathrm{e}}$, we deduce from the previous equality that $\hat{\tau}^{\Theta_\lambda}$ is distributed as $\hat{T}^{(\lambda),\mathrm{e}}$ conditionally on its root being of type s. Then, forgetting about the types, we deduce that τ^{Θ_λ} is distributed as $T^{(\lambda),*}$. □

When λ goes to $-\infty$, we get that the measure $g_\lambda(\theta)\, d\theta$ converges weakly to the Dirac mass at 0. We deduce that Θ_λ converges in distribution towards 0 as λ goes to $-\infty$. We then recover from Proposition 8.7 and Corollary 7.1 the convergence in distribution of $T^{(\lambda),*}$, that is of $T^{(\lambda),\mathrm{e}}$ conditionally on the non-extinction event, towards τ^0 given in Lemma 8.5.

If $\mathbb{E}[\mathrm{e}^{\lambda_c W}] = +\infty$ (and thus $\lambda_c > 0$) or equivalently $f(R_c) = +\infty$, then when λ goes up to λ_c we get that Θ_λ converges in distribution towards $+\infty$. We deduce from Lemma 8.1 the following corollary.

Corollary 8.8. *Let p be a non-degenerate super-critical offspring distribution whose generating function blows-up (that is $f(R_c) = +\infty$). Then, if $(\tau_\theta, \theta \in [0, \infty))$ converges in distribution as θ goes to infinity, then the limit is the distribution of τ^∞.*

Remark 8.9. If $R_c = +\infty$, then the tree τ^∞ has all its nodes with degree $\mathfrak{b} \in \bar{\mathbb{N}}$. Since the distribution of τ^∞ is maximal in the convex set of probability distributions on $\mathbb{T}_\infty$, we get that the distribution of τ^∞ is the limit in distribution of a sub-sequence $(\tau_{\theta_n}, n \in \mathbb{N})$ with $\lim_{n\to\infty} \theta_n = +\infty$.

We are able to prove the stronger result on the convergence in distribution of $(\tau_\theta, \theta \in [0, \infty))$ as θ goes to infinity in the particular case of the geometric offspring distribution (in this case λ_c is positive finite, $\mathbb{E}[\mathrm{e}^{\lambda_c W}] = +\infty$ and $\mathfrak{b} = \infty$), see [1]. The next proposition, which is a direct consequence of the convergence of $\rho_{\theta,r}$ as $\theta \to +\infty$ given in Lemma 4.4, asserts that it also holds if the offspring distribution has a finite support which is the so-called Harris case (in this case $\mathfrak{b} < \infty$ and $\lambda_c = +\infty$). Otherwise, the general case is open.

Proposition 8.10. *Let p be a non-degenerate super-critical offspring distribution with finite support, that is $\mathfrak{b} < +\infty$ (Harris case). Then we have the following convergence in distribution:*

$$\tau^\theta \xrightarrow[\theta\to\infty]{(d)} \tau^\infty.$$

8.4. **A remark on an other trees family.** We provide in this section an alternative description of $T^{(\lambda)}$ using a two-type GW tree $\hat{T}^{(\lambda),\mathrm{n}}$.

We assume that $\lambda_c > 0$. Notice that the sequence $(\zeta_h(\lambda), h \in \mathbb{N})$ defined in (33) is non-increasing and $\zeta_h(\lambda) > 1$ for all $h \in \mathbb{N}$, $\lambda \in (0, \lambda_c]$. Furthermore, as $R_c > 1$, we get that $f^{(\ell)}(1)$ is finite for all $\ell \in \mathbb{N}$. For $h \in \mathbb{N}$ and $\lambda \in (0, \lambda_c]$ such that $\zeta_h(\lambda)$ is finite, we define the probability $\hat{p}_h^{(\lambda),\mathrm{n}}$ as $\hat{p}_h^{(\lambda),\mathrm{e}}$ in (41) but with $\mathfrak{c}$ replaced by 1. That is for $\ell \in \mathbb{N}^*$:

$$\hat{p}_h^{(\lambda),\mathrm{n}}(\ell) = \frac{(\zeta_{h+1}(\lambda) - 1)^\ell}{(\zeta_h(\lambda) - 1)} \frac{f^{(\ell)}(1)}{\ell!}. \tag{44}$$

For $\ell \in \mathbb{N}$ such that $\ell \leq \mathfrak{b}$, we recall the ℓth-size biased probability distribution of p defined in (14). We define a two type random tree $\hat{T}^{(\lambda),\mathrm{n}}$ in the next definition and write $T^{(\lambda),\mathrm{n}} = \mathrm{Ske}(\hat{T}^{(\lambda),\mathrm{n}})$ as the tree $\hat{T}^{(\lambda),\mathrm{n}}$ when one forgets the types of the vertices of $\hat{T}^{(\lambda),\mathrm{n}}$.

Definition 8.11. *Let p be a non-degenerate super-critical offspring distribution such that $\lambda_c > 0$. Let $\lambda \in (0, \lambda_c]$. We define a labeled random tree $\hat{T}^{(\lambda),\mathrm{n}}$, whose vertices are either of type* s *(for survivor) or of type* n *(for normal).*

(i) *If $\zeta_0(\lambda) = \zeta_1(\lambda) = +\infty$, then $\hat{T}^{(\lambda),\mathrm{n}}$ is the regular $\mathfrak{b}$-ary tree and all its vertices are of type* s *(and thus there is no vertex of type* n*).*

(ii) *If $\zeta_1(\lambda) < +\infty$, the random tree $\hat{T}^{(\lambda),\mathrm{n}}$ is defined as follows:*
- *For a vertex, the number of offsprings of each type and their positions depend, conditionally on the vertices of lower or same height, only on its own type (branching property).*
- *The root is of type* s *with probability $(\zeta_0(\lambda) - 1)/\zeta_0(\lambda)$. This probability is set to 1 if $\zeta_0(\lambda) = +\infty$.*
- *A vertex of type* n *produces only vertices of type* n *with super-critical offspring distribution p.*
- *Recall that only $\zeta_0(\lambda)$ might be infinite. Let $h \in \mathbb{N}$ such that $\zeta_h(\lambda)$ is finite. A vertex $u \in \hat{T}^{(\lambda),\mathrm{n}}$ at level h of type* s *produces $\kappa^{\mathrm{s}}(u)$ vertices of type* s *with probability distribution $\hat{p}_h^{(\lambda),\mathrm{n}}$ and $\kappa^{\mathrm{n}}(u)$ vertices of type* n *such that conditionally on $\kappa^{\mathrm{s}}(u) = s_u \geq 1$, $k_u(T^{(\lambda),\mathrm{n}}) = \kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{n}}(u)$ has distribution $p_{[s_u]}$, defined in (14), and the s_u individuals of type* s *are chosen uniformly at random among the $k_u(T^{(\lambda),\mathrm{n}})$ children. More precisely if we denote by $\mathscr{S}_h = \{u \in T^{(\lambda),\mathrm{n}};\ |u| = h$ and u is of type* s$\}$ *the set of vertices of $\hat{T}^{(\lambda),\mathrm{n}}$ with type* s *at level $h \in \mathbb{N}$, and we have for $u \in \mathscr{S}_h$: for all $k_u \in \mathbb{N}^*$, $s_u \in \{1, \dots, k_u\}$, and $A_u \subset \{1, \dots, k_u\}$ such that $\sharp A_u = s_u$,*

$$\mathbb{P}\left(\kappa^{\mathrm{s}}(u) + \kappa^{\mathrm{n}}(u) = k_u \text{ and } \mathscr{S}_{h+1} \cap \{u1, \dots, uk_u\} = uA_u \,|\, r_h(T^{(\lambda),\mathrm{n}}), \mathscr{S}_h\right)$$
$$= \hat{p}_h^{(\lambda),\mathrm{n}}(s_u) \frac{1}{\binom{k_u}{s_u}} \mathfrak{p}_{[su]}(k_u) = \frac{(\zeta_{h+1}(\lambda) - 1)^{s_u}}{\zeta_h(\lambda) - 1} \mathfrak{c}^{k_u - s_u} p(k_u).$$

If $\zeta_0(\lambda) = +\infty$, then the root, which is of type s *a.s., has infinitely many children of type* s *and* n*, each children being, independently from the other, of type* s *with probability $(\zeta_1(\lambda) - 1)/\zeta_1(\lambda)$. That is for $k_0 \in \mathbb{N}^*$ and $S_1 \subset \{1, \dots, k_0\}$:*

$$\mathbb{P}\left(\mathscr{S}_1 \cap \{1, \dots, k_0\} = S_1\right) = \left(\frac{\zeta_1(\lambda) - 1}{\zeta_1(\lambda)}\right)^{\sharp S_1} \left(\frac{1}{\zeta_1(\lambda)}\right)^{k_0 - \sharp S_1}.$$

The main difference with $\hat{T}^{(\lambda),\mathrm{e}}$ is that the individuals of type s in $\hat{T}^{(\lambda),\mathrm{n}}$, if any, form a backbone on which are grafted, if $\mathfrak{a}=0$, independent GW trees distributed as τ (instead of τ conditionally on the extinction event $\mathcal{E}$ in $\hat{T}^{(\lambda),\mathrm{e}}$).

The following result states that the random tree $T^{(\lambda)}$ can also be seen as the skeleton of this new two-type GW tree. Its proof, which follows the proof of Lemma 8.4, is left to the reader.

Lemma 8.12. *Let p be a non-degenerate super-critical offspring distribution such that $\lambda_c > 0$. Then, for $\lambda \in (0, \lambda_c]$, the tree $T^{(\lambda),\mathrm{n}}$ is distributed as $T^{(\lambda)}$.*

Remark 8.13. Recall that $\zeta_0(\lambda) > 1$ for $\lambda \in (0, \lambda_c]$. For $\lambda \in (0, \lambda_c]$, such that $\zeta_0(\lambda)$ is finite, we consider the function h_λ defined by:

$$h_\lambda(\theta) = \frac{1}{\zeta_0(\lambda) - 1}\, w(\theta) \left(\mathrm{e}^{\lambda\theta} - 1\right) \mathbf{1}_{(0,+\infty)}(\theta).$$

Since, by definition, $\int h_\lambda = 1$, we deduce that h_λ is a probability density. Let Θ'_λ be a random variable with density h_λ. We consider the random tree $\tau^{\Theta'_\lambda}$ and the random two-type tree $\hat{\tau}^{\Theta'_\lambda}$, which conditionally on $\{\Theta'_\lambda = \theta\}$ are distributed respectively as τ^θ and $\hat{\tau}^\theta$. Computation similar as in (43) gives that for $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$, with $k = z_h(\mathbf{t})$, and $S_h \subset \{u \in \mathbf{t};\, |u| = h\}$ with S_h non empty and $\ell = \sharp S_h$:

$$\mathbb{P}(r_h(\tau^{\Theta'_\lambda}) = \mathbf{t}, \mathscr{S}_h = S_h) = \mathbb{P}(r_h(\tau) = \mathbf{t})\, \mathfrak{c}^{k-\ell} \frac{(\zeta_h(\lambda) - \mathfrak{c})^\ell - (1-\mathfrak{c})^\ell}{\zeta_0(\lambda) - 1}.$$

Similar computations as in (42) give that:

$$\mathbb{P}(r_h(T^{(\lambda),\mathrm{n}}) = \mathbf{t}, \mathscr{S}_h = S_h) = \mathbb{P}(r_h(\tau) = \mathbf{t})\, \frac{(\zeta_h(\lambda) - 1)^\ell}{\zeta_0(\lambda)}.$$

Summing over all non-empty subsets S_h of $\{u \in \mathbf{t};\, |u| = h\}$, gives that:

$$\mathbb{P}(r_h(\tau^{\Theta'_\lambda}) = \mathbf{t}) = \mathbb{P}(r_h(\tau) = \mathbf{t})\, \frac{\zeta_h(\lambda)^k - 1}{\zeta_0(\lambda) - 1} = \mathbb{P}(r_h(T^{(\lambda),\mathrm{n}}) = \mathbf{t} |\, \text{root is of type s}).$$

Thus the random tree $\tau^{\Theta'_\lambda}$ is distributed as $T^{(\lambda),\mathrm{n}}$ conditionally on the root being of type s.

9. Convergence of conditioned sub-critical GW tree

In this section, we consider a sub-critical GW tree τ with general non-degenerate offspring distribution $p = (p(n), n \in \mathbb{N})$ with finite mean $\mu \in (0, 1)$. To avoid trivial cases, we assume that $p(0) + p(1) < 1$. We denote by f the generating function of p. We assume that there exists $\kappa > 1$ such that $f(\kappa) = \kappa$ and $f'(\kappa) < +\infty$. Since f is strictly convex, κ, when it exists, is unique. Those assumptions are trivially satisfied if the radius of convergence of f is infinite. This is also the case for geometric offspring distribution studied in [1].

Define $\bar{f}(t) = f(\kappa t)/\kappa$ for $t \in [0, 1]$ and note that $\bar{f}$ is the generating function of a super-critical offspring distribution $\bar{p} = (\bar{p}(n), n \in \mathbb{N})$ with $\bar{p}(n) = \kappa^{n-1} p(n)$. The mean $\bar{\mu}$ of $\bar{p}$ is equal to $f'(\kappa)$; the fixed point $\bar{\mathfrak{c}} \in (0, 1)$ of $\bar{f}$ is given by $\bar{\mathfrak{c}} = 1/\kappa$; and $\bar{f}'(\bar{\mathfrak{c}}) = \mu$.

We have that $\bar{\mathfrak{p}}$ defined by (13) (with p replaced by $\bar{p}$) is equal to p by construction. Notice that we are in the Schröder case and that p is of type $(L_0, 0)$ as $\bar{p}(0) > 0$. Let $\bar{\tau}$ be the corresponding super-critical GW tree. It is elementary to check that for $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}$, we have with $k = z_h(\mathbf{t})$:

$$\mathbb{P}(r_h(\tau) = \mathbf{t}) = \kappa^{k-1} \mathbb{P}(r_h(\bar{\tau}) = \mathbf{t}). \tag{45}$$

Recall that $Z_n = z_n(\tau)$, and set $\bar{Z}_n = z_n(\bar{\tau})$. Following Section 4, let $(c_n, n \in \mathbb{N})$ be a sequence with $c_0 > 0$ such that $\left(\kappa^{Z_n} \mathrm{e}^{-Z_n/c_n}, n \in \mathbb{N}\right)$ or equivalently $\left(\mathrm{e}^{-\bar{Z}_n/c_n}, n \in \mathbb{N}\right)$ is a martingale. This sequence is increasing positive and unbounded. Furthermore, the sequence $(c_{n+1}/c_n, n \in \mathbb{N})$ increases towards $\bar{\mu} = f'(\kappa)$.

We consider a sequence $(a_n, n \in \mathbb{N}^*)$ of integers such that $\mathbb{P}(Z_n = a_n) > 0$ (see Remark 1.3). We denote by τ_n (resp. $\bar{\tau}_n$) a GW tree distributed as τ (resp. $\bar{\tau}$) conditionally on $\{Z_n = a_n\}$ (resp. $\{\bar{Z}_n = a_n\}$). Clearly if $a_n = 0$ for n large enough, then $(\tau_n, n \in \mathbb{N}^*)$ converges in distribution towards τ. So only the case a_n positive for $n \in \mathbb{N}^*$ is of interest.

It is straightforward to deduce from (45) that for $n \geq h \geq 1$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$:

$$\mathbb{P}(r_h(\tau_n) = \mathbf{t}) = \mathbb{P}(r_h(\bar{\tau}_n) = \mathbf{t}). \tag{46}$$

Let $\theta \in (0, +\infty)$. Let $\bar{\tau}^\theta$ be defined as τ^θ in Definition 5.1 where p has to be replaced by $\bar{p}$, and $\mathfrak{p}$ is then equal to p. When $\mathfrak{b}$, the upper bound of the support of p, is finite, we denote by $\bar{\tau}^\infty$ the deterministic regular $\mathfrak{b}$-ary tree. Let $\bar{\tau}^0$ be defined as the Kesten tree τ^0 in Definition 3.3 where $\mathfrak{p}$ is equal to p. We deduce from Propositions 6.2, 6.5 and 6.3, (46) and the characterization (7) of the convergence in $\mathbb{T}_{\mathrm{f}}$ the following result.

Proposition 9.1. *Let p be a non-degenerate sub-critical offspring distribution with generating function f such that $\mathfrak{b} \geq 2$ and suppose that there exists (a unique) $\kappa > 1$ such that $f(\kappa) = \kappa$ and $f'(\kappa) < +\infty$. Let $\theta \in [0, +\infty)$. Assume that $\lim_{n\to\infty} a_n/c_n = \theta$, $a_n > 0$ and τ_n is well defined for all $n \in \mathbb{N}^*$. Then, we have the following convergence in distribution:*

$$\tau_n \xrightarrow[n\to\infty]{\textit{(d)}} \bar{\tau}^\theta. \tag{47}$$

If $\mathfrak{b}$ is finite, then (47) holds also for $\theta = \infty$.

In the sub-critical regime, the local convergence of τ_n and the identification of the limit if any when 1 is the only root of the equation $f(\kappa) = \kappa$ is an open question.

10. Ancillary results

We adapt the proof of Theorem 1 in [16]. Recall that W, conditionally on $\{W > 0\}$ has a positive continuous density w on $(0, +\infty)$. We shall use the following well known result.

Lemma 10.1. *Let X be a real random variable with a continuous density. Let $a < b$ be elements of $\{\lambda \in \mathbb{R};\ \mathbb{E}[\mathrm{e}^{\lambda X}] < +\infty\}$. For $z \in \mathbb{C}$ such that $\Re(z) \in K = [a, b]$, the Laplace transform $g(z) = \mathbb{E}[\mathrm{e}^{zX}]$ is well defined and we have:*

$$\lim_{|t|\to+\infty} \sup_{u\in K} \frac{|g(u+it)|}{g(u)} = 0.$$

Let $t_0 > 0$. There exists $\eta \in (0,1)$ such that for all $u \in K$, $t \in \mathbb{R}$ with $|t| \geq t_0$, we have:

$$|g(u+it)| \leq (1-\eta) g(u). \tag{48}$$

Recall the function $\tilde{\varphi}(z) = \mathbb{E}[\mathrm{e}^{zW}]$ is well defined for $z \in \mathbb{C}$ such that $\Re(z) \in \mathcal{K} = \{\lambda \in \mathbb{R};\ \mathbb{E}[\mathrm{e}^{\lambda W}] < +\infty\}$. The next Lemma is a direct consequence of (48).

Lemma 10.2. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Let $a < 0 \leq b$ such that $K_0 := [a, b] \subset \mathcal{K}$. Let $t_0 > 0$. There exists $\eta \in (0,1)$ such that for all $u \in K_0$, $t \in \mathbb{R}$ with $|t| \geq t_0$:*

$$|\tilde{\varphi}(u+it)| \leq (1-\eta)\tilde{\varphi}(u). \tag{49}$$

Proof. Set $\mathcal{A} = \{(u,t);\ u \in K_0 \text{ and } |t| \geq t_0\}$. According to (48), with X replaced by W conditioned on $\{W > 0\}$, there exists $\eta' \in (0,1)$ such that $|\tilde{\varphi}(u+it) - \mathfrak{c}| \leq (1-\eta')(\tilde{\varphi}(u) - \mathfrak{c})$ for all $(u,t) \in \mathcal{A}$. Taking $\eta = \eta'(1 - \mathfrak{c}/\tilde{\varphi}(a)) \in (0,1)$ so that $\eta'\mathfrak{c} \leq (\eta' - \eta)\tilde{\varphi}(u)$ for all $u \in K_0$, we get for all $(u,t) \in \mathcal{A}$:

$$|\tilde{\varphi}(u+it)| \leq |\tilde{\varphi}(u+it) - \mathfrak{c}| + \mathfrak{c} \leq (1-\eta')(\tilde{\varphi}(u) - \mathfrak{c}) + \mathfrak{c} = (1-\eta')\tilde{\varphi}(u) + \eta'\mathfrak{c} \leq (1-\eta)\tilde{\varphi}(u).$$

This gives the result. □

The next lemma, see Lemma 16 in [18], is used for the Fourier inversion formula of $w^{*\ell}$. Set

$$\mathcal{K}' = \{\lambda \in \mathbb{R};\ \tilde{\varphi}'(\lambda) < +\infty\}. \tag{50}$$

Notice that $\mathcal{K}' \subset \mathcal{K}$ and $\mathcal{K}' \bigcup \{\lambda_c\} = \mathcal{K}$.

Lemma 10.3. *Let p be a non-degenerate super-critical offspring distribution with finite mean. Let $a < 0 \leq b$ such that $K_0 := [a,b] \subset \mathcal{K}$. If $\ell \in \mathbb{N}^*$ is such that $\ell > 1/\alpha$, then we have:*

$$\sup_{u \in K_0} \int_{\mathbb{R}} |\tilde{\varphi}(u+it) - \mathfrak{c}|^{\ell}\, dt < +\infty. \tag{51}$$

If $\alpha < +\infty$ and if $K_0 \subset \mathcal{K}'$, then we have:

$$\sup_{u \in K_0} \int_{\mathbb{R}} |\tilde{\varphi}'(u+it)|\, dt < +\infty. \tag{52}$$

Notice that the proof of Lemma 10.3 insures that $\tilde{\varphi}(u+it) - \mathfrak{c}$ is not L^1 if $\ell \leq 1/\alpha$. This dichotomy appears already in the proof of Lemma 9 from [14]. Recall that, as p is super-critical, we write $\mathfrak{m} = f'(\mathfrak{c}) \in [0,1)$.

Proof. The inequality (51) in the Böttcher case is given in Lemma 16 in [18]. So, we now consider the Schröder case, that is $\mathfrak{m} > 0$. In this case, there exists an analytic function S defined on $\overset{\circ}{D} = \{z \in \mathbb{C},\ |z| < 1\}$ such that the convergence $\lim_{n\to+\infty} \mathfrak{m}^{-n}(f_n(z) - \mathfrak{c}) = \mathrm{S}(z)$ holds uniformly on any compact subset of $\overset{\circ}{D}$, see [6] Corollary 3.7.3[3]. Since the functions are analytic, we also deduce that $\lim_{n\to+\infty} \mathfrak{m}^{-n} f_n'(z) = \mathrm{S}'(z)$ holds uniformly on any compact subset of $\overset{\circ}{D}$. We deduce from (16) and Remark 4.2 that $\tilde{\varphi}(z) = f_k\left(\tilde{\varphi}(\mu^{-k}z)\right)$ and thus for $k \in \mathbb{N}^*$ and $z \in \mathbb{C}$ such that $\varphi'(\mu^{-k}\mathfrak{R}(z)) < +\infty$:

$$\tilde{\varphi}'(z) = \mu^{-k} f_k'\left(\tilde{\varphi}(\mu^{-k}z)\right) \varphi'(\mu^{-k}z). \tag{53}$$

There exists $\varepsilon \in (0,1)$, such that for all $z \in \overset{\circ}{D}$ with $|z - \mathfrak{c}| < \varepsilon(1-\mathfrak{c})$ and $k \in \mathbb{N}^*$, we have $|f_k(z) - \mathfrak{c}| \leq \mathfrak{m}^k/\varepsilon$ and $|f_k'(z)| \leq \mathfrak{m}^k/\varepsilon$. Since $0 \in K_0$, we get that if $u \in K_0$, then $u\mu^{-k} \in K_0$. Thanks to Lemma 10.1 (with X replaced by W conditionally on $\{W > 0\}$), we can take $k_0 \in \mathbb{N}$ large enough so that $|\tilde{\varphi}(\mu^{-k}u + it) - \mathfrak{c}| \leq \varepsilon(1-\mathfrak{c})$, and thus $\tilde{\varphi}(\mu^{-k}u + it) \in \overset{\circ}{D}$, for all

[3]Notice Corollary 3.7.3 stated for $z \in \mathfrak{c} + (1-\mathfrak{c})\overset{\circ}{D}$ in fact holds for $z \in \overset{\circ}{D}$ according to Lemma 3.7.2 in [6], as $\lim_{n\to\infty} f_n(z) = \mathfrak{c}$ for $z \in \overset{\circ}{D}$.

$k \geq k_0$, $u \in K_0$ and $t \geq \mu^{k_0}$. Then, for $k \geq k_0 \geq 0$ and $u \in K_0$, we get, with $\mu^k s = t$, that:

$$\begin{aligned}\int_{\mu^{k+k_0}}^{\mu^{k+k_0+1}} |\tilde{\varphi}(u+it) - \mathfrak{c}|^\ell \, dt &= \int_{\mu^{k+k_0}}^{\mu^{k+k_0+1}} |f_k(\tilde{\varphi}(\mu^{-k}(u+it))) - \mathfrak{c}|^\ell \, dt \\ &= \mu^k \int_{\mu^{k_0}}^{\mu^{k_0+1}} |f_k(\tilde{\varphi}(\mu^{-k}u + is)) - \mathfrak{c}|^\ell \, ds \\ &\leq \mu^{k_0+1} \varepsilon^{-\ell} (\mu \mathfrak{m}^\ell)^k, \end{aligned} \tag{54}$$

as well as, using (53),

$$\begin{aligned}\int_{\mu^{k+k_0}}^{\mu^{k+k_0+1}} |\tilde{\varphi}'(u+it)| \, dt &= \mu^{-k} \int_{\mu^{k+k_0}}^{\mu^{k+k_0+1}} |f_k'(\tilde{\varphi}(\mu^{-k}(u+it)))| \, |\tilde{\varphi}'(\mu^{-k}(u+it))| \, dt \\ &= \int_{\mu^{k_0}}^{\mu^{k_0+1}} |f_k'(\tilde{\varphi}(\mu^{-k}u + is))| \, |\tilde{\varphi}'(\mu^{-k}u + is)| \, ds \\ &\leq \mu^{k_0+1} \varepsilon^{-1} \mathfrak{m}^k \sup_{u \in K_0} \tilde{\varphi}'(u). \end{aligned} \tag{55}$$

We deduce from (55) that the integral $\int_\mathbb{R} |\tilde{\varphi}'(u+it)| \, dt$ is uniformly bounded for $u \in K_0$ as $\sup_{u \in K_0} \tilde{\varphi}'(u)$ is finite since $K_0 \subset \mathcal{K}'$, and from (54) that the integral $\int_\mathbb{R} |\tilde{\varphi}(u+it) - \mathfrak{c}|^\ell \, dt$ is uniformly bounded for $u \in K_0$ as soon as $\mu \mathfrak{m}^\ell = \mu^{1-\ell\alpha} < 1$ that is $\ell > 1/\alpha$. □

We give a similar result on the integrability of $\tilde{\varphi}_j$, the Laplace transform of $-W_j = -Z_j/c_j$. See (166) in [18] and a variant of Lemmas 2 and 3 in [14], see also Lemmas 8 and 9 in [17]. By construction the process $M = (M_n = \mathrm{e}^{-W_n}, n \in \mathbb{N})$ is a positive bounded martingale with respect to the filtration $(\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n), n \in \mathbb{N})$. It is closed as it converges a.s. towards $M_\infty = \mathrm{e}^{-W}$. Let g be a convex non-negative function defined on $(0, +\infty)$. We deduce that $N^g = (N^g_n = g(M_n), n \in \mathbb{N})$ is a positive sub-martingale which converges a.s. towards $N^g_\infty = g(M_\infty)$. By Jensen inequality, we get that $N^g_n \leq \mathbb{E}[g(M_\infty)|\mathcal{F}_n]$. If $\mathbb{E}[g(M_\infty)] < +\infty$, then we get that: N^g is uniformly integrable, N^g converges in L^1 towards N^g_∞, $\lim_{n\to\infty} \mathbb{E}[N^g_n] = \sup_{n \in \mathbb{N}} \mathbb{E}[N^g_n] = \mathbb{E}[N^g_\infty] < +\infty$. For $\lambda \in \mathcal{K}$, consider the positive convex function $g(x) = x^{-\lambda}$ defined on $(0,1)$ and set $\tilde{\varphi}_n(\lambda) = N^g_n = \mathbb{E}[\mathrm{e}^{\lambda W_n}]$. We deduce that $\lim_{n\to+\infty} \tilde{\varphi}_n(\lambda) = \sup_{n \in \mathbb{N}} \tilde{\varphi}_n(\lambda) = \tilde{\varphi}(\lambda)$. Using monotone convergence, we get that $\tilde{\varphi}_n(z) = \mathbb{E}[\mathrm{e}^{zW_n}]$ converges uniformly on compacts subsets of $\{z \in \mathbb{C}; \Re(z) \in \mathcal{K}\}$ towards $\tilde{\varphi}(z)$ as n goes to infinity.

For $\lambda \in \mathcal{K}'$, consider the positive convex function $\mathfrak{g}(x) = -\log(x) x^{-\lambda}$ defined on $(0,1)$ and notice that $\tilde{\varphi}_n'(\lambda) = N^\mathfrak{g}_n$. Arguing as for g, we get that $\tilde{\varphi}_n'(z)$ converges uniformly on compacts subsets of $\{z \in \mathbb{C}; \Re(z) \in \mathcal{K}'\}$ and that for $\lambda \in \mathcal{K}'$:

$$\lim_{n\to+\infty} \tilde{\varphi}_n'(\lambda) = \sup_{n \in \mathbb{N}} \tilde{\varphi}_n'(\lambda) = \tilde{\varphi}'(\lambda). \tag{56}$$

Lemma 10.4. *Let p be a non-degenerate super-critical offspring distribution with finite mean and type (L_0, r_0). Let $a < 0 \leq b$ such that $K_0 := [a,b] \subset \mathcal{K}$. Let $t_1 \in (0, \pi c_0/L_0)$. There exists $\delta \in (0,1)$ such that for all $u \in K_0$, $t \in \mathbb{R}$, $n \in \mathbb{N}^*$, with $t_1 \leq |t| \leq \pi c_n/L_0$:*

$$|\tilde{\varphi}_n(u+it)| \leq (1-\delta) \tilde{\varphi}_n(u). \tag{57}$$

If $\alpha > 1$, we have:

$$\sup_{u \in K_0, n \in \mathbb{N}^*} \int_{[\pm \pi c_n/L_0]} |\tilde{\varphi}_n(u+it) - \mathfrak{c}| \, dt < +\infty. \tag{58}$$

If $\alpha < +\infty$ and $K_0 \subset \mathcal{K}'$, we have:

$$\sup_{u \in K_0, n \in \mathbb{N}^*} \int_{[\pm \pi c_n/L_0]} |\tilde{\varphi}_n'(u+it)| \, dt < +\infty. \tag{59}$$

Proof. Let $t_1 \in (0, \pi c_0/L_0)$. Because of the periodicity L_0, we deduce that for all $n \in \mathbb{N}^*$, $u \in K_0$, $0 < |t| < 2\pi c_n/L_0$, we have $|\tilde{\varphi}_n(u+it)|/\tilde{\varphi}_n(u) < 1$. Thanks to Lemma 10.2, there exists $\eta \in (0,1)$, such that for all $u \in K_0$, $|t| \geq t_1$, we have $|\tilde{\varphi}(u+it)| \leq (1-\eta)\tilde{\varphi}(u)$. Using the uniform convergence on compact subsets of $\{z \in \mathbb{C}; \Re(z) \in \mathcal{K}\}$ of $\tilde{\varphi}_n$ towards $\tilde{\varphi}$ and $\tilde{\varphi} > \mathfrak{c}$ on K_0, we deduce that for all $t_2 > t_1$, there exists $\eta' \in (0,1)$ such that for all $u \in K_0$, $n \in \mathbb{N}^*$, $\min(t_2, \pi c_n/L_0) \geq |t| \geq t_1$, we have:

$$|\tilde{\varphi}_n(u+it)| \leq (1-\eta')\tilde{\varphi}_n(u). \tag{60}$$

Set $t_1 = \pi c_0/L_0$, $t_2 = \pi c_0 \mu/L_0$ and $J_n = [\pi c_0/L_0, \pi c_0 c_n/L_0 c_{n-1}] \subset [t_1, \min(t_2, \pi c_n/L_0)]$. Using the uniform convergence on compacts of $\{z \in \mathbb{C}, \Re(z) \in \mathcal{K}\}$ of $\tilde{\varphi}_n$ towards $\tilde{\varphi}$ and $\tilde{\varphi}(0) = 1$, we deduce from (60) that there exists $\varepsilon > 0$ and $r_0 \in (0,1)$ such that for all $u \in K_0$ with $|u| \leq \varepsilon$, $n \in \mathbb{N}^*$, $|t| \in J_n$, we have $|\tilde{\varphi}_n(u+it)| \leq r_0 < 1$. Thus, there exists $k_0 \in \mathbb{N}$ such that $\sup_{K_0} |u| c_{n-k_0}/c_n \leq \varepsilon$ for all $n \geq k_0$. Thus for all $k, n \in \mathbb{N}^*$ with $k + k_0 \leq n$, $u \in K_0$, $|t| \in J_n$, we have:

$$\left| \tilde{\varphi}_k \left(u \frac{c_k}{c_n} + it \right) \right| \leq r_0 < 1. \tag{61}$$

We consider now the Schröder case, that is $\mathfrak{m} > 0$. According to the beginning of the proof of Lemma 10.3, there exists a finite constant B such that for all $z \in \mathbb{C}$ such that $|z| \leq r_0$ and $n \in \mathbb{N}^*$, we have:

$$|f_n(z) - \mathfrak{c}| \leq B\mathfrak{m}^n \quad \text{and} \quad |f_n'(z)| \leq B\mathfrak{m}^n. \tag{62}$$

For $k \in \{1, \ldots, n\}$, set $J_{n,k} = \{t \in \mathbb{R}, \pi c_n c_0/L_0 c_k \leq |t| \leq \pi c_n c_0/L_0 c_{k-1}\}$, so that $t \in J_{n,k}$ implies $|t| c_k/c_n \in J_n$ as the sequence $(c_k/c_{k-1}, k \in \mathbb{N}^*)$ is non-decreasing. For $k, n \in \mathbb{N}^*$ with $k + k_0 \leq n$, $u \in K_0$, $t \in J_{n,k}$, we deduce from (61) and (62) that:

$$|\tilde{\varphi}_n(u+it) - \mathfrak{c}| = \left| f_{n-k} \left(\tilde{\varphi}_k \left(u \frac{c_k}{c_n} + it \frac{c_k}{c_n} \right) \right) - \mathfrak{c} \right| \leq B\mathfrak{m}^{n-k}. \tag{63}$$

and, with $R_0 = \sup_{n \in \mathbb{N}^*} \sup_{u \in K_0} |\tilde{\varphi}_n'(u)|$, that:

$$\left|\tilde{\varphi}_n'(u+it)\right| = \frac{c_k}{c_n} \left| f_{n-k}' \left(\tilde{\varphi}_k \left(u \frac{c_k}{c_n} + it \frac{c_k}{c_n} \right) \right) \right| \left| \tilde{\varphi}_k' \left(u \frac{c_k}{c_n} + it \frac{c_k}{c_n} \right) \right| \leq \frac{c_k}{c_n} B\mathfrak{m}^{n-k} R_0. \tag{64}$$

Notice that $R_0 = \sup_{u \in K_0} \tilde{\varphi}'(u)$ and it is finite if $K_0 \subset \mathcal{K}'$.

As $c_n/c_{k-1} \leq \mu^{n-k+1}$, we get that $|J_{n,k}| \leq \pi c_0 c_n/L_0 c_{k-1} \leq \pi c_0 \mu^{n-k+1}/L_0$. This implies that for $k + k_0 \leq n$:

$$\int_{J_{n,k}} |\tilde{\varphi}_n(u+it) - \mathfrak{c}| \, dt \leq \frac{B\pi c_0 \mu}{L_0} (\mu\mathfrak{m})^{n-k} \quad \text{and} \quad \int_{J_{n,k}} \left|\tilde{\varphi}_n'(u+it)\right| \, dt \leq R_0 \frac{B\pi c_0 \mu}{L_0} \mathfrak{m}^{n-k}.$$

Since $[\pm \pi c_n/L_0] \subset [\pm \pi c_0 \mu^{k_0}/L_0] \cup \bigcup_{k=1}^{n-k_0} J_{n,k}$, we deduce that for all $u \in K_0$:

$$\int_{[\pm \pi c_n/L_0]} |\tilde{\varphi}_n(u+it) - \mathfrak{c}| \, dt \leq \frac{\pi c_0}{L_0} \left(\mu^{k_0} \sup_{n \in \mathbb{N}^*} \tilde{\varphi}_n(\sup K_0) + \mu^{k_0} \mathfrak{c} + B\mu \sum_{k=1}^{n-k_0} (\mu\mathfrak{m})^{n-k} \right) \tag{65}$$

and

$$\int_{[\pm\pi c_n/L_0]} |\tilde{\varphi}'_n(u+it)|\, dt \leq \frac{\pi c_0 R_0}{L_0} \left(\mu^{k_0} \sup_{n\in\mathbb{N}^*} \tilde{\varphi}'_n(\sup K_0) + B\mu \sum_{k=1}^{n-k_0} \mathfrak{m}^{n-k} \right). \tag{66}$$

The the upper bound (65) gives (58) when $\alpha > 1$ that is $\mu\mathfrak{m} < 1$, and the upper bound (66) gives (59) when $\alpha < +\infty$ and $K_0 \subset \mathcal{K}'$.

We now prove (58) in the Böttcher case, that is $\mathfrak{m} = 0$ and $\alpha = +\infty$. Notice then that $\sup_{|z|\leq r_0} |f_n(z)| \leq B\varepsilon_0^n$ for any $n \in \mathbb{N}^*$ and $\varepsilon_0 > 0$ with some finite constant B depending only on ε_0. Then we obtain (58) using similar arguments as in the Schröder case.

We now prove (57) in the Schröder case. There exists $\eta'' \in (0, 1/2)$ such that $\mathfrak{c} < (1 - 2\eta'')^2\tilde{\varphi}(a)$. We can choose an integer $k'_0 \geq k_0$ such that $\mathfrak{c} + B\mathfrak{m}^{k'_0} < (1-\eta'')^2\tilde{\varphi}(a)$. We can also choose $n_0 \geq k'_0$ large enough so that $c_{n_0} > c_0\mu^{k'_0}$ and $\inf_{n\geq n_0} \tilde{\varphi}_n(a) \geq (1-\eta'')\tilde{\varphi}(a)$. Notice that for $n \geq k'_0$:

$$\left[\frac{\pi c_0 \mu^{k'_0}}{L_0}, \frac{\pi c_n}{L_0}\right] \subset \left[\frac{\pi c_n c_0}{L_0 c_{n-k'_0}}, \frac{\pi c_n}{L_0}\right] = \bigcup_{k=1}^{n-k'_0} J_{n,k}.$$

Using (63), we get that for $k \in \mathbb{N}^*$, $n \geq n_0$ with $k + k'_0 \leq n$, $u \in K_0$, $t \in J_{n,k}$:

$$|\tilde{\varphi}_n(u+it)| \leq \mathfrak{c} + B\mathfrak{m}^{n-k} \leq \mathfrak{c} + B\mathfrak{m}^{k'_0} \leq (1-\eta'')^2\tilde{\varphi}(a) \leq (1-\eta'')\tilde{\varphi}_n(a) \leq (1-\eta'')\tilde{\varphi}_n(u).$$

This gives that $|\tilde{\varphi}_n(u+it)| \leq (1-\eta'')\tilde{\varphi}_n(u)$ for all $u \in K_0$, $t \in \left[\frac{\pi c_0\mu^{k'_0}}{L_0}, \frac{\pi c_n}{L_0}\right]$ and $n \geq n_0$. This and (60) with $t_2 = \pi c_{n_0}/L_0 > \pi c_0 \mu^{k'_0}/L_0$ complete the proof of (57) in the Schröder case.

The proof of (57) in the Böttcher case is similar and left to the reader. □

11. Results in the Harris case

We present detailed proofs of the results, because even if they correspond to an adaptation of the results known in the Böttcher case (see [17] and [18]), we believe that the adaptation is not straightforward since in particular the Fourier inversion of $w^{*\ell}$ is not valid if $\ell\alpha \leq 1$. We keep notations from Sections 2.3 and 4. Recall $\mathfrak{b}$ defined in (10) is the supremum of the support of the offspring distribution p. We assume $\mathfrak{b} < \infty$ (Harris case). Following [16] or [10], we define the (right) Böttcher constant $\beta_H \in (1, +\infty)$ by:

$$\mathfrak{b} = \mu^{\beta_H}.$$

11.1. **Preliminaries.** Since $\mathfrak{b}$ is finite, the radius of convergence R_c of f is infinite. According to Remark 4.2, we deduce that $\lambda_c = +\infty$, that is W has all its exponential moments and that for every $z \in \mathbb{C}$, with $\tilde{\varphi}(z) = \mathbb{E}[\mathrm{e}^{zW}] = \varphi(-z)$:

$$\tilde{\varphi}(z) = f(\tilde{\varphi}(z/\mu)). \tag{67}$$

We define the function $\tilde{b}$ on its domain by:

$$\tilde{b}(z) = \log(z) + \sum_{n=0}^{+\infty} \mathfrak{b}^{-n-1} \log\left(\frac{f_{n+1}(z)}{f_n(z)^{\mathfrak{b}}}\right). \tag{68}$$

According to Lemma 2.5 in [16], for every $\delta \in (0,1)$, there exists a constant $\theta = \theta(\delta) \in (0,\pi)$ such that $\tilde{b}$ is analytic on the open set:

$$\tilde{\mathcal{D}}(\delta) = \{z \in \mathbb{C};\, 1+\delta < |z| < \delta^{-1},\, |\arg(z)| < \theta\}. \tag{69}$$

Notice that the function $\tilde{b}$ is analytic and positive on $(1,\infty)$ and satisfies on $(1,\infty)$:

$$\tilde{b} \circ f = \mathfrak{b}\, \tilde{b}. \tag{70}$$

According to Lemma 2.6 in [16], the function $\tilde{b}$ satisfies:

$$(s\tilde{b}'(s))' > 0 \quad \text{on } (1,\infty), \quad \lim_{s\to 1+} s\tilde{b}'(s) = 0 \quad \text{and} \quad \lim_{s\to+\infty} s\tilde{b}'(s) = 1. \tag{71}$$

In particular, the function $\tilde{b}$ is increasing on $(1,+\infty)$.

We set $\tilde{\psi} = \tilde{b}\circ\tilde{\varphi}$ on $(0,+\infty)$. We directly recover Proposition 1 in [10], where it is assumed that $\mathfrak{c} = 0$. (We could have used directly the results from [10] using the generating function $\bar{f}$ given by the so-called Sevastyanov transform of f: $\bar{f}(z) = [f(\mathfrak{c} + (1-\mathfrak{c})z) - \mathfrak{c}]/(1-\mathfrak{c})$, where $\bar{f}'(1) = \mu$, $\bar{f}'(0) = f'(\mathfrak{c})$ and (67) also holds with f replaced by $\bar{f}$. But this approach breaks down, when considering the upper large deviation for Z_n, see Section 11.4.)

Lemma 11.1. *The function $\tilde{\psi}$ is analytic, increasing and strictly convex on $(0,+\infty)$, and*

$$\tilde{\psi}(s) = \tilde{\psi}(\mu s)/\mathfrak{b} \quad \text{on } (0,+\infty), \quad \lim_{s\to 0+} \tilde{\psi}'(s) = 0 \quad \text{and} \quad \lim_{s\to+\infty} \tilde{\psi}'(s) = +\infty. \tag{72}$$

Proof. Since $\tilde{\varphi}$ is analytic on $\mathbb{C}$ and $\tilde{\varphi}((0,+\infty)) = (1,+\infty)$, we get that $\tilde{\psi}$ is analytic on $(0,+\infty)$. It is clear that $\tilde{\psi}$ is increasing as the composition of two increasing functions. Moreover, using (71) as well as $\tilde{\varphi}'(s)^2 < \tilde{\varphi}''(s)\tilde{\varphi}(s)$ thanks to Cauchy-Schwartz inequality, we have for every $s \in (0,+\infty)$,

$$\begin{aligned}\tilde{\psi}''(s) &= \tilde{\varphi}''(s)\tilde{b}'(\tilde{\varphi}(s)) + \tilde{\varphi}'(s)^2\tilde{b}''(\tilde{\varphi}(s))\\ &\geq \frac{\tilde{\varphi}'(s)^2}{\tilde{\varphi}(s)}\left(\tilde{b}'(\tilde{\varphi}(s)) + \tilde{\varphi}(s)\tilde{b}''(\tilde{\varphi}(s))\right) > 0.\end{aligned}$$

We deduce that $\tilde{\psi}$ is strictly convex on $(0,+\infty)$. The functional equation $\tilde{\psi}(s) = \tilde{\psi}(\mu s)/\mathfrak{b}$ is a direct consequence of (67) and (70). Then use that W has an unbounded support to get that $\lim_{s\to+\infty} \tilde{\varphi}'(s)/\tilde{\varphi}(s) = +\infty$ and deduce the limits of $\tilde{\psi}'$ using (71). □

Recall Definition (69) of $\tilde{\mathcal{D}}(\delta)$. According to Lemma 2.5 in [16], there exists $\varepsilon = \varepsilon(\delta) \in (0, \tilde{b}(\delta))$ such that for all $z \in \tilde{\mathcal{D}}(\delta)$, we have:

$$f_n(z) = p(\mathfrak{b})^{-1/(\mathfrak{b}-1)}\, \mathrm{e}^{\mathfrak{b}^n\tilde{b}(z)}\left(1 + O(\mathrm{e}^{-\varepsilon\mathfrak{b}^n})\right). \tag{73}$$

We have the following result (see Lemma 13 in [18]).

Lemma 11.2. *For all $s \in (1,+\infty)$ and all $n \in \mathbb{N}^*$, we have:*

$$f_n(s) < p(\mathfrak{b})^{-1/(\mathfrak{b}-1)} \exp\left\{\mathfrak{b}^n\tilde{b}(s)\right\}.$$

Proof. We set

$$\tilde{b}_N(z) = \frac{1-\mathfrak{b}^{-N}}{\mathfrak{b}-1}\log(p(\mathfrak{b})) + \log(z) + \sum_{n=0}^{N-1} \mathfrak{b}^{-n-1}\log\left(\frac{f_{n+1}(z)}{p(\mathfrak{b})f_n(z)^{\mathfrak{b}}}\right)$$

for $z \in \bigcup_{\delta>0} \tilde{\mathcal{D}}(\delta)$. Notice that $\tilde{b}_N(s) = \mathfrak{b}^{-N} \log(f_N(s))$ for all $s > 0$. For $s > 0$, we have:

$$\mathfrak{b}^{-N} \log(f_N(s)) = \tilde{b}_N(s) = \tilde{b}(s) - \frac{\mathfrak{b}^{-N}}{\mathfrak{b}-1} \log(p(\mathfrak{b})) - \sum_{n\geq N} \mathfrak{b}^{-n-1} \log\left(\frac{f_{n+1}(s)}{p(\mathfrak{b}) f_n(s)^{\mathfrak{b}}}\right)$$

that is

$$\log(f_N(s)) = \mathfrak{b}^N \tilde{b}(s) - \frac{1}{\mathfrak{b}-1} \log(p(\mathfrak{b})) - \mathfrak{b}^N \sum_{n\geq N} \mathfrak{b}^{-n-1} \log\left(\frac{f_{n+1}(s)}{p(\mathfrak{b}) f_n(s)^{\mathfrak{b}}}\right).$$

For $s > 1$, we have $p(\mathfrak{b}) f_n(s)^{\mathfrak{b}} < f_{n+1}(s)$ so that:

$$\log(f_N(s)) < \mathfrak{b}^N \tilde{b}(s) - \frac{1}{\mathfrak{b}-1} \log(p(\mathfrak{b})).$$

This gives the result. □

11.2. **Right tail of w.** We denote by $\tilde{g}$ the inverse of $\tilde{\psi}'$, which is one to one on $(0, +\infty)$ by Lemma 11.1. For a given $v > 0$, the maximum of $uv - \tilde{\psi}(u)$ for $u \geq 0$ is uniquely reached at $\tilde{g}(v)$:

$$\max_{u\geq 0} \left(uv - \tilde{\psi}(u)\right) = \tilde{g}(v)v - \tilde{\psi}(\tilde{g}(v)). \tag{74}$$

We define the function $\tilde{M}$ for $v \in (0, +\infty)$ by:

$$\tilde{M}(v) = v^{-\beta_H/(\beta_H-1)} \max_{u\geq 0} \left(uv - \tilde{\psi}(u)\right). \tag{75}$$

According to Proposition 2 in [10], the function $\tilde{M}$ is analytic on $(0, +\infty)$. It is positive and multiplicatively periodic with period $\mu^{\beta_H-1} = \mathfrak{b}/\mu$, thanks to the functional equation in (72) and the definition of β_H, (see also Proposition 3 in [10]).

Mimicking the proof of Theorem 1 in [18] (see also Remark 3 therein), we set for $x \in [\mathfrak{b}/\mu, \infty)$:

$$\tilde{r}(x) = \left\lfloor \frac{\log(x)}{\log(\mathfrak{b}/\mu)} \right\rfloor \quad \text{and} \quad \tilde{y}(x) = x \left(\frac{\mu}{\mathfrak{b}}\right)^{r(x)} = x\mu^{-r(x)(\beta_H-1)} = x\mathfrak{b}^{-r(x)(\beta_H-1)/\beta_H}, \tag{76}$$

so that $\tilde{r}(x) \geq 0$ and $\tilde{y}(x) \in [1, \mathfrak{b}/\mu)$. Notice that $\tilde{r}(x) \to +\infty$ as $x \to +\infty$. Let $\ell \in \mathbb{N}^*$. We define the positive functions for $y > 0$:

$$\tilde{\mathcal{M}}_{1,\ell}(y) = \frac{p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)}}{\sqrt{2\pi\ell\tilde{\sigma}^2(y/\ell)}}\, y^{(\beta_H-2)/2(\beta_H-1)} \quad \text{and} \quad \tilde{\mathcal{M}}_{2,\ell}(y) = \tilde{\mathcal{M}}_{1,\ell}(y) \frac{y^{1/(\beta_H-1)}}{\tilde{g}(y/\ell)},$$

where $\tilde{\sigma}^2(y) = \tilde{\psi}''(\tilde{g}(y)) > 0$. For $\ell \in \mathbb{N}^*$ and $x \in [\mathfrak{b}/\mu, +\infty)$, we set:

$$\tilde{M}_{1,\ell}(x) = \tilde{\mathcal{M}}_{1,\ell}(\tilde{y}(x)) \quad \text{and} \quad \tilde{M}_{2,\ell}(x) = \tilde{\mathcal{M}}_{2,\ell}(\tilde{y}(x)).$$

By construction $x \mapsto \tilde{y}(x)$ is multiplicative periodic with period $\mu/\mathfrak{b} = \mathfrak{b}^{\beta_H-1}$. We deduce that, for fixed $\ell \in \mathbb{N}^*$, the functions $\tilde{M}_{1,\ell}$ and $\tilde{M}_{2,\ell}$ are multiplicative periodic with period $\mu/\mathfrak{b}$, positive, bounded and bounded away from 0.

We first state an upper bound on $w^{*\ell}$ whose proof is postponed to Section 11.5.

Lemma 11.3. *Let p be a non-degenerate super-critical offspring distribution with $\mathfrak{b} < +\infty$. For all $u_1 \geq 0$, there exists a finite constant C such that for all $\ell \in \mathbb{N}^*$, $x \geq \mathfrak{b}/\mu$ and $u \in [0, u_1]$, we have with $r = \tilde{r}(x)$ and $y = \tilde{y}(x)$:*

$$w^{*\ell}(x) \leq \frac{C\ell}{x} \mathfrak{b}^r \,\mathrm{e}^{-uy\mathfrak{b}^r}\, f_r(\tilde{\varphi}(u))^\ell. \tag{77}$$

We now state a slightly more general result than Remark 3 in [18]. (Notice in Remark 3 in [18] that there is a misprint in (21) and (22) where the power of x in the exponential should be negative.)

Lemma 11.4. *Let p be a non-degenerate super-critical offspring distribution with $\mathfrak{b} < +\infty$. Let $\ell \in \mathbb{N}^*$. As $x \nearrow +\infty$, we have:*

$$w^{*\ell}(x) \sim \tilde{M}_{1,\ell}(x)\, x^{(2-\beta_H)/2(\beta_H-1)} \exp\left\{-\ell^{-1/(\beta_H-1)}\, x^{\beta_H/(\beta_H-1)} \tilde{M}(x/\ell)\right\}, \tag{78}$$

$$w_\ell(x) \sim w^{*\ell}(x), \tag{79}$$

$$\mathbb{P}_\ell(W \geq x) \sim \tilde{M}_{2,\ell}(x)\, x^{-\beta_H/2(\beta_H-1)} \exp\left\{-\ell^{-1/(\beta_H-1)}\, x^{\beta_H/(\beta_H-1)} \tilde{M}(x/\ell)\right\}. \tag{80}$$

Using Lemma 3.6.11 in [6], we could derive similar formula as (78) for the j-th derivative of w, for $j < \alpha$. The proof of Lemma 11.4 is given in Sections 11.6 and 11.7.

11.3. Proof of Lemma 4.4 in the Harris case.

Let $\ell \in \mathbb{N}^*$. Using (21) and that $f^{(\mathfrak{b})}(\mathfrak{c})/\mathfrak{b}! = p(\mathfrak{b})$, we get for $x \geq \mathfrak{b}/\mu$:

$$\frac{1}{\mu}\, w^{*\ell}(x/\mu) = \sum_{s=(s_1,\dots,s_\ell)\in(\mathbb{N}^*)^\ell} w(x)^{*|s|_1} \prod_{i=1}^{\ell} \frac{f^{(s_i)}(\mathfrak{c})}{s_i!} = p(\mathfrak{b})^\ell w^{*\mathfrak{b}\ell}(x) + R_\ell(x), \tag{81}$$

where

$$R_\ell(x) = \sum_{s=(s_1,\dots,s_\ell)\in(\mathbb{N}^*)^\ell} \mathbf{1}_{\{|s|_1<\ell\mathfrak{b}\}}\, w(x)^{*|s|_1} \prod_{i=1}^{\ell} \frac{f^{(s_i)}(\mathfrak{c})}{s_i!}.$$

Using (77), we get for $u > 0$ with $r = \tilde{r}(x)$ and $y = \tilde{y}(x)$ defined in (76):

$$\begin{aligned} R_\ell(x) &\leq C\, \mathrm{e}^{-uy\mathfrak{b}^r} \sum_{s=(s_1,\dots,s_\ell)\in(\mathbb{N}^*)^\ell} \mathbf{1}_{\{|s|_1<\ell\mathfrak{b}\}}\, \mathfrak{b}^{|s|_1} f_r(\tilde{\varphi}(u))^{|s|_1} \prod_{i=1}^{\ell} \frac{s_i}{x} \frac{f^{(s_i)}(\mathfrak{c})}{s_i!} \\ &\leq \frac{C'}{x^\ell}\, \mathrm{e}^{-uy\mathfrak{b}^r}\; f_r(\tilde{\varphi}(u))^{\ell\mathfrak{b}-1}, \end{aligned}$$

for some finite constant C' (depending on u_1 and independent of x and $u \in [0, u_1]$). Using Lemma 11.2, we get that for all $u > 0$, $n \in \mathbb{N}^*$:

$$f_n(\tilde{\varphi}(u)) \leq p(\mathfrak{b})^{-1/(\mathfrak{b}-1)}\; \exp\left\{\mathfrak{b}^n \tilde{\psi}(u)\right\}.$$

Since $\tilde{\varphi}(u) \geq 1$, this gives with some constant C'' (depending on u_1 and independent of x and $u \in [0, u_1]$):

$$R_\ell(x) \leq \frac{C''}{x^\ell}\, \mathrm{e}^{\Gamma(x,u)} \quad \text{with} \quad \Gamma(x,u) = (\mathfrak{b}\ell - 1)\mathfrak{b}^r \tilde{\psi}(u) - uy\mathfrak{b}^r.$$

We set $u^* = \tilde{g}(y/\mathfrak{b}\ell)$. We get:

$$\begin{aligned}\Gamma(x,u^*) &= \mathfrak{b}^{r+1}\ell\left[\tilde{\psi}(u^*) - u^*\frac{y}{\mathfrak{b}\ell}\right] - \mathfrak{b}^r\tilde{\psi}(u^*)\\ &= -\mathfrak{b}^{r+1}\ell\left(\frac{y}{\mathfrak{b}\ell}\right)^{\beta_H/(\beta_H-1)}\tilde{M}\left(\frac{y}{\mathfrak{b}\ell}\right) - \mathfrak{b}^r\tilde{\psi}(u^*)\\ &= -\ell^{-1/(\beta_H-1)}\mathfrak{b}^{-1/(\beta_H-1)}x^{\beta_H/(\beta_H-1)}\tilde{M}\left(\frac{x}{\mu\ell}\right) - \mathfrak{b}^r\tilde{\psi}(u^*)\\ &= -\ell^{1/(\beta_H-1)}\left(\frac{x}{\mu}\right)^{\beta_H/(\beta_H-1)}\tilde{M}\left(\frac{x}{\mu\ell}\right) - \mathfrak{b}^r|\tilde{\psi}(u^*)|,\end{aligned}$$

where we used (74) and (75) for the second equality; that $y = x\mathfrak{b}^{-r(\beta_H-1)/\beta_H}$, $\tilde{M}$ is multiplicative periodic with period $\mathfrak{b}/\mu$ for the third one; and $\mathfrak{b} = \mu^{\beta_H}$ and $\tilde{\psi}$ is positive for the last one. For $x \in [\mathfrak{b}/\mu, +\infty)$, we have $(y/\mathfrak{b}\ell) \in [1/\mathfrak{b}\ell, 1/\mu\ell)$ and thus, as ℓ is fixed and $\tilde{g}$ continuous positive, $u^* = \tilde{g}(y/\mathfrak{b}\ell)$ belongs to an interval, say $[a,b]$, with $0 < a < b < +\infty$. This implies that $c_0 = \inf_{\{x\in[\mathfrak{b}/\mu,+\infty)\}}|\tilde{\psi}(u^*)| > 0$. Notice also that $c_2 = \inf_{\{x>0\}}\tilde{M}_{1,\ell}(x)$ is positive as $\tilde{M}_{1,\ell}$ is bounded away from 0. Thus, using (78), we deduce that:

$$R_\ell(x) \leq \frac{C''}{c_2}w^{*\ell}(x/\mu)\,\mathrm{e}^{-\ell\log(x)-\mathfrak{b}^r c_0 - \frac{2-\beta_H}{2(\beta_H-1)}\log(x/\mu)}$$

for x large enough. Recall $r = \tilde{r}(x)$ defined in (76). As $x \to +\infty$ we have $r = \tilde{r}(x) \to +\infty$ and $\log(x) \sim \tilde{r}(x)\log(\mathfrak{b}/\mu)$. Thus, we obtain $R_\ell(x) = o(w^{*\ell}(x/\mu))$ as $x \to +\infty$. Plugging this in (81) we get that:

$$\lim_{x\to+\infty}\mu\frac{w^{*\mathfrak{b}\ell}(x)}{w^{*\ell}(x/\mu)}\,p(\mathfrak{b})^\ell = 1.$$

From the definition of $\rho_{\theta,\ell}$ in (20), we deduce that $\lim_{\theta\to+\infty}\rho_{\theta,\ell}(\mathfrak{b},\dots\mathfrak{b}) = 1$. This ends the proof of Lemma 4.4 in the Harris case.

11.4. **Upper large deviations for Z_n.** Recall Definition (17) of $\mathcal{K}$ and notations from Section 10, and in particular Definition (50) of $\mathcal{K}'$. In the Harris case, we have $\mathcal{K} = \mathcal{K}' = \mathbb{R}$. We recall that for $j \in \mathbb{N}^*$, $\tilde{\varphi}_j(z) = \mathbb{E}[\mathrm{e}^{zW_j}] = f_j(\mathrm{e}^{z/c_j})$, with $W_j = Z_j/c_j$, is well defined for $z \in \mathbb{C}$ and that $\tilde{\varphi}_j$ converges uniformly on the compacts of $\mathbb{C}$ towards $\tilde{\varphi}$ as j goes to infinity. Elementary computations give that $\lim_{u\to+\infty}\tilde{\varphi}_j'(u)/\tilde{\varphi}_j(u) = \mathfrak{b}^j/c_j$.

We consider the functions $\tilde{\psi}_j = \tilde{b}\circ\tilde{\varphi}_j$ defined on some open neighborhood of $(0,+\infty)$ in $\mathbb{C}$ for $j \in \mathbb{N}^*$. Following Lemma 11.1, it is easy to check that the functions $\tilde{\psi}_j$ are analytic on $(0,+\infty)$, positive, increasing, strictly convex and that:

$$\lim_{x\to 0+}\tilde{\psi}_j'(x) = 0 \quad\text{and}\quad \lim_{x\to+\infty}\tilde{\psi}_j'(x) = \frac{\mathfrak{b}^j}{c_j}\cdot$$

Let $\tilde{g}_j$ be the inverse of $\tilde{\psi}_j'$ defined on $(0,\mathfrak{b}^j/c_j)$. In particular, for a given positive $v < \mathfrak{b}^j/c_j$, the minimum of $\tilde{\psi}_j(u) - uv$ for $u \geq 0$ is uniquely reached at $\tilde{g}_j(v)$. Using that $\tilde{\psi}_j$ converges uniformly, on compacts sub-sets of a neighborhood in $\mathbb{C}$ of $(0,+\infty)$, towards $\tilde{\psi}$, that $\tilde{b}$ and thus $\tilde{\psi}_j$ and $\tilde{\psi}$ are analytic, we get that for any compact of $(0,+\infty)$ and j large enough, the strictly convex functions $\tilde{\psi}_j$ and their derivatives converge uniformly towards the strictly convex function $\tilde{\psi}$ and its derivatives. We deduce that for any compact K of $(0,+\infty)$ and j large enough (more precisely j such that $\mathfrak{b}^j/c_j > \sup(K)$), $\tilde{g}_j$ is well defined on K and converges uniformly towards $\tilde{g}$ on K.

We consider the following general setting. Let $\ell \in \mathbb{N}^*$ and $a_n \in [\ell c_n/c_0, \ell \mathfrak{b}^n)$ such that $\limsup_{n\to\infty} a_n/\ell\mathfrak{b}^n < 1$. Since $\mathfrak{b} > \mu > c_{r+1}/c_r$ for all $r \in \mathbb{N}$, we deduce that the sequence $(c_{n-l}\mathfrak{b}^l, 0 \leq l \leq n)$ is increasing. Therefore, the integer $l_n = \sup\{l \in \{0, \dots, n\}, c_{n-l}\ell\mathfrak{b}^l \leq c_0 a_n\}$ is well-defined and strictly less than n. Set $j_n = n - l_n \geq 1$ and y_n such that:

$$a_n = y_n \, c_{j_n} \, \ell \mathfrak{b}^{l_n}, \tag{82}$$

so that $y_n \in [1/c_0, \mathfrak{b}c_{j_n-1}/c_0 c_{j_n})$. Notice that the conditions $\lim_{n\to\infty} a_n/c_n = +\infty$ and $a_n < \ell\mathfrak{b}^n$ imply that $\lim_{n\to\infty} l_n = +\infty$. The sequence $(j_n, n \in \mathbb{N}^*)$ may be bounded or not.

As $c_{r+1}/c_r < \mathfrak{b}$ for all $r \in \mathbb{N}$, we deduce that $y_n < \mathfrak{b}c_{j_n-1}/c_0 c_{j_n} < \mathfrak{b}^{j_n}/c_{j_n}$. Thus, we can define $\tilde{u}^*_{n,\ell} = \tilde{g}_{j_n}(y_n)$ and $\tilde{\sigma}^2_{n,\ell} = \tilde{\psi}''_{j_n}(\tilde{u}^*_{n,\ell}) > 0$.

Lemma 11.5. *Let p be a non-degenerate super-critical offspring distribution with $\mathfrak{b} < \infty$ and type (L_0, r_0). Let $\ell \in \mathbb{N}^*$. Assume that $\lim_{n\to\infty} a_n/c_n = \infty$ and $\limsup_{n\to\infty} a_n/\ell\mathfrak{b}^n < 1$. Then, we have, with $\lim_{n\to\infty} \tilde{\varepsilon}_{n,\ell} = 0$:*

$$\mathbb{P}_\ell(Z_n = a_n) = \frac{L_0 \, p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)}}{c_{j_n}\sqrt{2\pi \, \ell\mathfrak{b}^{l_n} \, \tilde{\sigma}^2_{n,\ell}}} \exp\left\{\ell\mathfrak{b}^{l_n}(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) + \tilde{u}^*_{n,\ell} y_n)\right\} (1 + \tilde{\varepsilon}_{n,\ell}) \mathbf{1}_{\{a_n = \ell r_0^n \, (\mathrm{mod} \; L_0)\}}.$$

The proof, detailed in Section 11.8 is in the spirit of the proof of (175) in [18]. We end this section with the following strong ratio limit.

Lemma 11.6. *Let p be a non-degenerate super-critical offspring distribution with $\mathfrak{b} < \infty$ and type (L_0, r_0). Let $\ell \in \mathbb{N}^*$. Assume that $\lim_{n\to\infty} a_n/c_n = \infty$, $\limsup_{n\to\infty} a_n/\ell\mathfrak{b}^n < 1$, and $a_n = \ell r_0^n (\mathrm{mod} \; L_0)$ for all $n \in \mathbb{N}^*$. Then, we have:*

$$\lim_{n\to\infty} \frac{\mathbb{P}_{\ell\mathfrak{b}^h}(Z_{n-h} = a_n)}{\mathbb{P}_\ell(Z_n = a_n)} = p(\mathfrak{b})^{-(\mathfrak{b}^h - 1)\ell/(\mathfrak{b}-1)}. \tag{83}$$

Proof. Let $\ell \in \mathbb{N}^*$. Assume that $a_n \in [\ell c_n/c_0, \ell\mathfrak{b}^n)$ and $a_n = \ell r_0^n (\mathrm{mod} \; L_0)$ for all $n \in \mathbb{N}^*$ and $\limsup_{n\to\infty} a_n/\ell\mathfrak{b}^n < 1$. An estimation of $\mathbb{P}_\ell(Z_n = a_n)$ is given in Lemma 11.5. We now give an estimation of $\mathbb{P}_{\ell'}(Z_{n'} = a_n)$ with $n' = n - h$ for some $h \in \mathbb{N}^*$ and $\ell' = \mathfrak{b}^h \ell$. Recall (82) and the definition of l_n, j_n and y_n. We have:

$$a_n = y'_n c_{j'_n} \, \ell' \mathfrak{b}^{l'_n} = y'_n c_{j'_n} \, \ell \mathfrak{b}^{l'_n + h},$$

with $j'_n + l'_n = n' = n - h$ and $l'_n = \sup\{l \in \{0, \dots, n' = n - h\}, c_{n-h-l}\ell\mathfrak{b}^{l+h} \leq c_0 a_n\}$. From the definition of l'_n, we deduce that $l'_n = l_n - h$ so that $\ell'\mathfrak{b}^{l'_n} = \ell\mathfrak{b}^{l_n}$, $j'_n = j_n$ and thus $y'_n = y_n$. This gives that $\tilde{g}_{j_n}(y_n) = \tilde{g}_{j'_n}(y'_n)$ and thus $\tilde{u}^*_{n',\ell'} = \tilde{u}^*_{n,\ell}$ as well as $\tilde{\sigma}^2_{n',\ell'} = \tilde{\sigma}^2_{n,\ell}$. Thanks to Remark 1.3, we have $\mathbb{P}_{\ell\mathfrak{b}^h}(Z_{n-h} = a_n) > 0$ and $\mathbb{P}_\ell(Z_n = a_n) > 0$ for n large. We deduce (83) from Lemma 11.5. $\square$

11.5. **Proof of Lemma 11.3.** Let $\ell \in \mathbb{N}^*$ be fixed. We deduce from Lemma 10.3 and the Fourier inversion formula for $xw^{*\ell}(x)$ that for $x > 0$, $v \in \mathbb{R}$:

$$w^{*\ell}(x) = -\frac{i\ell}{2\pi x} \int_{\mathbb{R}} \tilde{\varphi}'(v + is) \, (\tilde{\varphi}(v + is) - \mathfrak{c})^{\ell-1} \, \mathrm{e}^{-(v+is)x} \, ds. \tag{84}$$

We now follow closely the proofs from (120) to (148) of [18]. Recall notations for $\tilde{r}(x)$ and $\tilde{y}(x)$ given in (76). Using (53) and (67), we get with $r = \tilde{r}(x)$, $y = \tilde{y}(x)$ and setting $u = \mu^{-r} v$

and $t = \mu^{-r}s$:

$$
\begin{aligned}
w^{*\ell}(x) &= -\frac{i\ell\mu^{-r}}{2\pi x}\int_{\mathbb{R}} \tilde{\varphi}'\left(\frac{v+is}{\mu^r}\right) f_r'\left(\tilde{\varphi}\left(\frac{v+is}{\mu^r}\right)\right)\left(f_r\left(\tilde{\varphi}\left(\frac{v+is}{\mu^r}\right)\right)-\mathfrak{c}\right)^{\ell-1} \mathrm{e}^{-(v+is)x}\, ds\\
&= -\frac{i\ell}{2\pi x}\int_{\mathbb{R}} H(u+it)\, dt, \qquad (85)
\end{aligned}
$$

where

$$
H(z) = \tilde{\varphi}'(z)\, f_r'(\tilde{\varphi}(z))\, (f_r(\tilde{\varphi}(z)) - \mathfrak{c})^{\ell-1}\, \mathrm{e}^{-z\mathfrak{b}^r y}\,. \qquad (86)
$$

Since $\tilde{\varphi}(z) - \mathfrak{c} = \mathbb{E}[\mathrm{e}^{zW}\, \mathbf{1}_{\{W>0\}}]$, we deduce that $|\tilde{\varphi}(z) - \mathfrak{c}| \leq \tilde{\varphi}(\Re(z)) - \mathfrak{c}$. The Stevastyanov transformation of the generating function f given by $\bar{f}(z) = [f(\mathfrak{c} + (1-\mathfrak{c})z) - \mathfrak{c}]/[1-\mathfrak{c}]$ is a generating function, and the r-th iterate of $\bar{f}$ is $\bar{f}_r(z) = [f_r(\mathfrak{c} + (1-\mathfrak{c})z) - \mathfrak{c}]/[1-\mathfrak{c}]$. Since $\bar{f}_r$ is a generating function, we get that $|\bar{f}_r(z)| \leq \bar{f}_r(|z|)$ and thus $|f_r(\mathfrak{c}+z) - \mathfrak{c}| \leq f_r(\mathfrak{c}+|z|) - \mathfrak{c}$. Using this last equality with z replaced by $\tilde{\varphi}(z) - \mathfrak{c}$, we get that:

$$
|f_r(\tilde{\varphi}(z)) - \mathfrak{c}| \leq f_r(\mathfrak{c} + |\tilde{\varphi}(z) - \mathfrak{c}|) - \mathfrak{c} \leq f_r(\tilde{\varphi}(\Re(z))) - \mathfrak{c} \leq f_r(\tilde{\varphi}(\Re(z))). \qquad (87)
$$

Since $|f_r'(z)| \leq f_r'(|z|) \leq \mathfrak{b}^r f_r(|z|)/|z|$, we get:

$$
|H(z)| \leq \frac{1}{\tilde{\varphi}(\Re(z))}\left|\tilde{\varphi}'(z)\right| \mathfrak{b}^r f_r\left(\tilde{\varphi}(\Re(z))\right)^{\ell} \mathrm{e}^{-\Re(z)\mathfrak{b}^r y}\,.
$$

Since $\tilde{\varphi}(u) \geq 1$ and $C := \sup_{u\leq u_1} \int |\tilde{\varphi}'(u+it)|\, dt < +\infty$, thanks to (52), we deduce that:

$$
\int_{\mathbb{R}} H(u+it)\, dt \leq C\mathfrak{b}^r f_r\left(\tilde{\varphi}(u)\right)^{\ell} \mathrm{e}^{-u\mathfrak{b}^r y}\,.
$$

Then use (85) to conclude.

11.6. **Proof of (78) in Lemma 11.4.** We keep notations from Section 11.5. Set $u_0 = \tilde{g}(1/\ell)$, $u_1 = \tilde{g}(\mathfrak{b}/(\ell\mu))$ and $K = [u_0, u_1]$. Since $u_0 > 0$, we have $\tilde{\varphi}(u_0) > 1$. Let $\delta \in (0,1)$ be such that $1 + \delta < \tilde{\varphi}(u_0) < \tilde{\varphi}(u_1) < \delta^{-1}$. From the continuity of $\tilde{\varphi}$ on $\mathbb{C}$, and the fact that $\tilde{\varphi}(K) \subset \tilde{\mathcal{D}}(\delta)$, we deduce there exists $t_0 > 0$ such that for all $(u,t) \in K' := K \times [-t_0, t_0]$, we have $\tilde{\varphi}(u+it) \in \tilde{\mathcal{D}}(\delta)$, and thus $\tilde{\psi}$ is analytic on an open neighborhood of $\{u+it;\ (u,t) \in K'\}$. Since $\tilde{\psi}(u) > 0$ and $\tilde{\psi}''(u) > 0$ for $u > 0$, we can take t_0 small enough so that $\Re(\tilde{\psi}(u+it)) > 0$ for $(u,t) \in K'$ and:

$$
t_0 \sup_{(v,s)\in K'} |\tilde{\psi}'''(v+is)| \leq \inf_{v\in K} \tilde{\psi}''(v). \qquad (88)
$$

Recall H defined in (86). We shall study the asymptotics of $\int_{\mathbb{R}} H(u+it)\, dt$ for large x. Condition (88) will be used later on to study the main part of $\int_{|t|\leq t_0} H(u+it)\, dt$.

First step: the tail part. We first consider the tail part:

$$
I(t_0) = \left|\int_{|t|\geq t_0} H(u+it)\, dt\right|.
$$

As $\tilde{\varphi}(u_0) > 1$, we can take η small enough so that $(1-\eta)\tilde{\varphi}(u_0) + \eta\mathfrak{c} > 1$ and (49) holds on $\mathcal{A} = \{(u,t);\ u \in K \text{ and } |t| \geq t_0\}$. Using the first inequality in (87), we get for all $(u,t) \in \mathcal{A}$:

$$
|f_r(\tilde{\varphi}(u+it)) - \mathfrak{c}| \leq f_r((1-\eta)\tilde{\varphi}(u)) - \mathfrak{c} \leq f_r((1-\eta)\tilde{\varphi}(u)).
$$

We get for all $(u,t) \in \mathcal{A}$ that $|\tilde{\varphi}(u+it)| \leq |\tilde{\varphi}(u+it) - \mathfrak{c}| + \mathfrak{c} \leq (1-\eta)\tilde{\varphi}(u)$ and, using $|f_r'(z)| \leq f_r'(|z|) \leq \mathfrak{b}^r f_r(|z|)/|z|$, that:

$$\left|f_r'(\tilde{\varphi}(u+it))\right| \leq \left|f_r'\left((1-\eta)\tilde{\varphi}(u)\right)\right| \leq \mathfrak{b}^r \frac{f_r((1-\eta)\tilde{\varphi}(u))}{(1-\eta)\tilde{\varphi}(u)}.$$

Using (86) and then Lemma 11.2, we deduce that for all $(u,t) \in \mathcal{A}$:

$$\begin{aligned} |H(u+it)| &\leq \frac{1}{(1-\eta)\tilde{\varphi}(u)} \left|\tilde{\varphi}'(u+it)\right| \mathfrak{b}^r f_r\left((1-\eta)\tilde{\varphi}(u)\right)^\ell \mathrm{e}^{-u\mathfrak{b}^r y} \\ &\leq \frac{1}{(1-\eta)\tilde{\varphi}(u_0)} \left|\tilde{\varphi}'(u+it)\right| \mathfrak{b}^r p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)} \mathrm{e}^{\ell\mathfrak{b}^r \tilde{b}((1-\eta)\tilde{\varphi}(u)) - u\mathfrak{b}^r y}. \end{aligned}$$

Since $\tilde{b}$ is increasing, there exists $\varepsilon' > 0$ (depending on u_0, u_1 and t_0) such that for $u \in K$,

$$\tilde{b}\left((1-\eta)\tilde{\varphi}(u)\right) \leq \tilde{b}(\tilde{\varphi}(u)) - \varepsilon' = \tilde{\psi}(u) - \varepsilon'.$$

We get that for all $(u,t) \in \mathcal{A}$:

$$|H(u+it)| \leq \frac{p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)}}{(1-\eta')\tilde{\varphi}(u_0)} \left|\tilde{\varphi}'(u+it)\right| \mathfrak{b}^r \mathrm{e}^{\ell\mathfrak{b}^r\tilde{\psi}(u) - u\mathfrak{b}^r y - \ell\mathfrak{b}^r\varepsilon'}.$$

Using (52) in Lemma 10.3, we get, for some finite constant c (depending on u_0, u_1, t_1 and ℓ), that for all $u \in K$ and $x > 0$:

$$I(t_0) \leq c\, \mathfrak{b}^r \mathrm{e}^{\ell\mathfrak{b}^r\tilde{\psi}(u) - u\mathfrak{b}^r y - \ell\mathfrak{b}^r\varepsilon'}. \tag{89}$$

Second step: the main part. We now consider the main part $J(t_0) = \int_{|t|\leq t_0} H(u+it)\, dt$. An integration by part gives:

$$J(t_0) = \frac{1}{\ell}\left[\left(f_r\left(\tilde{\varphi}(u+it)\right) - \mathfrak{c}\right)^\ell \mathrm{e}^{-(u+it)y\mathfrak{b}^r}\right]_{t=-t_0}^{t=t_0} + \frac{iy\mathfrak{b}^r}{\ell} J_1(t_0),$$

with

$$J_1(t_0) = \int_{[\pm t_0]} \left(f_r\left(\tilde{\varphi}(u+it)\right) - \mathfrak{c}\right)^\ell \mathrm{e}^{-(u+it)y\mathfrak{b}^r}\, dt.$$

Arguing as in the first step, we get:

$$\left|J(t_0) - \frac{iy\mathfrak{b}^r}{\ell} J_1(t_0)\right| \leq \frac{p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)}}{\ell} \mathrm{e}^{\ell\mathfrak{b}^r\tilde{\psi}(u) - u\mathfrak{b}^r y - \ell\mathfrak{b}^r\varepsilon'}. \tag{90}$$

Now $J_1(t_0)$ is handled as in [18] from (128) to (139). By definition of δ and t_0, we get that $\tilde{\varphi}(u+it) \in \tilde{\mathcal{D}}(\delta)$ for $(u,t) \in K'$. Use (73), $\mathfrak{R}(\tilde{\psi}(u+it)) > 0$ for $(u,t) \in K'$ and that $\lim_{r\to+\infty} |f_r(z)| = +\infty$ on $\tilde{\mathcal{D}}(\delta)$, to get there exists $\varepsilon > 0$ such that, uniformly in $u \in K$:

$$J_1(t_0) = p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)} \left(1 + O(\mathrm{e}^{-\varepsilon\mathfrak{b}^r})\right) D(u), \tag{91}$$

with

$$D(u) = \int_{-t_0}^{t_0} \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(u+it) - (u+it)y\right)}\, dt.$$

We have for $(u,t) \in K'$:

$$\tilde{\psi}(u+it) = \tilde{\psi}(u) + it\tilde{\psi}'(u) - \frac{t^2}{2}\tilde{\psi}''(u) + h(t,u),$$

with $|h(t,u)| \leq t^3 C_3^+/6$, and $C_3^+ = \sup_{(v,s)\in K'} |\tilde{\psi}'''(v+is)| < +\infty$. Let $C_2^- = \inf_{v\in K} |\tilde{\psi}''(v)|$ which is a positive constant as $\tilde{\psi}$ is increasing and strictly convex on $(0,+\infty)$. Recall that by definition of t_0, see (88), we have $t_0 C_3^+ \leq C_2^-$.

We define $\tilde{u}^*_\ell$ as $\tilde{g}(y/\ell)$, so that $\tilde{u}^*_\ell \in [u_0, u_1]$ and we set $\tilde{\sigma}^2_\ell = \tilde{\psi}''(\tilde{u}^*_\ell)$. We get:

$$\ell\tilde{\psi}(\tilde{u}^*_\ell + it) - (\tilde{u}^*_\ell + it)y = \ell\tilde{\psi}(\tilde{u}^*_\ell) - \tilde{u}^*_\ell y - \frac{t^2}{2}\ell\tilde{\sigma}^2_\ell + \ell h(t, \tilde{u}^*_\ell),$$

with $|h(t, \tilde{u}^*_\ell)| \leq t^3 C_3^+/6$ and $|h(t, \tilde{u}^*_\ell)| \leq t^2\tilde{\sigma}^2_\ell/6$ for $t \in [-t_0, t_0]$. For x large enough (and thus r large enough), we consider the decomposition $D(\tilde{u}^*_\ell) = D_1 + D_2$ with:

$$D_1 = \int_{-r\mathfrak{b}^{-r}}^{r\mathfrak{b}^{-r}} \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell+it)-(\tilde{u}^*_\ell+it)y\right)}\, dt.$$

Using that $|h(t, \tilde{u}^*_\ell)| \leq r^3\mathfrak{b}^{-3r/2}C_3^+/6$ for $|t| \leq r\mathfrak{b}^{-r/2}$, we get with $s = \sqrt{\ell\mathfrak{b}^r\tilde{\sigma}^2_\ell}\, t$:

$$\begin{aligned}
D_1 &= \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell)-\tilde{u}^*_\ell y\right)} \int_{-r\mathfrak{b}^{-r/2}}^{r\mathfrak{b}^{-r/2}} \mathrm{e}^{-\ell\mathfrak{b}^r\tilde{\sigma}^2_\ell t^2/2+\ell\mathfrak{b}^r h(t,\tilde{u}^*_\ell)}\, dt \\
&= \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell)-\tilde{u}^*_\ell y\right)} \int_{-r\mathfrak{b}^{-r/2}}^{r\mathfrak{b}^{-r/2}} \mathrm{e}^{-\ell\mathfrak{b}^r\tilde{\sigma}^2_\ell t^2/2}\, dt\, \left(1 + O(r^3\mathfrak{b}^{-r/2})\right) \\
&= \frac{1}{\sqrt{\ell\mathfrak{b}^r\tilde{\sigma}^2_\ell}}\, \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell)-\tilde{u}^*_\ell y\right)} \int_{-r\tilde{\sigma}_\ell\sqrt{\ell}}^{r\tilde{\sigma}_\ell\sqrt{\ell}} \mathrm{e}^{-s^2/2}\, ds\, \left(1 + O(r^3\mathfrak{b}^{-r/2})\right) \\
&= \mathcal{I} \times \left(1 + O(r^3\mathfrak{b}^{-r/2})\right),
\end{aligned}$$

with

$$\mathcal{I} = \frac{\sqrt{2\pi}}{\sqrt{\ell\mathfrak{b}^r\tilde{\sigma}^2_\ell}} \exp\left\{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell) - \tilde{u}^*_\ell y\right)\right\}.$$

We now give an upper bound on $|D_2|$. Since $|h(t, \tilde{u}^*_\ell)| \leq t^2\tilde{\sigma}^2_\ell/6$, we deduce that for $t \in [-t_0, t_0]$:

$$\Re\left(\ell\tilde{\psi}(\tilde{u}^*_\ell + it) - (\tilde{u}^*_\ell + it)y\right) \leq \ell\tilde{\psi}(\tilde{u}^*_\ell) - u^*_\ell y - \ell\frac{t^2}{3}\tilde{\sigma}^2_\ell.$$

This implies that:

$$\begin{aligned}
|D_2| &\leq \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell)-\tilde{u}^*_\ell y\right)} \int_{|t|\in[r\mathfrak{b}^{-r/2},t_0]} \mathrm{e}^{-\ell\mathfrak{b}^r\tilde{\sigma}^2_\ell t^2/3}\, dt \\
&\leq 2t_0\, \mathrm{e}^{\mathfrak{b}^r\left(\ell\tilde{\psi}(\tilde{u}^*_\ell)-\tilde{u}^*_\ell y\right)}\, \mathrm{e}^{-\ell r^2\tilde{\sigma}^2_\ell/3} \\
&= \mathcal{I} \times O(r^3\mathfrak{b}^{-r/2}).
\end{aligned}$$

This gives that $D(\tilde{u}^*_\ell) = \mathcal{I} \times \left(1 + O(r^3\mathfrak{b}^{-r/2})\right)$. Use (91), (89), (90) to get that:

$$\int_{\mathbb{R}} H(\tilde{u}^*_\ell + it)\, dt = \frac{iy\mathfrak{b}^r}{\ell}\, p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)}\, \mathcal{I} \times \left(1 + O(r^3\mathfrak{b}^{-r/2})\right).$$

Then use (85), the definition of $\tilde{u}^*_\ell$, which implies that $\tilde{u}^*_\ell(y/\ell) - \tilde{\psi}(\tilde{u}^*_\ell) = \max_{u\geq 0}((uy/\ell) - \tilde{\psi}(u)) = (y/\ell)^{\beta_H/(\beta_H-1)}\tilde{M}(y/\ell)$ with $y = y(x)$, and then the periodicity of $\tilde{M}$ to conclude.

11.7. **Proof of (79) and (80) in Lemma 11.4.** From (19), we get $w_\ell(x) = w^{*\ell}(x) + R(x)$, with:

$$R(x) = \sum_{j=1}^{\ell-1} \binom{\ell}{j} \mathfrak{c}^{\ell-j} w^{*j}(x).$$

Using (77) and then Lemma 11.2, we deduce there exits a finite constant c such that for all $x \geq \mathfrak{b}/\mu$ and $u \in K$:

$$R(x) \leq \frac{c}{x} \mathfrak{b}^r \,\mathrm{e}^{-uy\mathfrak{b}^r} f_r(\tilde{\varphi}(u))^{\ell-1} \leq \frac{c}{x} p(\mathfrak{b})^{-(\ell-1)/(\mathfrak{b}-1)} \mathfrak{b}^r \,\mathrm{e}^{(\ell-1)\mathfrak{b}^r \tilde{\psi}(u) - uy\mathfrak{b}^r}.$$

Taking $u = \tilde{u}^*_\ell$ and $\mathcal{I}$ defined in Section 11.6, we get that $R(x) = \mathcal{I} \times O(\mathrm{e}^{-\mathfrak{b}^r \tilde{\psi}(\tilde{u}^*_\ell)/2}) = o(w^{*\ell}(x))$. This implies that $w_\ell(x) \sim w^{*\ell}(x)$ as x goes to infinity. This gives (79).

An exact computation using (19) and (84) leads to:

$$w_\ell(x) = -\frac{i\ell\mathfrak{c}}{2\pi x} \int_{\mathbb{R}} \tilde{\varphi}'(v+is)\, \tilde{\varphi}(v+is)^{\ell-1} \,\mathrm{e}^{-(v+is)x} \; ds.$$

By definition, we have $\mathbb{P}_\ell(W \geq x) = \int_x^{+\infty} w_\ell(x')\, dx'$. Arguing as in Section 11.6, with in particular the integration by part (in s) for the main part, it is easy to get that:

$$\mathbb{P}_\ell(W \geq x) \sim -\frac{i\ell\mathfrak{c}}{2\pi x} \int_{\mathbb{R}} \frac{\tilde{\varphi}'(v+is)}{v+is}\, \tilde{\varphi}(v+is)^{\ell-1} \,\mathrm{e}^{-(v+is)x} \; ds$$

as well as (80). The details are left to the reader.

11.8. **Proof of Lemma 11.5.** Recall $a_n = y_n c_{j_n} \ell \mathfrak{b}^{l_n} > 0$. Using Fourier inversion formula, we have for $v > 0$:

$$\begin{aligned}
\mathbb{P}_\ell(Z_n = a_n) &= \frac{L_0}{2\pi} \int_{[\pm\pi/L_0]} f_n\left(\mathrm{e}^{v+is}\right)^\ell \mathrm{e}^{-(v+is)a_n} \; ds \\
&= \frac{L_0}{2\pi} \int_{[\pm\pi/L_0]} \left(f_n\left(\mathrm{e}^{v+is}\right)^\ell - \mathfrak{c}^\ell\right) \mathrm{e}^{-(v+is)a_n} \; ds
\end{aligned}$$

since either $\mathfrak{a} \geq 1$ and thus $\mathfrak{c} = 0$, or $\mathfrak{a} = 0$ and $a_n = 0 \pmod{L_0}$. Setting $v = u/c_{j_n} > 0$, $s = t/c_{j_n}$ and $H_{l,j}(z) = f_l\left(\tilde{\varphi}_j(z)\right)^\ell - \mathfrak{c}^\ell$, we get using $l_n + j_n = n$:

$$\mathbb{P}_\ell(Z_n = a_n) = \frac{L_0}{2\pi c_{j_n}} \int_{[\pm c_{j_n}\pi/L_0]} H_{l_n,j_n}(u+it)\, \mathrm{e}^{-(u+it)y_n \ell \mathfrak{b}^{l_n}} \; dt. \tag{92}$$

We now explicit the range of the possible choice for u we shall consider. Without loss of generality, we can assume that there exists $\delta_0 > 0$ such that $\sup_{n\in\mathbb{N}^*} a_n/\ell\mathfrak{b}^n < 1 - \delta_0$. The restriction to $\mathbb{R}$ of the domain of definition of $\tilde{g}_j$ is $D_j = (0, \mathfrak{b}^j/c_j)$. Set $F_j = [1/c_0, \mathfrak{b}c_{j-1}/c_0 c_j]$ for $j \geq 2$ and $F_1 = [1/c_0, (1-\delta_0)\mathfrak{b}/c_1]$. From the uniform convergence of $\tilde{g}_j$ towards $\tilde{g}$ on compact sets of $(0,\infty)$ and the fact that $F_j \subset D_j$ for all $j \in \mathbb{N}^*$ and $\bigcup_{j\in\mathbb{N}^*} F_j \subset [1/c_0, \mathfrak{b}/c_1]$, we deduce that there exists $0 < u_0 < u_1 < +\infty$ such that for all $j \in \mathbb{N}^*$ and all $y \in F_j$, we have $\tilde{g}_j(y) \in K := [u_0, u_1]$. Since $y_n \in F_{j_n}$, we deduce that the sequence $(\tilde{u}^*_{n,\ell}, n \in \mathbb{N}^*)$ belongs to K.

11.8.1. *Preliminary upper bounds.* Using the continuity of $\tilde{\varphi}_j$ and their uniform convergence towards $\tilde{\varphi}$ as j goes to infinity, we get that there exists $t_0 > 0$, $\delta \in (0,1)$ such that for all $(u,t) \in K' := K \times [-t_0, t_0]$ and $j \in \mathbb{N}^*$, we have $\tilde{\varphi}_j(u+it) \in \tilde{\mathcal{D}}(\delta)$ and $m_0 = \inf\{\mathfrak{R}(\tilde{\psi}_j(u+it);\ (u,t) \in K', j \in \mathbb{N}^*\} > 0$. We set $\tilde{C}_3^+ = \sup_{j\in\mathbb{N}^*} \sup_{(u,t)\in K'} |\tilde{\psi}_j'''(u+it)|$ which is a finite constant since the derivative of $\tilde{\psi}_j$ converges uniformly on K' towards the derivative of $\tilde{\psi}$. Let $\tilde{C}_2^- = \inf_{j\in\mathbb{N}^*} \inf_{u\in K} |\tilde{\psi}_j''(u)|$ which is a positive constant since the derivative of $\tilde{\psi}_j$ converges uniformly on K towards the derivative of $\tilde{\psi}$ and that $\tilde{\psi}_j$ as well as $\tilde{\psi}$ are increasing and strictly convex on $(0,+\infty)$. Taking a smaller t_0 if necessary, we can assume that:

$$t_0 \tilde{C}_3^+ \leq \tilde{C}_2^-. \tag{93}$$

We deduce from (73) and the definition of $\tilde{\psi}_j$ and $\tilde{\varphi}_j$ that there exits $\varepsilon > 0$ and a finite constant C such that for all $l, j \in \mathbb{N}^*$, $(u,t) \in K'$:

$$f_l(\tilde{\varphi}_j(u+it)) = p(\mathfrak{b})^{-1/(\mathfrak{b}-1)} \, \mathrm{e}^{\mathfrak{b}^l \tilde{\psi}_j(u+it)} \, (1 + R(u,t,l,j))$$

and $\sup_{(u,t)\in K', j\in\mathbb{N}^*} |R(u,t,l,j)| \leq C\, \mathrm{e}^{-\varepsilon \mathfrak{b}^l}$. Since $m = \inf\{\tilde{\psi}_j(u);\ u \in K, j \in \mathbb{N}^*\} > 0$, taking ε smaller than m if necessary, we get that:

$$H_{l,j}(u+it) = p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)} \, \mathrm{e}^{\ell \mathfrak{b}^l \tilde{\psi}_j(u+it)} \left(1 + R'(u,t,l,j)\right)$$

and $\sup_{(u,t)\in K', j\in\mathbb{N}^*} |R'(u,t,l,j)| \leq C' \, \mathrm{e}^{-\varepsilon \mathfrak{b}^l}$ for some finite constant C'. Since $f_l(\tilde{\varphi}(u)) > 1 > \mathfrak{c}$ for $u > 0$, we deduce from Lemma 11.2 that for all $l, j \in \mathbb{N}^*$, $u \in (0, +\infty)$:

$$0 < H_{l,j}(u) \leq f_l(\tilde{\varphi}_j(u))^\ell \leq p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)} \exp\left\{\ell \mathfrak{b}^l \tilde{\psi}_j(u)\right\}. \tag{94}$$

11.8.2. *The tail part.* We first bound the tail of the integral which appears in (92):

$$I_{l,j}(t_0) = \left| \int_{|t|\in[t_0, c_j\pi/L_0]} H_{l,j}(u+it) \, \mathrm{e}^{-(u+it)y\ell\mathfrak{b}^l} \; dt \right|,$$

where y belongs to $[1/c_0, \mathfrak{b}c_{j-1}/c_0c_j)$. Using an integration by parts, we get:

$$\int_{|t|\in[t_0, c_j\pi/L_0]} H_{l,j}(u+it) \, \mathrm{e}^{-(u+it)y\ell\mathfrak{b}^l} \; dt = I_1^{+1} - I_1^{-1} + I_2,$$

where, for $\epsilon \in \{+1, -1\}$

$$I_1^\epsilon = \left[iH_{j,l}(u+it) \frac{\mathrm{e}^{-(u+it)y\ell\mathfrak{b}^l}}{y\ell\mathfrak{b}^l} \right]_{\delta t_0}^{\delta \frac{c_j \pi}{L_0}} \quad \text{and} \quad I_2 = -i \int_{|t|\in[t_0, c_j\pi/L_0]} H'_{l,j}(u+it) \frac{\mathrm{e}^{-(u+it)y\ell\mathfrak{b}^l}}{y\ell\mathfrak{b}^l} \, dt.$$

Set $\mathcal{A}_j = \{(u,t) \in \mathbb{R}^2;\ u \in K, t_0 \leq |t| \leq c_j\pi/L_0\}$. According to (57), there exists $\delta \in (0,1)$ such that for all $j \in \mathbb{N}^*$ and $(u,t) \in \mathcal{A}_j$:

$$|\tilde{\varphi}_j(u+it)| \leq (1-\delta)\tilde{\varphi}_j(u). \tag{95}$$

Taking δ small enough, we can assume that $m_1 = \inf\{(1-\delta)\tilde{\varphi}_j(u);\ j \in \mathbb{N}^*, u \in K\} > 1$. We have $H_{l,j}(z) = g(1) - g(0) = \int_0^1 g'(s)\, ds$, with $g(s) = f_l(s\tilde{\varphi}_j(z) + (1-s)\mathfrak{c})^\ell$. We get:

$$|g'(s)| \leq |\tilde{\varphi}_j(z) - \mathfrak{c}| \, \ell f_l(s(1-\delta)\tilde{\varphi}_j(u) + (1-s)\mathfrak{c})^{\ell-1} f_l'(s(1-\delta)\tilde{\varphi}_j(u) + (1-s)\mathfrak{c}).$$

We deduce that for all $l, j \in \mathbb{N}^*$ and $z = u + it$ with $(u,t) \in \mathcal{A}_j$:

$$|H_{l,j}(z)| \leq |\tilde{\varphi}_j(z) - \mathfrak{c}| \, \frac{f_l((1-\delta)\tilde{\varphi}_j(u))^\ell - \mathfrak{c}^\ell}{1-\mathfrak{c}} \leq 2\tilde{\varphi}_j(u) \frac{f_l((1-\delta)\tilde{\varphi}_j(u))^\ell}{1-\mathfrak{c}}.$$

Using Lemma 11.2, we get there exists a constant C such that for all $l, j \in \mathbb{N}^*$ and $(u,t) \in \mathcal{A}_j$:

$$|H_{l,j}(z)| \leq C \exp\left\{\ell \mathfrak{b}^l \tilde{b}((1-\delta)\tilde{\varphi}_j(u))\right\}.$$

Using that $\tilde{b}$ is analytic and increasing on $(1, +\infty)$ and $m_1 > 1$, we deduce that there exists $\varepsilon' > 0$ such that for all $j \in \mathbb{N}^*$, $u \in K$:

$$\tilde{b}((1-\delta)\tilde{\varphi}_j(u)) \leq \tilde{\psi}_j(u) - \varepsilon'.$$

We deduce that for all $l, j \in \mathbb{N}^*$, $(u,t) \in \mathcal{A}_j$:

$$|H_{l,j}(u+it)|\, dt \leq C \exp\left\{\ell \mathfrak{b}^l \tilde{\psi}(u) - \ell \mathfrak{b}^l \varepsilon'\right\}.$$

This gives that for all $u \in K$, $l, j \in \mathbb{N}^*$:

$$|I_1^{\pm 1}| \leq \frac{2C}{y \ell \mathfrak{b}^l} \, \mathrm{e}^{\ell \mathfrak{b}^l (\tilde{\psi}(u) - uy) - \ell \mathfrak{b}^l \varepsilon'}. \tag{96}$$

We have $H'_{l,j}(z) = \ell \tilde{\varphi}'_j(z) f'_l(\tilde{\varphi}_j(z))\, f_l(\tilde{\varphi}_j(z))^{\ell-1}$. For $(u,t) \in \mathcal{A}_j$, we have using (56), (95) and $f'_l(|z|) \leq \mathfrak{b}^l f_l(|z|)/|z|$:

$$|H'_{l,j}(u+it)| \leq \frac{\ell}{m_1} \tilde{|}\varphi'_j(u+it)| \mathfrak{b}^l f_l\left((1-\delta)\tilde{\varphi}_j(u)\right)^{\ell}.$$

Arguing as in the upper bound on $I_1^{\pm}$, we get there exists a finite constant C such that for all $l, j \in \mathbb{N}^*$, $u \in K$

$$|I_2| \leq \frac{C}{y} \, \mathrm{e}^{\ell \mathfrak{b}^l (\tilde{\psi}(u) - uy) - \ell \mathfrak{b}^l \varepsilon'} \int_{[\pm c_j \pi / L_0]} |\tilde{\varphi}'_j(u+it)|\, dt.$$

Then use (59), to conclude that $|I_2| \leq (C/y)\, \mathrm{e}^{\ell \mathfrak{b}^l (\tilde{\psi}(u) - uy) - \ell \mathfrak{b}^l \varepsilon'}$ for some finite constant C. This and (96) give there exists a finite constant C such that for all $l, j \in \mathbb{N}^*$, $u \in K$:

$$I_{l,j}(t_0) \leq \frac{C}{y} \, \mathrm{e}^{\ell \mathfrak{b}^l (\tilde{\psi}(u) - uy) - \ell \mathfrak{b}^l \varepsilon'}. \tag{97}$$

11.8.3. *The main part.* The main part is handled as in [18] from (168) to (172), see also [16]. For $(u,t) \in K'$, we have $\tilde{\varphi}_j(u+it) \in \tilde{\mathcal{D}}(\delta)$ and we deduce from (73), that there exists $\varepsilon > 0$ such that for $(u,t) \in K'$, $l, j \in \mathbb{N}^*$:

$$\int_{-t_0}^{t_0} H_{l,j}(u+it)\, \mathrm{e}^{-(u+it) y \ell \mathfrak{b}^l}\, dt = p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)} \left(1 + O(\mathrm{e}^{-\varepsilon \mathfrak{b}^l})\right) D(j,l,u),$$

and

$$D(j,l,u) = \int_{-t_0}^{t_0} \mathrm{e}^{\ell \mathfrak{b}^l \left(\tilde{\psi}_j(u+it) - (u+it) y\right)}\, dt.$$

where $O(\mathrm{e}^{-\varepsilon \mathfrak{b}^l}) = R(u,t,j,l,y)$ and there exists some finite constant C such that for all $l \in \mathbb{N}^*$, we have $\sup_{j \in \mathbb{N}^*} \sup_{y \in F_j, (u,t) \in K'} |R(u,t,j,l,y)| \leq C\, \mathrm{e}^{-\varepsilon \mathfrak{b}^l}$. We have for $(u,t) \in K'$:

$$\tilde{\psi}_j(u+it) = \tilde{\psi}_j(u) + it \tilde{\psi}'_j(u) - \frac{t^2}{2} \tilde{\psi}''_j(u) + h_j(t,u),$$

with $|h_j(t,u)| \leq t^3 \tilde{C}_3^+/6$, Recall that $\tilde{u}^*_{n,\ell}$ belongs to K. With the definition of $\tilde{u}^*_{n,\ell}$, we get that:

$$\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell} + it) - (u^*_{n,\ell} + it) y_n = \tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - u^*_{n,\ell} y_n - \frac{t^2}{2} \tilde{\sigma}^2_{n,\ell} + h_{j_n}(t, \tilde{u}^*_{n,\ell}),$$

with $\tilde{\sigma}^2_{n,\ell} = \tilde{\psi}''_{j_n}(\tilde{u}^*_{n,\ell})$, $|h_{j_n}(t, \tilde{u}^*_{n,\ell})| \leq t^3 \tilde{C}^+_3/6$. We consider the decomposition $D(j_n, l_n, \tilde{u}^*_{n,\ell}) = D_1 + D_2$ with:

$$D_1 = \int_{-l_n \mathfrak{b}^{-l_n/2}}^{l_n \mathfrak{b}^{-l_n/2}} \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}+it) - (\tilde{u}^*_{n,\ell}+it)y\right)} \; dt.$$

Using that $|h_{j_n}(t, \tilde{u}^*_{n,\ell})| \leq l_n^3 \mathfrak{b}^{-3l_n/2} \tilde{C}^+_3/6$ for $|t| \leq l_n \mathfrak{b}^{-l_n/2}$, we get:

$$\begin{aligned}
D_1 &= \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n\right)} \int_{-l_n \mathfrak{b}^{-l_n/2}}^{l_n \mathfrak{b}^{-l_n/2}} \mathrm{e}^{-\ell \mathfrak{b}^{l_n} \tilde{\sigma}^2_{n,\ell} t^2/2 + \ell \mathfrak{b}^{l_n} h_{j_n}(t, \tilde{u}^*_{n,\ell})} \; dt \\
&= \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n\right)} \int_{-l_n \mathfrak{b}^{-l_n/2}}^{l_n \mathfrak{b}^{-3l_n/2}} \mathrm{e}^{-\ell \mathfrak{b}^{l_n} \tilde{\sigma}^2_{n,\ell} t^2/2} \; dt \; \left(1 + O(l_n^3 \mathfrak{b}^{-3l_n/2})\right) \\
&= \frac{1}{\sqrt{\ell \mathfrak{b}^{l_n} \tilde{\sigma}^2_{n,\ell}}} \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n\right)} \int_{-l_n \tilde{\sigma}_{n,\ell} \sqrt{\ell}}^{l_n \tilde{\sigma}_{n,\ell} \sqrt{\ell}} \mathrm{e}^{-s^2/2} \; ds \; \left(1 + O(l_n^3 \mathfrak{b}^{-3l_n/2})\right) \\
&= \mathcal{I}_n \times \left(1 + O(l_n^3 \mathfrak{b}^{-3l_n/2})\right),
\end{aligned}$$

with

$$\mathcal{I}_n = \frac{\sqrt{2\pi}}{\sqrt{\ell \mathfrak{b}^{l_n} \tilde{\sigma}^2_{n,\ell}}} \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n\right)}.$$

We now give an upper bound for $|D_2|$. Thanks to (93), we have $|h_{j_n}(t, \tilde{u}^*_{n,\ell})| \leq t^2 \tilde{\sigma}^2_{n,\ell}/6$ for $t \in [-t_0, t_0]$. We deduce that for $t \in [-t_0, t_0]$:

$$\Re\left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell} + it) - (\tilde{u}^*_{n,\ell} + it) y_n\right) \leq \tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n - \frac{t^2}{3} \tilde{\sigma}^2_{n,\ell}.$$

This implies that:

$$\begin{aligned}
|D_2| &\leq \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n\right)} \int_{|t| \in [l_n \mathfrak{b}^{-l_n/2}, t_0]} \mathrm{e}^{-\ell \mathfrak{b}^{l_n} \tilde{\sigma}^2_{n,\ell} t^2/3} \; dt \\
&\leq 2t_0 \, \mathrm{e}^{\ell \mathfrak{b}^{l_n} \left(\tilde{\psi}_{j_n}(\tilde{u}^*_{n,\ell}) - \tilde{u}^*_{n,\ell} y_n\right)} \, \mathrm{e}^{-\ell l_n^2 \tilde{\sigma}^2_{n,\ell}/3} \\
&= \mathcal{I}_n \times O(l_n^3 \mathfrak{b}^{-3l_n/2}).
\end{aligned}$$

11.8.4. *Conclusion.* To conclude, we deduce from (97) with $y = y_n$ that:

$$\int_{|t| \in [t_0, c_{j_n}\pi/L_0]} |H_{l_n, j_n}(\tilde{u}^*_{n,\ell} + it) \, \mathrm{e}^{-(\tilde{u}^*_{n,\ell} + it) y_n \ell \mathfrak{b}^{l_n}}| \, dt = \mathcal{I}_n \times O(\mathrm{e}^{-\varepsilon \mathfrak{b}^{l_n}/2})$$

This implies that:

$$\int_{-\frac{c_{j_n}\pi}{L_0}}^{\frac{c_{j_n}\pi}{L_0}} H_{l_n, j_n}(\tilde{u}^*_{n,\ell} + it) \, \mathrm{e}^{-(\tilde{u}^*_{n,\ell} + it) y_n \ell \mathfrak{b}^{l_n}} \; dt = p(\mathfrak{b})^{-\ell/(\mathfrak{b}-1)} \mathcal{I}_n \times \left(1 + O(l_n^3 \mathfrak{b}^{-3l_n/2})\right).$$

Then use (92) to conclude.

12. Results in the Bötcher case

We present mostly the results without proof as their correspond either to a slight generalization of [17] and [18] or can be proven by mimicking the proof in the Harris case presented in Section 11. Recall the Böttcher constant $\beta \in (0, 1)$ is defined by $\mathfrak{a} = \mu^\beta$, where $\mathfrak{a}$ is the minimum of the support of p. We assume $\mathfrak{a} \geq 2$.

12.1. **Preliminaries.** We define the function b on its domain which is a subset of $\{z \in \mathbb{C};\, 0 < |z| < 1\}$ by:

$$b(z) = \log(z) + \sum_{n=0}^{\infty} \mathfrak{a}^{-n-1} \log\left(\frac{f_{n+1}(z)}{f_n(z)^{\mathfrak{a}}}\right). \tag{98}$$

According to Lemma 10 in [18], for every $\delta \in (0,1)$, there exists a constant $\theta = \theta(\delta) \in (0,\pi)$ such that b is analytic on the open set:

$$\mathcal{D}(\delta,\theta) = \{z \in \mathbb{C};\, 0 < |z| < 1-\delta,\, |\arg(z)| < \theta\}. \tag{99}$$

On $(0,1)$, the function b is analytic, negative and satisfies $b \circ f = \mathfrak{a} b$. We also have, see Lemma 14 in [18] that:

$$(sb'(s))' > 0 \text{ for } s \in (0,1), \quad \lim_{s \nearrow 1} sb'(s) = +\infty \quad \text{and} \quad \lim_{s \searrow 0} sb'(s) = 1.$$

Recall that φ denotes the Laplace transform of W. We also consider the function $\psi = b \circ \varphi$ defined on $(0,+\infty)$. According to Lemma 17 in [18], the function ψ is analytic on $(0,+\infty)$ strictly decreasing, strictly convex and such that:

$$\lim_{x \to 0+} \psi'(x) = -\infty \quad \text{and} \quad \lim_{x \to +\infty} \psi'(x) = 0.$$

Let g be the inverse of $-\psi'$. In particular, for a given $v > 0$, the minimum of $\psi(u) + uv$ for $u \geq 0$ is uniquely reached at $g(v)$:

$$\min_{u \geq 0} (\psi(u) + uv) = \psi(g(v)) + g(v)v. \tag{100}$$

12.2. **Left tail of w.** We define the function M for $v \in (0,+\infty)$ by:

$$M(v) = -v^{\beta/(1-\beta)} \min_{u \geq 0} (\psi(u) + uv). \tag{101}$$

The function M is analytic on $(0,+\infty)$, see Proposition 3 in [10], positive and multiplicatively periodic with period $\mu^{1-\beta}$. For $x \in (0, \mathfrak{a}/\mu]$, we set:

$$r(x) = \left\lfloor \frac{\log(x)}{\log(\mathfrak{a}/\mu)} \right\rfloor \quad \text{and} \quad y(x) = x \left(\frac{\mu}{\mathfrak{a}}\right)^{r(x)}, \tag{102}$$

so that $y(x) \in (\mathfrak{a}/\mu, 1]$. For $\ell \in \mathbb{N}^*$ and $y > 0$, we define the positive functions:

$$\mathcal{M}_{1,\ell}(y) = \frac{p(\mathfrak{a})^{-\ell/(\mathfrak{a}-1)}}{\sqrt{2\pi\ell\sigma^2(y/\ell)}}\, y^{(2-\beta)/2(1-\beta)} \quad \text{and} \quad \mathcal{M}_{2,\ell}(y) = \mathcal{M}_{1,\ell}(y) \frac{y^{-1/(1-\beta)}}{g(y/\ell)},$$

where $\sigma^2(y) = \psi''(g(y)) > 0$. For $\ell \in \mathbb{N}^*$ and $x \in (0, \mathfrak{a}/\mu]$, we set:

$$M_{1,\ell}(x) = \mathcal{M}_{1,\ell}(y(x)) \quad \text{and} \quad M_{2,\ell}(x) = \mathcal{M}_{2,\ell}(y(x)).$$

By construction $x \mapsto y(x)$ is multiplicative periodic with period $\mathfrak{a}/\mu = \mathfrak{a}^{1-\beta}$. We deduce that $M_{1,\ell}$ and $M_{2,\ell}$ are multiplicative periodic with period $\mathfrak{a}/\mu = \mathfrak{a}^{1-\beta}$, positive, bounded and bounded away from 0.

Let $\mathbb{P}_\ell$ be the distribution of $\sum_{i=1}^{\ell} W_i$, with $(W_i, i \in \mathbb{N}^*)$ independent random variables distributed as W. Since $\mathfrak{a} > 0$ and thus $\mathfrak{c} = 0$, we get that W has density w and that $\sum_{i=1}^{\ell} W_i$ has density $w^{*\ell}$. Mimicking very closely the proof in [18] stated for $\ell = 1$, it is not very difficult to check the following result. The verification is left to the reader.

Lemma 12.1. *Let p be a non-degenerate super-critical offspring distribution with finite mean and $\mathfrak{a} \geq 2$. Let $\ell \in \mathbb{N}^*$. As $x \searrow 0$, we have:*

$$w^{*\ell}(x) \sim w_\ell(x) \sim M_{1,\ell}(x)\, x^{(\beta-2)/2(1-\beta)} \exp\left\{-\ell^{1/(1-\beta)}\, x^{-\beta/(1-\beta)} M(x/\ell)\right\}, \tag{103}$$

$$\mathbb{P}_\ell(W \leq x) \sim M_{2,\ell}(x)\, x^{\beta/2(1-\beta)} \exp\left\{-\ell^{1/(1-\beta)}\, x^{-\beta/(1-\beta)} M(x/\ell)\right\}. \tag{104}$$

Using (118), (119), (122) (with f replaced by f_ℓ), (123) and (78) in [18], we also get the following upper bound, see also Lemma 11.3 in the Harris case.

Corollary 12.2. *Let p be a non-degenerate super-critical offspring distribution with finite mean and $\mathfrak{a} \geq 2$. There exists a finite constant C such that for all $\ell \in \mathbb{N}^*$, $x > 0$ and $u \geq 0$, we have with $r = r(x)$, $y = y(x)$:*

$$w^{*\ell}(x) \leq C\mu^r \frac{\mathrm{e}^{uy\mathfrak{a}^r}}{\varphi(u)}\, f_r(\varphi(u))^\ell. \tag{105}$$

12.3. **Proof of Lemma 4.4 in the Böttcher case.** Mimicking the arguments given in Section 11.3, it is easy, using Corollary 12.2 to get that:

$$\lim_{x\to 0+} \mu \frac{w^{*\mathfrak{a}\ell}(x)}{w^{*\ell}(x/\mu)}\, p(\mathfrak{a})^\ell = 1.$$

From the definition of $\rho_{\theta,\ell}$ in (20), we deduce that $\lim_{\theta\to 0+} \rho_{\theta,\ell}(\mathfrak{a}, \ldots \mathfrak{a}) = 1$. This ends the proof of Lemma 4.4 in the Böttcher case.

12.4. **Lower large deviations for Z_n.** For $j \in \mathbb{N}^*$, let φ_j denote the Laplace transform of $W_j = Z_j/c_j$: $\varphi_j(u) = \mathbb{E}[\mathrm{e}^{-uW_j}] = f_j(\mathrm{e}^{-u/c_j})$ for $u \in \mathbb{C}_+$, where $\mathbb{C}_+ = \{u \in \mathbb{C}, \Re(u) \geq 0\}$. Notice that φ_j converges uniformly on the compacts of $\mathbb{C}_+$ towards φ, the Laplace transform of W, as j goes to infinity. We also have that $\varphi_j'(u)/\varphi_j(u) = -\mathbb{E}[W_j\, \mathrm{e}^{-uW_j}]/\mathbb{E}[\mathrm{e}^{-uW_j}]$ so that $\lim_{u\to+\infty} \varphi_j'(u)/\varphi_j(u) = -\mathfrak{a}^j/c_j$.

We consider the functions $\psi_j = b \circ \varphi_j$ defined on $(0, +\infty)$ for $j \in \mathbb{N}^*$ and the function $\psi = b \circ \varphi$. According to Lemma 17 in [18], the function ψ is analytic on $(0, +\infty)$ strictly decreasing, strictly convex and such that $\lim_{x\to 0+} \psi'(x) = -\infty$ and $\lim_{x\to+\infty} \psi'(x) = 0$. Mimicking the proof of Lemma 17 in [18], it is easy to check that the functions ψ_j are analytic on $(0, +\infty)$ strictly decreasing, strictly convex and such that:

$$\lim_{x\to 0+} \psi_j'(x) = -\infty \quad \text{and} \quad \lim_{x\to+\infty} \psi_j'(x) = -\frac{\mathfrak{a}^j}{c_j}\cdot$$

Let g_j (resp. g) be the inverse of $-\psi_j'$ (resp. $-\psi'$) on $(\mathfrak{a}^j/c_j, +\infty)$ (resp. on $(0, +\infty)$). In particular, for a given $v > \mathfrak{a}^j/c_j$, the minimum of $\psi_j(u) + uv$ for $u \geq 0$ is uniquely reached at $g_j(v)$. Using that ψ_j converges uniformly on compact of $\mathbb{C}_+$ towards ψ, that b and thus ψ_j and ψ are analytic, we get that for any compact of $(0, +\infty)$, the strictly convex functions ψ_j and their derivatives converge uniformly towards the strictly convex function ψ and its derivatives. We deduce that for any compact of $(0, +\infty)$, g_j converges uniformly towards g.

We consider the following general setting. Let $\ell \in \mathbb{N}^*$ and $a_n \in (\ell\mathfrak{a}^n, \ell c_n/c_0]$ such that $\liminf_{n\to\infty} a_n/\ell\mathfrak{a}^n > 1$. Since $\mathfrak{a} < c_{r+1}/c_r < \mu$ for all $r \in \mathbb{N}$, we deduce that the sequence $(c_{n-l}\mathfrak{a}^l,\, 0 \leq l \leq n)$ is decreasing. Therefore, the integer $l_n = \sup\{l \in \{0, \ldots, n\},\, c_{n-l}\ell\mathfrak{a}^l \geq c_0 a_n\}$ is well-defined and strictly less than n. Set $j_n = n - l_n \geq 1$ and

$$a_n = y_n\, c_{j_n}\, \ell\mathfrak{a}^{l_n},$$

with $y_n \in (\mathfrak{a}c_{j_n-1}/c_0c_{j_n}, 1/c_0]$. Notice that the conditions $\lim_{n\to\infty} a_n/c_n = 0$ and $a_n > \ell\mathfrak{a}^n$ imply that $\lim_{n\to\infty} l_n = +\infty$. The sequence $(j_n, n \in \mathbb{N}^*)$ may be bounded or not.

As $\mathfrak{a} < c_{r+1}/c_r$ for all $r \in \mathbb{N}$, we deduce that $y_n > \mathfrak{a}c_{j_n-1}/c_0c_{j_n} > \mathfrak{a}^{j_n}/c_{j_n}$. Thus, we can define $u^*_{n,\ell} = g_{j_n}(y_n)$ and $\sigma^2_{n,\ell} = \psi''_{j_n}(u^*_{n,\ell})$. Mimicking very closely the proof of (175) in [18] (which is stated for $\ell = 1$ and $\lim_{n\to\infty} j_n = \infty$), it is not very difficult to check the following slightly more general result. The verification, which can also be seen as a direct adaptation of the detailed proof of Lemma 11.5, is left to the reader.

Lemma 12.3. *Let p be a non-degenerate super-critical offspring distribution with finite mean, $\mathfrak{a} \geq 2$ and type (L_0, r_0). Let $\ell \in \mathbb{N}^*$. Assume that $\lim_{n\to\infty} a_n/c_n = 0$ and $\liminf_{n\to\infty} a_n/\ell\mathfrak{a}^n > 1$. Then, we have, with $\lim_{n\to\infty} \varepsilon_{n,\ell} = 0$:*

$$\mathbb{P}_\ell(Z_n = a_n) = \frac{L_0\, p(\mathfrak{a})^{-\ell/(\mathfrak{a}-1)}}{c_{j_n}\sqrt{2\pi\, \ell\mathfrak{a}^{l_n}\, \sigma^2_{n,\ell}}} \exp\left\{\ell\mathfrak{a}^{l_n}(\psi_{j_n}(u^*_{n,\ell}) + u^*_{n,\ell}y_n)\right\}(1 + \varepsilon_{n,\ell}(1))\mathbf{1}_{\{a_n = \ell r_0^n (\mathrm{mod}\ L_0)\}}.$$

We end this section with the following strong ratio limit, whose proof is similar to the proof of Lemma 11.6.

Lemma 12.4. *Let p be a non-degenerate super-critical offspring distribution with finite mean and $\mathfrak{a} \geq 2$. Assume that $\lim_{n\to\infty} a_n/c_n = 0$, $\liminf_{n\to\infty} a_n/\ell\mathfrak{a}^n > 1$ and $a_n = \ell r_0^n (\mathrm{mod}\ L_0)$ for all $n \in \mathbb{N}^*$. Then, we have:*

$$\lim_{n\to\infty} \frac{\mathbb{P}_{\ell\mathfrak{a}^h}(Z_{n-h} = a_n)}{\mathbb{P}_\ell(Z_n = a_n)} = p(\mathfrak{a})^{-(\mathfrak{a}^h-1)\ell/(\mathfrak{a}-1)}. \tag{106}$$

12.5. **Proof of Proposition 6.5 in the Böttcher case.** For $h \in \mathbb{N}$, we have $\mathbb{P}(r_h(\tau) = r_h(\mathbf{t}_\mathfrak{a})) = p(\mathfrak{a})^{(\mathfrak{a}^h-1)/(\mathfrak{a}-1)}$. We deduce from (12) and the convergence characterization (7), using that $\mathbf{t}_\mathfrak{a}$ has a.s. an infinite height, that the proof of Proposition 6.5 is complete as soon as we prove the following strong ratio limit.

Lemma 12.5. *Let p be a non-degenerate super-critical offspring distribution with finite mean and such that $\mathfrak{a} \geq 2$. Assume that $\lim_{n\to+\infty} a_n/c_n = 0$ and that $\mathbb{P}(Z_n = a_n) > 0$ for every $n \in \mathbb{N}$ (which implies that $a_n \geq \mathfrak{a}^n$). Then, we have for $h, k \in \mathbb{N}^*$:*

$$\lim_{n\to\infty} \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}(Z_n = a_n)} = p(\mathfrak{a})^{-(\mathfrak{a}^h-1)/(\mathfrak{a}-1)}\mathbf{1}_{\{k=\mathfrak{a}^h\}}. \tag{107}$$

In fact, it is enough to prove (107) for $k = \mathfrak{a}^h$ as $\mathbb{P}(Z_h = \mathfrak{a}^h) = p(\mathfrak{a})^{-(\mathfrak{a}^h-1)/(\mathfrak{a}-1)}$. It is also enough to consider the two cases: $\lim_{n\to\infty} a_n/\mathfrak{a}^n = 1$ and $\liminf_{n\to\infty} a_n/\mathfrak{a}^n > 1$.

The case $\lim_{n\to\infty} a_n/\mathfrak{a}^n = 1$ is handled as in the Harris case, see the first part of the proof of Proposition 6.3 in Section 6.2. The case $\liminf_{n\to\infty} a_n/\mathfrak{a}^n > 1$ is a consequence of Lemma 12.4.

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