VACCINATING ACCORDING TO THE MAXIMAL ENDEMIC EQUILIBRIUM ACHIEVES HERD IMMUNITY

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ABSTRACT. We consider the simple epidemiological SIS model for a general heterogeneous population introduced by Lajmanovich and Yorke (1976) in finite dimension, and its infinite dimensional generalization we introduced in previous works. In this model the basic reproducing number R_0 is given by the spectral radius of an integral operator. If $R_0 > 1$, then there exists a maximal endemic equilibrium. In this very general heterogeneous SIS model, we prove that vaccinating according to the profile of this maximal endemic equilibrium ensures herd immunity. Moreover, this vaccination strategy is critical: the resulting effective reproduction number is exactly equal to one. As an application, we estimate that if $R_0 = 2$ in an age-structured community with mixing rates fitted to social activity, applying this strategy would require approximately 29% less vaccine doses than the strategy which consists in vaccinating uniformly a proportion $1 - 1/R_0$ of the population.

From a dynamical systems point of view, we prove that the non-maximality of an equilibrium g is equivalent to its linear instability in the original dynamics, and to the linear instability of the disease-free state in the modified dynamics where we vaccinate according to g.

1. INTRODUCTION

Increasing the prevalence of immunity from contagious disease in a population limits the circulation of the infection among the individuals who lack immunity. This so-called "herd effect" plays a fundamental role in epidemiology, for example it has had a major impact in the eradication of smallpox and rinderpest or the near eradication of poliomyelitis [9]. Our aim is to present a targeted vaccination strategy based on the heterogeneity of the infection spreading in the population which allows to eradicate the epidemic. We consider for simplicity the deterministic infinite-dimensional SIS model (with S=Susceptible and I=Infectious) and the effect of a perfect vaccine. However, we take into account a very general model for the heterogeneous population based on the infinite-dimensional model introduced in [2], that encompasses the meta-population SIS models developed by Lajmanovich and Yorke in their pioneer paper [11] or SIS model on graphs. More precisely, the probability $u_t(x)$ of an individual of type $x \in \Omega$ to be infected at time t is the solution of the (infinite dimensional) ordinary differential equation:

(1)
$$\partial_t u_t(x) = (1 - u_t(x)) \int_{\Omega} k(x, y) u_t(y) \mu(\mathrm{d}y) - \gamma(x) u_t(x) \quad \text{for } t \ge 0 \text{ and } x \in \Omega,$$

where k is the transmission rate kernel of the disease, γ is the recovery rate function and $\mu(dy)$ is the probability for an individual taken at random to be of type $y \in \Omega$; see Equation (11).

In an homogeneous population, the *basic reproduction number* of an infection, denoted by R_0 , is defined as the number of secondary cases one individual generates on average over the course of its infectious period, in an otherwise uninfected (susceptible) population. Intuitively, the disease should die out if $R_0 < 1$ and invade the population if $R_0 > 1$. For the heterogenous

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generalization of many classical models in epidemiology (including the heterogeneous SIS model), it is still possible to define a meaningful basic reproduction number R_0 , as the number of secondary cases generated by a *typical* infectious individual when all other individuals are uninfected and the threshold phenomenon occurs [7]. In the setting of [2], the reproduction number R_0 then corresponds to the spectral radius of the next-generation operator defined as the integral operator associated to the kernel $k(x, y)/\gamma(y)$.

After a vaccination campaign, let the vaccination strategy η denote the (non necessarily homogeneous) proportion of the **non-vaccinated** population, and let the effective reproduction number $R_e(\eta)$ denote the corresponding reproduction number of the non-vaccinated population. Following [2, Section 5.3.], the effective reproduction number $R_e(\eta)$ is given by the spectral radius of the effective next-generation operator defined as the integral operator associated to the kernel $k(x, y)\eta(y)/\gamma(y)$, where $\eta(y)$ is the proportion of individuals of type y which are not vaccinated.

The vaccination strategy η is called *critical* if $R_e(\eta) = 1$. Assuming $R_0 > 1$, suppose now that only a proportion η^{uni} of the population can catch the disease, the rest being perfectly immunized. An infected individual will now only generate $\eta^{\text{uni}}R_0$ new cases, since a proportion $1 - \eta^{\text{uni}}$ of previously successful infections will be prevented. Therefore, the new *effective reproduction number* is equal to $R_e(\eta^{\text{uni}}) = \eta^{\text{uni}}R_0$. This fact led to the recognition by Smith in 1970 [13] and Dietz in 1975 [8] of a simple threshold theorem: the incidence of an infection declines if the proportion of non-immune individuals is reduced below $\eta^{\text{uni}}_{\text{crit}} = 1/R_0$. This effect is called *herd immunity*, and the corresponding proportion $1 - \eta^{\text{uni}}_{\text{crit}}$ of people that have to be vaccinated is called the *herd immunity threshold* [14, 15].

2. CRITICAL VACCINATION GIVEN BY THE ENDEMIC EQUILIBRIUM

However, herd immunity can also be achieved using a non-uniform vaccination strategy when the population is heterogeneous. For example, the discussion of vaccination control of gonorrhea in [10, Section 4.5] suggests that it may be better to prioritize the vaccination of people that have already caught the disease: this leads us to consider a vaccination strategy guided by the equilibrium state. For the SIS model in heterogeneous population with $R_0 > 1$, there exists a maximal endemic equilibrium, say \mathfrak{g} , where $\mathfrak{g}(x)$ represents the fraction of infected people in the group with feature x. In other words, the function \mathfrak{g} is the maximal [0, 1]-valued solution g of:

(2)
$$(1 - g(x)) \int_{\Omega} k(x, y) g(y) \mu(\mathrm{d}y) = \gamma(x)g(x) \quad \text{for } x \in \Omega.$$

Let us mention that if there exist isolated subpopulations, it is possible to have other endemic equilibria, *i.e.*, solutions to Equation (2) that are not equal to 0 for all x. Irreducibility conditions on the kernel k ensure however the uniqueness of the endemic equilibrium [2, 11]. Consider the vaccination strategy, denoted by η^{equi} , corresponding to vaccinating a fraction $\mathfrak{g}(x)$ of people in the group with feature x, for all groups. In our mathematical framework, this amounts to setting:

(3)
$$\eta^{\text{equi}}(x) = 1 - \mathfrak{g}(x) \text{ for } x \in \Omega.$$

The following result ensures that this strategy reaches herd immunity; see Theorem 4.1 in Section 4 for a precise mathematical statement.

Theorem. In the heterogeneous SIS model with non-zero maximal endemic equilibrium, the vaccination strategy η^{equi} is critical:

(4)
$$R_e(\eta^{\text{equi}}) = 1$$

Let us stress that implementing the critical vaccination strategy η^{equi} can be achieved without estimating the transmission rate kernel and the recovery rate.

The proof of the theorem relies on the study of the spectral bound of the linearized operator associated to equation (1) near an equilibrium. When $R_0 > 1$, this spectral bound is nonpositive at the maximal equilibrium and positive at all other equilibria; see Proposition 4.4 (ii). Thus, the non-maximality of an equilibrium is equivalent to its linear instability in the original dynamics. We also prove the linear instability of the disease-free state in the modified dynamics where we vaccinate according to a non maximal equilibrium; see Proposition 4.4 (iv).

3. DISCUSSION

We expect the results obtained here for the SIS model to be generic, in the sense that similar behaviours should also be observed in more realistic and complex models in epidemiology for non-homogeneous populations: when an endemic equilibrium exists, vaccinating the population according to the maximal endemic profile should protect the population from the disease.

We refer to [6] for a general framework for cost comparison of vaccination strategies and the notions of "best" and "worst" vaccination strategies; see also [3, 4, 5] for further comments and various examples of optimal vaccinations.

Consider a general cost function C which measures the cost for the society of a vaccination strategy (production and diffusion). A simple and natural choice is the uniform cost given by the overall proportion of vaccinated individuals:

(5)
$$C(\eta) = \int_{\Omega} (\mathbb{1} - \eta) \,\mathrm{d}\mu = 1 - \int_{\Omega} \eta \,\mathrm{d}\mu$$

We have that $C(\eta_{\text{crit}}^{\text{uni}})$ is equal to the herd immunity threshold $1 - 1/R_0$ while $C(\eta_{\text{equi}}^{\text{equi}})$ is equal to the proportion of people in the endemic state in an SIS infection $\int_{\Omega} \mathfrak{g} d\mu$. It is not possible to determine which strategy is cheaper in general. However, in the following examples, we are able to compare their costs for mixing structures that are redundant in the epidemiologic litterature.

Example 3.1 (Homogeneous mixing). If the population is homogeneous (which corresponds to the one-dimensional SIS model where Ω is a singleton), then the maximal equilibrium is constant equal to $1 - 1/R_0$. It follows that $C(\eta_{\text{crit}}^{\text{uni}}) = C(\eta_{\text{equi}}^{\text{equi}})$.

Example 3.2 (Proportionate mixing structure with two subpopulations). The proportionate mixing is a classical mixing structure introduced by [12] and used in many different epidemio-logical models. It assumes that the number of adequate contacts between two subpopulations is proportional to the relative activity levels of the two subpopulations. Thus individuals in more active subpopulations will have more adequate contacts. Let us consider the simple case where there are only two subpopulations. Then the contact matrix is given by:

$$K = \begin{pmatrix} a^2 & ab\\ ab & b^2 \end{pmatrix}$$

where a and b are positive constants that correspond to the activity levels of the first and second subpopulations respectively. Denote by μ_1 and μ_2 their respective relative size, suppose that the recovery rate γ is equal to 1 for both subpopulations, and assume without loss of generality that $a \geq b$. In this case, we get that:

$$R_0 = a^2 \mu_1 + b^2 \mu_2$$
, $R_e(\eta) = a^2 \eta_1 \mu_1 + b^2 \eta_2 \mu_2$ and $C(\eta) = 1 - (\eta_1 \mu_1 + \eta_2 \mu_2)$

for the vaccination strategy $\eta = (\eta_1, \eta_2)$. If $R_0 > 1$, then the (unique) non-zero equilibrium satisfies:

$$(1 - \mathfrak{g}_i) \sum_{j=1}^{2} K_{i,j} \mathfrak{g}_j \mu_j = \mathfrak{g}_i \quad \text{for } i = 1, 2,$$

and the corresponding vaccination strategy $\eta^{\text{equi}} = 1 - \mathfrak{g}$ is given by:

$$\eta^{\text{equi}} = \left(\frac{1}{1+ac}, \frac{1}{1+bc}\right),\,$$

where $c \in [(1 - R_0)/a, (1 - R_0)/b]$ is the unique positive solution of the second-order equation $R_e(\eta^{\text{equi}}) = 1$. It is elementary to check that in this case $C(\eta^{\text{equi}}) < C(\eta^{\text{uni}})$, with an equality if and only if a = b. However, the critical vaccination strategy with minimal cost, say η^{opt} , corresponds to vaccinating in priority the population with the highest activity rate, that is, if a > b:

$$\eta^{\text{opt}} = \left(\frac{1 - \min(1, b^2 \mu_2)}{a^2 \mu_1}, \frac{1}{\max(1, b^2 \mu_2)}\right)$$

Example 3.3 (Age and activity structure). In [1], Britton, Ball and Trapman study an SEIR model, where immunity can be obtained through infection. Using parameters derived from real-world data, these authors noticed that the disease-induced herd immunity level can, for some models, be substantially lower than the classical herd immunity threshold $1 - 1/R_0$. This can be reformulated in term of targeted vaccination strategies: prioritizing the individuals that are more likely to get infected in a SEIR epidemic may be more efficient than distributing uniformly the vaccine in the population.

We use the same age and activity structures to determine which strategy between η^{equi} and $\eta_{\text{crit}}^{\text{uni}}$ is less costly. More precisely, the community is categorized into six age groups and contact rates between them are derived from an empirical study of social contacts [16]. For the activity structure, individuals are categorized into three different activity levels, which are arbitrary and chosen for illustration purposes: 50% of each age cohort have normal activity, 25% have low activity corresponding to half as many contacts compared with normal activity, and 25% have high activity corresponding to twice as many contacts as normal activity. Note that when the population is only structured by activity, the mixing is proportionate. Assuming that the recovery rate is constant equal to 1, we solved numerically Equation (2) and computed in Table 1 the cost of the uniform and the equilibrium strategies for different values of R_0 and different population structures. In Table 2, the fractions of vaccinated individuals in the different age activity groups when following the strategy η^{equi} . This is done by assuming $R_0 = 2$. Note than in this case, only three subpopulations need to be vaccinated at a level higher than $1 - 1/R_0$.

	$R_0 = 2$		$R_0 = 2.5$			$R_0 = 3$	
	$C(\eta^{ m equi})$	$C(\eta^{\mathrm{uni}})$	$\overline{C(\eta^{ ext{equi}})}$	$C(\eta^{\rm uni})$		$C(\eta^{\rm equi})$	$C(\eta^{\rm uni})$
Homogeneous	50	50	60	60		66.7	66.7
Age structure	46.6	50	56.7	60		63.9	66.7
Activity structure	40.1	50	50	60		57	66.7
Age and activity structure	35.7	50	45.2	60		52.2	66.7

TABLE 1. Cost of the equilibrium vaccination compared to the herd immunity level for different population structures. Numbers correspond to percentage.

TABLE 2. Fraction of vaccinated individuals in different groups for the strategy η^{equi} . The population structure includes both age and activity. Numbers correspond to percentage. These values assume that $R_0 = 2$, so that the uniform critical vaccination consists in vaccinating 50% of the population: only three groups require more vaccine in the targeted strategy than in the uniform strategy.

Age group	Low activity	Average activity	High activity
0–5 years	12.0	21.4	35.3
6-12 years	18.5	31.2	47.5
13–19 years	22.9	37.3	54.3
20-39 years	29.1	45.1	62.1
40-59 years	20.9	34.6	51.4
≥ 60 years	12.4	22.1	36.2

4. Proof

The differential equations governing the epidemic dynamics in meta-population SIS models were developed in paper [11] in finite dimension and generalized in [2].

4.1. The heterogeneous SIS model. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space, where $x \in \Omega$ represents a feature and the probability measure $\mu(dx)$ represents the fraction of the population with feature x. The parameters of the SIS model are given by a *recovery rate function* γ , which is a positive bounded measurable function defined on Ω , and a *transmission rate kernel* k, which is a non-negative measurable function defined on Ω^2 .

In accordance with [2], we consider for a kernel k on Ω and $q \in (1, +\infty)$ its norm:

$$\|\mathbf{k}\|_{\infty,q} = \sup_{x \in \Omega} \left(\int_{\Omega} \mathbf{k}(x,y)^{q} \,\mu(\mathrm{d}y) \right)^{1/q}$$

For a kernel k on Ω such that $\|\mathbf{k}\|_{\infty,q}$ is finite for some $q \in (1, +\infty)$, we define the integral operator $\mathcal{T}_{\mathbf{k}}$ on the set \mathscr{L}^{∞} of bounded measurable real-valued function on Ω by:

(6)
$$\mathcal{T}_{\mathbf{k}}(g)(x) = \int_{\Omega} \mathbf{k}(x, y)g(y)\,\mu(\mathrm{d}y) \quad \text{for } g \in \mathscr{L}^{\infty} \text{ and } x \in \Omega.$$

By convention, for f, g two non-negative measurable functions defined on Ω and k a kernel on Ω , we denote by fkg the kernel on Ω defined by:

(7)
$$fkg: (x,y) \mapsto f(x)k(x,y)g(y).$$

We shall consider the kernel $\mathbf{k} = k/\gamma$ (corresponding to $k\gamma^{-1}$, which differs in general from $\gamma^{-1}k$), which is thus defined by:

(8)
$$\mathbf{k}(x,y) = k(x,y)\,\gamma(y)^{-1}.$$

We shall assume that:

(9)
$$\|\mathbf{k}\|_{\infty,q} < \infty \quad \text{for some } q \in (1, +\infty).$$

The integral operator $\mathcal{T}_{\mathbf{k}}$ is the so called *next-generation operator*.

Let $\Delta = \{f \in \mathscr{L}^{\infty} : 0 \le f \le 1\}$ be the subset of non-negative functions bounded by 1, and let $\mathbb{1} \in \Delta$ be the constant function equal to 1. The SIS dynamics considered in [2] follows the vector field F defined on Δ by:

(10)
$$F(g) = (\mathbb{1} - g)\mathcal{T}_k(g) - \gamma g.$$

More precisely, we consider $u = (u_t, t \in \mathbb{R})$, where $u_t \in \Delta$ for all $t \in \mathbb{R}_+$ such that:

(11)
$$\overline{\partial_t u_t = F(u_t)} \quad \text{for } t \in \mathbb{R}_+,$$

with initial condition $u_0 \in \Delta$. The value $u_t(x)$ models the probability that an individual of feature x is infected at time t; it is proved in [2] that such a solution u exists and is unique.

An equilibrium of (11) is a function $g \in \Delta$ such that F(g) = 0. According to [2], there exists a maximal equilibrium \mathfrak{g} , *i.e.*, an equilibrium such that all other equilibria $h \in \Delta$ are dominated by \mathfrak{g} : $h \leq \mathfrak{g}$. It is to this maximal equilibrium that the process stabilizes when started from a situation where all the population is infected, that is, if $u_0 = 1$, then we have:

$$\lim_{t \to \infty} u_t = \mathfrak{g}.$$

For T a bounded operator on \mathscr{L}^{∞} endowed with its usual supremum norm, we define by $||T||_{\mathscr{L}^{\infty}}$ its operator norm and its spectral radius is given by:

$$\rho(T) = \lim_{n \to \infty} \| T^n \|_{\mathscr{L}^{\infty}}^{1/n}$$

The *reproduction number* R_0 associated to the SIS model given by (11) is the spectral radius of the next-generation operator:

(12)
$$R_0 = \rho(\mathcal{T}_{\mathbf{k}}),$$

If $R_0 \leq 1$ (sub-critical and critical case), then u_t converges pointwise to 0 when $t \to \infty$. In particular, the maximal equilibrium \mathfrak{g} is equal to 0 everywhere. If $R_0 > 1$ (super-critical case), then 0 is still an equilibrium but different from the maximal equilibrium \mathfrak{g} , as $\int_{\Omega} \mathfrak{g} d\mu > 0$.

4.2. Vaccination strategies. A vaccination strategy η of a vaccine with perfect efficiency is an element of Δ , where $\eta(x)$ represents the proportion of **non-vaccinated** individuals with feature x. Notice that $\eta \, d\mu$ corresponds in a sense to the effective population. In particular, the "strategy" that consists in vaccinating no one (resp. everybody) corresponds to $\eta = 1$, the constant function equal to 1, (resp. $\eta = 0$, the constant function equal to 0).

Recall the definition of the kernel $f\mathbf{k}g$ from (7). For $\eta \in \Delta$, the kernel $k\eta = k\eta/\gamma$ has finite norm $\|\cdot\|_{\infty,q}$, so we can consider the bounded positive operators $\mathcal{T}_{\mathbf{k}\eta}$ and $\mathcal{T}_{k\eta}$ on \mathscr{L}^{∞} . According to [2, Section 5.3.], the SIS equation with vaccination strategy η is given by (11), where F is replaced by F_{η} defined by:

(13)
$$F_{\eta}(g) = (\mathbb{1} - g)\mathcal{T}_{k\eta}(g) - \gamma g.$$

We denote by $u^{\eta} = (u_t^{\eta}, t \ge 0)$ the corresponding solution with initial condition $u_0^{\eta} \in \Delta$. We recall that $u_t^{\eta}(x)$ represents the probability for an non-vaccinated individual of feature x to be infected at time t. We define the *effective reproduction number* $R_e(\eta)$ associated to the vaccination strategy η as the spectral radius of the effective next-generation operator $\mathcal{T}_{\mathbf{k}n}$:

(14)
$$R_e(\eta) = \rho(\mathcal{T}_{\mathbf{k}\eta}).$$

For example, for the trivial vaccination strategies we get $R_e(1) = R_0$ and $R_e(0) = 0$.

We denote by \mathfrak{g}_{η} the corresponding maximal equilibrium. In particular, we have:

(15)
$$F_{\eta}(\mathfrak{g}_{\eta}) = 0$$

In particular, we have:

$$R_e(\mathbb{1}) = R_0 \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_{\mathbb{1}}.$$

4.3. Critical vaccination strategies. If $R_0 \ge 1$, then a vaccination strategy η is called *critical* if it achieves precisely herd immunity, that is $R_e(\eta) = 1$.

As the spectral radius is positively homogeneous (that is, $\rho(\lambda t) = \lambda \rho(T)$ for $\lambda \ge 0$), we also get, when $R_0 \ge 1$, that the uniform strategy that corresponds to the constant function:

$$\eta_{\rm crit}^{\rm uni} = \frac{1}{R_0}\mathbb{1}$$

is critical, as $R_e(\eta_{\text{crit}}^{\text{uni}}) = 1$. This is consistent with results obtained in the homogeneous model.

As hinted in [10, Section 4.5] for vaccination control of gonorrhea, it is interesting to consider vaccinating people with feature x with probability $\mathfrak{g}(x)$. This corresponds to the strategy based on the maximal equilibrium:

$$\eta^{\text{equi}} = \mathbb{1} - \mathfrak{g}.$$

The following result entails that this strategy is critical and thus achieves herd immunity. Recall that in the (infinite dimensional) SIS model (10) on the probability space $(\Omega, \mathscr{F}, \mu)$ the recovery rate function γ is positive and bounded, the transmission rate k is non-negative and the norm $\|\mathbf{k}\|_{\infty,q}$ of the kernel $\mathbf{k} = k/\gamma$ is finite for some $q \in (1, +\infty)$.

Theorem 4.1 (The maximal equilibrium yields a critical vaccination). Consider the SIS model (10) under the boundedness assumption (9). If $R_0 \ge 1$, then the vaccination strategy η^{equi} is critical, that is, $R_e(\eta^{\text{equi}}) = 1$.

This result will be proved below as a part of Proposition 4.4.

4.4. **Proof of Theorem 4.1.** For an operator A, we denote by A^{\top} its adjoint. We first give a preliminary lemma. For the convenience of the reader, we only use references to the results recalled in [2] for positive operators on Banach spaces. In particular, we shall use that if k and k' are two (non-negative) kernels on Ω with finite norms $\|\cdot\|_{\infty,q}$ for some $q \in (1, +\infty)$, then we have that:

(16)
$$\mathbf{k} \ge \mathbf{k}' \implies \rho(\mathcal{T}_{\mathbf{k}}) \ge \rho(\mathcal{T}_{\mathbf{k}'}),$$

see for example [2, Theorem 3.5(i)] as the operator $\mathcal{T}_{\mathbf{k}} - \mathcal{T}_{\mathbf{k}'}$ is positive. We shall also used that for two bounded operators T and S on \mathscr{L}^{∞} :

(17) $\rho(TS) = \rho(ST).$

We first state two technical lemmas.

Lemma 4.2. Let k be a non-negative kernel on Ω such that $\|\mathbf{k}\|_{\infty,q}$ is finite for some $q \in (1, +\infty)$ and and consider the positive bounded linear integral operator $\mathcal{T}_{\mathbf{k}}$ on \mathscr{L}^{∞} . If there exists $g \in \mathscr{L}^{\infty}_{+}$, with $\int_{\Omega} g \, d\mu > 0$ and $\lambda > 0$ satisfying:

$$\mathcal{T}_{\mathbf{k}}(g)(x) > \lambda g(x), \quad for \ all \quad x \ such \ that \ g(x) > 0,$$

then we have $\rho(\mathcal{T}_k) > \lambda$.

Proof. We simply write \mathcal{T} for \mathcal{T}_k . Let $A = \{g > 0\}$ be the support of the function g. Let \mathcal{T}' be the bounded operator defined by $\mathcal{T}'(f) = \mathbb{1}_A \mathcal{T}(\mathbb{1}_A f)$. Since $\mathcal{T}'(g) = \mathbb{1}_A \mathcal{T}(\mathbb{1}_A g) = \mathbb{1}_A \mathcal{T}(g) > \lambda g$, we deduce from the Collatz-Wielandt formula [2, Proposition 3.6] that $\rho(\mathcal{T}') \ge \lambda > 0$. According to [2, Lemma 3.7 (v)], there exists $v \in L^q_+ \setminus \{0\}$, seen as an element of the topological dual of \mathscr{L}^∞ , a left Perron eigenfunction of \mathcal{T}' , that is such that $(\mathcal{T}')^\top(v) = \rho(\mathcal{T}')v$. In particular, we have $v = \mathbb{1}_A v$ and thus $\int_A v \, d\mu > 0$ and $\int_\Omega vg \, d\mu > 0$. We obtain:

$$(\rho(\mathcal{T}') - \lambda) \langle v, g \rangle = \langle v, \mathcal{T}'(g) - \lambda g \rangle > 0$$

As $\mathcal{T}' = \mathcal{T}_{\mathbf{k}'}$ with $\mathbf{k}' = \mathbb{1}_A \mathbf{k} \mathbb{1}_A \leq \mathbf{k}$, we deduce from (16) that $\rho(\mathcal{T}) \geq \rho(\mathcal{T}') > \lambda$.

Lemma 4.3. Consider the SIS model (10) under the boundedness assumption (9). Let $\eta, g \in \Delta$. If $F_{\eta}(g) \geq 0$, then we have $g \leq \mathfrak{g}_{\eta}$.

Proof. Consider the solution u_t of the SIS model $\partial_t u_t = F_\eta(u_t)$ with vaccination η and initial condition $u_0 = g$. According to [2, Proposition 2.10], this solution is non-decreasing since $F_\eta(g) \ge 0$. According to [2, Proposition 2.13], the pointwise limit of u_t is an equilibrium. As this limit is dominated by the maximal equilibrium \mathfrak{g}_η and since u_t is non-decreasing, this proves that $g \le \mathfrak{g}_\eta$.

The next result characterizes the maximal equilibrium \mathfrak{g} among all equilibria by various spectral properties; Theorem 4.1 may be viewed as a corollary to this characterization. Recall

that $R_0 = R_e(1)$, and that the vector field F is defined by (10). Let DF[h] denote the bounded linear operator on \mathscr{L}^{∞} of the derivative of the map $f \mapsto F(f)$ defined on \mathscr{L}^{∞} at point h:

$$DF[h](g) = (1-h)\mathcal{T}_k(g) - (\gamma + \mathcal{T}_k(h))g \text{ for } h, g \in \mathscr{L}^{\infty}$$

Let s(A) denote the spectral bound of the bounded operator A, see Equation (33) in [2].

Proposition 4.4 (Equivalent conditions for maximality). Consider the SIS model (10) under the boundedness assumption (9). Let h in Δ be an equilibrium, that is, F(h) = 0. The following properties are equivalent:

(i) $h = \mathfrak{g},$ (ii) $s(DF[h]) \le 0,$ (iii) $R_e((1-h)^2) \le 1.$ (iv) $s(DF_{(1-h)}[\mathbb{O}]) \le 0.$ (v) $R_e(1-h) \le 1.$

Furthermore, $\mathfrak{g} = \mathbb{O}$ if and only if $R_0 \leq 1$, and if $\mathfrak{g} \neq \mathbb{O}$, then it is critical: $R_e(1 - \mathfrak{g}) = 1$.

Remark 4.5 (On stability). From a dynamical systems point of view, this proposition links together two different stability properties. The (classically equivalent) conditions (ii) and (iii) state that for the original dynamics given by (11) with vector field F, the equilibrium h is not linearly unstable. Similarly, conditions (iv) and (v) both state that in the vaccinated dynamics given by the modified vector field F_{1-h} defined by (13), the disease-free equilibrium 0 is not linearly unstable.

In particular, in the original dynamics given by (11), equilibria that are not maximal are necessarily linearly unstable.

Proof. Let $h \in \Delta$ be an equilibrium, that is F(h) = 0.

The equivalence between (iv) and (v) is a direct consequence of [2, Proposition 4.2].

Let us show the equivalence between (ii) and (iii). According to the same [2, Proposition 4.2], $s(DF[h]) \leq 0$ if and only if:

$$\rho\left(\mathcal{T}_{\mathbf{k}}\right) \leq 1 \quad \text{with} \quad \mathbf{k}(x, y) = (1 - h(x)) \frac{k(x, y)}{\gamma(y) + \mathcal{T}_{\mathbf{k}}(h)(y)}$$

Since F(h) = 0, we have $(1 - h)/\gamma = 1/(\gamma + \mathcal{T}_k(h))$. This gives:

(18)
$$k(x,y) = (1-h(x))\frac{k(x,y)(1-h(y))}{\gamma(y)}$$

and thus $\mathcal{T}_{\mathbf{k}} = M_{1-h} \mathcal{T}_{k/\gamma} M_{1-h}$, where M_f is the multiplication operator by f. Recall the definition (14) of R_e . We deduce from (17) that:

(19)
$$\rho(\mathcal{T}_{\mathbf{k}}) = \rho\left(\mathcal{T}_{k/\gamma}M_{(1-h)^2}\right) = R_e((1-h)^2).$$

This gives the equivalence between (ii) and (iii).

We prove that (i) implies (v). Suppose that $R_e(1-h) > 1$. Thanks to (17), we have $\rho(M_{1-h}\mathcal{T}_{k/\gamma}) = \rho(\mathcal{T}_{k/\gamma}M_{1-h}) = R_e(1-h) > 1$. According to [2, Lemma 3.7 (v)], there exists $v \in L^q_+ \setminus \{0\}$ a left Perron eigenfunction of $\mathcal{T}_{(1-h)k/\gamma}$, that is $\mathcal{T}_{(1-h)k/\gamma}^\top(v) = R_e(1-h)v$. Using F(h) = 0, and thus $(1-h)\mathcal{T}_k(h) = \gamma h$, for the last equality, we have:

$$R_e(1-h) \langle v, \gamma h \rangle = \langle v, (1-h)\mathcal{T}_{k/\gamma}(\gamma h) \rangle = \langle v, \gamma h \rangle.$$

We get $\langle v, \gamma h \rangle = 0$ and thus $\langle v, \mathbb{1}_A \rangle = 0$, where $A = \{h > 0\}$ denote the support of the function h. Since $\mathcal{T}_{(1-h)k/\gamma}^{\top}(v) = R_e(1-h)v$ and setting v' = (1-h)v (so that $v' = v \mu$ -almost surely on A^c), we deduce that:

$$\mathcal{T}_{k'/\gamma}^{\top}(v') = R_e(1-h)v',$$

where $k' = \mathbb{1}_{A^c} k \mathbb{1}_{A^c}$. This implies that $\rho(\mathcal{T}_{k'/\gamma}) \geq R_e(1-h)$. Since k' = (1-h)k' and $k-k' \geq 0$, we get that $\mathcal{T}_{k/\gamma} - \mathcal{T}_{k'/\gamma}$ is a positive operator. Using (16) for the inequality as $(1-h)k'/\gamma \leq (1-h)k/\gamma$, we deduce that $\rho(\mathcal{T}_{k'/\gamma}) = \rho(M_{1-h}\mathcal{T}_{k'/\gamma}) \leq \rho(M_{1-h}\mathcal{T}_{k/\gamma}) = R_e(1-h)$. Thus, the spectral radius of $\mathcal{T}_{k'/\gamma}$ is equal to $R_e(1-h)$. According to [2, Proposition 4.2], since $\rho(\mathcal{T}_{k'/\gamma}) > 1$, there exists $w \in \mathscr{L}^{\infty}_+ \setminus \{0\}$ and $\lambda > 0$ such that:

$$\mathcal{T}_{k'}(w) - \gamma w = \lambda w.$$

This also implies that w = 0 on $A = \{h > 0\}$, that is wh = 0 and thus $w\mathcal{T}_k(h) = 0$ as $\mathcal{T}_k(h) = \gamma h/(1-h)$. Using that F(h) = 0, $\mathcal{T}_k(w) = \mathcal{T}_{k'}(w) = (\gamma + \lambda)w$ and $h\mathcal{T}_k(w) = 0$, we obtain:

$$F(h+w) = w(\lambda - \mathcal{T}_k(w)).$$

Taking $\varepsilon > 0$ small enough so that $\varepsilon \mathcal{T}_k(w) \leq \lambda/2$ and $\varepsilon w \leq 1$, we get $h + \varepsilon w \in \Delta$ and $F(h + \varepsilon w) \geq 0$. Then use Lemma 4.3 to deduce that $h + \varepsilon w \leq \mathfrak{g}$ and thus $h \neq \mathfrak{g}$.

To see that (v) implies (iii), notice that $(1-h) \ge (1-h)^2$, and then use (16) to deduce that $\rho(\mathcal{T}_{\mathbf{k}(1-h)}) \ge \rho(\mathcal{T}_{\mathbf{k}(1-h)^2})$ and thus $R_e(1-h) \ge R_e((1-h)^2)$.

We prove that (iii) implies (i). Notice that F(g) = 0 and $g \in \Delta$ implies that g < 1. Assume that $h \neq \mathfrak{g}$. Notice that $\gamma/(1-h) = \gamma + \mathcal{T}_k(h)$, so that $\gamma(\mathfrak{g}-h)/(1-h) \in \mathscr{L}^{\infty}_+$. An elementary computation, using $F(h) = F(\mathfrak{g}) = 0$ and k defined in (18), gives:

$$\mathcal{T}_{k}\left(\gamma\frac{\mathfrak{g}-h}{1-h}\right) = (1-h)\mathcal{T}_{k}\left(\mathfrak{g}-h\right) = \gamma\frac{\mathfrak{g}-h}{1-\mathfrak{g}} = \frac{1-h}{1-\mathfrak{g}}\gamma\frac{\mathfrak{g}-h}{1-\mathfrak{g}}.$$

Since $h \neq \mathfrak{g}$ and $h \leq \mathfrak{g}$, we deduce that $(1-h)/(1-\mathfrak{g}) \geq 1$, with strict inequality on $\{\mathfrak{g} - h > 0\}$ which is a set of positive measure. We deduce from Lemma 4.2 (with k replaced by $k\gamma$) that $\rho(\mathcal{T}_k) > 1$. Then use (19) to conclude.

To conclude notice that $\mathfrak{g} = 0 \iff R_0 \leq 1$ is a consequence of the equivalence between (i) and (v) with h = 0 and $R_0 = R_e(\mathbb{1})$.

Using that $F(\mathfrak{g}) = 0$, we get $\mathcal{T}_k(\mathfrak{g}) = \gamma \mathfrak{g}/(1-\mathfrak{g})$. We deduce that $\mathcal{T}_{k(1-\mathfrak{g})/\gamma}(\mathcal{T}_k(\mathfrak{g})) = \mathcal{T}_k(\mathfrak{g})$. If $\mathfrak{g} \neq 0$, we get $\mathcal{T}_k(\mathfrak{g}) \neq 0$ (on a set of positive μ -measure). This implies that $R_e(1-\mathfrak{g}) \geq 1$. Then use (v) to deduce that $R_e(1-\mathfrak{g}) = 1$ if $\mathfrak{g} \neq 0$.

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