β-COALESCENTS AND STABLE GALTON-WATSON TREES

ROMAIN ABRAHAM AND JEAN-FRANÇOIS DELMAS

Abstract. Representation of coalescent process using pruning of trees has been used by Goldschmidt and Martin for the Bolthausen-Sznitman coalescent and by Abraham and Delmas for the $\beta(3/2, 1/2)$-coalescent. By considering a pruning procedure on stable Galton-Watson tree with n labeled leaves, we give a representation of the discrete $\beta(1+\alpha, 1-\alpha)$-coalescent, with $\alpha \in [1/2, 1)$ starting from the trivial partition of the n first integers. The construction can also be made directly on the stable continuum Lévy tree, with parameter $1/\alpha$, simultaneously for all n. This representation allows to use results on the asymptotic number of coalescence events to get the asymptotic number of cuts in stable Galton-Watson tree (with infinite variance for the reproduction law) needed to isolate the root. Using convergence of the stable Galton-Watson tree conditioned to have infinitely many leaves, one can get the asymptotic distribution of blocks in the last coalescence event in the $\beta(1+\alpha, 1-\alpha)$-coalescent.

1. Introduction

1.1. **Framework.** The idea of constructing coalescent processes by pruning discrete trees arises first in [23] where the Bolthausen-Sznitman coalescent is constructed by a uniform pruning of the branches of a random recursive tree, see also [30] and [20] for applications of such a representation. The same kind of ideas has been used in [4] to construct a $\beta(3/1, 1/2)$-coalescent process using a uniform pruning of the branches of a uniform random binary tree. This construction is also closely related to Aldous's continuum random tree. The goal of this paper is to extend this result by applying a pruning at nodes (introduced in [1] in a continuous setting and in [7] in a discrete setting) to a stable Lévy tree, obtaining a $\beta(1+\alpha, 1-\alpha)$-coalescent process, with $1/2 \leq \alpha < 1$.

Let Λ be a finite measure on $[0, 1]$. A Λ-coalescent $(\Pi(t), t \geq 0)$ is a Markov process which takes values in the set of partitions of $\mathbb{N}^* = \{1, 2, \dots\}$ introduced in [29] for coalescent processes with possible multiple collisions. It is defined via the transition rates of its restriction $\Pi^{[n]} = (\Pi^{[n]}(t), t \geq 0)$ to the n first integers: if $\Pi^{[n]}(t)$ is composed of b blocks, then k $(2 \leq k \leq b)$ fixed blocks coalesce at rate:

$$\lambda_{b,k} = \int_0^1 u^{k-2}(1-u)^{b-k}\Lambda(du). \tag{1}$$

In particular a coalescence event happens at rate:

$$\lambda_b = \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k}. \tag{2}$$

We also define the discrete process $\Pi^{[n]}_{\text{dis}} = (\Pi^{[n]}_{\text{dis}}(k), k \in \mathbb{N})$ as the different successive states of the process $\Pi^{[n]}$ until it reaches the absorbing state (which is the trivial partition consisting in one block) and afterward the discrete process remains constant.

As examples of Λ-coalescents, let us mention:

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- the Kingman's coalescent with $\Lambda(dx) = \delta_0(dx)$, see [27],
- the Bolthausen-Sznitman coalescent with $\Lambda(dx) = \mathbf{1}_{(0,1)}(x)dx$, see [13],
- the β-coalescents with $\Lambda(dx)$ is (up to a multiplicative constant) the $\beta(a, b)$ distribution. In the case of the $\beta(1+\alpha, 1-\alpha)$-coalescent, that is $\Lambda(dx) = (x/(1-x))^\alpha\, dx$, see [12, 10] for $-1 < \alpha < 0$. The case $\alpha = 0$ corresponds to the Bolthausen-Sznitman coalescent, while the limit case $\alpha = -1$ formally corresponds to the Kingman's coalescent. For the $\beta(1+\alpha, -\alpha)$-coalescent, with $-1 < \alpha < 0$ see [19].

We refer to the survey [11] for further results on coalescent processes.

Let $\alpha \in [1/2, 1)$. We consider a Galton-Watson (GW) tree T with offspring distribution characterized by its generating function for $r \in [0, 1]$:

$$g(r) = r + \alpha(1-r)^{1/\alpha}. \tag{3}$$

This GW tree arises as the shape of the sub-tree of a stable Lévy tree with index $\gamma = 1/\alpha$ generated by leaves chosen in a Poissonian manner, see [16], Theorem 3.2.1. We shall call these random trees the stable GW trees with parameter γ. We denote by $\mathbf{P}$ the distribution of T. If x is a node of T we denote by $k_x(T)$ the number of offsprings of x. Since $g'(0) = 0$, we get that a.s. $k_x(T) \neq 1$ for all $x \in T$. We denote by $\mathbf{P}_n$ the law of T conditioned to have exactly n leaves (a leaf is a node with no offspring). Under $\mathbf{P}_n$, we label the leaves of T from 1 to n uniformly at random, independently of T, and then we consider the following pruning procedure which is derived from [8], see Section 2.2. Choose an internal node x_1 (which has at least 2 children) at random with probability:

$$\frac{k_{x_1}(T) - 1}{L(T) - 1}.$$

We cut that node and keep the part $T_{(1)}$ of the tree that contains the root, and keep x_1 which is now a leaf of $T_{(1)}$ and we label x_1 by the block (i.e. the sequence) of labels of the leaves "above" x_1. We then iterate the procedure on the tree $T_{(1)}$ and so on until the root is cut (see Figure (1)).

This pruning procedure defines a discrete time process $\Pi^{[n]}_{\text{GW}} = (\Pi^{[n]}_{\text{GW}}(k), k \in \mathbb{N})$ taking values in the set of partitions of the n first integers, $\Pi^{[n]}_{\text{GW}}(k)$ being the set of labels of the leaves of the tree $T_{(k)}$ obtained after the k-th cut.

1.2. **Main result.** The process $\Pi^{[n]}_{\text{GW}}$ is then a coalescent process starting from the trivial partition consisting of singletons and blocks merge together as time goes by. Its law is given in the next theorem.

Theorem 1.1. *We set $\alpha = \frac{1}{\gamma} \in [1/2, 1)$. The process $\Pi^{[n]}_{GW}$ is distributed under $\mathbf{P}_n$ as $\Pi^{[n]}_{dis}$ for the $\beta(1+\alpha, 1-\alpha)$-coalescent with coalescent measure:*

$$\Lambda(dx) = \left(\frac{x}{1-x}\right)^\alpha dx. \tag{4}$$

Notice that the process $\Pi^{[n]}_{\text{dis}}$ is discrete in time and thus characterizes the coalescent measure up to a multiplicative constant.

One major drawback of this construction is that we define the process for fixed n and not simultaneously for all n. However, as in [4], we can construct directly the process $(\Pi(\theta), \theta \geq 0)$ taking values in the set of partitions of the integers using the pruning of a Lévy continuum random tree. More precisely, we consider the weighted stable Lévy tree $(\mathcal{T}, d, \mathbf{m}^{\mathcal{T}})$ associated

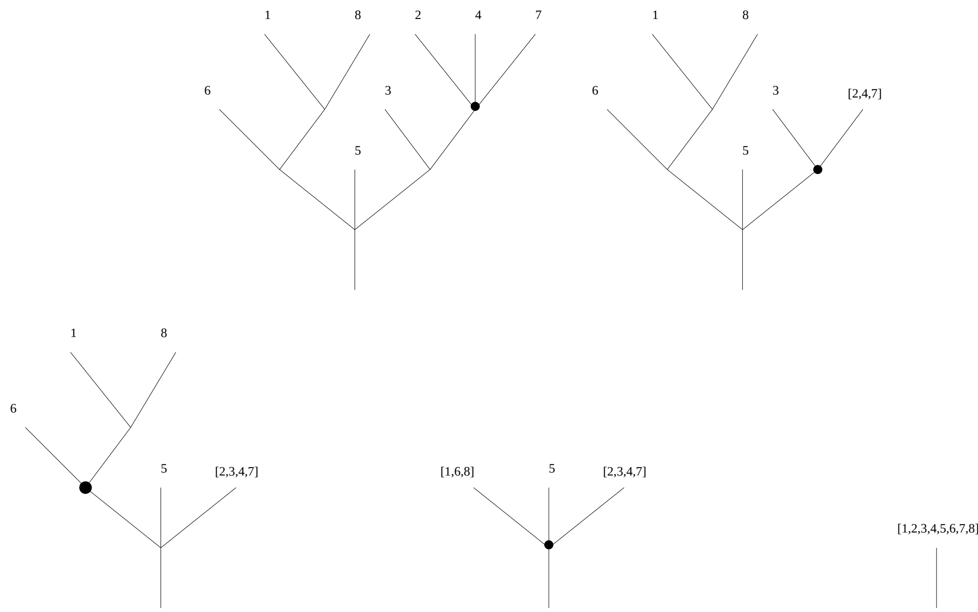

FIGURE 1. The pruning at node of a given tree. The bold internal node corresponds to the next chosen node.

with the branching mechanism $\psi(\lambda) = \lambda^\gamma$ for $\gamma \in (1, 2)$ (the case $\gamma = 2$ is studied in [4] and requires a different pruning). We recall that $\mathcal{T}$ is a real tree and that $\mathbf{m}^\mathcal{T}$ correspond to a uniform measure on the leaves of $\mathcal{T}$, see [16], [17] and also [9] more specifically for the space of weighted real trees. We work under the so-called normalized excursion measure $\mathbb{N}^{(1)}$ under which $\mathbf{m}^\mathcal{T}$ is a probability measure. We consider given $\mathcal{T}$ the pruning defined in [1]: to each branching point x of $\mathcal{T}$ we can associate a "mass" Δ_x of this node, which intuitively represents the size of its progeny, and a random variable E_x which is exponentially distributed with parameter Δ_x. This random variable represents the time at which the node x is cut. When we cut such a node, we remove the sub-tree above it. Let $\mathcal{T}_\theta$ denote the continuum random sub-tree obtained at time $\theta \geq 0$. We define a coalescent process using the usual paintbox procedure. Let $(U_i, i \in \mathbb{N}^*)$ be independent random variables with distribution $\mathbf{m}^\mathcal{T}$ under $\mathbb{N}^{(1)}$. We define a partition of $\mathbb{N}^*$ at time θ, $\Pi_{\text{Lévy}}(\theta)$ by saying that two integers i and j belong to the same block of $\Pi_{\text{Lévy}}(\theta)$ if and only if the random variables U_i and U_j have a leaf of $\mathcal{T}_\theta$ as a common ancestor. Intuitively this means that U_i and U_j belong to the same sub-tree attached above $\mathcal{T}_\theta$. This defines a coalescent process $\Pi_{\text{Lévy}} = (\Pi_{\text{Lévy}}(\theta), \theta \geq 0)$. We are now interested in its discrete (in time) restriction to the n first integers. Let $\Pi^{[n]}_{\text{Lévy}} = (\Pi^{[n]}_{\text{Lévy}}(k), k \in \mathbb{N})$ be the discrete process associated with $\Pi_{\text{Lévy}}$ restricted to the n first integers until it reaches the absorbing state (which is the trivial partition consisting in one block) and which afterward remains constant.

By construction, and thanks to Theorem 3.2.1 in [16], we can deduce that under $\mathbb{N}^{(1)}$, the discrete coalescent process $\Pi^{[n]}_{\text{Lévy}}$ is distributed as $\Pi^{[n]}_{\text{GW}}$ under $\mathbf{P}_n$. In fact, we have the following stronger result.

Theorem 1.2. *We set $\alpha = \frac{1}{\gamma} \in (1/2, 1)$. Under $\mathbb{N}^{(1)}$, the processes $(\Pi^{[n]}_{Lévy}, n \in \mathbb{N}^*)$ associated with the Lévy tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ is distributed as $(\Pi^{[n]}_{dis}, n \in \mathbb{N}^*)$ associated with the Lévy measure $\Lambda(dx) = (x/1-x)^\alpha\, dx$.*

We conjecture that the process $\Pi_{\text{Lévy}}$ under $\mathbb{N}^{(1)}$ is up to random time change a $\beta(1+\alpha, 1-\alpha)$-coalescent.

Remark 1.3. Lets us remark that the $\beta(1+\alpha, 1-\alpha)$-coalescent we obtain is also a $\beta(2-a, a)$-coalescent (with $a = 1-\alpha$) as in [10] but with a different range for a. The difference between the two cases is that in [10] $\alpha \in (-1, 0)$ and the coalescent process comes down from infinity (i.e. for every positive time θ, the partition $\Pi(\theta)$ contains only a finite number of blocks) whereas in our case $\alpha \in (1/2, 1)$ the process always contains an infinite number of singletons (also called "dust").

1.3. **Number of cuts need to isolate the root in a stable GW tree.** We now give an application of our representation using results on β-coalescent to get the asymptotic number of cuts needed to isolate the root in a stable GW tree with n leaves. Notice that the reproduction law for stable GW tree has an infinite variance for $\alpha \in (1/2, 1)$, whereas it is finite if $\alpha = 1/2$. For GW trees with finite variance for the reproduction law, consider the cutting procedure given by choosing a node at random and removing the trees attached to this node not containing the root. Let Z'_n denote the number of cuts needed to isolate the root when the GW tree has n leaves. The limit Z' of Z'_n is given in [25] for the convergence in distribution and in [6] for an a.s. convergence in the case of binary trees. In particular, Z' is distributed as the height of a random leaf of the normalized Lévy tree with $\gamma = 2$ that is up to some scaling factor of the Aldous' continuum random tree.

Let Z_n be the number of cuts, using the procedure developed is Section 1.1, needed to isolate the root of a stable GW tree:

$$Z_n = \inf\{k;\ \Pi^{[n]}_{\text{GW}}(k) = \{\{1, \dots, n\}\}\}.$$

Notice that for r-ary trees, since all the internal nodes have the same degree the cutting procedure given in Section 1.1, corresponds to choose an internal node uniformly, which is the cutting procedure in [25].

From [21, 24, 22] on the asymptotics of the number of coalescence events in β-coalescent, we can then deduce the following result which extends part of the result in [25] to GW tree with infinite variance of the reproduction law.

Proposition 1.4. *Let $\alpha = 1/\gamma \in [1/2, 1)$. We have the following convergence in distribution:*

$$n^{\alpha-1} Z_n \xrightarrow[n\to+\infty]{(d)} Z,$$

with the distribution of Z characterized by, for $n \in \mathbb{N}^*$:

$$\mathbb{E}\left[Z^n\right] = \alpha^n \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma((n+1)(1-\alpha))}.$$

The distribution of Z corresponds to the expected limit distribution in the Conjecture of [3] for the number of cuts needed to isolate the root in general GW trees. (Notice that in the conjecture, one choose an internal node $x \in T$ with probability proportional to $k_x(T)$ whereas in Section 1.1 one choose an internal node $x \in T$ with probability proportional to $k_x(T) - 1$.) In particular, Z is distributed as the height of a random leaf of the normalized Lévy tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$.

The proof of the Proposition is given in Section 5

1.4. **Number of blocks in the last coalescence event.** Using the pruning of GW tree conditioned to have an infinite number of leaves (which is very close to Kesten result on GW tree conditionally on the non extinction) we get the asymptotic of the number B_n of blocks involved is the last coalescence event of $\Pi^{[n]}$.

The proof of the following Proposition is given in Section 6.

Proposition 1.5. *Let $\alpha = 1/\gamma \in [1/2, 1)$. We have the following convergence in distribution:*

$$B_n \xrightarrow[n\to+\infty]{(d)} B,$$

with the distribution of B given by its generating function $\varphi_\alpha(r) = \mathbb{E}\left[r^B\right]$, with for $r \in [0,1]$:

$$\varphi_\alpha(r) = (1-\alpha)r \int_0^1 \frac{dx}{1-(1-x)^\alpha} \left(\frac{1}{(1-rx)^\alpha} - 1 \right). \tag{5}$$

See also [4] for more results in this direction when $\alpha = 1/2$ including the number of singletons involved in the last coalescence event as well as a closed form for $\varphi_{1/2}$.

One can also check, using results from Section 6, that in the last coalescence event of the n first integers one block is of order n, while the other blocks are of order 1.

Remark 1.6. For $\alpha = 0$ (that is let α goes to 0 in (5)), we get:

$$\varphi_0(r) = r \int_0^1 \frac{\log(1-rx)}{\log(1-x)}\, dx.$$

Using elementary computations, one can see that $\varphi_0(r) = r \sum_{n\in\mathbb{N}^*} q_n r^n$ with for $n \in \mathbb{N}^*$:

$$q_n = -\frac{1}{n}\int_0^1 \frac{x^n}{\log(1-x)}\, dx = \frac{1}{n}\int_0^{+\infty} \left(1-\mathrm{e}^{-v}\right)^n \mathrm{e}^{-v}\ \frac{dv}{v} = \frac{1}{n}\sum_{k=1}^n \binom{n}{k} (-1)^{k+1} \log(k+1).$$

Therefore φ_0 is the generating function of the asymptotic number of blocks of the last coalescence event in the Bolthausen-Sznitman coalescent whose distribution is given in Theorem 3.1 and Proposition 3.2 of [23]. Then notice that the Bolthausen-Sznitman coalescent corresponds indeed to the $\beta(1+\alpha, 1-\alpha)$-coalescent (with coalescent measure given by (4)) with $\alpha = 0$.

For $\alpha = -1$, we get $\varphi_{-1}(r) = r^2$. Notice that φ_{-1} is trivially the generating function of the number of blocks of the last (in fact all) coalescence event in the Kingman's coalescent, as all the coalescence events are binary. Furthermore, Kingman's coalescent can be seen as the limit of the $\beta(1+\alpha, 1-\alpha)$-coalescent as α goes down to -1.

We wonder if φ_α is the generating function of the asymptotic number of blocks of the last coalescence event in the $\beta(1+\alpha, 1-\alpha)$-coalescent (with coalescent measure given by (4)) with $\alpha \in (-1, 1)$. Notice also that multiplying the coalescent measure by a constant doesn't change the distribution of the number of blocks in the last coalescence event. Notice also that $\varphi'_\alpha(1) = +\infty$ for $\alpha \geq 0$ (when the coalescent doesn't come down from infinity) and $\varphi'_\alpha(1) < +\infty$ for $\alpha < 0$ (when the coalescent comes down from infinity).

1.5. **Organization of the paper.** Section 2 gives a representation of the pruning at node procedure for GW tree in continuous time motivated by [8]. This procedure corresponds in fact to the one presented in Introduction, Section 1.1. Section 3 is devoted to the proof of Theorem 1.1. Section 4 devoted to the proof of Theorem 1.2 is more technical as it relies on continuum random Lévy trees and the pruning of such trees as developed in [1]. Eventually Sections 5 and 6 are devoted to the proofs of Propositions 1.4 and 1.5.

2. Pruning at node of discrete GW trees

2.1. **Discrete trees.** Let us recall here the formalism for ordered discrete trees. We set

$$\mathcal{U} = \bigcup_{n\geq 0} (\mathbb{N}^*)^n$$

the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. For $u \in \mathcal{U}$ let $|u|$ be the length or generation of u defined as the integer n such that $u \in (\mathbb{N}^*)^n$. If u and v are two sequences of $\mathcal{U}$, we denote by uv the concatenation of the two sequences, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$. The set of ancestors of u is the set:

$$A_u = \{v \in \mathcal{U}; \text{there exists } w \in \mathcal{U} \text{ such that } u = vw\}. \tag{6}$$

A discrete tree $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies:

- $\emptyset \in \mathbf{t}$,
- If $u \in \mathbf{t}$, then $A_u \subset \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists a non-negative integer $k_u(\mathbf{t})$ such that, for all positive integer i, $ui \in \mathbf{t}$ iff $1 \leq i \leq k_u(\mathbf{t})$.

The integer $k_u(\mathbf{t})$ represents the number of offsprings of the node u in the tree $\mathbf{t}$. We define $\mathcal{L}(\mathbf{t})$ the set of leaves of $\mathbf{t}$ and $\mathcal{N}(\mathbf{t})$ the set of internal nodes of $\mathbf{t}$ as:

$$\mathcal{L}(\mathbf{t}) = \{u \in \mathbf{t},\ k_u(\mathbf{t}) = 0\} \quad \text{and} \quad \mathcal{N}(\mathbf{t}) = \mathbf{t} \setminus \mathcal{L}(\mathbf{t}).$$

Let $L(\mathbf{t}) = \text{Card}\,(\mathcal{L}(\mathbf{t}))$ be the number of leaves of the tree $\mathbf{t}$, and notice that:

$$L(\mathbf{t}) - 1 = \sum_{u\in\mathcal{N}(\mathbf{t})} (k_u(\mathbf{t}) - 1). \tag{7}$$

We denote by $\mathbb{T}$ the set of discrete trees and by $\mathbb{T}_n = \{\mathbf{t} \in \mathbb{T}; L(\mathbf{t}) = n\}$ the set of discrete trees with n leaves.

2.2. **A discrete tree-valued process.** We consider the pruning procedure developed in [7]. Let $\mathbf{t} \in \mathbb{T}$. Under some probability measure $\mathbf{P}^{\mathbf{t}}$, we consider a family $(\xi_u, u \in \mathcal{U})$ of independent non-negative real random variables (possibly infinite) such that $\mathbf{P}^{\mathbf{t}}$-a.s. for $u \notin \mathbf{t}$ or $u \in \mathbf{t}$, such that $k_u(\mathbf{t}) \in \{0, 1\}$, $\xi_u = +\infty$ and for $u \in \mathbf{t}$ such that $k_u(\mathbf{t}) \geq 2$:

$$\mathbf{P}^{\mathbf{t}}(\xi_u \geq \theta) = (1 + \theta)^{1-k_u(\mathbf{t})}.$$

At time θ, we define the pruned tree $\mathcal{P}_\theta(\mathbf{t})$ as the sub-tree given by:

$$\mathcal{P}_\theta(\mathbf{t}) = \{u \in \mathbf{t};\ \xi_v > \theta \text{ for all } v \in A_u\}.$$

For $u \in \mathcal{N}(\mathbf{t})$, let D_u be the event that u is marked first, that is:

$$D_u = \{\xi_u = \min_{v\in\mathcal{N}(\mathbf{t})} \xi_v\}.$$

Lemma 2.1. *Let $u \in \mathcal{N}(\mathbf{t})$. We have:*

$$\mathbf{P}^{\mathbf{t}}(D_u) = \frac{k_u(\mathbf{t}) - 1}{L(\mathbf{t}) - 1}\cdot$$

This lemma implies that the cutting procedure given in Section 1.1, corresponds to the successive states of the process $(\mathcal{P}_\theta(\mathbf{t}), \theta \geq 0)$.

Proof. We have, using (7) for the last equality:

$$\begin{aligned}\mathbf{P}^{\mathbf{t}}(D_u) &= \mathbf{P}^{\mathbf{t}}(\xi_u \le \xi_v \quad \forall v \ne u, v \in \mathcal{N}(\mathcal{T})) \\ &= \mathbf{E}^{\mathbf{t}}\left[(1+\xi_u)^{-\sum_{v\ne u, v\in\mathcal{N}(\mathbf{t})}(k_v(\mathbf{t})-1)}\right] \\ &= (k_u(\mathbf{t})-1)\int_{[0,+\infty)} (1+\theta)^{-\sum_{v\in\mathcal{N}(\mathbf{t})}(k_v(\mathbf{t})-1)-1}\, d\theta \\ &= \frac{k_u(\mathbf{t})-1}{\sum_{v\in\mathcal{N}(\mathbf{t})}(k_v(\mathbf{t})-1)} \\ &= \frac{k_u(\mathbf{t})-1}{L(\mathbf{t})-1}.\end{aligned}$$

□

2.3. **Construction of the partition-valued process $\Pi^{[n]}_{\mathbf{GW}}$.** Let $\alpha \in [1/2, 1)$. Recall that function g defined by (3) is the generating function of a probability measure ν_g on $\mathbb{N}$. We denote by $G_g(dT)$ the distribution on $\mathbb{T}$ of the critical GW tree with offspring distribution ν_g. We will denote by $\mathbf{P}$ the probability measure on $\mathbb{T} \times [0, +\infty]^{\mathcal{U}}$:

$$\mathbf{P}(dT, d\xi) = G_g(dT)\mathbf{P}^T(d\xi).$$

Under $\mathbf{P}$, the random tree T is a GW tree whose reproduction law ν_g has generating function g given by (3). According to Propositions 2.1 and 3.2 in [8], $(\mathcal{P}_\theta(T), \theta \ge 0)$ is a Markov process and $\mathcal{P}_\theta(T)$ is a GW tree whose reproduction law has generating function g_θ, with:

$$g_\theta(r) = 1 + (1+\theta)\left[g\left(\frac{r}{1+\theta}\right) - g\left(\frac{1}{1+\theta}\right)\right].$$

Notice that:

$$g_\theta(r) = r + \alpha\frac{(1-r+\theta)^{1/\alpha} - \theta^{1/\alpha}}{(1+\theta)^{(1/\alpha)-1}}. \tag{8}$$

For every positive integer n, we set:

$$\mathbf{P}_n(\bullet) = \mathbf{P}(\bullet \mid L(T) = n).$$

Under $\mathbf{P}_n$, the distribution of the tree T is given by the following formula (see [16], Theorem 3.3.3, or [28]), for $\mathbf{t} \in \mathbb{T}_n$:

$$\mathbf{P}_n(T = \mathbf{t}) = n!\left(\prod_{v\in\mathcal{N}(\mathbf{t})}\frac{p_{k_v(\mathbf{t})}}{k_v(\mathbf{t})!}\right)\frac{\alpha^{n-1}\Gamma(1-\alpha)}{\Gamma(n-\alpha)} \tag{9}$$

where $p_1 = 0$ and, for $k \ge 2$, $p_k = |(1-\gamma)(2-\gamma)\cdots(k-\gamma)|$.

Let $n \in \mathbb{N}^*$. Let T be a random tree distributed as $\mathbf{P}_n$. Conditionally on T, we define a uniform random labeling $U_1, \dots, U_n$ of the leaves of T, independently of the variables $(\xi_u, u \in T)$. Recall the set of ancestors defined in (6) and the pruning procedure $\mathcal{P}_\theta$ introduced in Section 2.2. We define the equivalence relation $\mathcal{R}^{[n]}_\theta$ on $\{1, 2, \dots, n\}$ by: $i\mathcal{R}^{[n]}_\theta j$ if $A_{U_i} \bigcap A_{U_j} \bigcap \mathcal{L}(\mathcal{P}_\theta(T))$ is non empty, that is U_i and U_j have a leaf of $\mathcal{P}_\theta(T)$ as common ancestor. Then, for every $\theta \ge 0$, let $\hat{\Pi}^{[n]}_{\mathrm{GW}}(\theta)$ be the equivalence classes of the equivalence relation $\mathcal{R}^{[n]}_\theta$ of the n first integers. Let $\Pi^{[n]}_{\mathrm{GW}} = (\Pi^{[n]}_{\mathrm{GW}}(k), k \in \mathbb{N})$ be the discrete process associated with $\hat{\Pi}^{[n]}_{\mathrm{GW}} = (\hat{\Pi}^{[n]}_{\mathrm{GW}}(\theta), \theta \ge 0)$ until it reaches the absorbing state (which is the trivial partition consisting in one block) and afterward the discrete process remains constant.

We end this section with an elementary lemma which will be used in the proof of Theorem 1.1.

Lemma 2.2. *We have for $n \geq 2$:*

$$\mathbf{E}_n \left[k_\emptyset(T) - 1\right] = \frac{1-\alpha}{\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \frac{\Gamma(n-1+\alpha)}{\Gamma(n-\alpha)} \cdot \tag{10}$$

Proof. We consider the generating function of $(k_\emptyset(T), L(T))$ under $\mathbf{P}$, that is $H(s,t) = \mathbf{E}\left[s^{k_\emptyset(T)} t^{L(T)}\right]$. Using the branching property of GW trees, we have:

$$H(s,t) = \mathbf{E}\left[s^{k_\emptyset(T)} \mathbf{E}[t^{L(T)}]^{k_\emptyset(T)} \mathbf{1}_{\{k_\emptyset(T) \neq 0\}}\right] + t\mathbf{P}(k_\emptyset(T) = 0). \tag{11}$$

Notice that $g(s) = \mathbf{E}\left[s^{k_\emptyset(T)}\right] = H(s,1)$. We set $h(t) = H(1,t) = \mathbf{E}\left[t^{L(T)}\right]$ the generating function of $L(T)$. So that (11) becomes:

$$H(s,t) = g(s\, h(t)) - g(0)(1-t). \tag{12}$$

Taking $s = 1$ in (12), we get:

$$g(h(t)) - h(t) = g(0)(1-t). \tag{13}$$

Using expression (3), we get:

$$h(t) = 1 - (1-t)^\alpha \quad \text{and} \quad H(s,t) = s\, h(t) + \alpha(1 - s\, h(t))^{1/\alpha} - \alpha(1-t).$$

We deduce that:

$$\begin{aligned}
\mathbf{E}\left[k_\emptyset(T) t^{L(T)}\right] = \frac{\partial H}{\partial s}(1,t) &= h(t) - h(t)(1-h(t))^{(1/\alpha)-1} \\
&= \mathbf{E}\left[t^{L(T)}\right] - \left[1 - (1-t)^\alpha\right](1-t)^{1-\alpha} \\
&= \mathbf{E}\left[t^{L(T)}\right] - (1-t)^{1-\alpha} + 1 - t.
\end{aligned}$$

This gives:

$$\mathbf{E}\left[(k_\emptyset(T) - 1) t^{L(T)}\right] = -(1-t)^{1-\alpha} + 1 - t.$$

For $n \geq 2$, we get:

$$\begin{aligned}
\mathbf{E}\left[(k_\emptyset(T) - 1)\mathbf{1}_{\{L(T)=n\}}\right] &= \frac{1}{n!}\left(\frac{d^n}{dt^n}\mathbf{E}\left[(k_\emptyset(T) - 1)t^{L(T)}\right]\right)_{|t=0} \\
&= \frac{1}{n!}(1-\alpha)\prod_{k=0}^{n-2}(\alpha + k) \\
&= \frac{1}{n!}(1-\alpha)\frac{\Gamma(n-1+\alpha)}{\Gamma(\alpha)} \cdot
\end{aligned}$$

We also get for $n \geq 2$:

$$\mathbf{P}(L(T) = n) = \frac{1}{n!} h^{(n)}(0) = \frac{1}{n!}\alpha \prod_{k=1}^{n-1}(k - \alpha) = \frac{1}{n!}\alpha \frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)} \cdot$$

We deduce that:

$$\mathbf{E}_n\left[k_\emptyset(T) - 1\right] = \frac{\mathbf{E}\left[(k_\emptyset(T) - 1)\mathbf{1}_{\{L(T)=n\}}\right]}{\mathbf{P}(L(T) = n)} = \frac{1-\alpha}{\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\frac{\Gamma(n-1+\alpha)}{\Gamma(n-\alpha)} \cdot$$

□

3. Proof of Theorem 1.1

Let $\alpha \in [1/2, 1)$ and Λ given by (4). Notice that the probability that the first coalescence event for $\Pi_{\mathrm{dis}}^{[n]}$ corresponds to the collision of k given blocks is $\lambda_{n,k}/\lambda_n$, with $\lambda_{n,k}$ and λ_n given respectively by (1) and (2).

Theorem 1.1 is a direct consequence of Lemma 3.3 which states that the probability that the first coalescence event for $\Pi_{\mathrm{GW}}^{[n]}$ corresponds to the collision of k given blocks is $\lambda_{n,k}/\lambda_n$, and of Lemma 3.4, which states that after the first coalescence event, the law of the pruned tree under $\mathbf{P}_n$ conditionally given that it has k leaves is exactly $\mathbf{P}_k$.

The proof of Lemme 3.3 (resp. 3.4) is given in Section 3.1 (resp. 3.2).

3.1. **Computation of the coalescence rates.** We first give an intermediate lemma. For $\alpha \in (0, 1)$ and $\lambda > \alpha - 1$, we set:

$$\phi_{1+\alpha,1-\alpha}(\lambda) = \int_0^1 \left(1 - (1-x)^{\lambda}\right) x^{\alpha-2}(1-x)^{-\alpha}\, dx. \tag{14}$$

Lemma 3.1. *For $\alpha \in (0, 1)$ and $\lambda > \alpha - 1$, we have:*

$$\phi_{1+\alpha,1-\alpha}(\lambda) = \lambda \frac{\Gamma(\alpha)\Gamma(\lambda+1-\alpha)}{(1-\alpha)\Gamma(\lambda+1)}. \tag{15}$$

Notice that for $\lambda > 0$, (15) reduces to:

$$\phi_{1+\alpha,1-\alpha}(\lambda) = \frac{\Gamma(\alpha)\Gamma(\lambda+1-\alpha)}{(1-\alpha)\Gamma(\lambda)}. \tag{16}$$

Proof. We set:

$$I = \int_0^1 \left((1-u)^{-\alpha} - 1\right)\, u^{\alpha-2}\, du.$$

Notice that I is finite and $\phi_{1+\alpha,1-\alpha}(\alpha) = I$. For $\lambda > \alpha$, using an integration by part, we have:

$$\begin{aligned}
\phi_{1+\alpha,1-\alpha}(\lambda) &= \int_0^1 \left(1 - (1-x)^{\lambda}\right) x^{\alpha-2}(1-x)^{-\alpha}\, dx \\
&= \int_0^1 \left((1-x)^{-\alpha} - 1\right) x^{\alpha-2}\, dx + \int_0^1 \left(1 - (1-x)^{\lambda-\alpha}\right) x^{\alpha-2} dx \\
&= I - \frac{1}{1-\alpha} + \frac{\lambda-\alpha}{1-\alpha} \int_0^1 (1-x)^{\lambda-\alpha-1} x^{\alpha-1}\, dx \\
&= I - \frac{1}{1-\alpha} + \frac{\Gamma(\alpha)\Gamma(\lambda+1-\alpha)}{(1-\alpha)\Gamma(\lambda)}.
\end{aligned}$$

We now compute I. For $\lambda = 1$, we also have:

$$\phi_{1+\alpha,1-\alpha}(1) = \int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\, dx = \Gamma(\alpha)\Gamma(1-\alpha).$$

We deduce that:

$$I - \frac{1}{1-\alpha} + \frac{\Gamma(\alpha)\Gamma(2-\alpha)}{(1-\alpha)\Gamma(1)} = \phi_{1+\alpha,1-\alpha}(1) = \Gamma(\alpha)\Gamma(1-\alpha).$$

This readily implies that $I = 1/(1-\alpha)$ and thus (15) holds for $\lambda \geq \alpha$. Then uses that the right-hand sides of (14) and (15) are analytic for $\lambda > \alpha - 1$ to get that (15) also holds for $\lambda > \alpha - 1$. □

Recall $\lambda_{n,k}$ and λ_n are given respectively by (1) and (2), for Λ given by (4).

Lemma 3.2. *Let $\alpha \in [1/2, 1)$. We have for $2 \leq k \leq n$:*

$$\frac{\lambda_{n,k}}{\lambda_n} = \frac{1-\alpha}{\Gamma(\alpha+1)} \frac{\Gamma(k+\alpha-1)\Gamma(n-k-\alpha+1)}{\Gamma(n-\alpha)} \frac{1}{n-1}. \tag{17}$$

Proof. We have

$$\begin{aligned}\lambda_{n,k} = \int_0^1 u^{k-2}(1-u)^{n-k}\Lambda(du) &= \int_0^1 u^{k-2+\alpha}(1-u)^{n-k-\alpha} du \\ &= \beta(k+\alpha-1, n-k-\alpha+1) \\ &= \frac{\Gamma(k+\alpha-1)\Gamma(n-k-\alpha+1)}{\Gamma(n)},\end{aligned}$$

and

$$\lambda_n = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} = \int_0^1 (1-(1-u)^n - nu(1-u)^{n-1})u^{-2}\Lambda(du).$$

Then using notation (14) and (16), we deduce that:

$$\begin{aligned}\lambda_n &= \phi_{1+\alpha,1-\alpha}(n) - n\int_0^1 u^{\alpha-1}(1-u)^{n-1-\alpha}\, du \\ &= \frac{\Gamma(\alpha)\Gamma(n+1-\alpha)}{(1-\alpha)\Gamma(n)} - n\frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)} \\ &= \left(\frac{n-\alpha}{1-\alpha} - n\right)\frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)} \\ &= (n-1)\frac{\alpha}{1-\alpha}\frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)}.\end{aligned}$$

The expression obtained for $\lambda_{n,k}$ then gives the result. □

If $\mathbf{t}_1$ and $\mathbf{t}_2$ are two discrete trees and $u \in \mathcal{L}(\mathbf{t}_1)$ is a leaf of $\mathbf{t}_1$, we shall denote by $\mathbf{t}_1 \circledast_u \mathbf{t}_2$ the tree obtained by grafting the tree $\mathbf{t}_2$ on the leaf u of $\mathbf{t}_1$, that is:

$$\mathbf{t}_1 \circledast_u \mathbf{t}_2 = \{v,\ v \in \mathbf{t}_1\} \cup \{uv,\ v \in \mathbf{t}_2\}. \tag{18}$$

Lemma 3.3. *Let $\alpha \in [1/2, 1)$. The probability under $\mathbf{P}_n$ that the first coalescence event in $\Pi_{GW}^{[n]}$ is the coalescence of k given integers into one block is $\lambda_{n,k}/\lambda_n$.*

Proof. Let A_k be the event that the first coalescence event corresponds to the k first integers merging together. By echangeability, the lemma is proved as soon as we check that $\mathbf{P}_n(A_k) = \lambda_{n,k}/\lambda_n$.

The event A_k is realized, if and only if:

- The initial tree T is of the form $\mathbf{t}_1 \circledast_u \mathbf{t}_2$ for some $\mathbf{t}_2 \in \mathbb{T}_k$ and $\mathbf{t}_1 \in \mathbb{T}_{n-k+1}$ and $u \in \mathcal{L}(\mathbf{t}_1)$.
- The leaves of $\mathbf{t}_2$ are labeled from 1 to k (and therefore, the leaves of $\mathbf{t}_1$ except u are labeled from $k+1$ to n). This occurs with probability $\frac{k!(n-k)!}{n!}$.
- The first chosen node of $\mathbf{t}_1 \circledast_u \mathbf{t}_2$ is u. This occurs according to Lemma 2.1 with probability $\frac{k_\emptyset(\mathbf{t}_2)-1}{n-1}$.

Thus, using (9) for the probability of having a given tree, we have:

$$\begin{aligned}
\mathbf{P}_n(A_k) &= \sum_{\substack{\mathbf{t}_1\in\mathbb{T}_{n-k+1}\\ \mathbf{t}_2\in\mathbb{T}_k\\ u\in\mathcal{L}(\mathbf{t}_1)}} \mathbf{P}_n(T=\mathbf{t}_1\circledast_u \mathbf{t}_2)\frac{k!(n-k)!}{n!}\frac{k_\emptyset(\mathbf{t}_2)-1}{n-1}\\
&= \sum_{\substack{\mathbf{t}_1\in\mathbb{T}_{n-k+1}\\ \mathbf{t}_2\in\mathbb{T}_k\\ u\in\mathcal{L}(\mathbf{t}_1)}} n!\left(\prod_{v\in\mathcal{N}(\mathbf{t}_1\circledast_u\mathbf{t}_2)}\frac{p_{k_v(\mathbf{t}_1\circledast_u\mathbf{t}_2)}}{k_v(\mathbf{t}_1\circledast_u\mathbf{t}_2)!}\right)\frac{\alpha^{n-1}\Gamma(1-\alpha)}{\Gamma(n-\alpha)}\frac{k!(n-k)!}{n!}\frac{k_\emptyset(\mathbf{t}_2)-1}{n-1}\\
&= (n-k+1)\sum_{\substack{\mathbf{t}_1\in\mathbb{T}_{n-k+1}\\ \mathbf{t}_2\in\mathbb{T}_k}}\frac{n!}{k!(n-k+1)!}\mathbf{P}_{n-k+1}(T=\mathbf{t}_1)\mathbf{P}_k(T=\mathbf{t}_2)\\
&\qquad \frac{\alpha^{n-1}\Gamma(1-\alpha)}{\Gamma(n-\alpha)}\frac{\Gamma(n-k-\alpha+1)}{\alpha^{n-k}\Gamma(1-\alpha)}\frac{\Gamma(k-\alpha)}{\alpha^{k-1}\Gamma(1-\alpha)}\frac{k!(n-k)!}{n!}\frac{k_\emptyset(\mathbf{t}_2)-1}{n-1}\\
&= \frac{\Gamma(n-k-\alpha+1)\Gamma(k-\alpha)}{\Gamma(n-\alpha)\Gamma(1-\alpha)}\frac{1}{n-1}\mathbf{E}_k\left[k_\emptyset(T)-1\right].
\end{aligned}$$

We then use Lemma 2.2 and Lemma 3.2 to conclude. □

3.2. **Law of the tree after the first coalescence event.** Let S be the time of the first coalescence event and recall that $\mathcal{P}_S(T)$ denote the pruned tree at the first coalescence event.

Lemma 3.4. *Let* $\mathbf{t}\in\mathbb{T}_k$*. We have:*

$$\mathbf{P}_n(\mathcal{P}_S(T)=\mathbf{t}\,\big|\,L(\mathcal{P}_S(T))=k)=\mathbf{P}_k(T=\mathbf{t}). \tag{19}$$

Proof. Let $\mathbf{t}\in\mathbb{T}_k$. We obtain $\mathbf{t}$ just after the first coalescence event if T is of the form $\mathbf{t}\circledast_u\mathbf{s}$ for some $s\in\mathbb{T}_{n-k+1}$, $u\in\mathcal{L}(\mathbf{t})$ and u is the first chosen internal node. This gives:

$$\begin{aligned}
\mathbf{P}_n(\mathcal{P}_S(T)=\mathbf{t}) &= \sum_{\substack{u\in\mathcal{L}(\mathbf{t})\\ \mathbf{s}\in\mathbb{T}_{n-k+1}}}\mathbf{P}_n(T=\mathbf{t}\circledast_u\mathbf{s})\frac{k_\emptyset(\mathbf{s})-1}{n-1}\\
&= k\sum_{\mathbf{s}\in\mathbb{T}_{n-k+1}} n!\left(\prod_{v\in\mathcal{N}(\mathbf{t})}\frac{p_{k_v(\mathbf{t})}}{k_v(\mathbf{t})!}\prod_{v\in\mathcal{N}(\mathbf{s})}\frac{p_{k_v(\mathbf{s})}}{k_v(\mathbf{s})!}\right)\frac{\alpha^{n-1}\Gamma(1-\alpha)}{\Gamma(n-\alpha)}\frac{k_\emptyset(\mathbf{s})-1}{n-1}\\
&= k\sum_{\mathbf{s}\in\mathbb{T}_{n-k+1}}\frac{n!}{k!(n-k+1)!}\mathbf{P}_k(T=\mathbf{t})\mathbf{P}_{n-k+1}(T=\mathbf{s})\\
&\qquad \frac{\alpha^{n-1}\Gamma(1-\alpha)}{\Gamma(n-\alpha)}\frac{\Gamma(k-\alpha)}{\alpha^{k-1}\Gamma(1-\alpha)}\frac{\Gamma(n-k+1-\alpha)}{\alpha^{n-k}\Gamma(1-\alpha)}\frac{k_\emptyset(\mathbf{s})-1}{n-1}\\
&= \frac{n!}{(k-1)!(n-k+1)!}\frac{\Gamma(n-k+1-\alpha)\Gamma(k-\alpha)}{\Gamma(n-\alpha)\Gamma(1-\alpha)}\\
&\qquad \frac{1}{n-1}\mathbf{E}_{n-k+1}[k_\emptyset(T)-1]\mathbf{P}_k(T=\mathbf{t}).
\end{aligned}$$

As the term in front of $\mathbf{P}_k(T=\mathbf{t})$ does not depend on $\mathbf{t}$, it has to be equal to $\mathbf{P}_n(L(\mathcal{P}_S(T))=k)$ and therefore (19) holds. □

4. Pruning of rooted real trees and proof of Theorem 1.2

The aim of this section is to use the pruning procedure for Lévy trees developed in [1] to give a consistent representation of the family of coalescent processes $(\hat{\Pi}^{[n]}_{\mathrm{GW}}, n \in \mathbb{N}^*)$, see Corollary 4.4 and thus deduce Theorem 1.2.

4.1. **The CRT framework.**

4.1.1. *Real trees.* Real trees have been introduced first in the field of geometric group theory (see for instance [14]) and then used later for defining continuum random trees (the framework first appeared in [18]). A real tree is a complete metric space $(\mathcal{T}, d)$ satisfying the following two properties for every $x, y \in \mathcal{T}$:

- (unique geodesic) There is a unique isometric map $f_{x,y}$ from $[0, d(x,y)]$ into $\mathcal{T}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x,y)) = y$.
- (no loop) If φ is a continuous injective map from $[0,1]$ into $\mathcal{T}$ such that $\varphi(0) = x$ and $\varphi(1) = y$, then
$$\varphi([0,1]) = f_{x,y}([0, d(x,y)]).$$

A rooted real tree is a real tree with a distinguished vertex denoted $\emptyset$ and called the root.

For every $x, y \in \mathcal{T}$, we denote by $[\![x, y]\!]$ the range of the map $f_{x,y}$ (i.e. the only path in the tree that links x to y) and we set $[\![x, y[\![= [\![x, y]\!] \setminus \{y\}$.

If $\mathcal{T}$ is a rooted real tree, for $x \in \mathcal{T}$, we define the degree of x, denoted by n_x, as the number of connected components of $\mathcal{T} \setminus \{x\}$. The leaves of $\mathcal{T}$ is $\mathcal{L}(\mathcal{T}) = \{x \in \mathcal{T} \setminus \{\emptyset\};\, n_x = 1\}$. If $n_x \geq 3$, we say that x is a branching point of $\mathcal{T}$. We denote by $\mathcal{B}_{\mathrm{br}}(\mathcal{T})$ the set of branching points of $\mathcal{T}$. The height of $\mathcal{T}$ is $H_{\max}(\mathcal{T}) = \sup\{d(\emptyset, x);\ x \in \mathcal{T}\}$. Let $(x_i, i \in I)$ be a family of elements of $\mathcal{T}$, we define their most recent common ancestor denoted by $\mathrm{MRCA}(x_i, i \in I)$ as the element x of $\mathcal{T}$ such that $[\![\emptyset, x]\!] = \bigcap_{i \in I} [\![\emptyset, x_i]\!]$.

A weighted rooted real tree $(\mathcal{T}, d, \mathbf{m})$ is a rooted real tree $(\mathcal{T}, d)$ endowed with a mass measure $\mathbf{m}$ on $\mathcal{T}$.

4.1.2. *Stable Lévy tree.* Set $\psi(\lambda) = \lambda^\gamma$ with $\gamma \in (1,2)$. We refer to [17] and [9] for the existence of a measure $\mathbb{N}[d\mathcal{T}]$ on the set of weighted locally compact rooted real tree such that $\mathcal{T}$ is under $\mathbb{N}[d\mathcal{T}]$ a Lévy tree associated with the branching mechanism ψ. For the Lévy tree $(\mathcal{T}, d, \mathbf{m})$, $\mathbb{N}[d\mathcal{T}]$ -a.e., the mass measure has support $\mathcal{L}(\mathcal{T})$ and has no atom. Furthermore, $\mathbb{N}[d\mathcal{T}]$-a.e., all the branching points of the tree are of infinite degree. Following [17], there exists a local time process $(\ell^a, a \geq 0)$ with values on finite measures on $\mathcal{T}$, which is càdlàg for the weak topology on finite measures on $\mathcal{T}$ and such that $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:
$$\mathbf{m}(dx) = \int_0^\infty \ell^a(dx)\, da,$$
$\ell^0 = 0$, $\inf\{a > 0; \ell^a = 0\} = \sup\{a \geq 0; \ell^a \neq 0\} = H_{\max}(\mathcal{T})$ and for every fixed $a \geq 0$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e. the measure ℓ^a is supported on $\{x \in \mathcal{T}; d(\emptyset, x) = a\}$ and the real valued process $(\langle \ell^a, 1\rangle, a \geq 0)$ is distributed as a continuous state branching process (CSBP) with branching mechanism ψ under its canonical measure. In particular, as the total size of a critical CSBP is finite, we get that $\mathbb{N}$-a.e. $\sigma = \mathbf{m}(\mathcal{T})$ is finite.

The set $\{d(\emptyset, x),\ x \in \mathrm{Br}(\mathcal{T})\}$ coincides $\mathbb{N}^\psi$-a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, $\mathbb{N}^\psi$-a.e., for every such discontinuity time b, there is a unique $x \in \mathcal{B}_{\mathrm{br}}(\mathcal{T})$ such that $d(\emptyset, x) = b$ and $\Delta_x > 0$, such that:
$$\ell^b = \ell^{b-} + \Delta_x \delta_x,$$

where $\Delta_x > 0$ is called the mass of the node x. Intuitively Δ_x represent the mass of the progeny of x.

The scaling property of the stable Lévy implies that there exists a well defined probability measure $\mathbb{N}^{(1)}$ defined as the measure $\mathbb{N}$ conditioned on $\{\sigma = 1\}$. The probability measure is also referred as the normalized excursion measure for Lévy trees.

4.2. **The partition-valued process.** Set $\psi(\lambda) = \lambda^\gamma$ with $\gamma \in (1, 2)$.

4.2.1. *Pruning of the stable Lévy tree.* We consider the pruning procedure introduced in [1] (this procedure is defined when there is no Brownian part in the Lévy process with index given by the branching mechanism ψ). Under $\mathbb{N}$ or $\mathbb{N}^{(1)}$, conditionally given $\mathcal{T}$, we consider a family $(E_x, x \in \mathcal{B}_{\mathrm{br}}(\mathcal{T}))$ of independent real random variables such that the random variable E_x is exponentially distributed with parameter Δ_x. This random variable represents the time at which the branching point x is marked. For every $\theta > 0$, we set

$$\mathcal{T}_\theta = \{x \in \mathcal{T},\ \forall y \in [\![\emptyset, x[\![,\ E_y \geq \theta\}.$$

The set $\mathcal{T}_\theta$ is still a real tree which represents the tree $\mathcal{T}$ pruned at time θ: we cut $\mathcal{T}$ at the points that are marked before time θ and keep the connected component of the tree that contains the root. We set $\mathcal{T}_0 = \mathcal{T}$. By [1], Theorem 1.5, the tree $\mathcal{T}_\theta$ is distributed under $\mathbb{N}$ as a Lévy tree with branching mechanism ψ_θ defined by:

$$\psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta).$$

Moreover, by [2], the process $(\mathcal{T}_\theta, \theta \geq 0)$ is under $\mathbb{N}$ a Markov process.

4.2.2. *Definition of the partition-valued process.* Under $\mathbb{N}$ or $\mathbb{N}^{(1)}$, conditionally on $\mathcal{T}$, let $(F_i, i \in \mathbb{N}^*)$ be independent random variables on $\mathcal{T}$ distributed according to the probability mass measure $\mathbf{m}/\mathbf{m}(\mathcal{T})$, and independent of the marks $(E_x, x \in \mathcal{B}_{\mathrm{br}}(\mathcal{T}))$. Notice that $\mathbb{N}$-a.e. or $\mathbb{N}^{(1)}$-a.s. $(F_i, i \in \mathbb{N}^*)$ are leaves of $\mathcal{T}$. For $\theta \geq 0$, we define the equivalence relation $\mathcal{R}_\theta^{\text{Lévy}}$ on $\mathbb{N}^*$ by: $i\mathcal{R}_\theta^{\text{Lévy}}j$ if $[\![\emptyset, F_i]\!] \bigcap [\![\emptyset, F_j]\!] \bigcap \mathcal{L}(\mathcal{T}_\theta)$ is non empty, that is F_i and F_j have a leaf of $\mathcal{T}_\theta$ as common ancestor. This is very close to the definition of the equivalence relation $\mathcal{R}_\theta^{[\mathrm{n}]}$ defined in Section 2.3. We denote by $\Pi_{\text{Lévy}}(\theta)$ the partition of $\mathbb{N}^*$ formed by the equivalence classes of $\mathcal{R}_\theta^{\text{Lévy}}$ and set $\Pi_{\text{Lévy}} = (\Pi_{\text{Lévy}}(\theta), \theta \geq 0)$.

4.3. **Lévy sub-trees.**

4.3.1. *Skeleton of finite real tree.* Let $\hat{\mathbf{t}}$ be a real tree with finite height and a finite number of leaves, such that the leaves $(f_i, i \in I(\hat{\mathbf{t}}))$ are indexed by a totally ordered set $I(\hat{\mathbf{t}})$. We define the skeleton $\tilde{\mathbf{t}}$ of the tree $\hat{\mathbf{t}}$ as the discrete tree (belonging to $\mathbb{T}$) where we forget the edge lengths. As the trees in $\mathbb{T}$ are ordered, we must be a bit more rigorous for the definition of $\tilde{\mathbf{t}}$.

The skeleton $\tilde{\mathbf{t}}$ of the real tree with ordered leaves $\big(\hat{\mathbf{t}}, (f_i, i \in I(\hat{\mathbf{t}}))\big)$ is defined recursively as follows. We define $k_\emptyset(\tilde{\mathbf{t}})$ as the degree of $\mathrm{MRCA}(f_i, i \in I(\hat{\mathbf{t}}))$ the ancestor of all the leaves of $\hat{\mathbf{t}}$. If $k_\emptyset(\tilde{\mathbf{t}}) = 0$, then $\tilde{\mathbf{t}}$ is reduced to $\emptyset$. In this case $\hat{\mathbf{t}}$ has one leaf, let f be its label, and the discrete tree $\tilde{\mathbf{t}}$ has thus one leaf to which we give the label f. If $k_\emptyset(\tilde{\mathbf{t}}) > 0$, then we consider the $k_\emptyset(\tilde{\mathbf{t}})$ connected components of $\hat{\mathbf{t}} \setminus \{\mathrm{MRCA}(f_i, i \in I(\hat{\mathbf{t}}))\}$ that do not contain the root and label them from 1 to $k_\emptyset(\tilde{\mathbf{t}})$ according to the lowest label of the leaves of $\hat{\mathbf{t}}$ which belongs to them. This gives an ordered family $(\hat{\mathbf{t}}_k, k \in \{1, \dots, k_\emptyset(\tilde{\mathbf{t}})\})$ of real trees, and let $\mathrm{MRCA}(f_i, i \in I(\hat{\mathbf{t}}))\})$ be the root of each one. For $k \in \{1, \dots, k_\emptyset(\tilde{\mathbf{t}})\}$, let

$I(\hat{\mathbf{t}}_k) = \{i \in I(\hat{\mathbf{t}}); f_i \in \hat{\mathbf{t}}_k)$ be the labels of the leaves of $\hat{\mathbf{t}}_k$ and the discrete tree $\tilde{\mathbf{t}}_k$ is the skeleton of $\big(\hat{\mathbf{t}}_k, (f_i, i \in I(\hat{\mathbf{t}}_k))\big)$.

Notice that $\tilde{\mathbf{t}}$ is finite, $k_u(\tilde{\mathbf{t}}) \neq 1$ for all $u \in \tilde{\mathbf{t}}$, and $\hat{\mathbf{t}}$ and $\tilde{\mathbf{t}}$ have the same number of leaves. In the previous construction to a leaf f_i of $\hat{\mathbf{t}}$ with label i corresponds a unique leaf e_i of $\tilde{\mathbf{t}}$ with label i. For $u \in \tilde{\mathbf{t}}$, we define $\tilde{\mathbf{t}}_u$ the sub-tree of $\tilde{\mathbf{t}}$ attached to the node u i.e.

$$\tilde{\mathbf{t}}_u = \{w \in \mathcal{U},\ uv \in \tilde{\mathbf{t}}\},$$

and let $I_u = \{i; e_i \in \tilde{\mathbf{t}}_u\}$. Define $\hat{\mathbf{t}}_u$ as $\mathbf{s}_u = \hat{\mathbf{t}} \backslash \bigcup_{i \not\in I_u} [\![\emptyset, f_i]\!]$ to which we add the root $\emptyset_u = \overline{\mathbf{s}_u} \backslash \mathbf{s}_u$, and $I(\hat{\mathbf{t}}_u) = \{i; e_i \in \tilde{\mathbf{t}}_u\}$. Notice that by construction $\tilde{\mathbf{t}}_u$ is the skeleton of $\big(\hat{\mathbf{t}}_u, (f_i; i \in I(\hat{\mathbf{t}}_u))\big)$. We say that $u \in \tilde{\mathbf{t}}$ are the individuals of $\hat{\mathbf{t}}$, and define their lifetime as the length h_u of the geodesic $B(u) = [\![\emptyset_u, \mathrm{MRCA}(f_i, i \in I(\hat{\mathbf{t}}_u))]\!]$. We say the corresponding node in $\hat{\mathbf{t}}$ of $u \in \tilde{\mathbf{t}}$ is $C(u) = \mathrm{MRCA}(f_i, i \in I(\hat{\mathbf{t}}_u))$.

Notice it is easy to reconstruct $\hat{\mathbf{t}}$ from $\tilde{\mathbf{t}}$ and the family of lifetime $(h_u, u \in \tilde{\mathbf{t}})$.

4.3.2. *Coalescence of Lévy tree and GW tree.* Let M be, under $\mathbb{N}$ or $\mathbb{N}^{(1)}$ conditionally on $\mathcal{T}$, a Poisson random variable with finite mean $\sigma = \mathbf{m}(\mathcal{T})$. On $\{M \geq 1\}$, let $\hat{T}_0$ be the real sub-tree of $\mathcal{T}$ generated by the root and $(F_i, 1 \leq i \leq M)$:

$$\hat{T}_0 = \bigcup_{1 \leq i \leq M} [\![\emptyset, F_i]\!].$$

Since $\mathbf{m}$ has support $\mathcal{L}(\mathcal{T})$ and has no atom, we deduce that $(F_i, 1 \leq i \leq M)$ are distinct and are the leaves of $\hat{T}_0$.

We denote by $\tilde{T}_0$ the skeleton of $\hat{T}_0$ with the labeled leaves $(F_i, 1 \leq i \leq M)$. According to [16], Theorem 3.2.1, the tree $\hat{T}_0$ is distributed under $\mathbb{N}[\,\cdot \mid M \geq 1]$ as a continuous GW tree (i.e. a GW tree with edge-lengths) such that

- The discrete tree $\tilde{T}_0$ is a GW tree with reproduction law characterized by its generating function g defined by (3) with $\alpha = 1/\gamma$.
- Lifetimes of individuals $(h_u, u \in \tilde{T}_0)$ are independent random variables with exponential distribution with parameter γ.

Remark 4.1. Using the scaling property of the Lévy tree, we have that the distributions of $\hat{T}_0$ under $\mathbb{N}[\,\cdot \mid M = n]$ and under $\mathbb{N}^{(1)}[\,\cdot \mid M = n]$ are the same.

We now consider the marks that define the pruned tree $\mathcal{T}_\theta$ and we denote by $\hat{T}_\theta$ the tree $\hat{T}_0$ pruned on the same marks, in other words, we set

$$\hat{T}_\theta = \hat{T}_0 \cap \mathcal{T}_\theta.$$

Let $\hat{\Pi}^{[n]}_{\text{Lévy}}$ be the restriction of $\Pi_{\text{Lévy}}$ to the n first integer. By construction, if A_θ is an element of $\hat{\Pi}^{[n]}_{\text{Lévy}}$, then there exists a leaf x of $\hat{T}_\theta$ such that x belongs to the sub-tree $\bigcup_{i \in A_\theta} [\![\emptyset, F_i]\!]$, and x is the only leaf of $\hat{T}_\theta$ with this property. We set A_θ for the label of x, and we consider the order of the elements of $\tilde{\Pi}^{[n]}_{\text{Lévy}}$ given by the order of their smallest integer. We set $I_\theta = I(\hat{T}_\theta)$ for the labels of the leaves of $\hat{T}_\theta$ and $(F^\theta_i, i \in I_\theta)$ for the leaves of $\hat{T}_\theta$.

We denote by $\tilde{T}_\theta$ the skeleton of $\hat{T}_\theta$ with the labeled leaves $(F^\theta_i, i \in I(\theta))$. According to [8], Proposition 4.1, the tree $\hat{T}_\theta$ is distributed under $\mathbb{N}[\,\cdot \mid M \geq 1]$ as a continuous GW tree such that

- $\tilde{T}_\theta$ is a GW tree with offspring distribution characterized by its generating function g_θ given in (8) with $\alpha = 1/\gamma$.

- The lifetimes of individuals $(h_u, u \in \hat{T}_\theta)$ are independent random variable with exponential distribution with parameter $\psi'_\theta(1) = \gamma(1+\theta)^{\gamma-1}$.

Proposition 4.2. *The process $(\tilde{T}_\theta, \theta \geq 0)$ is distributed under $\mathbb{N}[\,\cdot \,\big|\, M \geq 1]$ as the process $(\mathcal{P}_\theta(T), \theta \geq 0)$ under $\mathbf{P}$.*

Proof. Let $\theta > 0$. Theorem 6.1 of [8] describes how $\hat{T}_\theta$ is obtained from $\hat{T}_0$:

- A branching point x of $\hat{T}_0$ with $k_x = k_x(\hat{T}_0)$ children is marked at time τ_x with distribution given by:
$$\mathbb{N}[\tau_x \geq \theta \,\big|\, \hat{T}_0] = -\int_\theta^{+\infty} \frac{\psi^{(k_x+1)}(1+z)}{\psi^{(k_x)}(1)} dz = \frac{\psi^{(k_x)}(1+\theta)}{\psi^{(k_x)}(1)} = \left(\frac{1}{1+\theta}\right)^{k_x-\gamma}.$$
- A branch B of length h is marked at time τ_B with distribution given by:
$$\mathbb{N}[\tau_B \geq \theta \,\big|\, \hat{T}_0] = \exp\left(-h\int_0^\theta \psi''(1+z)dz\right) = \mathrm{e}^{-\big(\psi'(1+\theta)-\psi'(1)\big)h}.$$

Then the tree $\hat{T}_0$ is cut according to the marks present at time θ and the tree $\hat{T}_\theta$ is the connected component that contains the root. Therefore, the tree $\tilde{T}_\theta$ is obtained from the tree $\tilde{T}_0$ by a pruning at node. A node $u \in \tilde{T}_0$ is marked if the corresponding node $C(u) \in \hat{T}_0$ is marked at time θ in the previous procedure OR the branch $B(u)$ with length h_u is marked. So the node u of $\tilde{T}_0$ is marked at time $\zeta_u = \tau_{C(u)} \wedge \tau_{B(u)}$ and using that the edge lengths of $\hat{T}_0$ are independent and exponentially distributed with parameter $\gamma = \psi'(1)$, we have with $k_u = k_u(\hat{T}_0)$:
$$\begin{aligned}
\mathbb{N}[\zeta_u \geq \theta \,\big|\, \tilde{T}_0] &= \mathbb{N}[\tau_{C(u)} \geq \theta \,\big|\, \tilde{T}_0]\; \mathbb{N}[\tau_{B(u)} \geq \theta \,\big|\, \tilde{T}_0] \\
&= \left(\frac{1}{1+\theta}\right)^{k_u-\gamma} \int_0^{+\infty} dh\,\gamma\,\mathrm{e}^{-\gamma h}\,\mathrm{e}^{-\big(\psi'(1+\theta)-\gamma\big)h} \\
&= \left(\frac{1}{1+\theta}\right)^{k_u-\gamma} \left(\frac{1}{1+\theta}\right)^{\gamma-1} \\
&= \left(\frac{1}{1+\theta}\right)^{k_u-1}.
\end{aligned}$$
Since the cutting time $\tau_{C(u)}$ and $\tau_{B(u)}$ are independent for all internal nodes u, we recover the discrete pruning procedure that defines the process $(\mathcal{P}_\theta(T), \theta \geq 0)$ under $\mathbf{P}$. To conclude notice that $\tilde{T}_0$ and T are GW tree with offspring distribution characterized by its generating function g. □

4.4. **Proof Theorem 1.2.** The next corollary states that the pruning procedure for stable GW tree developed in [7] and the pruning procedure for Lévy trees developed in [1] and applied in [8] to sub-trees with finite number of leaves coincide.

Corollary 4.3. *Let $n \in \mathbb{N}$. The process $(\tilde{T}_\theta, \theta \geq 0)$ is distributed under $\mathbb{N}[\,\cdot \,\big|\, M = n]$ as the process $(\mathcal{P}_\theta(T), \theta \geq 0)$ under $\mathbf{P}_n$.*

Proof. This is a direct consequence of Proposition 4.2 and the fact that $M = L(\tilde{T}_0)$. □

Theorem 1.2 follows directly from Theorem 1.1 and of the following corollary, which is a direct consequence of Corollary 4.3. Recall that $\hat{\Pi}^{[n]}_{\text{Lévy}}$ is the restriction of $\Pi_{\text{Lévy}}$ defined in Section 4.2.2 to the n first integers.

Corollary 4.4. *The process $\hat{\Pi}^{[n]}_{Lévy}$ is under $\mathbb{N}^{(1)}$ distributed as $\hat{\Pi}^{[n]}_{GW}$ under $\mathbf{P}_n$.*

5. Proof of Proposition 1.4

We recall results from [21, 24, 22]. Let X_n be the number of coalescence events for a $\beta(a,b)$-coalescent. For $1 < a < 2$ and $b > 0$, we have that:

$$\frac{2-a}{\Gamma(a)} n^{a-2} X_n$$

converges in distribution towards

$$W_{a,b} = \int_0^\infty dt\ \mathrm{e}^{-(2-a)S_{a,b}(t)},$$

where $S_{a,b}$ is a subordinator with Laplace exponent $\phi_{a,b}$ given by:

$$\phi_{a,b}(\lambda) = \int_0^1 \Big(1-(1-x)^\lambda\Big) x^{a-3}(1-x)^{b-1}\, dx.$$

Notice that this notation is consistent with (14). Since Z_n is distributed as X_n with $a = 1+\alpha$ and $b = 1-\alpha$. We deduce that:

$$n^{\alpha-1} Z_n \xrightarrow[n\to+\infty]{(d)} Z,$$

with Z distributed as $\frac{\Gamma(1+\alpha)}{1-\alpha} W_{1+\alpha,1-\alpha}$.

Using Lemma 3.1, we compute the moments of Z:

$$\begin{aligned}
\mathbb{E}\left[W^n_{1+\alpha,1-\alpha}\right] &= n! \int_{0\leq t_1 \leq \cdots \leq t_n} \mathbb{E}\left[\mathrm{e}^{-(1-\alpha)\sum_{k=1}^n S_{1-\alpha,1+\alpha}(t_k)}\right]\, dt_1\cdots dt_n \\
&= n! \int_{0\leq r_1,\cdots,0\leq r_n} \prod_{k=1}^n \mathbb{E}\left[\mathrm{e}^{-(1-\alpha)kS_{1-\alpha,1+\alpha}(r_k)}\right]\, dr_1\cdots dr_n \\
&= \frac{n!}{\prod_{k=1}^n \phi_{1+\alpha,1-\alpha}(k(1-\alpha))} \\
&= \left(\frac{1-\alpha}{\Gamma(\alpha)}\right)^n \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma((n+1)(1-\alpha))}.
\end{aligned}$$

We deduce that:

$$\mathbb{E}\left[Z^n\right] = \alpha^n \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma((n+1)(1-\alpha))}.$$

6. Number of blocks in the last coalescence event

We consider the number of blocks B_n involved in the last coalescence event of $\Pi^{[n]}_{\text{dis}}$. In order to stress the dependence in n, we shall denote by T_n the GW tree T under $\mathbf{P}_n$. We also write $\xi_u(T_n)$ for ξ_u to stress the dependence of the marks introduced in Section 2.2 as a function of the underlying tree T_n. Notice that the time $\xi_\emptyset(T_n)$ at which the root of T_n is marked correspond to the last coalescence event associated with T_n. Thanks to Theorem 1.1, B_n is distributed as the number of leaves of the pruned tree obtained from T_n just before the last coalescence event, that is:

$$B_n \overset{(\mathrm{d})}{=} L(\mathcal{P}_{\xi_\emptyset(T_n)-}(T_n)). \tag{20}$$

6.1. **Local limit.** The method used in [4] when $\alpha = 1/2$ relies on the Aldous's CRT, which is the (global) limit of T_n when the length of the branch of T_n are rescaled by $1/\sqrt{n}$, see [15]. Since Lévy's trees are more difficult to handle, we choose here to use the local limit of T_n, which is the Kesten's tree T^*, according to [5].

Recall that ν_g is the distribution with generating function g given in (3) and that ν_g is critical as $g'(1) = 1$. We recall the distribution of the Kesten's tree T^* associated with the critical reproduction law ν_g, see [26]. Let ν_g^* be the corresponding size-biased distribution: $\nu_g^*(k) = k\nu_g(k)$ for all $k \in \mathbb{N}$. For $h \in \mathbb{N}$, we consider the truncation operator r_h on $\mathbb{T}$ defined as:

$$r_h\mathbf{t} = \{u \in \mathbf{t}; |u| \leq h\}.$$

The distribution of T^* is as follows. Almost surely, T^* contains a unique infinite path i.e. a unique infinite sequence $(V_k, k \in \mathbb{N}^*)$ of positive integers such that, for every $h \in \mathbb{N}$, $V_1 \cdots V_h \in T^*$, with the convention that $V_1 \cdots V_h = \emptyset$ if $h = 0$. The joint distribution of $(V_k, k \in \mathbb{N}^*)$ and T^* is determined recursively as follows: for each $h \in \mathbb{N}$, conditionally given $(V_1, \ldots, V_h)$ and r_hT^*, we have:

- The number of children $(k_v(T^*),\ v \in T^*,\ |v| = h)$ are independent and distributed according to ν_g if $v \neq V_1 \cdots V_h$ and according to ν_g^* if $v = V_1 \cdots V_h$.
- Given also the numbers of children $(k_v(T^*),\ v \in T^*,\ |v| = h)$, the vertex V_{h+1} is uniformly distributed on the set of integers $\left\{1, \ldots, \sum_{v \in T^*,\ |v|=h} k_v(T^*)\right\}$.

Recall that the height of a discrete tree $\mathbf{t} \in \mathbb{T}$ is $H_{\max}(\mathbf{t}) = \sup\{|u|, u \in \mathbf{t}\}$. For $h \in \mathbb{N}^*$, we have that for all $\mathbf{t} \in \mathbb{T}$ with height h and $u \in \mathbf{t}$ with $|u| = h$:

$$\mathbb{P}(r_hT^* = \mathbf{t},\, V_1 \cdots V_h = u) = \mathbf{P}_n(r_hT_n = \mathbf{t}).$$

The local limit convergence of critical GW trees, see [5], implies that, for all $h \in \mathbb{N}^*$, $\mathbf{t} \in \mathbb{T}$ with height h:

$$\lim_{n \to +\infty} \mathbf{P}_n(r_hT_n = \mathbf{t}) = \mathbb{P}(r_hT^* = \mathbf{t}).$$

For $\mathbf{t} \in \mathbb{T}$, and $u \in \mathbf{t}$, we consider the sub-tree $\mathbf{t}_u$ attached at u defined by:

$$\mathbf{t}_u = \{w \in \mathcal{U}; uw \in \mathbf{t}\}.$$

By construction of the marks, we deduce the following result.

Lemma 6.1. *We have, for all* $\mathbf{t} \in \mathbb{T}$*:*

$$\lim_{n \to +\infty} \mathbf{P}_n(\mathcal{P}_{\xi_\emptyset(T_n)-}(T_n) = \mathbf{t}) = \mathbb{P}(\bar{T} = \mathbf{t}),$$

where $\bar{T}$ *is such that:*

- $k_\emptyset(\bar{T})$ *has distribution* ν_g^*.
- *Conditionally on* $k_\emptyset(\bar{T})$, ξ *is a random variable such that* $\mathbb{P}(\xi \geq \theta) = (1+\theta)^{1-k_\emptyset(\bar{T})}$ *for all* $\theta \geq 0$.
- *Conditionally on* $k_\emptyset(\bar{T})$ *and* ξ, V_1 *is a uniform random variable on* $\{1, \ldots, k_\emptyset(\bar{T})\}$.
- *Conditionally on* $k_\emptyset(\bar{T})$, ξ *and* V_1, $(\bar{T}_u, u \in \{1, \ldots, k_\emptyset(\bar{T})\})$ *are independent random trees distributed such that for* $u \neq V_1$, T_u *is distributed as* $\mathcal{P}_\xi(T)$ *with* T *a GW tree with reproduction law* ν_g, *and* T_{V_1} *is distributed as* $\mathcal{P}_\xi(T^*)$, *with* T^* *distributed as the Kesten's tree associated with the reproduction law* ν_g.

Notice that by construction, $\bar{T}$ is finite.

6.2. **Proof of Proposition 1.5.** We deduce from (20), Lemma 6.1 and the fact that $\bar{T}$ is a.s. finite, that B_n converge in distribution to $B = L(\bar{T})$. From Lemma 6.1, we have that B is distributed as

$$L(\mathcal{P}_\xi(T^*)) + \sum_{k=1}^{k_\emptyset - 1} L(\mathcal{P}_\xi(T_k)),$$

where $k_\emptyset$ has distribution ν_g^*, ξ has density $(k_\emptyset - 1)(1+\theta)^{-k_\emptyset} \mathbf{1}_{\{\theta \geq 0\}}$, T^* is independent and distributed as the Kesten's tree associated with ν_g, and $(T_k, k \in \mathbb{N}^*)$ are independent and distributed as a Galton-Watson tree T with reproduction law ν_g. We deduce that:

$$\mathbb{E}\left[r^B\right] = \mathbb{E}\left[N(N-1)\int_0^{+\infty} (1+\theta)^{-N} d\theta \, \mathbb{E}\left[r^{L_\theta}\right]^{N-1} \mathbb{E}\left[r^{L_\theta^*}\right]\right],$$

where N has distribution ν_g, L_θ is the number of leaves of $\mathcal{P}_\theta(T)$ and L_θ^* is the number of leaves of $\mathcal{P}_\theta(T^*)$.

Let h_θ be the generating function of L_θ and h_θ^* be the generating function of L_θ^*. We have:

$$\mathbb{E}\left[r^B\right] = \int_0^{+\infty} \frac{d\theta}{(1+\theta)^2} \, g''\left(\frac{h_\theta(r)}{1+\theta}\right) h_\theta(r) h_\theta^*(r).$$

Recall that $\mathcal{P}_\theta(T)$ is a GW tree whose reproduction law has generating function g_θ given by (8). Similar arguments as in the proof of (13), yields that:

$$g_\theta(h_\theta(r)) - h_\theta(r) = g_\theta(0)(1-r). \tag{21}$$

We deduce from (8) that:

$$g_\theta''(r) = g''\left(\frac{r}{1+\theta}\right) \frac{1}{1+\theta} \cdot$$

We deduce from (21) that:

$$(1 - g_\theta'(h_\theta(r))) = \frac{g_\theta(0)}{h_\theta'(r)} \quad \text{and} \quad g_\theta''(h_\theta(r)) = (1 - g_\theta'(h_\theta(r))) \frac{h_\theta''(r)}{(h_\theta'(r))^2} \cdot \tag{22}$$

We obtain:

$$g''\left(\frac{h_\theta(r)}{1+\theta}\right) \frac{1}{1+\theta} = g_\theta(0) \frac{h_\theta''(r)}{(h_\theta'(r))^3} \cdot$$

We now compute h_θ^*. According to Remark 3.7 in [8], we have for $\mathbf{t} \in \mathbb{T}$:

$$\mathbf{P}(\mathcal{P}_\theta(T^*) = \mathbf{t}) = L(\mathbf{t}) \, \frac{1 - g_\theta'(1)}{g_\theta'(0)} \, \mathbf{P}(\mathcal{P}_\theta(T) = \mathbf{t}).$$

We deduce that:

$$\begin{aligned} h_\theta^*(r) = \mathbf{E}\left[r^{L_\theta^*}\right] &= \sum_{\mathbf{t} \in \mathbb{T}} r^{L(\mathbf{t})} \mathbf{P}(\mathcal{P}_\theta(T^*) = \mathbf{t}) \\ &= \frac{1 - g_\theta'(1)}{g_\theta(0)} \sum_{\mathbf{t} \in \mathbb{T}} L(\mathbf{t}) r^{L(\mathbf{t})} \mathbf{P}(\mathcal{P}_\theta(T) = \mathbf{t}) \\ &= r \, \frac{h_\theta'(r)}{h_\theta'(1)}, \end{aligned}$$

where we used the first equality in (22) with $r = 1$ and $h_\theta(1) = 1$. We get:

$$\mathbb{E}\left[r^B\right] = r \int_0^{+\infty} \frac{d\theta}{1+\theta} \, \frac{g_\theta(0)}{h_\theta'(1)} \, \frac{h_\theta''(r)}{(h_\theta'(r))^2} \, h_\theta(r). \tag{23}$$

We have from (8) that:

$$g_\theta(0) = \alpha(1+\theta)\left[1 - \left(\frac{\theta}{1+\theta}\right)^{1/\alpha}\right].$$

We deduce from (21) that:

$$h_\theta(r) = (1+\theta)\left[1 - \left\{1 - r\left[1 - \left(\frac{\theta}{1+\theta}\right)^{1/\alpha}\right]\right\}^{\alpha}\right].$$

Then, the change of variable $x = 1 - (\theta/(1+\theta))^{1/\alpha}$ in (23) gives that φ_α, given in (5), is the generating function of B.

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Romain Abraham, Laboratoire MAPMO, CNRS, UMR 7349, Fédération Denis Poisson, FR 2964, Université d'Orléans, B.P. 6759, 45067 Orléans cedex 2, France.
E-mail address: `romain.abraham@univ-orleans.fr`

Jean-François Delmas, Université Paris-Est, École des Ponts, CERMICS, 6-8 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France.
E-mail address: `delmas@cermics.enpc.fr`