THE FOREST ASSOCIATED WITH THE RECORD PROCESS ON A LÉVY TREE

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Abstract. We perform a pruning procedure on a Lévy tree and instead of throwing away the removed sub-tree, we regraft it on a given branch (not related to the Lévy tree). We prove that the tree constructed by regrafting is distributed as the original Lévy tree, generalizing a result where only Aldous's tree is considered. As a consequence, we obtain that the quantity which represents in some sense the number of cuts needed to isolate the root of the tree, is distributed as the height of a leaf picked at random in the Lévy tree.

1. Introduction

Lévy trees arise as the scaling limits of Galton-Watson trees in the same way as continuous state branching processes (CSBP) are the scaling limits of Galton-Watson processes (see [13], Chapter 2). Hence, Lévy trees can be seen as the genealogical tree of some CSBP. Following [18], one can define a random variable $\mathcal{T}$ in the space of real trees (see [15, 14, 19, 5]) that describes the genealogy of a CSBP with branching mechanism ψ of the form:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r\right) \pi(dr) \quad \text{for } \lambda \geq 0,$$

with $\alpha \geq 0$, $\beta \geq 0$ and π a σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (r \wedge r^2)\pi(dr) < +\infty$. We assume that either $\beta > 0$ or $\pi((0, 1)) = +\infty$. In particular, the corresponding CSBP is sub-critical as $\psi'(0) = \alpha \geq 0$ (see [1] for the definition of a Lévy tree with super-critical branching mechanism). In order to use the setting of measured real trees developed in [4], we shall restrict our-self to compact Lévy tree, that is with ranching mechanism satisfying the Grey condition:

$$\int^{+\infty} \frac{dv}{\psi(v)} < +\infty.$$

However, we conjecture that our main result holds without the Grey condition.

In [6], a pruning mechanism has been constructed so that the Lévy tree with branching mechanism ψ pruned at rate $q > 0$ is a Lévy tree with branching mechanism ψ_q defined by:

$$\psi_q(\lambda) = \psi(\lambda + q) - \psi(q) \quad \text{for } \lambda \geq 0.$$

This pruning is performed by throwing marks on the tree in a Poissonian manner and by cutting the tree according to these marks. This pruning procedure allowed to construct a tree-valued Markov process [2] or to study the record process on Aldous's continuum random tree (CRT) [1] which is related to the number of cuts needed to reduce a Galton-Watson tree.

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This problem of cutting down a random tree arises first in [22]: consider a rooted discrete tree with n vertices, pick an edge uniformly at random and remove it together with the sub-tree attached to it and then iterate the procedure on the remaining tree until only the root is left. The question is "How many cuts are needed to isolate the root by this procedure" ? Asymptotics in law for this quantity are given in [22] when the tree is a Cayley tree (see also [9, 10] in this case where the problem is generalized to the isolation of several leaves and not only the root) and in [20] for conditioned (critical with finite variance) Galton-Watson trees. A.s. convergence has also been obtained in the latter case for a slightly different quantity in [1] using a special pruning procedure that we describe now.

Let $\mathcal{T}$ be a Lévy tree with branching mechanism ψ and $\mathbf{m}^{\mathcal{T}}(dx)$ its "mass measure" supported by the leaves of $\mathcal{T}$. We denote by $\mathbb{P}_r^\psi$ the distribution of the Lévy tree corresponding to the CSBP with branching mechanism ψ starting at r and by $\mathbb{N}^\psi$ the corresponding excursion measure also called canonical measure. The branching points of the Lévy tree are either binary or of infinite degree (see [14]) and to each infinite degree branching point x, one can associate a size Δ_x which measures in some sense the number of sub-trees attached to it (see (5) in Section 2.5). We then consider a measure $\mu^{\mathcal{T}}$ on $\mathcal{T}$ defined by:

$$\mu^{\mathcal{T}}(dy) = 2\beta\ell^{\mathcal{T}}(dy) + \sum_{x\in\mathrm{Br}_\infty(\mathcal{T})} \Delta_x\delta_x(dy),$$

where $\ell^{\mathcal{T}}$ is the length measure on the skeleton of the tree, $\mathrm{Br}_\infty(\mathcal{T})$ is the set of branching points of infinite degree and δ_x is the Dirac measure at point x. Aldous's CRT corresponds to the distribution of $\mathcal{T}$ under $\mathbb{N}^\psi$, with $\psi(\lambda) = \frac{1}{2}\lambda^2$, and conditionally on $\mathbf{m}^{\mathcal{T}}(\mathcal{T}) = 1$. In this case $\mathrm{Br}_\infty(\mathcal{T})$ is empty and thus $\mu^{\mathcal{T}}(dy) = \ell^{\mathcal{T}}(dy)$.

Then we consider, conditionally given $\mathcal{T}$, a Poisson point process $M^{\mathcal{T}}(d\theta, dy)$ of marks on the tree with intensity

$$\mathbf{1}_{[0,+\infty)}(\theta)d\theta\,\mu^{\mathcal{T}}(dy).$$

Parameter y indicates the location of the mark whereas θ represents the time at which it appears. For every $x \in \mathcal{T}$, we set $\theta(x)$ the first time θ at which a mark appears between x and the root. We consider Θ the average of this records over the Lévy tree:

$$\Theta = \int_{\mathcal{T}} \theta(x)\mathbf{m}^{\mathcal{T}}(dx).$$

It has been proven in [1], that when $\mathcal{T}$ is Aldous's CRT, if we denote by X_n the number of cuts needed to isolate the root in the sub-tree spanned by n leaves randomly chosen, then a.s. $\lim_{n\to+\infty} X_n/\sqrt{2n} = \Theta$. Moreover, the law of Θ in that case is a Rayleigh distribution (i.e. with density $x\,\mathrm{e}^{-x^2}\,\mathbf{1}_{\{x\geq 0\}}$). The distribution of Θ is also the law of the height of a leaf picked at random in Aldous's tree. This surprising relationship is explained by Addario-Berry, Broutin and Holmgren in [7], Theorem 10. The authors consider a branch with length Θ, and when a mark appears, the tree is cut and the sub-tree which does not contain the root is removed and grafted on this branch (the grafting position is described using some local time). Then the new tree obtained by this grafting procedure is again distributed as Aldous's tree.

The goal of this paper is to generalize this result to general Lévy trees. We consider a Lévy tree $\mathcal{T}$ under $\mathbb{N}^\psi$ and we perform the pruning procedure described above. When a mark appears, we remove the sub-tree attached to this mark and keep the sub-tree containing the root. We denote by $\mathcal{T}_q$ the resulting tree at time q i.e. the set of points of the initial tree $\mathcal{T}$

which have no marks between them and the root at time q:

$$\mathcal{T}_q = \{x \in \mathcal{T}; \theta(x) \geq q\}.$$

According to [6], $\mathcal{T}_q$ is a Lévy tree with branching mechanism ψ_q. We consider Θ_q the average of the records shifted by q over the Lévy tree $\mathcal{T}_q$:

$$\Theta_q = \int_{\mathcal{T}_q} (\theta(x) - q)\, \mathbf{m}^{\mathcal{T}}(dx).$$

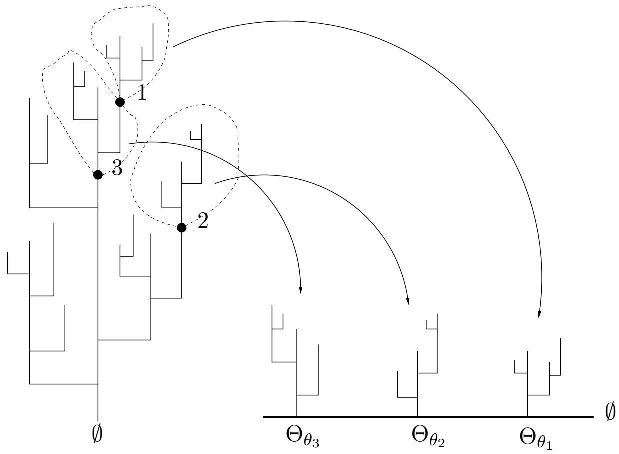

FIGURE 1. Pruning of a Lévy tree (left) and tree $\mathcal{T}^R$ obtained by regrafting on a branch (right). The marks are numbered according to their order of appearance.

We define an equivalent relation on the tree $\mathcal{T}$: $x \sim y$ if the function θ remains constant on the path between x and y. We consider the equivalent classes $(\mathcal{T}^i, i \in I^R)$ and denote by θ_i the common value of the function θ. In the pruning procedure described above, the tree $\mathcal{T}^i$ corresponds to the sub-tree which is removed at time θ_i and it is distributed according to $\mathbb{N}^{\psi_{\theta_i}}$. Then we consider a branch B^R of length Θ rooted at some end point, say $\emptyset$. The sub-tree $\mathcal{T}^i$ is grafted on B^R at distance Θ_{θ_i} from the root, see Figure 1. Let $\mathcal{T}^R$ denote this tree obtained by regrafting. Our main result, see Theorem 3.1, relies on Laplace transform computations and can be stated as follows.

Theorem. *Assume the Grey condition holds. Under $\mathbb{N}^\psi$, $(B^R, \mathcal{T}^R)$ is distributed as $(B, \mathcal{T})$ where B is a branch from the root $\emptyset$ to a leaf chosen at random on $\mathcal{T}$ according to the mass measure $\mathbf{m}^{\mathcal{T}}$.*

As a consequence, we get that Θ is distributed as the height H of a leaf randomly chosen in the Lévy tree. Using the Bismut decomposition of Lévy trees, we recover and extend to general Lévy trees Proposition 8.2 from [2] on the asymptotics of the masses of $(\mathcal{T}^i, i \in I^R)$. For $i \in I^R$, set $\sigma^i = \mathbf{m}^{\mathcal{T}}(\mathcal{T}^i)$.

Corollary. *Assume the Grey condition holds. $\mathbb{N}^\psi$-a.e., we have:*

$$\lim_{\varepsilon \to 0} \frac{1}{\mathbb{N}^\psi[\sigma > \varepsilon]} \sum_{i \in I^R} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} = \Theta.$$

Similar results hold for the convergence of $\frac{1}{\mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma \leq \varepsilon\}}]} \sum_{i \in I^R} \sigma^i \mathbf{1}_{\{\sigma^i \leq \varepsilon\}}$ to Θ, see Corollary 3.2. Those results generalize Proposition 8.3 from [2].

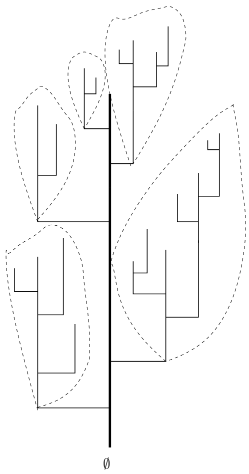

FIGURE 2. Bismut decomposition of a Lévy tree.

On one hand, according to [4], $\mathcal{T}^i$ has distribution $\mathbb{N}^{\psi_{\theta_i}}$, and thus for $\theta_i < \theta_j$, $\mathcal{T}^i$ is stochastically larger than $\mathcal{T}^j$ (that is one can build $\tilde{\mathcal{T}}^i$ distributed as $\mathcal{T}^i$ and $\tilde{\mathcal{T}}^j$ distributed as $\mathcal{T}^j$ such that $\tilde{\mathcal{T}}^j \subset \tilde{\mathcal{T}}^i$). On the other hand, in the Bismut decomposition of $\mathcal{T}$ (see Figure 2), the sub-trees which are grafted on the random branch B of length H are distributed according to $\mathbb{N}^\psi$ (the rate of grafting the sub-trees is also different). This paradox is due to the fact that, in the Bismut decomposition, the distribution of the sub-trees is given conditionally on the length H of the random branch B, whereas in the decomposition of $\mathcal{T}^R$, the distribution of the sub-tree rooted at level h is given conditionally on the sub-trees rooted below level h.

In the present work, we ignore the marks that fall on the sub-trees once they have been removed. However, we could use them to iterate our construction on each sub-trees $(\mathcal{T}^i, i \in I^R)$ and so on, in order to generalize to general Lévy trees the result obtained for Aldous's CRT by Bertoin and Miermont [10].

In view of the present work, we conjecture that similar results to [20] hold for infinite variance offspring distribution. Let X_n denote the number of cuts needed to isolate the root by pruning at edges a Galton-Watson trees conditioned to have n vertices. We also consider the pruning at vertices inspired by [3], which is the discrete analogue of the continuous pruning: pick an edge uniformly at random and remove **the vertex from which the edge comes from** together with the sub-tree attached to this vertex. Let $\tilde{X}_n$ be the number of cuts until the root is removed by this procedure for a Galton-Watson trees conditioned to have n vertices. According to [20], the number of cuts needed to remove the root for the pruning at vertices $(\tilde{X}_n)$ or to isolate the root for the pruning at edges (X_n) are asymptotically equivalent for finite variance offspring distribution. However, we expect a different behavior in the infinite variance case. Consider a critical Galton-Watson tree with offspring distribution in the domain of attraction of a stable law of index $\gamma \in (1, 2]$. According to [12] or [21], the (contour process of the) Galton-Watson tree conditioned to have total progeny n, properly rescaled by b_n, converge in distribution to (the contour process of) a Lévy tree under $\mathbb{N}^\psi\left[\cdot|\sigma=1\right]$, with $\psi(\lambda)=\lambda^\gamma$. Let L_n denote the length of the rescaled Galton-Watson tree conditioned to have total progeny n (this is half of the integral of the rescaled contour process). We conjecture that:

$$\frac{\tilde{X}_n}{L_n} \xrightarrow[n\to+\infty]{(d)} Z,$$

for some random variable Z distributed as the height of a random leaf chosen at random according to the mass measure under $\mathbb{N}^\psi[\cdot|\sigma=1]$. Set $a=(\gamma-1)/\gamma$. Using Laplace transform (see Theorem 2.1), we get that the height H of a leaf randomly chosen in the Lévy tree is distributed under $\mathbb{N}^\psi$ as $\sigma^a Z$, with Z and σ independent and the distribution of Z is characterized for $n\in\mathbb{N}$ by:

$$\mathbb{E}\left[Z^n\right]=\frac{1}{\gamma^n}\,\frac{\Gamma(a)\Gamma(n+1)}{\Gamma(a(n+1))}.$$

The paper is organized as follows. We collect results on Lévy trees in Section 2, with the Bismut decomposition is Section 2.7 and the pruning procedure in Section 2.8. The main result is then precisely stated in Section 3 and proved in Section 4.

2. Lévy trees and the forest obtained by pruning

2.1. **Notations.** Let (E,d) be a metric Polish space. For $x\in E$, δ_x denotes the Dirac measure at point x. For μ a Borel measure on E and f a non-negative measurable function, we set:

$$\langle\mu,f\rangle=\int f(x)\,\mu(dx)=\mu(f).$$

2.2. **Real trees.** We refer to [15] or [17] for a general presentation of random real trees. Informally, real trees are metric spaces without loops, locally isometric to the real line. More precisely, a metric space (T,d) is a real tree if the following properties are satisfied:

(1) For every $s,t\in T$, there is a unique isometric map $f_{s,t}$ from $[0,d(s,t)]$ to T such that $f_{s,t}(0)=s$ and $f_{s,t}(d(s,t))=t$.
(2) For every $s,t\in T$, if q is a continuous injective map from $[0,1]$ to T such that $q(0)=s$ and $q(1)=t$, then $q([0,1])=f_{s,t}([0,d(s,t)])$.

If $s,t\in T$, we will note $[\![s,t]\!]$ the range of the isometric map $f_{s,t}$ described above. We will also note $[\![s,t[\![$ for the set $[\![s,t]\!]\setminus\{t\}$.

We say that $(T,d,\emptyset)$ is a rooted real tree with root $\emptyset$ if (T,d) is a real tree and $\emptyset\in T$ is a distinguished vertex.

Let $(T,d,\emptyset)$ be a rooted real tree. If $x\in T$, the degree of x, $n(x)$, is the number of connected components of $T\setminus\{x\}$. The shall consider the set of leaves $\mathrm{Lf}(T)=\{x\in T\backslash\{\emptyset\},\, n(x)=1\}$, the set of branching points $\mathrm{Br}(T)=\{x\in T,\, n(x)\geq 3\}$ and the set of infinite branching points is $\mathrm{Br}_\infty(T)=\{x\in T,\, n(x)=\infty\}$. The skeleton of T is the set of points in the tree that aren't leaves: $\mathrm{Sk}(T)=T\backslash\mathrm{Lf}(T)$. The trace of the Borel σ-field of T restricted to $\mathrm{Sk}(T)$ is generated by the sets $[\![s,s']\!]$; $s,s'\in\mathrm{Sk}(T)$. Hence, one defines uniquely a σ-finite Borel measure ℓ^T on T, called length measure of T, such that:

$$\ell^T(\mathrm{Lf}(T))=0\quad\text{and}\quad\ell^T([\![s,s']\!])=d(s,s').$$

For every $x\in T$, $[\![\emptyset,x]\!]$ is interpreted as the ancestral line of vertex x in the tree. We define a partial order on T by setting $x\preccurlyeq y$ (x is an ancestor of y) if $x\in[\![\emptyset,y]\!]$. If $x,y\in T$, there exists a unique $z \in T$, called the Most Recent Common Ancestor (MRCA) of x and y, such that $[\![\emptyset, x]\!] \cap [\![\emptyset, y]\!] = [\![\emptyset, z]\!]$. We write $z = x \wedge y$.

2.3. **Measured rooted real trees.** We will denote by $\mathbb{T}$ the set of (measure-preserving and root-preserving isometry classes of) measured rooted real trees $(T, d, \emptyset, \mathbf{m})$ where $(T, d, \emptyset)$ is a locally compact rooted real tree and $\mathbf{m}$ is a locally finite measure on T. Sometimes, we will write $(T, d^T, \emptyset^T, \mathbf{m}^T)$ for $(T, d, \emptyset, \mathbf{m})$ to stress the dependence in T, or simply T when there is no confusion. One can define a distance on $\mathbb{T}$ such that endowed with this distance, $\mathbb{T}$ is a Polish space, see [5].

Let $T \in \mathbb{T}$. For $x \in T$, we set $h(x) = d(\emptyset, x)$ the height of x and $H_{\max}(T) = \sup_{x \in T} h(x)$ the height of the tree (possibly infinite). For $a \geq 0$, we set:

$$T(a) = \{x \in T,\, d(\emptyset, x) = a\} \quad \text{and} \quad \pi_a(T) = \{x \in T,\, d(\emptyset, x) \leq a\},$$

the restriction of the tree T at level a and the truncated tree T up to level a. We consider $\pi_a(T)$ with the induced distance, the root $\emptyset$ and the mass measure $\mathbf{m}^{\pi_a(T)}$ which is the restriction of $\mathbf{m}^T$ to $\pi_a(T)$, to get a measured rooted real tree. We denote by $(T^{i,\circ}, i \in I)$ the connected components of $T \setminus \pi_a(T)$. Let $\emptyset_i$ be the MRCA of all the vertices of $T^{i,\circ}$. We consider the real tree $T^i = T^{i,\circ} \cup \{\emptyset_i\}$ rooted at point $\emptyset_i$ with mass measure $\mathbf{m}^{T^i}$ defined as the restriction of $\mathbf{m}^T$ to T^i. We will consider the point measure on $T \times \mathbb{T}$:

$$\mathcal{N}_a^T = \sum_{i \in I} \delta_{(\emptyset_i, T^i)}.$$

2.4. **Grafting procedure.** We will define in this section a procedure by which we add (graft) measured rooted real trees on an existing measured rooted real trees. More precisely, let $T \in \mathbb{T}$ and let $((T_i, x_i), i \in I)$ be a finite or countable family of elements of $\mathbb{T} \times T$. We define the real tree obtained by grafting the trees T_i on T at point x_i. We set $\tilde{T} = T \sqcup \left(\bigsqcup_{i \in I} T_i \backslash \{\emptyset^{T_i}\}\right)$ where the symbol $\sqcup$ means that we choose for the sets T and $(T_i)_{i \in I}$ representatives of isometry classes in $\mathbb{T}$ which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set $\emptyset^{\tilde{T}} = \emptyset^T$. The set $\tilde{T}$ is endowed with the following metric $d^{\tilde{T}}$: if $s, t \in \tilde{T}$,

$$d^{\tilde{T}}(s,t) = \begin{cases} d^T(s,t) & \text{if } s, t \in T, \\ d^T(s, x_i) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } s \in T,\ t \in T_i \backslash \{\emptyset^{T_i}\}, \\ d^{T_i}(s,t) & \text{if } s, t \in T_i \backslash \{\emptyset^{T_i}\}, \\ d^T(x_i, x_j) + d^{T_j}(\emptyset^{T_j}, s) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } i \neq j \text{ and } s \in T_j \backslash \{\emptyset^{T_j}\},\ t \in T_i \backslash \{\emptyset^{T_i}\}. \end{cases}$$

We define the mass measure on $\tilde{T}$ by:

$$\mathbf{m}^{\tilde{T}} = \mathbf{m}^T + \sum_{i \in I} \left(\mathbf{1}_{T_i \backslash \{\emptyset^{T_i}\}} \mathbf{m}^{T_i} + \mathbf{m}^{T_i}(\{\emptyset^{T_i}\}) \delta_{x_i} \right),$$

where δ_x is the Dirac mass at point x. It is clear that the metric space $(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}})$ is still a rooted complete real tree. (Notice, it is not always true that $\tilde{T}$ remains locally compact or that $\mathbf{m}^{\tilde{T}}$ defines a locally finite measure on $\tilde{T}$). We will use the following notation:

$$(\tilde{T}, d^{\tilde{T}}, \emptyset^{\tilde{T}}, \mathbf{m}^{\tilde{T}}) = T \circledast_{i \in I} (T_i, x_i). \tag{1}$$

2.5. **Excursion measure of Lévy tree.** Let ψ be a critical branching mechanism defined by:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r\right) \pi(dr) \tag{2}$$

with $\alpha \geq 0$, $\beta \geq 0$ and π is a σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)}(r \wedge r^2)\pi(dr) < +\infty$ and $\langle \pi, 1\rangle = +\infty$ if $\beta = 0$. As $\psi'(0) = \alpha \geq 0$, the corresponding CSBP is sub-critical or critical. We also assume the Grey condition:

$$\int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty. \tag{3}$$

The Grey condition is equivalent to the a.s. finiteness of the extinction time of the CSBP. This assumption is used to ensure that the corresponding Lévy tree is locally compact. Let v be the unique non-negative solution of the equation:

$$\int_{v(a)}^{+\infty} \frac{d\lambda}{\psi(\lambda)} = a.$$

Results from [14] can be stated in the following form, see [4]. There exists a σ-finite measure $\mathbb{N}^\psi[d\mathcal{T}]$ on $\mathbb{T}$, or excursion measure of Lévy tree, with the following properties.

(i) **Height.** For all $a > 0$, $\mathbb{N}^\psi[H_{\max}(\mathcal{T}) > a] = v(a)$.

(ii) **Mass measure.** The mass measure $\mathbf{m}^\mathcal{T}$ is supported by $\mathrm{Lf}(\mathcal{T})$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.

(iii) **Local time.** There exists a $\mathcal{T}$-measure valued process $(\ell^a, a \geq 0)$ càdlàg for the weak topology on finite measure on $\mathcal{T}$ such that $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

$$\mathbf{m}^\mathcal{T}(dx) = \int_0^\infty \ell^a(dx)\, da, \tag{4}$$

$\ell^0 = 0$, $\inf\{a > 0; \ell^a = 0\} = \sup\{a \geq 0; \ell^a \neq 0\} = H_{\max}(\mathcal{T})$ and for every fixed $a \geq 0$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

- The measure ℓ^a is supported on $\mathcal{T}(a)$.
- We have for every bounded continuous function ϕ on $\mathcal{T}$:

$$\begin{aligned}\langle \ell^a, \phi\rangle &= \lim_{\epsilon \downarrow 0} \frac{1}{v(\epsilon)} \int \phi(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}') \geq \epsilon\}} \mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}') \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{v(\epsilon)} \int \phi(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}') \geq \epsilon\}} \mathcal{N}_{a-\epsilon}^\mathcal{T}(dx, d\mathcal{T}'), \text{ if } a > 0.\end{aligned}$$

Under $\mathbb{N}^\psi$, the real valued process $(\langle \ell^a, 1\rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism ψ under its canonical measure.

(iv) **Branching property.** For every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}')$ under $\mathbb{N}^\psi[d\mathcal{T} | H_{\max}(\mathcal{T}) > a]$, given $\pi_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(dx)\mathbb{N}^\psi[d\mathcal{T}']$.

(v) **Branching points.**
- $\mathbb{N}^\psi[d\mathcal{T}]$-a.e., the branching points of $\mathcal{T}$ are of degree 3 or $+\infty$.
- The set of binary branching points (i.e. of degree 3) is empty $\mathbb{N}^\psi$ a.e if $\beta = 0$ and is a countable dense subset of $\mathcal{T}$ if $\beta > 0$.
- The set $\mathrm{Br}_\infty(\mathcal{T})$ of infinite branching points is nonempty with $\mathbb{N}^\psi$-positive measure if and only if $\pi \neq 0$. If $\langle \pi, 1\rangle = +\infty$, the set $\mathrm{Br}_\infty(\mathcal{T})$ is $\mathbb{N}^\psi$-a.e. a countable dense subset of $\mathcal{T}$.

(vi) **Mass of the nodes.** The set $\{d(\emptyset, x),\ x \in \mathrm{Br}_\infty(\mathcal{T})\}$ coincides $\mathbb{N}^\psi$-a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, $\mathbb{N}^\psi$-a.e., for every such discontinuity time b, there is a unique $x_b \in \mathrm{Br}_\infty(\mathcal{T}) \cap \mathcal{T}(b)$ and $\Delta_b > 0$, such that:

$$\ell^b = \ell^{b-} + \Delta_b \delta_{x_b},$$

where $\Delta_b > 0$ is called the mass of the node x_b. Furthermore Δ_b can be obtained by the approximation:

$$\Delta_b = \lim_{\epsilon \to 0} \frac{1}{v(\epsilon)} n(x_b, \epsilon), \tag{5}$$

where $n(x_b, \epsilon) = \int \mathbf{1}_{\{x = x_b\}}(x) \mathbf{1}_{\{H_{max}(\mathcal{T}') > \epsilon\}} \mathcal{N}_b^{\mathcal{T}}(dx, d\mathcal{T}')$ is the number of sub-trees originating from x_b with height larger than ϵ.

In order to stress the dependence in $\mathcal{T}$, we may write $\ell^{a,\mathcal{T}}$ for ℓ^a.

We set $\sigma^{\mathcal{T}}$ or simply σ when there is no confusion, the total mass of the mass measure on $\mathcal{T}$:

$$\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T}). \tag{6}$$

In particular, as σ is distributed as the total mass of a CSBP under its canonical measure, we have that $\mathbb{N}^\psi$-a.s. $\sigma > 0$ and for $q > 0$:

$$\mathbb{N}^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = q, \quad \mathbb{N}^\psi \left[\sigma\, \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{1}{\psi'(q)} \quad \text{and} \quad \mathbb{N}^\psi \left[\sigma^2\, \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{\psi''(q)}{\psi'(q)^3}. \tag{7}$$

The last two equations hold for $q = 0$ if $\psi'(0) > 0$.

2.6. **Related measure on Lévy trees.** We define a probability measure on $\mathbb{T}$ as follow. Let $r > 0$ and $\sum_{k \in \mathcal{K}} \delta_{\mathcal{T}^k}$ be a Poisson point measure on $\mathbb{T}$ with intensity $r\mathbb{N}^\psi$. Consider $\emptyset$ as the trivial measured rooted real tree reduced to the root with null mass measure. Define $\mathcal{T} = \emptyset \circledast_{k \in \mathcal{K}} (\mathcal{T}^k, \emptyset)$. Using Property (i) as well as (7), one easily get that $\mathcal{T}$ is a measured compact rooted real tree, and thus belong to $\mathbb{T}$. We denote by $\mathbb{P}_r^\psi$ its distribution. Its corresponding local time is defined by $\ell^a = \sum_{k \in \mathcal{K}} \ell^{a, \mathcal{T}^k}$ and its total mass is defined by $\sigma = \sum_{k \in \mathcal{K}} \sigma^{\mathcal{T}^k}$. Under $\mathbb{P}_r^\psi$, the real valued process $(\langle \ell^a, 1 \rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism ψ with initial value r.

We consider the following measure on $\mathbb{T}$:

$$\mathbf{N}^\psi[d\mathcal{T}] = 2\beta \mathbb{N}^\psi[d\mathcal{T}] + \int_0^{+\infty} r\pi(dr)\, \mathbb{P}_r^\psi(d\mathcal{T}). \tag{8}$$

Elementary computations yield for $q > 0$:

$$\mathbf{N}^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = \psi'(q) - \psi'(0), \tag{9}$$

as well as

$$\mathbf{N}^\psi \left[\sigma\, \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{\psi''(q)}{\psi'(q)} \quad \text{and} \quad \mathbf{N}^\psi \left[\sigma^2\, \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{1}{\psi'(q)}\, \partial_q \left(\frac{-\psi''(q)}{\psi'(q)}\right). \tag{10}$$

The last two equalities also hold for $q = 0$ if $\psi'(0) > 0$.

2.7. **Bismut decomposition of a Lévy tree.** We first present a decomposition of $T \in \mathbb{T}$ according to a given vertex $x \in T$. We denote by $(T^{j,\circ}, j \in J_x)$ the connected components of $T \setminus [\![\emptyset, x]\!]$. For every $j \in J_x$, let x_j be the MRCA of $T^{j,\circ}$ and consider $T^j = T^{j,\circ} \bigcup \{x_j\}$ as an element of $\mathbb{T}$ with mass measure the mass measure of T restricted to $T^{j,\circ}$. In order to graft together all the sub-trees with the same MRCA, we consider the following equivalence relation on J_x:

$$j \sim j' \iff x_j = x_{j'}.$$

Let I_x^B be the set of equivalence classes. For $i \in I_x^B$, we set x_i for the common value of x_j with $j \in i$. We consider $\{x_i\}$ as an element of $\mathbb{T}$ with mass measure $\mathbf{m}^T(\{x_i\})\delta_{x_i}$. For $i \in I_x^B$, we consider the following element of T defined by:

$$T^{B,i} = \{x_i\} \circledast_{j\in i} (T^j, x_i).$$

Let $h_i = d(\emptyset, x_i)$. We consider the random point measure $\mathcal{M}_x^T$ on $\mathbb{R}_+ \times \mathbb{T}$ defined by:

$$\mathcal{M}_x^T = \sum_{i \in I_x^B} \delta_{(h_i, T^{B,i})}.$$

Under $\mathbb{N}^\psi$, conditionally on $\mathcal{T}$, let U be a $\mathcal{T}$-valued random variable, with distribution $\sigma^{-1}\,\mathbf{m}^\mathcal{T}$. In other words, conditionally on $\mathcal{T}$, U represents a random leaf randomly uniformly chosen. We define under $\mathbb{N}^\psi$ a non-negative random variable and a random point measure on $\mathbb{R}_+ \times \mathbb{T}$ as follow:

$$H = d^\mathcal{T}(\emptyset^\mathcal{T}, U) \quad \text{and} \quad \mathcal{Z}^B = \mathcal{M}_U^\mathcal{T}. \tag{11}$$

By construction, for every non-negative measurable function Φ on $\mathbb{R}_+ \times \mathbb{T}$ and for every $\lambda \geq 0$, $\rho \geq 0$, we have:

$$\mathbb{N}^\psi\left[\sigma\, \mathrm{e}^{-\lambda\sigma - \rho H - \langle \mathcal{Z}^B, \Phi\rangle}\right] = \mathbb{N}^\psi\left[\int_\mathcal{T} \mathbf{m}^\mathcal{T}(dx)\ \mathrm{e}^{-\lambda\sigma - \rho h(x) - \langle \mathcal{M}_x^\mathcal{T}, \Phi\rangle}\right].$$

As a direct consequence of Theorem 4.5 of [14], we get the following result.

Theorem 2.1. *For every non-negative measurable function Φ on $\mathbb{R}_+ \times \mathbb{T}$ and for every $\lambda \geq 0$, $\rho \geq 0$, we have:*

$$\mathbb{N}^\psi\left[\sigma\, \mathrm{e}^{-\lambda\sigma - \rho H - \langle \mathcal{Z}^B, \Phi\rangle}\right] = \int_0^{+\infty} da\, \mathrm{e}^{-\rho a} \exp\left(-\int_0^a g(\lambda, u)du\right),$$

where

$$g(\lambda, u) = \psi'(0) + \mathbf{N}^\psi\left[1 - \mathrm{e}^{-\lambda\sigma - \Phi(u, \mathcal{T})}\right]. \tag{12}$$

In other words, under $\mathbb{N}^\psi\,[\sigma,\, d\mathcal{T}]$, if we choose a leaf U uniformly (i.e. according to the normalized mass measure $\mathbf{m}^\mathcal{T}$), the height H of this leaf is distributed according to the density $da\, \mathrm{e}^{-\psi'(0)a}$ and, conditionally on H, the point measure $\mathcal{Z}^B$ is a Poisson point process on $[0, H]$ with intensity $\mathbf{N}^\psi[d\mathcal{T}]$.

2.8. **Pruning a Lévy tree.** A general pruning of a Lévy tree has been defined in [6]. We use a special case of this pruning depending on a one-dimensional parameter θ used first in [23] to define a fragmentation process of the tree.

More precisely, for a tree $\mathcal{T} \in \mathbb{T}$, we consider a mark process $M^{\mathcal{T}}(d\theta, dy)$ on the tree which is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}$ with intensity:

$$\mathbf{1}_{[0,+\infty)}(\theta)d\theta \left(2\beta\ell^{\mathcal{T}}(dy) + \sum_{x \in \mathrm{Br}_\infty(\mathcal{T})} \Delta_x \delta_x(dy) \right).$$

The atoms $(\theta_i, y_i)_{i\in I}$ of this measure can be seen as marks that arrive on the tree, y_i being the location of the mark and θ_i the "time" at which it appears. There are two kinds of marks: some are "uniformly" distributed on the skeleton of the tree (they correspond to the term $2\beta\ell^{\mathcal{T}}$ in the intensity) whereas the others lay on the infinite branching points of the tree, an infinite branching point y being first marked after an exponential time with parameter Δ_y.

For every $x \in \mathcal{T}$, we set:

$$\theta(x) = \inf\{\theta > 0,\ M^{\mathcal{T}}([0,\theta] \times [\![\emptyset, x]\!]) > 0\},$$

which is called the record process on the tree as defined in [1]. This correspond to the first time at which a mark arrives on $[\![\emptyset, x]\!]$. Using this record process, we define the pruned tree at time q as:

$$\mathcal{T}_q = \{x \in \mathcal{T},\ \theta(x) \geq q\}$$

with the induced metric, root $\emptyset$ and mass measure the restriction of the mass measure $\mathbf{m}^{\mathcal{T}}$. If one cuts the tree $\mathcal{T}$ at time θ_i at point y_i, then $\mathcal{T}_q$ is the sub-tree of $\mathcal{T}$ containing the root at time q.

Proposition 2.2. *([6], Theorem 1.1) For $q > 0$ fixed, the distribution of $\mathcal{T}_q$ under $\mathbb{N}^\psi$ is $\mathbb{N}^{\psi_q}$ with the branching mechanism ψ_q defined for $\lambda \geq 0$ by:*

$$\psi_q(\lambda) = \psi(\lambda + q) - \psi(q). \tag{13}$$

Furthermore, we have the following Girsanov transformation that links the measures $\mathbb{N}^\psi$ and $\mathbb{N}^{\psi_q}$, see [2]: for every $q \geq 0$ and every bounded function F on $\mathbb{T}$, we have:

$$\mathbb{N}^{\psi_q}[F(\mathcal{T})] = \mathbb{N}^\psi\left[F(\mathcal{T})\,\mathrm{e}^{-\psi(q)\sigma}\right]. \tag{14}$$

We deduce from definition (8) of $\mathbf{N}^\psi$, that for any measurable non-negative functionals F and $q \geq 0$:

$$\mathbf{N}^{\psi_q}[F(\mathcal{T})] = \mathbf{N}^\psi\left[F(\mathcal{T})\,\mathrm{e}^{-\psi(q)\sigma}\right]. \tag{15}$$

Making q vary allows us to define a tree-valued process $(\mathcal{T}_q, q \geq 0)$ which is a Markov process under $\mathbb{N}^\psi$, see [2]. The process $(\mathcal{T}_q, q \geq 0)$ is a non-increasing process (for the inclusion of trees), and is càdlàg. Its one-dimensional marginals are described in Proposition 2.2 whereas its transition probabilities are given by the so-called special Markov property (see [6] Theorem 4.2 or [2] Theorem 5.6). The time-reversed process is also a Markov process and its infinitesimal transitions are described in [4] using a point process whose definition we recall now. We set:

$$\{\theta_i, i \in I^R\}$$

the set of jumping times of the process $(\mathcal{T}_\theta, \theta \geq 0)$. For every $i \in I^R$, we set $\mathcal{T}^{i,\circ} = \mathcal{T}_{\theta_i -} \setminus \mathcal{T}_{\theta_i}$ and denote by x_i the MRCA of $\mathcal{T}^{i,\circ}$. For $i \in I^R$, we set:

$$\mathcal{T}^i = \mathcal{T}^{i,\circ} \cup \{x_i\}$$

which is a real tree with distance the induced distance, root x_i and mass measure the restriction of $\mathbf{m}^{\mathcal{T}}$ to $\mathcal{T}^i$. Finally, we define, conditionally on $\mathcal{T}_0$, the following random point measure on $\mathcal{T}_0 \times \mathbb{T} \times \mathbb{R}_+$:

$$\mathcal{N} = \sum_{i \in I^R} \delta_{(x_i, \mathcal{T}^i, \theta_i)}.$$

Theorem 2.3 ([4], Theorem 3.2 and Lemma 3.3)**.** *Under $\mathbb{N}^\psi$, the predictable compensator of the backward point process defined on $\mathbb{R}_+$ by:*

$$\theta \mapsto \mathbf{1}_{\{\theta \leq q'\}} \mathcal{N}(dx, d\mathcal{T}, dq')$$

with respect to the backward left-continuous filtration $\mathcal{F} = (\mathcal{F}_\theta, \theta \geq 0)$ defined by:

$$\mathcal{F}_\theta = \sigma((x_i, \mathcal{T}^i, \theta_i),\ i \in I^R, \theta_i \geq \theta) = \sigma(\mathcal{T}_{q-},\ q \geq \theta).$$

is given by:

$$\mu(dx, d\mathcal{T}, dq) = m^{\mathcal{T}_q}(dx) \mathbf{N}^{\psi_q}[d\mathcal{T}] \mathbf{1}_{\{q \geq 0\}} dq.$$

And for any non-negative predictable process ϕ with respect to the backward filtration $\mathcal{F}$, we have:

$$\mathbb{N}^\psi \left[\int \mathcal{N}(dx, d\mathcal{T}, dq) \phi(q, \mathcal{T}_q, \mathcal{T}_{q-}) \right] = \mathbb{N}^\psi \left[\int \mu(dx, dT, dq) \phi(q, \mathcal{T}_q, \mathcal{T}_q \circledast (T, x)) \right].$$

3. Statement of the main result

We keep the notations of the previous Section. First notice that for $i \in I^R$, $\theta(x) = \theta_i$ for every $x \in \mathcal{T}^i$. We set $\sigma_i = m^{\mathcal{T}}(\mathcal{T}^i) = \sigma_{\theta_i -} - \sigma_{\theta_i}$ and $\sigma_q = m^{\mathcal{T}}(\mathcal{T}_q)$ the total mass of $\mathcal{T}_q$. By construction, we have for every $q \geq 0$:

$$\sigma_q = \sum_{i \in I^R} \mathbf{1}_{\{\theta_i \geq q\}} \sigma_i.$$

We set:

$$\Theta_q = \int_{\mathcal{T}_q} (\theta(x) - q)\, m^{\mathcal{T}}(dx).$$

This quantity appears in [1] as the limit of the number of cuts on the Aldous' CRT to isolate the root. Since $\theta(x)$ is constant on $\mathcal{T}^i$, we get:

$$\Theta_q = \sum_{i \in I^R} \mathbf{1}_{\{\theta_i \geq q\}}\, (\theta_i - q) \sigma_i = \int_q^{+\infty} \sigma_r \, dr.$$

For simplicity, we write Θ for Θ_0 and σ for σ_0.

We consider the random point measure $\mathcal{Z}^R$ on $\mathbb{R}_+ \times \mathbb{T}$ defined by:

$$\mathcal{Z}^R = \sum_{i \in I^R} \delta_{(\Theta_{\theta_i}, \mathcal{T}^i)}. \tag{16}$$

Recall the definition of H and $\mathcal{Z}^B$ of Subsection 2.7.

The main result of the paper is the next Theorem that identifies the law of the pair $(H, \mathcal{Z}^B)$ and the pair $(\Theta, \mathcal{Z})$.

Theorem 3.1. *Assume the Grey condition holds. For every non-negative measurable function Φ on $\mathbb{R}_+ \times \mathbb{T}$, and every $\lambda > 0$, $\rho \geq 0$, we have:*

$$\mathbb{N}^\psi \left[\sigma \, \mathrm{e}^{-\lambda \sigma - \rho H - \langle \mathcal{Z}^B, \Phi \rangle} \right] = \mathbb{N}^\psi \left[\sigma \, \mathrm{e}^{-\lambda \sigma - \rho \Theta - \langle \mathcal{Z}^R, \Phi \rangle} \right].$$

In particular Θ is distributed as the height H of a leaf chosen according to the normalized mass measure on the Lévy tree.

Recall that $\lim_{\varepsilon\to 0} \mathbb{N}^\psi[\sigma > \varepsilon] = +\infty$ and $\lim_{\varepsilon\to 0} \mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma\leq\varepsilon\}}] = 0$, as well as:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma\leq\varepsilon\}}] = +\infty$$

thanks to Lemma 4.1 from [11] (which is stated for $\beta = 0$ but which also holds for $\beta > 0$). The next Corollary is a direct consequence of Theorem 3.1 and the properties of Poisson point measures for the Bismut decomposition (see Proposition 4.2 in [11] for a proof of similar results).

Corollary 3.2. *Assume the Grey condition holds.* $\mathbb{N}^\psi$*-a.e., we have:*

$$\lim_{\varepsilon\to 0} \frac{1}{\mathbb{N}^\psi[\sigma > \varepsilon]} \sum_{i\in I^R} \mathbf{1}_{\{\sigma^i\geq\varepsilon\}} = \Theta.$$

$\mathbb{N}^\psi$*-a.e., for any positive sequence* $(\varepsilon_n, n \geq 0)$ *converging to* 0*, there exists a susbsequence* $(\varepsilon_{n_k}, k \geq 0)$ *such that:*

$$\lim_{k\to+\infty} \frac{1}{\mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma\leq\varepsilon_{n_k}\}}]} \sum_{i\in I^R} \sigma^i \mathbf{1}_{\{\sigma^i\leq\varepsilon_{n_k}\}} = \Theta.$$

When ψ *is regularly varying at infinity with index* $\gamma \in (1, 2]$*,* $\mathbb{N}^\psi$*-a.e. we have:*

$$\lim_{\varepsilon\to 0} \frac{1}{\mathbb{N}^\psi[\sigma \mathbf{1}_{\{\sigma\leq\varepsilon\}}]} \sum_{i\in I^R} \sigma^i \mathbf{1}_{\{\sigma^i\leq\varepsilon\}} = \Theta.$$

4. Proof of the main result

4.1. **Preliminaries results.** We first state a basic Lemma.

Lemma 4.1. *Let* $\mathcal{N}_1 = \sum_{j\in J_1} \delta_{r_j,x_j}$ *be a point measure on* $[0, +\infty)$*. If* $\sum_{j\in J_1} x_j < +\infty$*, then for every* $r \geq 0$*, we have:*

$$1 - \exp\left(-\sum_{j\in J_1} \mathbf{1}_{\{r_j\geq r\}} x_j\right) = \sum_{j\in J_1} \mathbf{1}_{\{r_j\geq r\}} (1 - \mathrm{e}^{-x_j}) \exp\left(-\sum_{\ell\in J_1} \mathbf{1}_{\{r_\ell > r_j\}} x_\ell\right). \tag{17}$$

Proof. The result is obvious for J_1 finite. For the infinite case, for $\varepsilon > 0$ consider the finite set:

$$J_{1,\varepsilon} = \{j \in J_1,\ x_j \geq \varepsilon\}.$$

Apply Formula (17) with J_1 replaced by $J_{1,\varepsilon}$ and then conclude by letting ε tend to 0 thanks to monotone convergence and dominated convergence. □

Since $\mathcal{T}_q$ is distributed according to $\mathbb{N}^{\psi_q}$, we deduce from (7) that for $q > 0$:

$$\mathbb{N}^\psi[\sigma_q] = \mathbb{N}^{\psi_q}[\sigma] = \frac{1}{\psi'(q)}, \quad \mathbb{N}^\psi[\sigma_q^2] = \mathbb{N}^{\psi_q}[\sigma^2] = \frac{\psi''(q)}{\psi'(q)^3}. \tag{18}$$

4.2. **Laplace transform of** $(\sigma, \Theta, \mathcal{Z}^R)$**.**

Proposition 4.2. *Let* Φ *be a non-negative measurable function on* $\mathbb{R}_+ \times \mathbb{T}$*. Assume that* $\langle \mathcal{Z}^R, \Phi\rangle < +\infty$ $\mathbb{N}^\psi$*-a.e. and for all* $\lambda > 0$, $\sup_{u\geq 0} g(\lambda, u) < +\infty$ *with* g *defined by (12). Then, for all* $\lambda > 0$ *and* $\rho \geq 0$*, we have:*

$$\mathbb{N}^\psi \left[\sigma\big(\rho + g(\lambda, \Theta)\big)\, \mathrm{e}^{-\lambda\sigma-\rho\Theta-\langle \mathcal{Z}^R,\Phi\rangle}\right] = 1. \tag{19}$$

Proof. For every $\varepsilon > 0$, $q \geq 0$, we set:

$$\sigma_q^\varepsilon = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i\geq q\}}\mathbf{1}_{\{\sigma_i\geq\varepsilon\}}\sigma_i, \qquad \Theta_q^\varepsilon = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i\geq q\}}\mathbf{1}_{\{\sigma_i\geq\varepsilon\}}\sigma_i(\theta_i - q),$$

and

$$Z_q^\varepsilon = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i\geq q\}}\mathbf{1}_{\{\sigma_i\geq\varepsilon\}}\Phi(\Theta_{\theta_i}, \mathcal{T}_i), \qquad Z_q = \sum_{i\in I^R} \mathbf{1}_{\{\theta_i\geq q\}}\Phi(\Theta_{\theta_i}, \mathcal{T}_i),$$

so that $Z_0 = \langle \mathcal{Z}^R, \Phi\rangle$. For every $\varepsilon > 0$, $q > 0$, we set:

$$\varphi_q^\varepsilon(\lambda, \rho) = \mathbb{N}^\psi \left[1 - \exp(-\lambda\sigma_q^\varepsilon - \rho\Theta_q^\varepsilon - Z_q^\varepsilon)\right].$$

Since $\langle \mathcal{Z}^R, \Phi\rangle$ is finite, we get that Z_q^ε is finite. We use Lemma 4.1 to get:

$$\varphi_q^\varepsilon(\lambda, \rho) = \mathbb{N}^\psi \Bigg[\sum_{i\in I^R} \mathbf{1}_{\{\theta_i\geq q\}}\mathbf{1}_{\{\sigma_i\geq\varepsilon\}}\Big(1 - \exp\Big(-\big(\lambda + \rho(\theta_i - q)\big)\sigma_i - \Phi(\Theta_{\theta_i}, \mathcal{T}_i)\Big)\Big)$$
$$\left(\exp\left(-\sum_{\ell\in I^R} \mathbf{1}_{\{\theta_\ell>\theta_i\}}\mathbf{1}_{\{\sigma_\ell\geq\varepsilon\}}\Big(\big(\lambda + \rho(\theta_\ell - q)\big)\sigma_\ell + \Phi(\Theta_{\theta_\ell}, \mathcal{T}_\ell)\Big)\right)\right)\Bigg].$$

Then, if we use Theorem 2.3 (recall that $\sigma_q = m^{\mathcal{T}_q}(\mathcal{T}_q)$), we get:

$$\varphi_q^\varepsilon(\lambda, \rho) = \mathbb{N}^\psi \left[\int_q^{+\infty} dr\, \sigma_r\, G_r^\varepsilon(\lambda + \rho(r - q), \Theta_r)\ \exp\Big(-\big(\lambda + \rho(r - q)\big)\sigma_r^\varepsilon - \rho\Theta_r^\varepsilon - Z_r^\varepsilon\Big)\right],$$

with

$$G_r^\varepsilon(\kappa, t) = \mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma\geq\varepsilon\}}\left(1 - \mathrm{e}^{-\kappa\sigma-\Phi(t,\mathcal{T})}\right)\right].$$

Thanks to (10) and (15), we get:

$$0 \leq G_r^\varepsilon(\kappa, t) \leq \mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma\geq\varepsilon\}}\right] \leq \frac{1}{\varepsilon}\mathbf{N}^{\psi_r}[\sigma] = \frac{1}{\varepsilon}\frac{\psi''(r)}{\psi'(r)}. \tag{20}$$

Since ψ'' is non-increasing and ψ' is non-decreasing, we get that for fixed $q > 0$, the map $r \mapsto \partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right)$ is non-negative and bounded for $r > q$. We deduce from (10) and (15) that:

$$\mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma\geq\varepsilon\}}\sigma\, \mathrm{e}^{-\kappa\sigma-\Phi(t,\mathcal{T})}\right] \leq \frac{1}{\varepsilon}\mathbf{N}^{\psi_r}\left[\sigma^2\right] = \frac{1}{\varepsilon}\frac{1}{\psi'(r)}\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right).$$

We deduce that the map $\kappa \mapsto G_r^\varepsilon(\kappa, t)$ is $\mathcal{C}^1$ and:

$$0 \leq \partial_\kappa G_r^\varepsilon(\kappa, t) = \mathbf{N}^{\psi_r}\left[\mathbf{1}_{\{\sigma\geq\varepsilon\}}\sigma\, \mathrm{e}^{-\kappa\sigma-\Phi(t,\mathcal{T})}\right] \leq \frac{1}{\varepsilon}\frac{1}{\psi'(r)}\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right). \tag{21}$$

We set:

$$H_{r,\lambda}^\varepsilon(q) = \mathbb{N}^\psi\left[\sigma_r\, G_r^\varepsilon(\lambda + \rho(r - q), \Theta_r)\ \exp\Big(-\big(\lambda + \rho(r - q)\big)\sigma_r^\varepsilon - \rho\Theta_r^\varepsilon - Z_r^\varepsilon\Big)\right],$$

so that:

$$\varphi_q^\varepsilon(\lambda,\rho) = \int_q^{+\infty} H^\varepsilon_{r,\lambda}(q)\, dr.$$

Thanks to (20) and (18), we get $0 \leq H^\varepsilon_{r,\lambda}(q) \leq \varepsilon^{-1}\psi''(r)/\psi'(r)^2$. This implies in turn that $\varphi_q^\varepsilon(\lambda,\rho) \leq \varepsilon^{-1}/\psi'(q)$.

For $r>0$, $\kappa>0$, we set:

$$h_r^\varepsilon(\kappa) = \mathbb{N}^\psi\left[\sigma_r\ \left(\partial_\kappa G_r^\varepsilon(\kappa,\Theta_r) + \sigma_r^\varepsilon G_r^\varepsilon(\kappa,\Theta_r)\right)\, \mathrm{e}^{-\kappa\sigma_r^\varepsilon - \rho\Theta_r^\varepsilon - Z_r^\varepsilon}\right].$$

Since $\sigma_r^\varepsilon \leq \sigma_r$, we have, using (18):

$$\begin{aligned} 0 \leq h_r^\varepsilon(\kappa) &\leq \frac{1}{\varepsilon}\,\mathbb{N}^\psi\left[\sigma_r\, \frac{1}{\psi'(r)}\,\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right) + \sigma_r^2\frac{\psi''(r)}{\psi'(r)}\right] \\ &\leq \frac{1}{\varepsilon}\left[\frac{1}{\psi'(r)^2}\,\partial_r\left(\frac{-\psi''(r)}{\psi'(r)}\right) + \frac{\psi''(r)^2}{\psi'(r)^4}\right]. \end{aligned}$$

By monotonicity, we get:

$$\begin{aligned} &\int_{[q,+\infty)^2} duds\, \mathbf{1}_{\{u<s\}} h_s^\varepsilon(\lambda+\rho(s-u)) \\ &\qquad \leq \int_{[q,+\infty)^2} duds\, \mathbf{1}_{\{u<s\}}\, \frac{1}{\varepsilon}\left[\frac{1}{\psi'(s)^2}\,\partial_s\left(\frac{-\psi''(s)}{\psi'(s)}\right) + \frac{\psi''(s)^2}{\psi'(s)^4}\right] \\ &\qquad \leq \int_{[q,+\infty)^2} duds\ \frac{1}{\varepsilon}\,\mathbf{1}_{\{u<s\}}\left[\frac{1}{\psi'(u)^2}\,\partial_s\left(\frac{-\psi''(s)}{\psi'(s)}\right) + \frac{\psi''(u)}{\psi'(u)^2}\frac{\psi''(s)}{\psi'(s)^2}\right] \\ &\qquad = \frac{2}{\varepsilon}\int_{[q,+\infty)} du\, \frac{\psi''(u)}{\psi'(u)^3} \\ &\qquad = \frac{1}{\varepsilon}\,\frac{1}{\psi'(q)^2}. \end{aligned}$$

We deduce that the maps $u \mapsto H^\varepsilon_{s,\lambda}(u)$ and $\lambda \mapsto H^\varepsilon_{s,\lambda}(u)$ are $\mathcal{C}^1$ for $\lambda \geq 0$, $s \geq u \geq q$, with:

$$\partial_u H^\varepsilon_{s,\lambda}(u) = -\rho\partial_\lambda H^\varepsilon_{s,\lambda}(u) \quad \text{and} \quad \left|\partial_\lambda H^\varepsilon_{s,\lambda}(u)\right| \leq h_s^\varepsilon(\lambda+\rho(s-u)).$$

Thus we have $\int_{[q,+\infty)^2} duds\, \mathbf{1}_{\{u<s\}}\left|\partial_u H^\varepsilon_{s,\lambda}(u)\right| \leq \rho/\varepsilon\psi'(q)^2$. Then, elementary computation yields:

$$\varphi_q^\varepsilon(\lambda,\rho) = \int_q^{+\infty} H^\varepsilon_{r,\lambda}(q)\, dr = \int_q^{+\infty} du\, \left[H^\varepsilon_{u,\lambda}(u) - \int_u^{+\infty} ds\, \partial_u H^\varepsilon_{s,\lambda}(u)\right].$$

We deduce that the maps $q \mapsto \varphi_q^\varepsilon(\lambda,\rho)$ and $\lambda \mapsto \varphi_q^\varepsilon(\lambda,\rho)$ are $\mathcal{C}^1$ and:

$$\partial_q\varphi_q^\varepsilon(\lambda,\rho) = -H^\varepsilon_{q,\lambda}(q) + \int_q^{+\infty} ds\, \partial_u H^\varepsilon_{s,\lambda}(q) = -H^\varepsilon_{q,\lambda}(q) - \rho\partial_\lambda\int_q^{+\infty} ds\, H^\varepsilon_{s,\lambda}(q).$$

With $H^\varepsilon_{q,\lambda}(q) = \mathbb{N}^\psi\left[\sigma_q\, G_q^\varepsilon(\lambda,\Theta_q)\ \exp\Big(-\lambda\sigma_q^\varepsilon - \rho\Theta_q^\varepsilon - Z_q^\varepsilon\Big)\right]$, we deduce that:

$$\partial_q\varphi_q^\varepsilon(\lambda,\rho) = -\mathbb{N}^\psi\left[\sigma_q\, G_q^\varepsilon(\lambda,\Theta_q)\ \exp\Big(-\lambda\sigma_q^\varepsilon - \rho\Theta_q^\varepsilon - Z_q^\varepsilon\Big)\right] - \rho\partial_\lambda\varphi_q^\varepsilon(\lambda,\rho). \tag{22}$$

We also have:

$$\partial_\lambda\varphi_q^\varepsilon(\lambda,\rho) = \mathbb{N}^\psi\left[\sigma_q^\varepsilon \exp(-\lambda\sigma_q^\varepsilon - \rho\Theta_q^\varepsilon - Z_q^\varepsilon)\right]. \tag{23}$$

Moreover, thanks to Girsanov formula (14), we have:

$$\varphi_q^\varepsilon(\lambda, \rho) = \mathbb{N}^\psi \left[(1 - \exp(-\lambda\sigma_0^\varepsilon - \rho\Theta_0^\varepsilon - Z_0^\varepsilon))\, \mathrm{e}^{-\psi(q)\sigma}\right].$$

We deduce that:

$$\begin{aligned}\partial_q\varphi_q^\varepsilon(\lambda, \rho) &= -\psi'(q)\mathbb{N}^\psi \left[\sigma\,(1 - \exp(-\lambda\sigma_0^\varepsilon - \rho\Theta_0^\varepsilon - Z_0^\varepsilon))\, \mathrm{e}^{-\psi(q)\sigma}\right] \\ &= -1 + \psi'(q)\mathbb{N}^\psi \left[\sigma_q \exp(-\lambda\sigma_q^\varepsilon - \rho\Theta_q^\varepsilon - Z_q^\varepsilon)\right].\end{aligned}$$

We deduce from (22) and (23) that:

$$\mathbb{N}^\psi \left[\Big(\sigma_q(\psi'(q) + G_q^\varepsilon(\lambda, \Theta_q)) + \rho\sigma_q^\varepsilon\Big) \exp\left(-\lambda\sigma_q^\varepsilon - \rho\Theta_q^\varepsilon - Z_q^\varepsilon\right)\right] = 1. \tag{24}$$

Using Girsanov formula (15) and (9), we get:

$$G_q^\varepsilon(\lambda, t) \leq G_q^0(\lambda, t) = g(\lambda + \psi(q), t) - \psi'(0) - \mathbf{N}^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = g(\lambda + \psi(q), t) - \psi'(q).$$

We deduce that:

$$\sigma_q(\psi'(q) + G_q^\varepsilon(\lambda, \Theta_q)) + \rho\sigma_q^\varepsilon \leq \sigma_q(\sup_{t\geq 0} g(\lambda + \psi(q), t) + \rho).$$

By dominated convergence, letting ε decrease to 0 in (24), we deduce that:

$$\mathbb{N}^\psi \left[\sigma_q\Big(g(\lambda + \psi(q), \Theta) + \rho\Big) \exp\left(-\lambda\sigma_q - \rho\Theta_q - Z_q\right)\right] = 1.$$

Using Girsanov formula (14) once again, we get:

$$\mathbb{N}^\psi \left[\sigma\Big(g(\lambda + \psi(q), \Theta) + \rho\Big) \exp\left(-(\lambda + \psi(q))\sigma - \rho\Theta - \langle \mathcal{Z}^R, \Phi\rangle\right)\right] = 1.$$

Since $\lambda > 0$ and $q > 0$ are arbitrary, we deduce that (19) holds. □

We deduce the following Corollary which states that $(H, (\mathcal{T}^j, j \in I^B))$ and $(\Theta, (\mathcal{T}^i, i \in I^R))$ have the same distribution (but not yet (H, Z^B) and (Θ, Z^R) since the branching points are not taken into account).

Let γ be a non-negative measurable function defined on $\mathbb{T}$. For a measure $\mathcal{Z}$ on $\mathbb{R}_+ \times \mathbb{T}$, we shall abuse notation and write:

$$\langle \mathcal{Z}, \gamma\rangle = \int \gamma(T)\, \mathcal{Z}(dt, dT).$$

Corollary 4.3. *For every non-negative measurable function γ on $\mathbb{T}$ such that $\gamma(\mathcal{T}) = 0$ if $m^{\mathcal{T}}(\mathcal{T}) = 0$, and every $\lambda \geq 0$, $\rho \geq 0$, we have:*

$$\mathbb{N}^\psi \left[\sigma\, \mathrm{e}^{-\lambda\sigma - \rho H - \langle \mathcal{Z}^B, \gamma\rangle}\right] = \mathbb{N}^\psi \left[\sigma\, \mathrm{e}^{-\lambda\sigma - \rho\Theta - \langle \mathcal{Z}^R, \gamma\rangle}\right]. \tag{25}$$

Proof. Let $\lambda > 0$. Recall $\sigma = m^{\mathcal{T}}(\mathcal{T})$. First assume that $\gamma(\mathcal{T}) \leq c\sigma$ for some finite constant c. Taking $\Phi(t, \mathcal{T}) = \gamma(\mathcal{T})$ in Theorem 2.1 and using that $g(\lambda, u)$ doesn't depend on u, we get:

$$\mathbb{N}^\psi \left[\sigma\, \mathrm{e}^{-\lambda\sigma - \rho H - \langle \mathcal{Z}^B, \gamma\rangle}\right] = \frac{1}{\rho + g(\lambda, 0)}.$$

Notice that $\langle \mathcal{Z}^R, \Phi\rangle \leq c\sigma$ and thus hypothesis from Proposition 4.2 are in force. We deduce from Proposition 4.2 that:

$$\mathbb{N}^\psi \left[\sigma \exp\left(-\lambda\sigma - \rho\Theta - \langle \mathcal{Z}^R, \gamma\rangle\right)\right] = \frac{1}{\rho + g(\lambda, 0)}.$$

Thus equality (25) holds. Use monotone convergence to remove hypothesis $\lambda > 0$ and $\gamma(\mathcal{T}) \leq c\sigma$ for some finite constant c. $\square$

4.3. **Proof of Theorem 3.1.** Let Φ be a measurable non-negative function defined on the set $\mathbb{R}_+ \times \mathbb{T}$. Let us assume that for every $\mathcal{T} \in \mathbb{T}$, $t \mapsto \Phi(t, \mathcal{T})$ is continuous, $\langle \mathcal{Z}^R, \Phi\rangle$ is finite $\mathbb{N}^\psi$-a.s. and that the function g defined by (12) is bounded for any $\lambda > 0$ as a function of u. We set:

$$\Gamma^R(r,h) = \mathbb{N}^\psi\left[\mathrm{e}^{-\langle \mathcal{Z}^R,\Phi\rangle} \mid \sigma = r,\ \Theta = h\right].$$

We deduce from Proposition 4.2 and Corollary 4.3 that for every $\lambda > 0$, $\rho \geq 0$, we have:

$$\begin{aligned} 1 &= \mathbb{N}^\psi\left[\sigma\big(\rho + g(\lambda,\Theta)\big)\,\mathrm{e}^{-\lambda\sigma-\rho\Theta-\langle \mathcal{Z}^R,\Phi\rangle}\right] \\ &= \mathbb{N}^\psi\left[\sigma\big(\rho + g(\lambda,\Theta)\big)\,\mathrm{e}^{-\lambda\sigma-\rho\Theta}\,\Gamma^R(\sigma,\Theta)\right] \\ &= \mathbb{N}^\psi\left[\sigma\Big(\rho + g(\lambda,H)\Big)\,\mathrm{e}^{-\lambda\sigma-\rho H}\,\Gamma^R(\sigma,H)\right]. \end{aligned} \tag{26}$$

Let $\sum_{i\in I}\delta_{(h_i,\mathcal{T}_i)}$ be a Poisson measure with intensity $dh\,\mathbf{N}^\psi[d\mathcal{T}]$ under some probability measure P. For every $i \in I$, we set $\sigma_i = m^{\mathcal{T}_i}(\mathcal{T}_i)$. Then for every $h > 0$, we set:

$$\sigma(h) = \sum_{i\in I}\mathbf{1}_{\{h_i\leq h\}}\sigma_i.$$

Equation (26) and Theorem 2.1 imply that:

$$\int_0^{+\infty} dh\ \mathrm{e}^{-(\rho+\psi'(0))h}\,\mathrm{e}^{-G(h)}(\rho + g(\lambda,h)) = 1,$$

with:

$$G(h) = -\log\left(E\left[\mathrm{e}^{-\lambda\sigma(h)}\,\Gamma^R(\sigma(h),h)\right]\right).$$

We deduce that:

$$\int_0^{+\infty} dh\ \mathrm{e}^{-\rho h}\left[1 - \mathrm{e}^{-\psi'(0)h-G(h)}\right] = \int_0^{+\infty}\frac{1}{\rho}\,\mathrm{e}^{-\rho h}\ dA(h) = \int_0^{+\infty} dh\ \mathrm{e}^{-\rho h}\,A(h),$$

with:

$$A(h) = \int_0^h du\ \mathrm{e}^{-\psi'(0)u-G(u)}\,g(\lambda,u).$$

Since this holds for every $\rho \geq 0$, uniqueness of the Laplace transform implies that:

$$A(h) = 1 - \mathrm{e}^{-\psi'(0)h-G(h)} \quad \text{a.e.} \tag{27}$$

Since A is continuous, there exists a continuous function $\tilde{G}$ such that a.e. $\tilde{G} = G$. Since, $t \mapsto \Phi(t,\mathcal{T})$ is continuous, we get that, for every $\lambda \geq 0$, $u \mapsto g(\lambda,u)$ is continuous. Then A is of class $\mathcal{C}^1$ and so is $\tilde{G}$. Moreover, by differentiating (27), we get:

$$\psi'(0) + \tilde{G}'(h) = g(\lambda,h).$$

Since $A(0) = 0$, we get $\tilde{G}(0) = 0$, and thus $\psi'(0)h + \tilde{G}(h) = \int_0^h g(\lambda,u)du$. This implies that:

$$\int_0^h g(\lambda,u)du = G(h) + \psi'(0)h \quad \text{a.e.} \tag{28}$$

We have:

$$\begin{aligned}
\mathbb{N}^\psi\left[\sigma\,\mathrm{e}^{-\lambda\sigma-\rho H-\langle \mathcal{Z}^B,\Phi\rangle}\right] &= \int_0^{+\infty} dh\;\mathrm{e}^{-\rho h-\int_0^h g(\lambda,u)du}\\
&= \int_0^{+\infty} dh\;\mathrm{e}^{-(\rho+\psi'(0))h-G(h)}\\
&= \int_0^{+\infty} dh\;\mathrm{e}^{-(\rho+\psi'(0))h}\,E\left[\mathrm{e}^{-\lambda\sigma(h)}\,\Gamma^R(\sigma(h),h)\right]\\
&= \mathbb{N}^\psi\left[\sigma\,\mathrm{e}^{-\lambda\sigma-\rho H}\,\Gamma^R(\sigma,H)\right]\\
&= \mathbb{N}^\psi\left[\sigma\,\mathrm{e}^{-\lambda\sigma-\rho\Theta}\,\Gamma^R(\sigma,\Theta)\right]\\
&= \mathbb{N}^\psi\left[\sigma\,\mathrm{e}^{-\lambda\sigma-\rho\Theta-\langle \mathcal{Z}^R,\Phi\rangle}\right],
\end{aligned}$$

where we used Theorem 2.1 for the first and fourth equalities, (28) for the second, the definition of G for the third, Corollary 4.3 (which states that (σ, H) and (σ,Θ) have the same distribution under $\mathbb{N}^\psi$) for the fifth, and the definition of Γ^R for the last.

Then use monotone class theorem to remove the hypothesis on Φ and end the proof.

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