



Optimal vaccination: various (counter) intuitive examples

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Abstract

In previous articles, we formalized the problem of optimal allocation strategies for a (perfect) vaccine in an infinite-dimensional metapopulation model. The aim of the current paper is to illustrate this theoretical framework with multiple examples where one can derive the analytic expression of the optimal strategies. We discuss in particular the following points: whether or not it is possible to vaccinate optimally when the vaccine doses are given one at a time (greedy vaccination strategies); the effect of assortativity (that is, the tendency to have more contacts with similar individuals) on the shape of optimal vaccination strategies; the particular case where everybody has the same number of neighbors.

Keywords Kernel operator · Vaccination strategy · Effective reproduction number · Multi-objective optimization · Pareto frontier

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1 Introduction

1.1 Motivation

The basic reproduction number, denoted by R_0 , plays a fundamental role in epidemiology as it determines the long-term behavior of an epidemic. For a homogeneous model, it is defined as the number of secondary cases generated by an infected individual in an otherwise susceptible population. When this number is below 1, an infected individual causes less than one infection before its recovery in average; the disease therefore declines over time until it eventually dies out. On the contrary, when the reproduction number is greater than 1, the disease invades the population. It follows from this property that a proportion equal to $1 - 1/R_0$ of the population should be immunized in order to stop the outbreak. We refer the reader to the monograph of Keeling and Rohani (2008) for a reminder of these basic properties on the reproduction number.

In the heterogeneous generalization of classical compartmental models (Lajmanovich and Yorke 1976; Beretta and Capasso 1986; Delmas et al. 2022a), the population is stratified into homogeneous groups sharing the same characteristics (*e.g.*, time to recover from the disease and interaction with the other groups). For these so-called metapopulation models, it is still possible to define a meaningful reproduction number R_0 , as the number of secondary cases generated by a typical infectious individual when all other individuals are uninfected (Diekmann et al. 1990). The reproduction number can then be identified as the spectral radius of the so-called next-generation matrix (Van Den Driessche and Watmough 2002). This encompasses SIS, SIR and SEIR epidemic models; see Section 2 in (Delmas et al. 2022b) for a discussion. With this definition, it is still true that the outbreak dies out if R_0 is smaller than 1 and invades the population otherwise; see Thieme (1985); Hethcote and Thieme (1985); Van Den Driessche and Watmough (2002); Thieme (2011); Delmas et al. (2022a) for instance.

Suppose now that we have at our disposal a vaccine with *perfect efficacy*, that is, vaccinated individuals are completely immunized to the disease. After a vaccination campaign, let η denote the proportion of **non-vaccinated** individuals in the population: in inhomogeneous models, η depends *a priori* on the group as different groups may be vaccinated differently. We will call η a *vaccination strategy*. For any strategy η , let us denote by $R_e(\eta)$ the corresponding reproduction number of the non-vaccinated population, also called the *effective reproduction number*. In the metapopulation model, it can also be expressed as the spectral radius of the effective next generation matrix; see Equation (5) below. The choice of η naturally raises a question that may be expressed as the following informal constrained optimization problem:

$$\begin{cases} \text{Minimize:} & \text{the quantity of vaccine to administrate} \\ \text{subject to:} & \text{herd immunity is reached, that is, } R_e \leq 1. \end{cases} \quad (1)$$

For practical reasons, we will instead look at the problem the other way around. If the vaccine is only available in limited quantities, the decision makers could try to allocate the doses so as to maximize efficiency; a natural indicator of this efficiency is

the effective reproduction number. This reasoning leads to the following constrained problem:

$$\left\{ \begin{array}{ll} \textbf{Minimize:} & \text{the effective reproduction number } R_e \\ \textbf{subject to:} & \text{a given quantity of available vaccine.} \end{array} \right. \quad (2)$$

In accordance with (Delmas et al. 2021b), we will denote by R_{e^*} the value of this problem: it is a function of the quantity of available vaccine. The graph of this function is called the *Pareto frontier*. In order to measure how bad a vaccination strategy can be, we will also be interested in maximizing the effective reproduction number given a certain quantity of vaccine:

$$\left\{ \begin{array}{ll} \textbf{Maximise:} & \text{the effective reproduction number } R_e \\ \textbf{subject to:} & \text{a given quantity of available vaccine.} \end{array} \right. \quad (3)$$

The value function corresponding to this problem is denoted by R_e^* and its graph is called the *anti-Pareto frontier*. We will quantify the “quantity of available vaccine” for the vaccination strategy η by a cost $C(\eta)$. Roughly speaking the “best” (resp. “worst”) vaccination strategies are solutions to Problem (2) (resp. Problem (3)). Still following Delmas et al. (2021b), they will be called *Pareto optimal* (resp. *anti-Pareto optimal*) strategies.

The problem of optimal vaccine allocation has been studied mainly in the metapopulation setting where the population is divided into a finite number of subgroups with the same characteristics. Longini, Ackerman and Elverback were the first interested in the question of optimal vaccine distribution given a limited quantity of vaccine supply (Longini et al. 1978). Using the concept of next-generation matrix introduced by Diekmann, Heesterbeek and Metz Diekmann et al. (1990), Hill and Longini reformulated this problem thanks to the reproduction number (Hill and Longini 2003). Several theoretical and numerical studies followed focusing on Problem (1) and/or Problem (2) in the metapopulation setting (Goldstein et al. 2010; Poghotanyan et al. 2018; Duijzer et al. 2018; Hao et al. 2019). We also refer the reader to the introduction of Delmas et al. (2021b) for a detailed review of the bibliography.

In two previous works (Delmas et al. 2021b, a), we provided an infinite-dimensional framework generalizing the metapopulation model where Problems (2) and (3) are well-posed, justified that the optimizers are indeed Pareto optimal and studied in detail the Pareto and anti-Pareto frontiers. Since there is no closed form for the effective reproduction number, Problems (2) and (3) are hard to solve in full generality: our goal here is to exhibit examples where one can derive analytic expressions for the optimal vaccination strategies. The simple models we study give a gallery of examples and counter-examples to natural questions or conjectures, and may help understanding common rules of thumb for choosing vaccination policies. We will in particular be interested in the following three notions.

- (i) **Greedy parametrization of the frontiers.** For the decision maker it is important to know if global optimization and sequential optimization are the same as one cannot unvaccinate people and redistribute the vaccine once more doses become

available. More precisely, there is a natural order on the vaccination strategies: let us write $\eta' \leq \eta$ if all the people that are vaccinated when following the strategy η are also vaccinated when following the strategy η' . Let η be an optimal solution of (2) for cost $c = C(\eta)$, that is, $R_e(\eta) = R_{e^*}(c)$. If, for $c' > c$, we can find a strategy $\eta' \leq \eta$ such that $R_e(\eta') = R_{e^*}(c')$, then the optimization may be, at least in principle, found in a greedy way: giving sequentially each new dose of vaccine so as to minimize R_e gives, in the end, an optimal strategy for any quantity of vaccine. By analogy with the corresponding notion for algorithms we will say in this case that there exists a *greedy parametrization* of the Pareto frontier. The existence of such a greedy parametrization was already discussed by Cairns (1989) and is examined for each model throughout this paper.

- (ii) **Assortative/Disassortative network.** The second notion is a property of the network called *assortativity*: a network is called assortative when the nodes tend to attach to others that are similar in some way and *disassortative* otherwise. The assortativity or disassortativity of a network is an important property that helps to understand its topology. It has been observed that social networks are usually assortative while biological and technological networks are disassortative, see for example (Newman 2002). The optimal vaccination strategies can differ dramatically in the case of assortative versus disassortative mixing, see (Galeotti and Rogers 2013) for a study in a population composed of two groups. This question is in particular addressed in Sect. 4 for an elementary model with an arbitrary number of groups.
- (iii) **How to handle individuals with the same level of connection.** Targeting individuals that are the most connected is a common approach used to prevent an epidemic in a complex network (Pastor-Satorras and Vespignani 2002). In [17], we show that these strategies are optimal for the so-called monotonic kernel models, in which the individuals may be naturally ordered by a score related to their connectivity. When many individuals or groups are tied for the best score, either from the beginning or after some vaccine has been distributed, the optimal way of vaccinating them may be surprisingly varied according to the situation. This variety of answers appears already in the treatment of such individuals in the assortative/disassortative toy model developed in Sect. 4. To go further in this direction, a large part of the current paper, see Sects. 5–7, is devoted to regular or “constant degree” models where all individuals share the same degree. We shall in particular ask whether uniform vaccination strategies are either the “best” or the “worst” or even neither the “best” nor the “worst” possible strategies.

In most of the examples below, the next-generation matrices are symmetric. Although the optimization problems (2) and (3) make sense without symmetry assumptions (Delmas et al. 2021b), symmetry, or at least symmetrizability, is required for the convexity and concavity properties of the effective reproduction number R_e proved in Delmas et al. (2021a). Note that real world data provides in general symmetric or symmetrizable contact matrices; see for example the POLYMOD matrix in Mossong et al. (2008) and the theoretical model in Busenberg and Castillo-Chavez (1991).

1.2 Main results

Section 2 is dedicated to classical finite-dimensional metapopulation models. We present two simple models that, despite being seemingly very similar, display totally different behaviors: the asymmetric and symmetric circle graphs. For the first one, where individuals of the group i can only be infected by individuals of the group $i - 1$ and which corresponds to a next generation matrix given by:

$$K_{ij} = \mathbb{1}_{\{i=j+1 \pmod N\}},$$

with N the number of groups or nodes in the circle, we derive a greedy parametrization of the Pareto frontier. On the second one, where individuals of the group i can only be infected by individuals of the group $i - 1$ or $i + 1$ and which corresponds to a next generation matrix given by:

$$K_{ij} = \mathbb{1}_{\{i=j\pm 1 \pmod N\}},$$

we observe numerically that the Pareto frontier is much more complicated, and in particular cannot be parametrized greedily. Those two models are in fact constant degree models; the uniform vaccination strategies are the “worst” for the first model, and neither the “best” nor the “worst” strategies for the second.

After Sect. 3, where we recall the kernel setting used in Delmas et al. (2021b) for infinite dimensional models, we focus in Sect. 4 on the effect of assortativity on optimal vaccination strategies. We define a simple kernel model that may be assortative or disassortative depending on the sign of a parameter. In the discrete metapopulation model, the next generation matrix can be written (up to a multiplicative constant) as:

$$K_{ij} = (1 + \varepsilon \mathbb{1}_{\{i \neq j\}}) \mu_j,$$

where $\mu_j \geq 0$ represents the proportional size of group j . The model is assortative if $\varepsilon < 0$ (and $\varepsilon \geq -1$ so that the matrix K is non-negative) and disassortative if $\varepsilon > 0$. We describe completely the optimal vaccination strategies, see Theorem 4.2, and show that the best strategies for the assortative case are the worst ones if the mixing pattern is disassortative, and vice-versa. We also prove that all the Pareto and anti-Pareto frontiers admit greedy parametrizations, and that Pareto optimal strategies prioritize individuals that in some sense have the highest degree, that is, are the most connected.

In Sect. 5, we consider constant degree models, which are the analogue of regular graphs in the infinite-dimensional setting. In the discrete metapopulation model, the sums over each row and the sums over each column of the next generation matrix are constant. We prove, see Proposition 5.4, that if the effective reproduction function R_e is convex then the uniform strategies are the “best” and they give a greedy parametrization of the Pareto frontier; and that if R_e is concave, the uniform strategies are the “worst”. Section 6 is then devoted to a particular model of rank two, which corresponds in the discrete metapopulation model to a next generation matrix of the form:

$$K_{ij} = (1 + \varepsilon \alpha_i \alpha_j) \mu_j \quad \text{with} \quad \sum_j \alpha_j \mu_j = 0,$$

where ε may be $+1$ or -1 , and $\sup_i \alpha_i^2 \leq 1$, so that the matrix K is non-negative. The condition $\sum_j \alpha_j \mu_j = 0$ ensures that the model has a constant degree. In those cases, we give a complete description of the “best” and the “worst” vaccination strategies, the uniform one being “best” for $\varepsilon = +1$ and “worst” otherwise, see Proposition 6.2. In Sect. 6.5, we also provide an example of kernel (in infinite dimension) for which the set of optimal strategies has an infinite number of connected components. In this particular case, there is no greedy parametrization of the Pareto frontier.

As another application of the results of Sect. 5, we investigate in Sect. 7 geometric constant degree kernels defined on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Intuitively an individual at point x on the sphere is infected by an individual at point y with an intensity $k(x, y)$ depending on the distance between x and y . Those kernels appear in the graphon theory as limit of large dense random geometric graphs. We give a particular attention to the affine model in Sect. 7.3, where:

$$k(x, y) = 1 + \varepsilon \langle x, y \rangle, \quad \varepsilon \geq -1,$$

where $\langle x, y \rangle$ is the usual scalar product in the ambient space \mathbb{R}^d . Intuitively, for $\varepsilon > 0$, the infection propagates through the nearest neighbors: this may be seen as a kind of spatial assortativity. By contrast, for $\varepsilon < 0$ the infection propagates through the furthest individuals neighbors, in a spatially disassortative way. For this affine model, we completely describe the “best” and the “worst” vaccination strategies, see Proposition 7.9.

2 First examples in the discrete setting

In this section, we use the framework developed by Hill and Longini in Hill and Longini (2003) for metapopulation models and provide optimal vaccination strategies for two very simple examples. Despite their simplicity, these examples showcase a number of interesting behaviors, that will occur in a much more general setting, as we will see in the rest of the paper.

2.1 The reproduction number in metapopulation models

In metapopulation models, the population is divided into $N \geq 2$ different subpopulations and we suppose that individuals within a same subpopulation share the same characteristics. The different groups are labeled $1, 2, \dots, N$. We denote by $\mu_1, \mu_2, \dots, \mu_N$ their respective size (in proportion with respect to the total size) and we suppose that those do not change over time. By the linearization of the dynamic of the epidemic at the disease-free equilibrium, we obtain the so-called *next-generation matrix* K , see Van Den Driessche and Watmough (2002), which is a $N \times N$ matrix with non-negative coefficients. For a detailed discussion on the biological interpretation of the coefficients of the next-generation matrix, we refer the reader to (Delmas et al. 2022b, Section 2). We also refer to [14] for an extensive treatment of the two-dimensional case.

The basic reproduction number is equal to the spectral radius of the next-generation matrix:

$$R_0 = \rho(K), \tag{4}$$

where ρ denotes the spectral radius. Since the matrix K has non-negatives entries, the Perron-Frobenius theory implies that R_0 is also an eigenvalue of K . If $R_0 > 1$, the epidemic process grows away from zero infectives while if $R_0 < 1$, the disease cannot invade the population; see (Van Den Driessche and Watmough 2002 ,Theorem 2) .

We now introduce the effect of vaccination. Suppose that we have at our disposal a vaccine with perfect efficacy, *i.e.*, vaccinated individuals are completely immunized to the infection. We denote by $\eta = (\eta_1, \dots, \eta_N)$ the vector of the proportions of **non-vaccinated** individuals in the different groups. We shall call η a vaccination strategy and denote by $\Delta = [0, 1]^N$ the set of all possible vaccination strategies. According to Delmas et al. (2022b, 2021b), the next-generation matrix corresponding to the dynamic with vaccination is equal to the matrix K multiplied by the matrix $\text{Diag}(\eta)$ on the right, where $\text{Diag}(\eta)$ is the $N \times N$ diagonal matrix with coefficients $\eta \in \Delta$. We call the spectral radius of this matrix the *effective reproduction number*:

$$R_e(\eta) = \rho(K \cdot \text{Diag}(\eta)) . \tag{5}$$

The effective reproduction number accounts for the vaccinated (and immunized) people in the population, as opposed to the basic reproduction number, which corresponds to a fully susceptible population. When nobody is vaccinated, that is $\eta = \mathbb{1} = (1, \dots, 1)$, $\text{Diag}(\eta)$ is equal to the identity matrix, the next-generation matrix is unchanged and $R_e(\eta) = R_e(\mathbb{1}) = R_0$.

We suppose that the *cost* of a vaccination strategy is, up to an irrelevant multiplicative constant, equal to the total proportion of vaccinated people and is therefore given by:

$$C(\eta) = \sum_{i=1}^N (1 - \eta_i)\mu_i = 1 - \sum_{i=1}^N \eta_i \mu_i, \tag{6}$$

where $\eta = (\eta_1, \dots, \eta_N) \in \Delta$. We refer to (Delmas et al. 2021b, Section 5.1, Remark 5.2) for considerations on more general cost functions.

Example 2.1 (Uniform vaccination) The uniform strategy of cost c consists in vaccinating the same proportion of people in each group: $\eta = (1 - c)\mathbb{1}$. By homogeneity of the spectral radius, the reproduction number $R_e(\eta)$ is then equal to $(1 - c)R_0$.

2.2 Optimal allocation of vaccine doses

As mentioned in the introduction and recalled in Sect. 2.1, reducing the reproduction number is fundamental in order to control and possibly eradicate the epidemic. However, the vaccine may only be available in a limited quantity, and/or the decision maker

may wish to limit the cost of the vaccination policy. This motivates our interest in the following related problem:

$$\begin{cases} \min & R_e(\eta), \\ \text{such that} & C(\eta) = c. \end{cases} \tag{7}$$

According to Delmas et al. (2021b), one can replace the constraint $\{C(\eta) = c\}$ by $\{C(\eta) \leq c\}$ without modifying the solutions. The opposite problem consists in finding out the *worst possible way* of allocating vaccine. While this does not seem at first sight to be as important, a good understanding of bad vaccination strategies may also provide rules of thumb in terms of anti-patterns. In order to estimate how bad a vaccination strategy can be, we therefore also consider the following problem:

$$\begin{cases} \max & R_e(\eta), \\ \text{such that} & C(\eta) = c. \end{cases} \tag{8}$$

According to Delmas et al. (2021b), one can replace the constraint $\{C(\eta) = c\}$ by $\{C(\eta) \geq c\}$ without modifying the solutions.

Since the coefficients of the matrix $K \cdot \text{Diag}(\eta)$ depend continuously on η , it is classical that its eigenvalues also depend continuously on η (see for example Horn and Johnson 2013, Appendix D) and in particular the function R_e is continuous on $\Delta = [0, 1]^N$. Since the function C is also continuous on Δ , the compactness of Δ ensures the existence of solutions for Problems (7) and (8). For $c \in [0, 1]$, $R_{e\star}(c)$ (resp. $R_e^\star(c)$) stands for the minimal (resp. maximal) value taken by R_e on the set of all vaccination strategies η such that $C(\eta) = c$:

$$R_{e\star}(c) = \min\{R_e(\eta) : \eta \in \Delta \text{ and } C(\eta) = c\}, \tag{9}$$

$$R_e^\star(c) = \max\{R_e(\eta) : \eta \in \Delta \text{ and } C(\eta) = c\}. \tag{10}$$

It is easy to check that the functions $R_{e\star}$ and R_e^\star are non increasing. Indeed, if η^1 and η^2 are two vaccination strategies such that $\eta^1 \leq \eta^2$ (where \leq stands for the pointwise order), then $R_e(\eta^1) \leq R_e(\eta^2)$ according to the Perron-Frobenius theory. This easily implies that $R_{e\star}$ and R_e^\star are non-increasing. We refer to Delmas et al. (2021b, 2022b) for more properties on those functions; in particular they are also continuous. For the vaccination strategy $\eta = 0 = (0, \dots, 0)$ (everybody is vaccinated) with cost $C(0) = 1$, the transmission of the disease in the population is completely stopped, *i.e.*, the reproduction number is equal to 0. In the examples below, we will see that for some next-generation matrices K , this may be achieved with a strategy η with cost $C(\eta) < 1$. Hence, let us denote by c_\star the minimal cost required to completely stop the transmission of the disease:

$$c_\star = \inf\{c \in [0, 1] : R_{e\star}(c) = 0\} = \inf\{C(\eta) : R_e(\eta) = 0\}. \tag{11}$$

In a similar fashion, we define by symmetry the maximal cost of totally inefficient vaccination strategies:

$$c^* = \sup\{c \in [0, 1] : R_e^*(c) = R_0\} = \sup\{C(\eta) : R_e(\eta) = R_0\}. \tag{12}$$

According to (Delmas et al. 2021b, Lemma 5.13(ii)), we have $c^* = 0$ if the matrix K is irreducible, *i.e.*, not similar via a permutation to a block upper triangular matrix. The two matrices considered below in this section are irreducible.

Following Delmas et al. (2021b), the *Pareto frontier* associated to the “best” vaccination strategies, solution to Problem (7), is defined by:

$$\mathcal{F} = \{(c, R_{e^*}(c)) : c \in [0, c^*]\}. \tag{13}$$

The set of “best” vaccination strategies, called *Pareto optimal* strategies, is defined by:

$$\mathcal{P} = \{\eta \in \Delta : (C(\eta), R_e(\eta)) \in \mathcal{F}\}. \tag{14}$$

When $c^* = 0$ (which will be the case for all the examples considered in this paper), the *anti-Pareto frontier* associated to the “worst” vaccination strategies, solution to Problem (8), is defined by:

$$\mathcal{F}^{\text{Anti}} = \{(c, R_e^*(c)) : c \in [0, 1]\}. \tag{15}$$

The set of “worst” vaccination strategies, called *anti-Pareto optimal* strategies, is defined by:

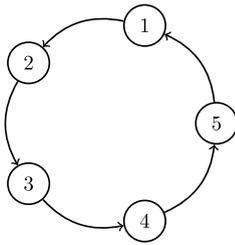
$$\mathcal{P}^{\text{Anti}} = \{\eta \in \Delta : (C(\eta), R_e(\eta)) \in \mathcal{F}^{\text{Anti}}\}. \tag{16}$$

The set of uniform strategies will play a role in the sequel:

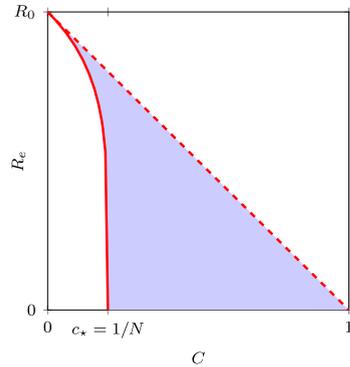
$$\mathcal{S}^{\text{uni}} = \{t\mathbb{1} : t \in [0, 1]\}. \tag{17}$$

We denote by $\mathbf{F} = \{(C(\eta), R_e(\eta)) : \eta \in \Delta\}$ the set of all possible outcomes. According to (Delmas et al. 2021b, Section 6.1), the set \mathbf{F} is a subset of $[0, 1] \times [0, R_0]$ delimited below by the graph of R_{e^*} and above by the graph of R_e^* ; it is compact, path connected and its complement is connected in \mathbb{R}^2 .

A *path* of vaccination strategies is a measurable function $\gamma : [a, b] \rightarrow \Delta$ where $a < b$. It is *monotone* if for all $a \leq s \leq t \leq b$ we have $\gamma(s) \geq \gamma(t)$, where \leq denotes the pointwise order. A *greedy parametrization* of the Pareto (resp. anti-Pareto) frontier is a monotone continuous path γ such that the image of $(C \circ \gamma, R_e \circ \gamma)$ is equal to \mathcal{F} (resp. $\mathcal{F}^{\text{Anti}}$). If such a path exists, then its image can be browsed by a greedy algorithm which performs infinitesimal locally optimal steps.



(A) Graphical representation of the transmission of the disease.



(B) Solid line: the Pareto frontier \mathcal{F} ; dashed line: the anti-Pareto frontier $\mathcal{F}^{\text{anti}}$ (which corresponds to the uniform strategies); blue region: all possible outcomes \mathbf{F} .

Fig. 1 Example of optimization for the fully asymmetric circle model with $N = 5$ subpopulations

Remark 2.2 Let K be the next-generation matrix and let $\lambda \in \mathbb{R}_+ \setminus \{0\}$. By homogeneity of the spectral radius, we have $\rho(\lambda K \cdot \text{Diag}(\eta)) = \lambda \rho(K \cdot \text{Diag}(\eta))$. Thus, the solutions of Problems (7) and (8) and the value of c_* are invariant by scaling of the matrix K . As for the functions R_{e^*} and R_e^* , they are scaled by the same quantity. Hence, in our study, the value of R_0 will not matter. Our main concern will be to find the best and the worst vaccination strategies for a given cost and compare them to the uniform strategy.

2.3 The fully asymmetric circle model

We consider a model of $N \geq 2$ equal subpopulations (i.e. $\mu_1 = \dots = \mu_N = 1/N$) where each subpopulation only contaminates the next one. The next-generation matrix, which is equal to the cyclic permutation matrix, and the effective next generation matrix are given by:

$$K = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ 1 & 0 & & & 0 \end{pmatrix} \quad \text{and} \quad K \cdot \text{Diag}(\eta) = \begin{pmatrix} 0 & \eta_2 & & & \\ & 0 & \eta_3 & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & \eta_N \\ \eta_1 & 0 & & & 0 \end{pmatrix}, \quad (18)$$

where $\eta = (\eta_1, \dots, \eta_N) \in \Delta = [0, 1]^N$. The next-generation matrix can be interpreted as the adjacency matrix of the fully asymmetric cyclic graph; see Fig. 1A.

By an elementary computation, the characteristic polynomial of the matrix $K \cdot \text{Diag}(\eta)$ is equal to $X^N - \prod_{1 \leq i \leq N} \eta_i$. Hence, the effective reproduction number can

be computed via an explicit formula; it corresponds to the geometric mean:

$$R_e(\eta) = \left(\prod_{i=1}^N \eta_i \right)^{1/N}. \tag{19}$$

The Pareto and anti-Pareto frontier are totally explicit for this elementary example, and given by the following proposition. For additional comments on this example; see also Example 5.9 below.

Proposition 2.3 (Asymmetric circle) *For the fully asymmetric circle model, we have:*

- (i) *The least quantity of vaccine necessary to completely stop the propagation of the disease is $c_\star = 1/N$. Pareto optimal strategies have a cost smaller than c_\star , and correspond to giving all the available vaccine to one subpopulation:*

$$\mathcal{P} = \left\{ \eta = (\eta_1, \dots, \eta_N) \in [0, 1]^N : \eta_i = 1 \text{ for all } i \text{ but at most one} \right\}.$$

The Pareto frontier is given by the graph of the function $R_{e\star}$ on $[0, c_\star]$, where $R_{e\star}$ is given by:

$$R_{e\star}(c) = (1 - Nc)_+^{1/N} \text{ for } c \in [0, 1].$$

- (ii) *The maximal cost of totally inefficient vaccination strategies is $c^\star = 0$. The anti-Pareto optimal strategies consist in vaccinating uniformly the population, i.e.:*

$$\mathcal{P}^{\text{Anti}} = \mathcal{S}^{\text{uni}}.$$

The anti-Pareto frontier is given by the graph of the function $R_e^\star : c \mapsto 1 - c$ on $[0, 1]$.

In Fig. 1B, we have plotted the Pareto and the anti-Pareto frontiers corresponding to asymmetric circle model with $N = 5$ subpopulations.

Remark 2.4 (Greedy parametrization) From Proposition 2.3, we see that there exists a greedy parametrization of the Pareto frontier, which consists in giving all the available vaccine to one subpopulation until its complete immunization. Similarly, the anti-Pareto frontier is greedily parametrized by the uniform strategies.

Proof We first prove (i). Suppose that $c \geq 1/N$. There is enough vaccine to protect entirely one of the groups and obtain $R_e(\eta) = 0$ thanks to Equation (19). This gives $c_\star \leq 1/N$ and $R_{e\star}(c) = 0$ for $c \geq 1/N$.

Let $0 \leq c < 1/N$. According to (Boyd and Vandenberghe 2004, Section 3.1.5), the map $\eta \mapsto R_e(\eta)$ is concave. According to Bauer’s maximum principle (Niculescu and Persson 2006, Corollary A.3.3), R_e attains its minimum on $\{\eta \in [0, 1]^N : C(\eta) = c\}$ at some extreme point of this set. These extreme points are strategies $\eta \in [0, 1]^N$ such that $\eta_i = 1 - Nc$ for some i and $\eta_j = 1$ for all $j \neq i$. Since R_e is a symmetric function

of its N variables, it takes the same value $(1 - Nc)^{1/N}$ on all these strategies, so they are all minimizing, which proves Point (i).

We give another elementary proof of (i) when $c < 1/N$. Let η be a solution of Problem (7). Assume without loss of generality that $\eta_1 \leq \dots \leq \eta_N$. Suppose for a moment that $\eta_2 < 1$, and let $\varepsilon > 0$ be small enough to ensure $\eta_1 > \varepsilon$ and $\eta_2 < 1 - \varepsilon$. Then the vaccination strategy $\tilde{\eta} = (\eta_1 - \varepsilon, \eta_2 + \varepsilon, \eta_3, \dots, \eta_N)$ is admissible, and:

$$R_e(\tilde{\eta})^N = R_e(\eta)^N - (\varepsilon(\eta_2 - \eta_1) + \varepsilon^2) \prod_{i=3}^N \eta_i < R_e(\eta)^N,$$

contradicting the optimality of η . Therefore the Pareto-optimal strategies have only one term different from 1, and must be equal to $((1 - Nc), 1, \dots, 1)$, up to a permutation of the indices.

Now, let us prove (ii). Let η be such that $C(\eta) = c$. According to the inequality of arithmetic and geometric means:

$$R_e(\eta) \leq \frac{\eta_1 + \dots + \eta_N}{N} = 1 - c.$$

By Example 2.1, the right hand side is equal to the effective reproduction number of the uniform vaccination at cost c . This ends the proof of the proposition. \square

2.4 Fully symmetric circle model

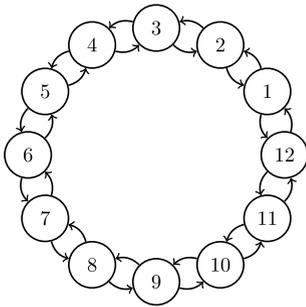
We now consider the case where each of the N subpopulation may infect both of their neighbours. The next-generation matrix and the effective next-generation matrix are given by:

$$K = \begin{pmatrix} 0 & 1 & & 0 & 1 \\ 1 & 0 & 1 & & 0 \\ & & 1 & \ddots & \ddots \\ 0 & & \ddots & \ddots & 0 & 1 \\ 1 & 0 & & 1 & 0 \end{pmatrix} \quad \text{and} \quad K \cdot \text{Diag}(\eta) = \begin{pmatrix} 0 & \eta_2 & & 0 & \eta_N \\ \eta_1 & 0 & \eta_3 & & 0 \\ & \eta_2 & \ddots & \ddots & \\ 0 & & \ddots & \ddots & 0 & \eta_N \\ \eta_1 & 0 & & \eta_{N-1} & 0 \end{pmatrix}. \quad (20)$$

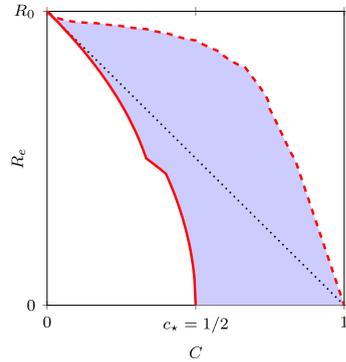
Again, we can represent this model as a graph; see Fig. 2A.

There is no closed-form formula to express R_e for $N \geq 5$ and the optimization is much harder than the asymmetric case. Since K is irreducible, we have $c^* = 0$. Our only analytical result for this model is the computation of c_\star .

Proposition 2.5 (Optimal strategy for stopping the transmission) *For the fully symmetric circle model, the strategy $\eta' = \mathbb{1}_{i \text{ even}}$ is Pareto optimal for the fully symmetric circle and $R_e(\eta') = 0$. In particular, c_\star is equal to $C(\eta') = \lceil N/2 \rceil / N$.*



(A) Graphical representation of the transmission of the disease.



(B) Solid line: the Pareto frontier \mathcal{F} ; dashed line: the anti-Pareto frontier $\mathcal{F}^{\text{Anti}}$; dotted line: outcomes of the uniform strategies; blue region: all possible outcomes \mathbf{F} .

Fig. 2 Example of optimization for the fully symmetric circle model with $N = 12$ subpopulations

Proof The term X^{N-2} of the characteristic polynomial of $K \cdot \text{Diag}(\eta)$ has a coefficient equal to the sum of all principal minors of size 2:

$$-(\eta_1 \eta_2 + \eta_2 \eta_3 + \dots + \eta_{N-1} \eta_N + \eta_N \eta_1). \tag{21}$$

If η is such that $NC(\eta) < \lceil N/2 \rceil$, then at least one of the term above is not equal to 0, proving that the sum is negative. Hence, there is at least one eigenvalue of $K \cdot \text{Diag}(\eta)$ different from 0, and $R_e(\eta) > 0$. We deduce that $c_* \geq \lceil N/2 \rceil / N$.

Now, let η' be such that $\eta'_i = 0$ for all odd i and $\eta'_i = 1$ for all even i , so that $C(\eta') = \lceil N/2 \rceil / N$. The matrix $K \cdot \text{Diag}(\eta')$ is nilpotent as its square is 0. Since the spectral radius of a nilpotent matrix is equal 0, we get $R_e(\eta') = 0$. This ends the proof of the proposition.

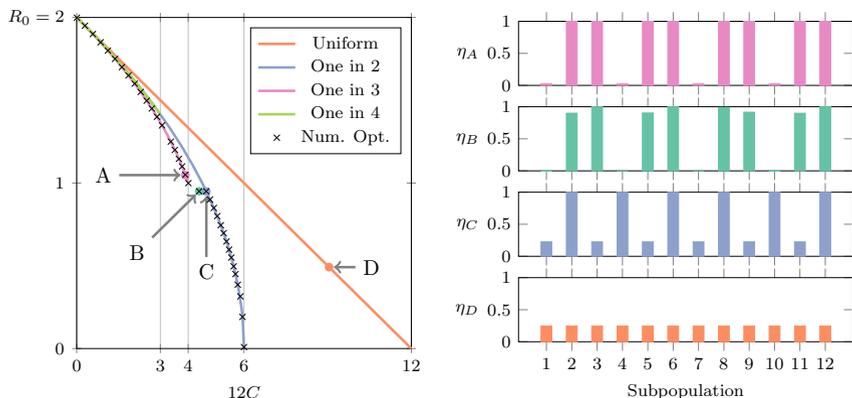
We can give another proof of the proposition: it is enough to notice that the nodes labelled with an odd number form a maximal independent set of the cyclic graph. Taking η' equal to the indicator function of this set, we deduce from (Delmas et al. Delmas et al. (2022b), Section 4.2) that η' is Pareto optimal, $R_e(\eta') = 0$ and $c_* = C(\eta')$. \square

We pursue the analysis of this model with numerical computations. We choose $N = 12$ subpopulations, and compute an approximate Pareto frontier, using the Borg multiobjective evolutionary algorithm.¹ The results are plotted in Fig. 3. We represent additionally the curves $(c, R(\eta(c)))$ where the vaccination strategy $\eta(c)$ for a given cost c are given by deterministic path of “meta-strategies”:

- **Uniform strategy:** distribute the vaccine uniformly to all N subpopulations;
- **“One in j ” strategy:** vaccinate one in j subpopulation, for $j = 2, 3, 4$.

Let us follow the scatter plot of R_{e*} in Fig. 3A, starting from the upper left.

¹ The algorithm is described in Hadka and Reed (2013); we use the version coded in the `BlackBoxOptim` package for the Julia programming language.



(A) Effective reproduction number R_e against the vaccination cost c for various meta-strategies. (B) Vaccination strategies corresponding to the four labelled points.

Fig. 3 Pareto frontier and computation of the outcomes for the paths of the four meta-strategies. Some meta-strategies $\{\eta_A, \eta_B, \eta_C, \eta_D\}$ are represented on the right with their corresponding outcome points A–D on the left

- (1) In the beginning nobody is vaccinated, and R_0 is equal to 2.
- (2) For small costs all strategies have similar efficiency. Zooming in shows that the (numerically) optimal strategies split the available vaccine equally between four subpopulations that are separated from each other by two subpopulations. This corresponds to the “one in 3” meta-strategies path. As represented in Fig. 3B, η_A with outcome point $A = (C(\eta_A), R_e(\eta_A))$ belongs to this path. In particular, note that disconnecting the graph is not Pareto optimal for $12c = 3$ as the disconnecting “one in 4” strategy gives values $R_e = \sqrt{2} \simeq 1.41$ opposed to the value $R_e \simeq 1.37$ for the “one in 3” strategy with same cost. However, note that, in agreement with (Delmas et al. 2022b, Proposition 5.3), this disconnecting “one in 4” strategy is also not anti-Pareto optimal, since it performs better than the uniform strategy with the same cost.
- (3) When $12c = 4$ the circle has been split in four “islands” of two interacting subpopulations. There is a small interval of values of c for which it is (numerically) optimal to split the additional vaccine uniformly between the four “islands”, and give it entirely to one subpopulation in each island: see point B and the associated strategy η_B .
- (4) Afterwards (see point C), it is in fact better to try and vaccinate all the (say) even numbered subpopulations. Therefore, the optimal vaccinations *do not vary monotonously* with respect to the amount of available vaccine; in other words, distributing vaccine in a greedy way is not optimal. This also suggests that, even though the frontier is continuous (in the objective space (c, r)), the set of optimal strategies may not be connected: the “one in two” vaccination strategy of point C cannot be linked to “no vaccination” strategy by a continuous path of optimal strategies. In particular, the Pareto frontier cannot be greedily parametrized. The disconnectedness of the set of optimal strategies will be established rigorously in Sect. 6 for another model.

- (5) For $12c = 6$, that is $c = c_*$ as stated in Proposition 2.5, it is possible to vaccinate completely all the (say) odd numbered subpopulations, thereby disconnecting the graph completely. The infection cannot spread at all.
- (6) Even though the problem is symmetric and all subpopulations play the same role, the proportional allocation of vaccine is far from optimal; on the contrary, the optimal allocations focus on some subpopulations.

Using the same numerical algorithm, we have also computed the anti-Pareto frontier for this model; see the dashed line in Fig. 2B. Although we do not give a formal proof, the anti-Pareto frontier seems to be perfectly given by the following greedy parametrization:

- (1) Distribute all the available vaccine supply to one group until it is completely immunized.
- (2) Once this group is fully vaccinated, distribute the vaccine doses to one of its neighbour.
- (3) Continue this procedure by vaccinating the neighbour of the last group that has been immunized.
- (4) When there are only two groups left, split the vaccine equally between these two.

3 The kernel model

In order to get a finer description of the heterogeneity, we could divide the population into a growing number of subgroups $N \rightarrow \infty$. The recent advances in graph limits theory (Backhausz and Szegedy 2020; Lovász 2012) justify describing the transmission of the disease by a kernel defined on a probability space. We already used this type of model in Delmas et al. (2021a, b, 2022a, b), in particular for an SIS dynamics, see also (Delmas et al. 2022b, Section 2) for other epidemic models.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space that represents the population: the individuals have features labeled by Ω and the infinitesimal size of the population with feature x is given by $\mu(dx)$. Let $L^2(\mu)$ (L^2 for short) be the space of real-valued measurable functions f defined on Ω such that $\|f\|_2 = (\int_{\Omega} f^2 d\mu)^{1/2}$ is finite, where functions which agree μ -a.s. are identified. Let $L^2_+ = \{f \in L^2 : f \geq 0\}$ be the subset of non-negative functions of L^2 . We define a *kernel* on Ω as a \mathbb{R}_+ -valued measurable function defined on $(\Omega^2, \mathcal{F}^{\otimes 2})$. We will only consider kernels with finite double-norm on L^2 :

$$\|k\|_{2,2} = \left(\int_{\Omega \times \Omega} k(x, y)^2 \mu(dx)\mu(dy) \right)^{1/2} < +\infty. \tag{22}$$

To a kernel k with finite double-norm on L^2 , we associate the integral operator T_k on L^2 defined by:

$$T_k(g)(x) = \int_{\Omega} k(x, y)g(y) \mu(dy) \text{ for } g \in L^2 \text{ and } x \in \Omega. \tag{23}$$

The operator T_k is bounded, and its operator norm $\|T_k\|_{L^2}$ satisfies:

$$\|T_k\|_{L^2} \leq \|k\|_{2,2}. \tag{24}$$

According to (Conway 1990, Proposition II.4.7), the operator T_k is actually compact. A kernel is said to be symmetric if $k(x, y) = k(y, x)$, $\mu(dx)\mu(dy)$ -almost surely. It is said to be *irreducible* if for all $A \in \mathcal{F}$, we have:

$$\int_{A \times A^c} k(x, y) \mu(dx)\mu(dy) = 0 \implies \mu(A) \in \{0, 1\}. \tag{25}$$

If k is not irreducible, it is called *reducible*.

By analogy with the discrete setting and also based on Delmas et al. (2022a, b), we define the basic reproduction number in this context thanks to the following formula:

$$R_0 = \rho(T_k), \tag{26}$$

where ρ stands for the spectral radius of an operator. According to the Krein-Rutman theorem, R_0 is an eigenvalue of T_k . Besides, there exists left and right eigenvectors associated to this eigenvalue in L^2_+ ; such functions are called Perron eigenfunctions.

For f, g two non-negative bounded measurable functions defined on Ω and k a kernel on Ω with finite double-norm on L^2 , we denote by $fk g$ the kernel on Ω defined by:

$$(fk g)(x, y) = f(x)k(x, y)g(y). \tag{27}$$

Since f and g are bounded, the kernel $fk g$ has also a finite double-norm on L^2 .

Denote by Δ the set of measurable functions defined on Ω taking values in $[0, 1]$. A function η in Δ represents a vaccination strategy: $\eta(x)$ represents the proportion of **non-vaccinated** individuals with feature x . In particular $\eta = \mathbb{1}$ (the constant function equal to 1) corresponds to the absence of vaccination and $\eta = \mathbb{0}$ (the constant function equal to 0) corresponds to the whole population being vaccinated. The uniform strategies are given by:

$$\eta^{\text{uni}} = t\mathbb{1}$$

for some $t \in [0, 1]$, and we denote by $\mathcal{S}^{\text{uni}} = \{t\mathbb{1} : t \in [0, 1]\}$ the set of uniform strategies.

The (uniform) cost of the vaccination strategy $\eta \in \Delta$ is given by the total proportion of vaccinated people, that is:

$$C(\eta) = \int_{\Omega} (1 - \eta) d\mu = 1 - \int_{\Omega} \eta d\mu. \tag{28}$$

The measure $\eta d\mu$ corresponds to the *effective population*, that is the individuals who effectively play a role in the dynamic of the epidemic. The effective reproduction

number is defined by:

$$R_e(\eta) = \rho(T_{k\eta}), \tag{29}$$

We consider the weak topology on Δ given by the trace of the weak topology on L^2 , so that with a slight abuse of notation we identify Δ with $\{\eta \in L^2 : 0 \leq \eta \leq 1\}$. According to Theorem 4.2 and Remark 3.2 in Delmas et al. (2021b), the function $R_e : \eta \mapsto R_e(\eta)$ is continuous on Δ with respect to the weak topology. The compactness of Δ for this topology implies the existence of solutions for Problems (7) and (8). We will conserve the same notation and definitions as in the discrete setting for: the value functions R_{e^*} and R_e^* , the minimal/maximal costs c_* and c^* , the various sets of strategies \mathcal{P} and $\mathcal{P}^{\text{Anti}}$, and the various frontiers \mathcal{F} and $\mathcal{F}^{\text{Anti}}$; see Eqs. (9)–(17) in Sect. 2.2.

We shall also use the following result from (Delmas et al. 2022b, Proposition 5.1) (recall that a vaccination strategy is defined up the a.s. equality).

Lemma 3.1 *Let k be a kernel on Ω with finite double-norm on L^2 such that a.s. $k > 0$. Then, we have $c^* = 0$, $c_* = 1$ and the strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost $c = 0$ (resp. $c = 1$).*

Example 3.2 (Discrete and continuous representations of a metapopulation model) We recall the natural correspondence between metapopulation models (discrete models) and kernel models (continuous models) from (Delmas et al. 2021b, Section 7.4.1). Consider a metapopulation model with N groups given by a finite set $\Omega_d = \{1, 2, \dots, N\}$ equipped with a probability measure μ_d giving the relative size of each group and a next generation matrix $K = (K_{ij}, i, j \in \Omega_d)$. The corresponding discrete kernel k_d on Ω_d is defined by:

$$K_{ij} = k_d(i, j)\mu_j \quad \text{where} \quad \mu_i = \mu_d(\{i\}). \tag{30}$$

Then, the matrix $K \cdot \text{Diag}(\eta)$ is the matrix representation of the endomorphism $T_{k_d\eta}$ in the canonical basis of \mathbb{R}^N .

Following Delmas et al. (2021b), we can also consider a continuous representation on the state space $\Omega_c = [0, 1)$ equipped with the Lebesgue measure μ_c . Let $I_1 = [0, \mu_1)$, $I_2 = [\mu_1, \mu_1 + \mu_2)$, ..., $I_N = [1 - \mu_N, 1)$, so that the intervals $(I_n, 1 \leq n \leq N)$ form a partition of Ω . Now define the kernel:

$$k_c(x, y) = \sum_{1 \leq i, j \leq N} k_d(i, j)\mathbb{1}_{I_i \times I_j}(x, y). \tag{31}$$

Denote by R_e^d and R_e^c the effective reproduction number in the discrete and continuous representation models. In the same manner, the uniform cost in each model is denoted by C^d and C^c . According to Delmas et al. (2021b), these functions are linked through the following relation:

$$R_e^d(\eta^d) = R_e^c(\eta^c), \quad \text{and} \quad C^d(\eta^d) = C^c(\eta^c),$$

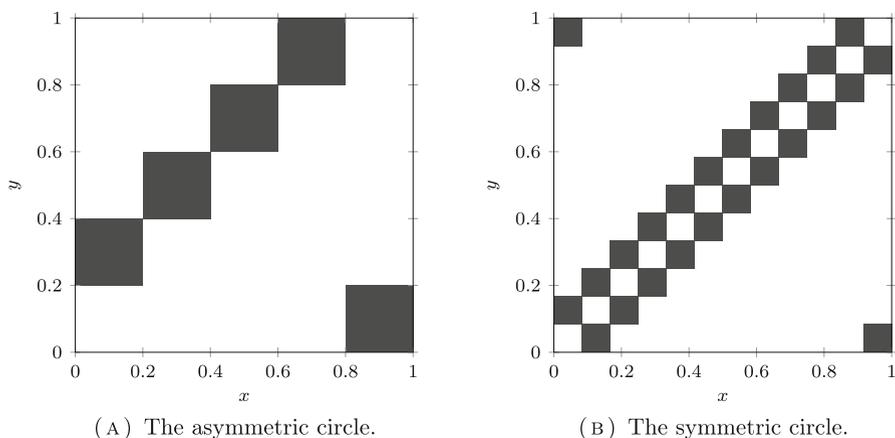


Fig. 4 Kernels k_c (equal to 0 in the white zone and to 1 in the black zone) on $\Omega_c = [0, 1]$ and μ_c the Lebesgue measure of the continuous model associated to discrete metapopulation models

for all $\eta^d : \Omega_d \rightarrow [0, 1]$ and $\eta^c : \Omega_c \rightarrow [0, 1]$ such that:

$$\eta^d(i) = \frac{1}{\mu_i} \int_{I_i} \eta^c d\mu_c \quad \text{for all } i \in \Omega_d.$$

Let us recall that the Pareto and anti-Pareto frontiers for the two models are the same.

In Fig. 4, we have plotted the kernels of the continuous models associated to the asymmetric and symmetric circles models from Sects. 2.3 and 2.4.

4 Assortative versus disassortative mixing

4.1 Motivation

We consider a population divided into an at most countable number of groups. Individuals within the same group interact with intensity a and individuals in different groups interact with intensity b . Hence, the model is entirely determined by the coefficients a and b and the size of the different groups. This simple model allows to study the effect of assortativity, that is, the tendency for individuals to connect with individuals belonging to their own subgroup. The mixing pattern is called *assortative* (higher interaction in the same subgroup) if $a > b$, and *disassortative* (lower interaction in the same subgroup) when $b > a$. Our results illustrate how different the optimal vaccination strategies can be between assortative and disassortative models, an effect that was previously studied by Galeotti and Rogers (2013) in a population composed of two groups.

When the population is equally split in a finite number of subgroups, and a is equal to 0, the next-generation matrix of this model corresponds, up to a multiplicative constant, to the adjacency matrix of a complete multipartite graph. Recall that an m -

partite graph is a graph that can be colored with m different colors, so that all edges have their two endpoints colored differently. When $m = 2$ these are the so-called bipartite graphs. A complete multipartite graph is a m -partite graph (for some $m \in \mathbb{N}^*$) in which there is an edge between every pair of vertices from different colors.

The complete multipartite graphs have interesting spectral properties. Indeed, Smith (1970) showed that a graph with at least one edge has its spectral radius as its only positive eigenvalue if and only if its non-isolated vertices induce a complete multipartite graph. In Esser and Harary (1980), Esser and Harary proved that two complete m -partite graphs with the same number of nodes are isomorphic if and only if they have the same spectral radius. More precisely, they obtained a comparison of the spectral radii of two complete m -partite graphs by comparing the sizes of the sets in their partitions through majorization; see (Esser and Harary 1980, Lemma 3).

The goal of this section is to generalize and complete these results and give a full picture of the Pareto and anti-Pareto frontiers for the assortative and the disassortative models.

4.2 Spectrum and convexity

We will use an integer intervals notation to represent the considered kernels. For $i, j \in \mathbb{N} \cup \{+\infty\}$, we set $\llbracket i, j \rrbracket$ (resp. $\llbracket i, j \llbracket$ for $[i, j] \cap (\mathbb{N} \cup \{+\infty\})$ (resp. $[i, j) \cap \mathbb{N}$). Let $N \in \llbracket 2, +\infty \rrbracket$ and $\Omega = \llbracket 1, N \rrbracket$ if N is finite and $\Omega = \llbracket 1, +\infty \llbracket$ otherwise. The set Ω is endowed with the discrete σ -algebra $\mathcal{F} = \mathcal{P}(\Omega)$ and a probability measure μ . To simplify the notations, we write μ_i for $\mu(\{i\})$ and $f_i = f(i)$ for a function f defined on Ω . Without loss of generality, we can suppose that $\mu_i \geq \mu_j > 0$ for all $i \leq j$ elements of Ω . We consider the kernel k defined for $i, j \in \Omega$ by:

$$k(i, j) = \begin{cases} a & \text{if } i = j, \\ b & \text{otherwise,} \end{cases} \tag{32}$$

where a and b are two non-negative real numbers.

If $b = 0$, then the kernel is reducible, and, thanks to (Delmas et al. 2021a, Lemma 5.3), the effective reproduction number is given by the following formula: $R_e(\eta) = a \max_{i \in \Omega} \eta_i \mu_i$, for all $\eta = (\eta_i, i \in \Omega) \in \Delta$. This is sufficient to treat this case and we have $c^* = 1 - \mu_1$.

From now on, we assume that $b > 0$. The next two results describe the spectrum of T_k in both the assortative and disassortative case. Notice the spectrum of T_k is real as k is symmetric. Recall that $R_0 = \rho(T_k)$.

Proposition 4.1 (Convexity/concavity of R_e) *Let k be given by (32), with $b > 0$ and $a \geq 0$.*

- (i) **Assortative model.** *If $a \geq b > 0$, then the operator T_k is positive semi-definite and the function R_e is convex.*
- (ii) **Disassortative model.** *If $b \geq a \geq 0$ and $b > 0$, then R_0 is the only positive eigenvalue of T_k , and it has multiplicity one. Furthermore, the function R_e is concave.*

In the following proof, we shall consider the symmetric matrix M_n of size $n \times n$, with $n \in \mathbb{N}^*$, given by:

$$M_n(i, j) = \begin{cases} a & \text{if } i = j, \\ b & \text{otherwise.} \end{cases}$$

The matrix M_n is the sum of b times the all-ones matrix and $a - b$ times the identity matrix. Thus, M_n has two distinct eigenvalues: $nb + a$ with multiplicity 1 and $a - b$ with multiplicity $n - 1$.

Proof We first prove (i). For any $g \in L^2$, we have:

$$\int_{\Omega \times \Omega} g(x)k(x, y)g(y) \mu(dx)\mu(dy) = a \sum_{i \in \Omega} g_i^2 \mu_i^2 + b \sum_{i \neq j} g_i g_j \mu_i \mu_j \geq b \|g\|_2^2.$$

This implies that T_k is positive semi-definite. Thus, as k is symmetric, the fonction R_e is convex, thanks to (Delmas et al. 2021a, Theorem 4.10).

We now prove (ii). We give a direct proof when N is finite, and use an approximation procedure for $N = \infty$. We first assume that N is finite. For $n \leq N$, let $v_n = \mathbb{1}_{\llbracket 1, n \rrbracket}$ and set $T_n = T_{v_n k v_n}$. The non-null eigenvalues of T_n (with their multiplicity) are the eigenvalues of the matrix $M_n \cdot \text{Diag}_n(\mu)$, where $\text{Diag}_n(\mu)$ is the diagonal $n \times n$ -matrix with (μ_1, \dots, μ_n) on the diagonal. Thanks to (Horn and Johnson 2013, Theorem 1.3.22), these are also the eigenvalues of the matrix $Q_n = \text{Diag}_n(\mu)^{1/2} \cdot M_n \cdot \text{Diag}_n(\mu)^{1/2}$. By Sylvester’s law of inertia (Horn and Johnson 2013, Theorem 4.5.8), the matrix Q_n has the same signature as the symmetric matrix M_n . In particular, since we have supposed $a - b \leq 0$, M_n has only one positive eigenvalue. Thus Q_n has only one positive eigenvalue: thanks to the Perron-Frobenius theory, it is its spectral radius. This concludes the proof when N is finite by choosing $n = N$.

If $N = \infty$, we consider the limit $n \rightarrow N$. Since:

$$\lim_{n \rightarrow \infty} \|k - v_n k v_n\|_{2,2} = 0,$$

the spectrum of T_n converges to the spectrum of T_k , with respect to the Hausdorff distance, and the multiplicity on the non-zero eigenvalues also converge, see (Delmas et al. 2021a, Lemma 2.4). This shows that $\rho(T_k)$ is the only positive eigenvalue of T_k , and it has multiplicity one. Since k is symmetric, we deduce the concavity of the function R_e from (Delmas et al. 2021a, Theorem 4.10). □

4.3 Explicit description of the Pareto and anti-Pareto frontiers

For $c \in [0, 1]$, we define an “horizontal vaccination” $\eta^h(c) \in \Delta$ with cost c in the following manner. Rather than defining directly the proportion of non-vaccinated people in each class, it will be convenient to define first the resulting effective population

size, which will be denoted by ξ . For all $\alpha \in [0, \mu_1]$, let $\xi^h(\alpha) \in \Delta$ be defined by:

$$\xi_i^h(\alpha) = \min(\alpha, \mu_i), \quad i \in \Omega. \tag{33}$$

For all $i \in \Omega$, $\xi_i^h(\alpha)$ is a non-decreasing and continuous function of α . The map $\alpha \mapsto \sum_i \xi_i^h(\alpha)$ is continuous and increasing from $[0, \mu_1]$ to $[0, 1]$, so for any $c \in [0, 1]$, there exists a unique α_c such that $\sum_i \xi_i^h(\alpha_c) = 1 - c$. We then define the horizontal vaccination profile $\eta^h(c) \in \Delta$ by:

$$\eta_i^h(c) = \xi_i^h(\alpha_c) / \mu_i, \quad i \in \Omega. \tag{34}$$

In words, it consists in vaccinating in such a way that the quantity of the non-vaccinated individuals $\xi_i^h = \eta_i \mu_i$ in each subpopulation is always less than the ‘‘horizontal’’ threshold α : see Fig. 5A. The cost of the vaccination strategy $\eta^h(c)$ is indeed c . Note that $\eta^h(0) = \mathbb{1}$ (no vaccination), whereas $\eta^h(1) = \mathbb{0}$ (full vaccination), and that the path $c \mapsto \eta^h(c)$ is greedy. We denote its range by \mathcal{P}_h .

For $c \in [0, 1]$, we define similarly a ‘‘vertical vaccination’’ $\eta^v(c) \in \Delta$ with cost c . First let us define for $\beta \in [0, N]$:

$$\xi_i^v(\beta) = \mu_i \cdot \min(1, (\beta + 1 - i)_+), \quad i \in \Omega. \tag{35}$$

The map $\beta \mapsto \sum_i \xi_i^v(\beta)$ is increasing and continuous from $[0, N]$ to $[0, 1]$, so for any $c \in [0, 1]$ there exists a unique β_c such that $\sum_i \xi_i^v(\beta_c) = 1 - c$. We then define the vertical vaccine profile $\eta^v(c)$ by:

$$\eta_i^v(c) = \xi_i^v(\beta_c) / \mu_i, \quad i \in \Omega. \tag{36}$$

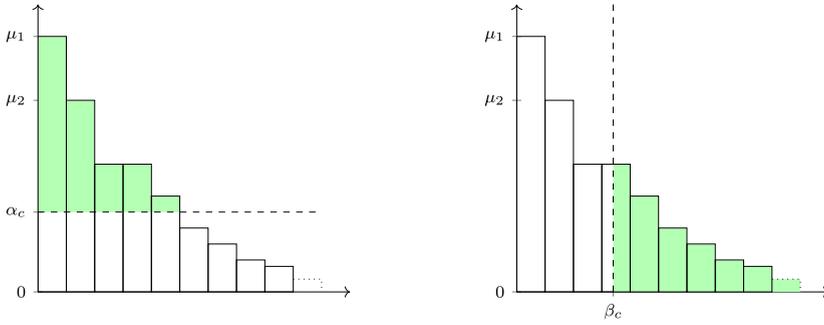
In words, if $\lceil \beta \rceil = \ell$, this consists in vaccinating all subpopulations j for $j > \ell$, and a fraction $\lceil \beta \rceil - \beta$ of the subpopulation ℓ , see Fig. 5B for a graphical representation. The cost of the vaccination strategy $\eta^v(c)$ is by construction equal to c .

For all $i \in \Omega$, $\eta_i^v(c)$ is a non-increasing and continuous function of c . Just as in the horizontal case, we have $\eta^v(0) = \mathbb{1}$ (no vaccination), $\eta^v(1) = \mathbb{0}$ (full vaccination), and the path $c \mapsto \eta^v(\beta(c))$ is also greedy. We denote its range by \mathcal{P}_v .

These two paths give a greedy parametrization of the Pareto and anti-Pareto frontiers for the assortative and disassortative models: more explicitly, we have the following result, whose proof can be found in Sect. 4.4.

Theorem 4.2 (Assortative vs disassortative) *Let k be given by (32), with $b > 0$ and $a \geq 0$.*

- (i) *Assortative model. If $a \geq b > 0$, then \mathcal{P}_v and \mathcal{P}_h are greedy parametrizations of the anti-Pareto and Pareto frontiers respectively.*
- (ii) *Disassortative model. If $b \geq a > 0$, then \mathcal{P}_v and \mathcal{P}_h are greedy parametrizations of the Pareto and anti-Pareto frontiers respectively.*
- (iii) *Complete multipartite model. If $a = 0$ and $b > 0$, then \mathcal{P}_h is a greedy parametrization of the anti-Pareto frontier and the subset of strategies $\eta \in \mathcal{P}_v$*



(A) Representation of the greedy path \mathcal{P}_h . (B) Representation of the greedy path \mathcal{P}_v .

Fig. 5 Greedy parametrization of the (anti-)Pareto front. The bar plot represents the measure μ . The proportion of green in each bar correspond to the proportion of vaccinated individuals in each subpopulation

such that $C(\eta) \leq 1 - \mu_0$ is a greedy parametrization of the Pareto frontier. In particular, we have $c_\star = 1 - \mu_1$ and $c^\star = 0$.

Notice that $c^\star = 0$ and $c_\star = 1$ in cases (i) and (ii) as k is positive thanks to Lemma 3.1.

Remark 4.3 (Highest Degree vaccination) The effective degree function of a symmetric kernel k at $\eta \in \Delta$ is the function deg_η defined on Ω by:

$$\text{deg}_\eta(x) = \int_{\Omega} k(x, y)\eta(y) \mu(dy). \tag{37}$$

When $\eta = \mathbb{1}$, it is simply called the degree of k and is denoted by deg . In our model, the effective degree of the subgroup i is given by

$$\text{deg}_\eta(i) = a\eta_i\mu_i + b \sum_{\ell \neq i} \eta_\ell \mu_\ell, \tag{38}$$

and thus the degree of the subgroup i is given by $\text{deg}(i) = (a - b)\mu_i + b$. As $\mu_i \geq \mu_j$ for $i < j$ elements of Ω , we deduce that the degree function is monotone: non-increasing in the assortative model and non-decreasing in the disassortative model. The group with the highest degree therefore corresponds to the largest group in the assortative model and the smallest group (if it exists) in the disassortative model.

Consider the assortative model where all the groups have different size, *i.e.*, $\mu_1 > \mu_2 > \dots$. Following the parametrization $c \mapsto \eta^h(c)$, starting from $c = 0$, will first decrease the effective size of the group 1 (the group with the highest degree) until it reaches the effective degree of group 2 (with the second highest degree). Once these two groups share the same effective degree which corresponds to reaching $\mu_1\eta_1^h = \mu_2$, they are vaccinated uniformly (that is, ensuring that they keep the same effective degree: using (38) this corresponds to $\mu_1\eta_1^h = \mu_2\eta_2^h$) until their effective degree is equal to the third highest degree, and so on and so forth.

In the disassortative model, the function deg_η remains (strictly) increasing when the vaccination strategies in \mathcal{P}_v are applied. In particular, if $\mu_1 > \mu_2 > \dots$, then the optimal strategies prioritize the groups with the higher effective degree until they are completely immunized. If multiple groups share the same degree, it is optimal to give all available doses to one group.

In conclusion, in both models, the optimal vaccination consists in vaccinating the groups with the highest effective degree in priority if this group is unique. But if multiple groups share the same degree (*i.e.*, have the same size), the optimal strategies differ between the assortative and the disassortative case. In the assortative case, groups with the same size must be vaccinated uniformly while in the disassortative case, all the vaccine doses shall be given to one group until it is completely vaccinated.

Example 4.4 (*Group sizes following a dyadic distribution*) Let $N = \infty$, $\Omega = \mathbb{N}^*$ and $\mu_i = 2^{-i}$ for all $i \in \Omega$. Following (Delmas et al. 2021b, Section 7.4.1), we will couple this discrete model with a continuum model for a better visualization on the figures. Let $\Omega_c = [0, 1)$ be equipped with the Borel σ -field \mathcal{F}_c and the Lebesgue measure μ_c . The set Ω_c is partitionned into a countable number of intervals $I_i = [1 - 2^{-i+1}, 1 - 2^{-i})$, for $i \in \mathbb{N}^*$, so that $\mu_c(I_i) = \mu_i$. The kernel of the continuous model corresponding to k in (32) is given by:

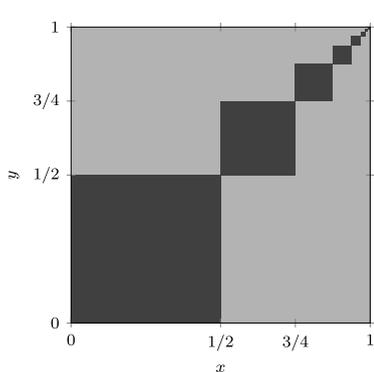
$$k_c = (a - b) \sum_{i \in \mathbb{N}^*} \mathbb{1}_{I_i \times I_i} + b\mathbb{1}. \tag{39}$$

The kernel k_c is plotted in Figs. 6A, 7A and 8A for different values of a and b corresponding respectively to the assortative, the disassortative and the complete multipartite case corresponding to points (i), (ii) and (iii) of Theorem 4.2 respectively. Their respective Pareto and anti-Pareto frontiers are plotted in Figs. 6B, 7B and 8B, using a finite-dimensional approximation of the kernel k and the power iteration method. In Fig. 8B, the value of c_* is equal to $1 - \mu_1 = 1/2$. With this continuous representation of the population, the set \mathcal{P}_v corresponds to the strategies of the form $\mathbb{1}_{[0,t)}$ for $t \in [0, 1]$.

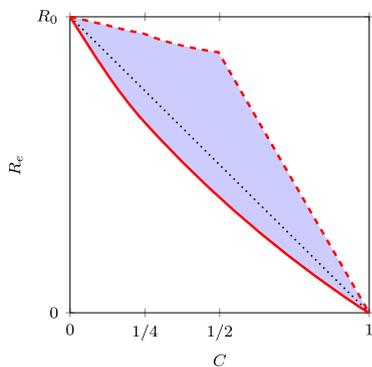
Notice that the Pareto frontier in the assortative case is convex. This is consistent with (Delmas et al. 2021b, Proposition 6.6) since the cost function is affine and R_e is convex when $a \geq b$; see Proposition 4.1 (i). In the same manner, the anti-Pareto frontier in the disassortative and the multipartite cases is concave. Once again, this is consistent with (Delmas et al. 2021b, Proposition 6.6) since the cost function is affine and R_e is concave when $b \geq a$; see Proposition 4.1 (ii).

4.4 Proof of Theorem 4.2

After recalling known facts of majorization theory, we first consider the finite dimension models, and then the general case by an approximation argument.

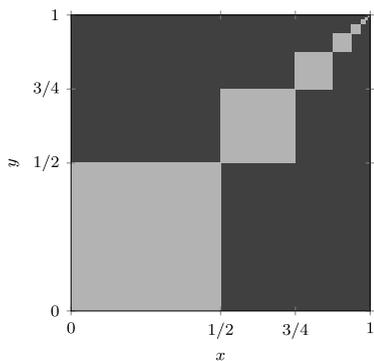


(A) Grayplot of the kernel k_c from (39) on $\Omega_c = [0, 1)$ with $k_c = a = 5$ on the dark gray zone, and $k_c = b = 2$ on the light gray zone.

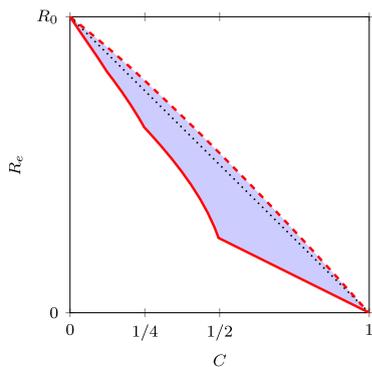


(B) Solid line: the Pareto frontier \mathcal{F} ; dashed line: the anti-Pareto frontier \mathcal{F}^{Anti} ; dotted line: path of the uniform strategies; blue region: all possible outcomes \mathbf{F} .

Fig. 6 An assortative model



(A) Grayplot of the kernel k_c from (39) on $\Omega_c = [0, 1)$ with $k_c = a = 2$ on the light gray zone, and $k_c = b = 5$ on the dark gray zone.



(B) Solid line: the Pareto frontier \mathcal{F} ; dashed line: the anti-Pareto frontier \mathcal{F}^{Anti} ; dotted line: path of the uniform strategies; blue region: all possible outcomes \mathbf{F} .

Fig. 7 A disassortative model

4.4.1 Majorization

In this section, we recall briefly some definitions and results from majorization theory, and refer to Arnold (1987); Marshall et al. (2011) for an extensive treatment of this topic.

Let $n \geq 1$ and $\xi, \chi \in \mathbb{R}_+^n$. We denote by ξ^\downarrow and χ^\downarrow their respective order statistics, that is the vectors in \mathbb{R}_+^n with the same components, but sorted in descending order.

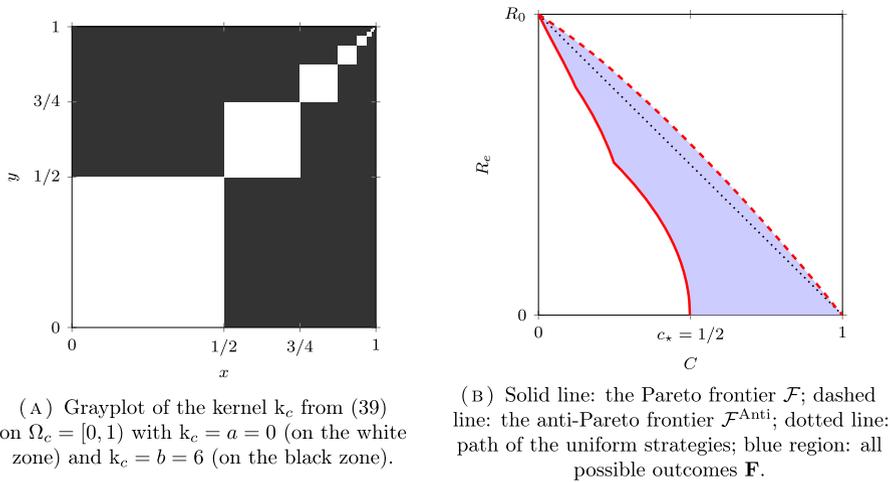


Fig. 8 An example of the complete multipartite model

We say that ξ is *majorized* by χ , and write $\xi < \chi$, if:

$$\sum_{j=1}^i \xi_j^\downarrow \leq \sum_{j=1}^i \chi_j^\downarrow \quad \text{for all } i \in \{1, \dots, n\}, \text{ and } \sum_{j=1}^n \xi_j = \sum_{j=1}^n \chi_j. \quad (40)$$

Among the various characterizations of majorization, we will use the following by Hardy, Littlewood and Pólya; see (Marshall et al. 2011, Proposition I.4.B.3):

$$\xi < \chi \iff \sum_{i=1}^n (\xi_i - t)_+ \leq \sum_{i=1}^n (\chi_i - t)_+ \quad \text{for all } t \in \mathbb{R}_+, \quad (41)$$

where $u_+ = \max(u, 0)$, for all $u \in \mathbb{R}$. A real-valued function Θ defined on \mathbb{R}_+^n is called *Schur-convex* if it is non-decreasing with respect to $<$, that is, $\xi < \chi$ implies $\Theta(\xi) \leq \Theta(\chi)$. A function Θ is called *Schur-concave* if $(-\Theta)$ is Schur-convex.

4.4.2 Schur-convexity and concavity of the spectral radius in finite dimension

We define the function Θ_n on \mathbb{R}_+^n by:

$$\Theta_n(\xi) = \rho(M_n \cdot \text{Diag}(\xi)),$$

where $\text{Diag}(\xi)$ is the diagonal $n \times n$ -matrix with ξ on the diagonal. By construction, for $\eta = (\eta_1, \dots, \eta_n, 0, \dots)$, we have:

$$R_e(\eta) = \Theta_n(\eta_1 \mu_1, \dots, \eta_n \mu_n). \quad (42)$$

The key property below will allow us to identify the optimizers.

Lemma 4.5 (Schur-concavity and Schur-convexity) *Let $b > 0$ and $a \geq 0$. The function Θ_n is Schur-convex if $a \geq b$, and Schur-concave if $a \leq b$.*

Proof Let us consider the disassortative case where $a \leq b$. By a classical result of majorization theory (Marshall et al. 2011, Proposition I.3.C.2.), it is enough to show that Θ_n is symmetric and concave.

To prove that Θ_n is symmetric, consider σ a permutation of $\{1, 2, \dots, n\}$ and P_σ the associated permutation matrix of size $n \times n$. Since $P_\sigma M_n P_\sigma^{-1} = M_n$, we deduce that $\Theta_n(\xi_\sigma) = \Theta_n(\xi)$, where ξ_σ is the σ -permutation of $\xi \in \mathbb{R}_+^n$. Thus Θ_n is symmetric.

We now prove that Θ_n is concave on \mathbb{R}_+^n . Since R_e is concave thanks to Proposition 4.1 (ii), we deduce from (42), that the function Θ_n is concave on $[0, \mu_1] \times \dots \times [0, \mu_n]$. Since Θ_n is homogeneous, it is actually concave on the whole domain \mathbb{R}_+^n . This concludes the proof when $a \leq b$.

The proof is the same for the assortative case $a \geq b$, replacing the reference to Proposition 4.1 (ii) by (i). □

4.4.3 Extreme vaccinations for fixed cost

Let us show that the horizontal and vertical vaccinations give extreme points for the preorder $<$ on finite sets, when the quantity of vaccine is fixed. Recall that ξ^h and ξ^v are defined in (33) and (35) respectively.

Proposition 4.6 (Extreme vaccinations) *Let $n \in \Omega$, $\beta \in [0, n)$ and $\alpha \in [0, \mu_1]$. Let $\xi^{v,n} = (\xi_1^v(\beta), \dots, \xi_n^v(\beta))$, and $\xi^{h,n} = (\xi_1^h(\alpha), \dots, \xi_n^h(\alpha))$. For any $\xi = (\xi_1, \dots, \xi_n) \in [0, \mu_1] \times \dots \times [0, \mu_n]$, we have:*

$$\left(\sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi_i^{v,n} \right) \implies \xi < \xi^{v,n}, \quad \text{and} \quad \left(\sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi_i^{h,n} \right) \implies \xi^{h,n} < \xi.$$

Proof Let $\xi \in [0, \mu_1] \times \dots \times [0, \mu_n]$ be such that $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi_i^{v,n}$. The reordered vector ξ^\downarrow clearly satisfies the same conditions, so without loss of generality we may assume that ξ is sorted in descending order. Using Equation (35), we get:

$$\sum_{i=1}^{\ell} \xi_i \leq \sum_{i=1}^{\ell} \mu_i = \sum_{i=1}^{\ell} \xi_i^{v,n}, \quad \text{for } 1 \leq \ell \leq \lfloor \beta \rfloor.$$

We also have:

$$\sum_{i=1}^{\ell} \xi_i \leq \sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi_i^{v,n} = \sum_{i=1}^{\ell} \xi_i^{v,n}, \quad \text{for } \ell > \lfloor \beta \rfloor.$$

Therefore, we get $\xi < \xi^{v,n}$, by the definition of $<$.

Similarly, let $\xi \in [0, \mu_1] \times \dots \times [0, \mu_n]$ be such that $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi_i^{h,n}$. If $t \geq \alpha$ then:

$$\sum_i (\xi_i^{h,n} - t)_+ = 0 \leq \sum_i (\xi_i - t)_+,$$

while if $t \in [0, \alpha)$, using the fact that $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi_i^{h,n}$, the expression $\xi_i^{h,n} = \min(\alpha, \mu_i)$, and the inequalities $\xi_i \leq \mu_i$, we get:

$$\begin{aligned} \sum_{i=1}^n (\xi_i^{h,n} - t)_+ &= \sum_{i=1}^n (\xi_i^{h,n} - t) + \sum_{i=1}^n (t - \xi_i^{h,n})_+ \\ &= \sum_{i=1}^n (\xi_i - t) + \sum_{i=1}^n (t - \mu_i)_+ \\ &\leq \sum_{i=1}^n (\xi_i - t) + \sum_{i=1}^n (t - \xi_i)_+ \\ &= \sum_{i=1}^n (\xi_i - t)_+. \end{aligned}$$

This gives $\xi^{h,n} < \xi$, by the characterization (41). □

4.4.4 "Vertical" Pareto optima in the disassortative case

We consider here the disassortative model $b \geq a \geq 0$ and $b > 0$. Let $c \in (0, 1)$ and $D(c) = \{\eta \in \Delta : C(\eta) = c\}$ be the set of vaccination strategies with cost c . We will solve the constrained optimization Problem (7) that corresponds to:

$$\begin{cases} \min & R_e(\eta), \\ \text{such that} & \eta \in D(c). \end{cases} \tag{43}$$

Recall the definitions of β_c and $\eta^v(c)$ given page 36. Let $\eta \in D(c)$. Let n be large enough so that $\sum_{j>n} \mu_j < 1 - c$ so that $\sum_{j \leq n} \eta_j \mu_j > 0$, and assume that $n > \beta_c$. Let $\eta^{(n)} \in \Delta$ be defined by:

$$\eta_i^{(n)} = \frac{\sum_{j \leq n} \eta_j^v(c) \mu_j}{\sum_{j \leq n} \eta_j \mu_j} \mathbb{1}_{\{i \leq n\}} \eta_i.$$

Note that since $C(\eta^v(c)) = c = C(\eta)$, we have $\lim_{n \rightarrow \infty} \eta^{(n)} = \eta$ (pointwise and in L^2). Let $\xi^n = (\eta_1^{(n)} \mu_1, \dots, \eta_n^{(n)} \mu_n)$ and $\xi^{v,n}$ be defined as in Proposition 4.6 with $\beta = \beta_c$. By construction, we have $\sum_{i=1}^n \xi_i^n = \sum_{i=1}^n \xi_i^{v,n}$, so by Proposition 4.6, we get $\xi^n < \xi^{v,n}$. This implies that:

$$R_e(\eta^{(n)}) = \Theta_n(\xi^n) \geq \Theta_n(\xi^{v,n}) = R_e(\eta^v(c)),$$

where the inequality follows from the Schur concavity of Θ_n in the disassortative case (see Lemma 4.5) and where the last equality holds as $n \geq \lceil \beta_c \rceil$. Since R_e is continuous and $\eta^{(n)}$ converges pointwise and in L^2 to η , we get $R_e(\eta) \geq R_e(\eta^v)$. This implies that η^v is a solution of Problem (43).

If $a > 0$, then k is positive everywhere, and we deduce from Lemma 3.1 that $c_\star = 1$. If $a = 0$, it is easy to prove that $\{0\}$ is a maximal independent set of k ; this gives that $c_\star = 1 - \mu_1$, thanks to (Delmas et al. 2022b, Remark 4.5). Since for all $c \in [0, c_\star)$ there exists $\eta \in \mathcal{P}_v$ such that $C(\eta) = c$, we also get that $\mathcal{P}_v \cap \{\eta \in \Delta : C(\eta) \leq c_\star\}$ is a parametrization of the Pareto frontier. This gives the parametrization of the Pareto frontier using \mathcal{P}_v from Theorem 4.2 (ii) and (iii).

4.4.5 “Horizontal” anti-Pareto optima in the disassortative case

We still consider $b \geq a \geq 0$ and $b > 0$. Let $c \in (0, 1)$. We now turn to the anti-Pareto frontier by studying the constrained maximization Problem (8) that corresponds to:

$$\begin{cases} \max & R_e(\eta), \\ \text{such that } & \eta \in D(c). \end{cases} \tag{44}$$

Recall the definitions of α_c and $\eta^h(c)$ given page 34. Let $\eta \in D(c)$. Let n be large enough so that $\sum_{j>n} \mu_j < 1 - c$ and thus $\sum_{j \leq n} \eta_j \mu_j > 0$. Define $\eta^{(n)} \in \Delta$ by:

$$\eta_i^{(n)} = \frac{\sum_{j \leq n} \eta_j^h(c) \mu_j}{\sum_{j \leq n} \eta_j \mu_j} \mathbb{1}_{\{i \leq n\}} \eta_i.$$

Let $\xi^n = (\eta_1^{(n)} \mu_1, \dots, \eta_n^{(n)} \mu_n)$ and let $\xi^{h,n}$ be defined as in Proposition 4.6 with $\alpha = \alpha_c$. By construction, we have $\sum_{i=1}^n \xi_i^n = \sum_{i=1}^n \xi_i^{h,n}$, so by Proposition 4.6, we obtain $\xi^{h,n} \prec \xi^n$. This implies that:

$$R_e(\eta^{(n)}) = \Theta_n(\xi^n) \leq \Theta_n(\xi^{h,n}) = R_e(\eta^h(c) \mathbb{1}_{\llbracket 1, n \rrbracket}),$$

where the inequality follows from the Schur concavity of Θ_n .

Now, as n goes to infinity $\eta^{(n)}$ converges pointwise and in L^2 to η , and $\eta^h(c) \mathbb{1}_{\llbracket 1, n \rrbracket}$ converges pointwise and in L^2 to $\eta^h(c)$, so by continuity of R_e we get $R_e(\eta) \leq R_e(\eta^h(c))$, and $\eta^h(c)$ is solution of the Problem (44) and is thus anti-Pareto optimal for $c \in (0, 1)$ as $c^\star = 0$. Since $c^\star = 0$, we also deduce from (Delmas et al. 2021b, Propostion 5.8 (iii)) that $\mathbb{0}$ and $\mathbb{1}$ are anti-Pareto optimal. Since for all $c \in [0, 1]$ there exists $\eta \in \mathcal{P}_h$ such that $C(\eta) = c$, we deduce that \mathcal{P}_h is a parametrization of the anti-Pareto frontier.

4.4.6 The assortative case

The case $a \geq b > 0$, corresponding to point (i) in Proposition 4.2, is handled similarly, replacing concavity by convexity, minima by maxima and vice versa.

5 Constant degree kernels and uniform vaccinations

5.1 Motivation

We have seen in the previous section an example of model where vaccinating individuals with the highest degree is the best strategy. A similar phenomenon is studied in [17], where under monotonicity arguments on the kernel, vaccinating individuals with the highest (resp. lowest) degree is Pareto (resp. anti-Pareto) optimal. However, in case multiple individuals share the same maximal degree, the optimal strategies differ completely between the assortative and the disassortative models: the Pareto optimal strategies for one model correspond to the anti-Pareto optimal strategies for the other and vice versa.

Motivated by this curious symmetry, we investigate in this section constant degree kernels, that is, the situation where all the individuals have the same number of connections. In Sect. 5.2, we define these kernels formally and give the main result on the optimality of the uniform strategies when R_e is either convex or concave, see Proposition 5.4. Section 5.3 is devoted to the proof of this main result. We study in more detail the optimal strategies in an example of constant degree symmetric kernels of rank two in Sect. 6. Eventually, we study in Sect. 7 geometric kernels on the sphere, which are constant degree kernels.

5.2 On the uniform strategies for constant degree kernels

In graph theory, a regular graph is a graph where all vertices have the same number of in-neighbors, and the same number of out-neighbors. In other words all vertices have the same in-degree and the same out-degree. Limits of undirected regular graphs have been studied in details by Backhausz and Szegedy (2020) and Kunszenti-Kovács et al. (2021). When the graphs are dense, their limit can be represented as a regular graphon, that is a symmetric kernel with a constant degree function.

Since we do not wish to assume symmetry, we give the following general definition. For a kernel k on Ω , we set, for all $z \in \Omega$ and $A \in \mathcal{F}$:

$$k(z, A) = \int_A k(z, y) \mu(dy) \quad \text{and} \quad k(A, z) = \int_A k(x, z) \mu(dx).$$

For $z \in \Omega$, its in-degree is $k(z, \Omega)$ and its out-degree is $k(\Omega, z)$.

Definition 5.1 (*Constant degree kernel*) A kernel k with a finite L^2 double-norm and positive spectral radius $R_0 > 0$ is called *constant degree* if all the in-degrees and all the out-degrees have the same value, that is, the maps $x \mapsto k(x, \Omega)$ and $y \mapsto k(\Omega, y)$ defined on Ω are constant, and thus equal.

Remark 5.2 Let k be a constant degree kernel with spectral radius $R_0 > 0$. Notice the condition “all the in-degrees and out-degrees have the same value” is also equivalent to $\mathbb{1}$ being a left and right eigenfunction of T_k . We now check that the corresponding eigenvalue is R_0 .

Let $h \in L^2_+(\Omega) \setminus \{0\}$ be a left Perron-eigenfunction. Denote by λ the eigenvalue associated to $\mathbb{1}$. Then, we have:

$$\lambda \int_{\Omega} h(x) \mu(dx) = \int_{\Omega} h(x)k(x, y)\mu(dx)\mu(dy) = R_0 \int_{\Omega} h(y) \mu(dy),$$

where the first equality follows from the regularity of k and from the fact that h is a left Perron-eigenfunction of T_k . Since h is non-negative and not equal to 0 almost everywhere, we get that $\lambda = R_0$ and $\mathbb{1}$ is a right Perron-eigenvector of T_k . With a similar proof, we show that $\mathbb{1}$ is a left Perron-eigenvector of T_k . In particular, if k is constant degree, then the reproduction number is given by:

$$R_0 = \int_{\Omega \times \Omega} k(x, y) \mu(dx)\mu(dy). \tag{45}$$

Example 5.3 We now give examples of constant degree kernels.

- (i) Let $G = (V, E)$ be a finite non-oriented simple graph, and μ the uniform probability measure on the vertices V . The degree of a vertex $x \in V$ is given by

$$\text{deg}(x) = \#\{y \in V : (x, y) \in E\}.$$

The graph G is constant degree if all its vertices have the same degree, say $d \geq 1$. Then the kernel defined on the finite space $\Omega = V$ by the adjacency matrix is constant degree with $R_0 = d$. Notice it is also symmetric.

- (ii) Let $G = (V, E)$ be a finite directed graph, and μ be the uniform probability measure on the vertices V . The in-degree of a vertex $x \in V$ is given by

$$\text{deg}_{\text{in}}(x) = \#\{y \in V : (y, x) \in E\},$$

and the out-degree is given by

$$\text{deg}_{\text{out}}(x) = \#\{y \in V : (x, y) \in E\}.$$

The graph G is regular if all its vertices have the same in-degree and out-degree, say $d \geq 1$. Then the kernel defined on the finite space $\Omega = V$ by the adjacency matrix is regular with $R_0 = d$. Notice it might not be symmetric.

- (iii) Let $\Omega = (\mathbb{R}/(2\pi\mathbb{Z}))^m$ be the m -dimensional torus endowed with its Borel σ -field \mathcal{F} and the normalized Lebesgue measure μ . Let f be a measurable square-integrable non-negative function defined on Ω . We consider the geometric kernel on Ω defined by:

$$k_f(x, y) = f(x - y).$$

The kernel k_f has a finite double-norm as $f \in L^2$. The operator T_{k_f} corresponds to the convolution by f , and its spectral radius is given by $R_0 = \int_{\Omega} f d\mu$. Then the kernel k_f is constant degree as soon as f is not equal to 0 almost surely.

This example is developed in Sect. 7 in the case $m = 1$ (corresponding to $d = 2$ therein), see in particular Examples 7.2 and 7.3.

- (iv) More generally, let (Ω, \cdot) be a compact topological group and let μ be its left Haar probability measure. Let f be non-negative square-integrable function on Ω . Then the kernel $k_f(x, y) = f(y^{-1} \cdot x)$ is constant degree.

We summarize our main result in the next proposition, whose proof is given in Sect. 5.3. We recall that a strategy is called uniform if it is constant over Ω .

Proposition 5.4 (Uniform strategies for constant degree kernels) *Let k be a constant degree kernel on Ω .*

- (i) *If the map R_e defined on Δ is convex, then all uniform strategies are Pareto optimal (i.e. $S^{\text{uni}} \subset \mathcal{P}$). Consequently, $c_\star = 1$, the Pareto frontier is the segment joining $(0, R_0)$ to $(1, 0)$, and for all $c \in [0, 1]$:*

$$R_{e\star}(c) = (1 - c)R_0.$$

- (ii) *If the map R_e defined on Δ is concave, then the kernel k is irreducible and all uniform strategies are anti-Pareto optimal (i.e. $S^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$). Consequently, $c^\star = 0$, the anti-Pareto frontier is the segment joining $(0, R_0)$ to $(1, 0)$, and for all $c \in [0, 1]$:*

$$R_e^\star(c) = (1 - c)R_0.$$

In (Delmas et al. 2021a, Section 4.2), we give sufficient condition on the spectrum of T_k to be either concave or convex. Combining this result with Proposition 5.4, we get the following corollary.

Corollary 5.5 *Let k be a constant degree symmetric kernel.*

- (i) *If the eigenvalues of T_k are non-negative, then the uniform vaccination strategies are Pareto optimal and $c_\star = 1$ (i.e. $S^{\text{uni}} \subset \mathcal{P}$).*
- (ii) *If R_0 is a simple eigenvalue of T_k and the others eigenvalues are non-positive, then the kernel k is irreducible, the uniform vaccination strategies are anti-Pareto optimal and $c^\star = 0$ (i.e. $S^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$).*

Remark 5.6 (Equivalent conditions) *Let k be a constant degree symmetric kernel. The eigenvalues of the operator T_k are non-negative if and only if T_k is semi-definite positive, that is:*

$$\int_{\Omega \times \Omega} k(x, y)g(x)g(y)\mu(dx)\mu(dy) \geq 0 \quad \text{for all } g \in L^2. \tag{46}$$

Similarly, the condition given in Corollary 5.5 (ii) that implies the concavity of R_e is equivalent to the semi-definite negativity of T_k on the orthogonal of $\mathbb{1}$:

$$\int_{\Omega \times \Omega} k(x, y)g(x)g(y)\mu(dx)\mu(dy) \leq 0 \quad \text{for all } g \in L^2 \quad \text{such that} \quad \int_{\Omega} g \, d\mu = 0. \tag{47}$$

Remark 5.7 (Comparison with a result from Poghotyan et al. (2018)) (Poghotyan et al. 2018, Theorem 4.7) obtained a similar result in finite dimension using a result from Friedland (1981): if the next-generation non-negative matrix K of size $N \times N$ satisfies the following conditions

- (i) $\sum_{j=1}^N K_{ij}$ does not depend on $i \in \llbracket 1, N \rrbracket$ (which corresponds the parameters a_i in (Poghotyan et al. 2018, Equation (2.4)) being all equal),
- (ii) $\mu_i K_{ij} = \mu_j K_{ji}$ for all $i, j \in \llbracket 1, N \rrbracket$ where μ_i denote the relative size of population i (which corresponds to (Poghotyan et al. 2018), Equation (2.4)),
- (iii) K is not singular and its inverse is an M-matrix (*i.e.*, its non-diagonal coefficients are non-positive),

then the uniform strategies are Pareto optimal (*i.e.*, they minimize the reproduction number among all strategies with same cost). Actually, this can be seen as a direct consequence of Corollary 5.5 (i). Indeed, the corresponding kernel k_d defined by (30) in the discrete probability space $\Omega = \llbracket 1, N \rrbracket$ endowed with the discrete probability measure μ_d also defined by (30) has constant degree thanks to Point (i) and is symmetric thanks to Point (ii). Since K^{-1} is an M-matrix, its real eigenvalues are positive according to (Berman and Plemmons, 1994, Chapter 6 Theorem 2.3). The eigenvalues of T_{k_d} and K are actually the same as K is the representation matrix of T_{k_d} in the canonic basis of \mathbb{R}^N . We conclude that the operator T_{k_d} is positive definite. Hence Corollary 5.5 (i) can be applied to recover that the uniform strategies are Pareto optimal.

However, Points (i) and (ii) together with the positive-definiteness of K do not imply Point (iii). As a counter-example, consider a population divided in $N = 3$ groups of same size (*i.e.*, $\mu_1 = \mu_2 = \mu_3 = 1/3$) and the following next-generation matrix:

$$K = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix} \quad \text{with inverse} \quad K^{-1} = \begin{pmatrix} 1.4 & -1.6 & 0.4 \\ -1.6 & 2.4 & -0.6 \\ 0.4 & -0.6 & 0.4 \end{pmatrix}.$$

Clearly Points (i) and (ii) hold and Point (iii) fails as K^{-1} is not an M-matrix. Nevertheless, the matrix K is definite positive as its eigenvalues $\sigma(K) = \{5, 2 + \sqrt{3}, 2 - \sqrt{3}\}$ are positive. And thus, thanks to Corollary 5.5 (i), we get that the uniform strategies are Pareto optimal. Hence, Corollary 5.5 (i) is a strict generalization of (Poghotyan et al. 2018, Theorem 4.7) even for finite metapopulation models.

Remark 5.8 We also refer the reader to the paper of Friedland and Karlin (1975): from the Inequality (7.10) therein, we can obtain Corollary 5.5 (i) when Ω is a compact set of \mathbb{R}^n , μ is a finite measure, k is a continuous symmetrizable kernel such that $k(x, x) > 0$ for all $x \in \Omega$. Further comments on related results may be found in (Delmas et al. 2021a, Section 4).

Below, we give examples of metapopulation models from the previous sections where Proposition 5.4 applies. For continuous models, we refer the reader to Sections 6 and 7.

Example 5.9 (*Fully asymmetric cycle model*) We consider the fully asymmetric circle model with $N \geq 3$ vertices developed in Sect. 2.3. Since the in and out degree of each vertex is exactly one, the adjacency matrix is constant degree according to Example 5.3 (ii).

The spectrum of the adjacency matrix is given by the N th roots of unity, so for $N \geq 3$ it does not lie in $\mathbb{R}_- \cup \{R_0\}$, so Corollary 5.5 does not apply. However, in this case the effective spectral radius R_e is given by formula (19), which corresponds to the geometric mean. According to (Boyd and Vandenberghe 2004, Section 3.1.5), the map $\eta \mapsto R_e(\eta)$ is concave, so Proposition 5.4 (ii) applies. This proves that the spectral condition given in Corollary 5.5 and in (Delmas et al. 2021a, Section 4.1) to get the concavity of R_e is only sufficient.

Example 5.10 (*Finite assortative and disassortative model*) Let $\Omega = \{1, 2, \dots, N\}$ and μ be the uniform probability on Ω . Let $a, b \in \mathbb{R}_+$. We consider the kernel from the models developed in Sect. 4:

$$k(i, j) = a\mathbb{1}_{i=j} + b\mathbb{1}_{i \neq j}.$$

Since μ is uniform, the kernel k is constant degree; provided its spectral radius is positive, i.e., a or b is positive.

In the assortative model $0 < b \leq a$, according to Proposition 4.1 (i), the eigenvalues of the symmetric operator T_k are non-negative. Hence, Corollary 5.5 (i) applies: the uniform strategies are Pareto optimal. This is consistent with Theorem 4.2 (i).

In the dissortative model, we have $0 \leq a \leq b$ and $b > 0$. According to Proposition 4.1 (ii), the eigenvalues of T_k different from its spectral radius are non-positive. Hence, Corollary 5.5 (ii) applies: the uniform strategies are anti-Pareto. This is consistent with Theorem 4.2 (ii) and (iii).

5.3 Proof of Proposition 5.4

By analogy with (Eaves et al. 1985), we consider the following definition.

Definition 5.11 (*Completely reducible kernels*) A kernel k is said to be *completely reducible* if there exist an at most countable index set I , and measurable sets Ω_0 and $(\Omega_i, i \in I)$, such that Ω is the disjoint union $\Omega = \Omega_0 \sqcup (\bigsqcup_{i \in I} \Omega_i)$, the kernel k decomposes as $k = \sum_{i \in I} \mathbb{1}_{\Omega_i} k \mathbb{1}_{\Omega_i}$ a.e., and, for all $i \in I$, the kernel k restricted to Ω_i is irreducible with positive spectral radius.

As in the discrete case for so-called line sum symmetric matrices, see (Eaves et al. 1985, Lemma 1), kernels for which for any x the out-degree is equal to the in-degree are necessarily completely reducible; the fact that these degrees do not depend on x impose further constraints.

Lemma 5.12 (*Complete reduction*) *If k is a constant degree kernel on Ω , then k is completely reducible. Furthermore, the set Ω_0 from Definition 5.11 is empty, the cardinal of the partition $(\Omega_i, i \in I)$ is equal to the multiplicity of R_0 and thus is finite; and, for all $i \in I$, the kernel k restricted to Ω_i is a constant degree irreducible kernel with spectral radius equal to R_0 .*

Proof We recall that a set $A \in \mathcal{F}$ is invariant if $k(A^c, A) = 0$, where for $A, B \in \mathcal{F}$:

$$k(B, A) = \int_{B \times A} k(x, y) \mu(dx)\mu(dy).$$

Since for each x , the in-degree $k(x, \Omega)$ is equal to the out-degree $k(\Omega, x)$, we get by integration $k(A, \Omega) = k(\Omega, A)$, so

$$k(A^c, A) = k(A^c, \Omega) - k(A^c, A^c) = k(\Omega, A^c) - k(A^c, A^c) = k(A, A^c).$$

Therefore if A is invariant, then so is its complement A^c . According to (Delmas et al. 2021a, Section 5) and more precisely Remark 5.1(viii), there exists then an at most countable partition of Ω made of Ω_0 and $(\Omega_i, i \in I)$ such that $k = \sum_{i \in I} k_i$, with $k_i = \mathbb{1}_{\Omega_i} k \mathbb{1}_{\Omega_i}$, $\mu(\Omega_i) > 0$ and k_i restricted to Ω_i is irreducible with positive spectral radius. Since $\mathbb{1}$ is an eigenvector of T_k associated to the eigenvalue R_0 and the sets Ω_0 and $(\Omega_i, i \in I)$ are pairwise disjoint, we deduce that Ω_0 is of zero measure and $\mathbb{1}_{\Omega_i}$ is an eigenvector of T_{k_i} with eigenvalue $R_0 > 0$, for all $i \in I$. Hence, all the kernels k_i restricted to Ω_i are irreducible constant degree kernels with spectral radius equal to R_0 . Thus, the cardinal of I is equal to the multiplicity of R_0 (for T_k). Since k has finite L^2 double-norm, the operator T_k is compact, and the multiplicity of $R_0 > 0$, and thus the cardinal of I , is finite. \square

Lemma 5.13 *Let k be a constant degree irreducible kernel on Ω . Then the uniform strategy is a critical point for R_e among all the strategies with the same cost in $(0, 1)$, and more precisely: for all η with the same cost in $(0, 1)$ as $\eta^{\text{uni}} \in \mathcal{S}^{\text{uni}}$ and $\varepsilon > 0$ small enough, we have:*

$$R_e((1 - \varepsilon)\eta^{\text{uni}} + \varepsilon\eta) = R_e(\eta^{\text{uni}}) + O(\varepsilon^2).$$

Proof Let η^{uni} be the uniform strategy with cost $c \in (0, 1)$. Since k is irreducible, we get that $(1 - c)R_0$ is a simple isolated eigenvalue of $k\eta^{\text{uni}}$, whose corresponding left and right eigenvector are $\mathbb{1}$ as $k\eta^{\text{uni}}$ is also constant degree. For $\eta \in \Delta$, we get that $T_{k((1-\varepsilon)\eta^{\text{uni}}+\varepsilon\eta)}$ converges to $T_{k\eta^{\text{uni}}}$ (in operator norm, thanks to (24)) as ε goes down to 0. Notice that:

$$\| T_{k(\eta^{\text{uni}}+\varepsilon(\eta-\eta^{\text{uni}}))} - T_{k\eta^{\text{uni}}} \|_{L^2}^2 = O(\varepsilon^2).$$

According to (KloECKner 2019, Theorem 2.6), we get that for any $\eta \in \Delta$ and $\varepsilon > 0$ small enough:

$$\begin{aligned} R_e((1 - \varepsilon)\eta^{\text{uni}} + \varepsilon\eta) - R_e(\eta^{\text{uni}}) &= \varepsilon \int_{\Omega} k(x, y)(\eta(y) - \eta^{\text{uni}}(y)) \mu(dx)\mu(dy) + O(\varepsilon^2) \\ &= \varepsilon R_0 \int_{\Omega} (\eta(y) - \eta^{\text{uni}}(y)) \mu(dy) + O(\varepsilon^2), \end{aligned}$$

where for the last equality we used that k is constant degree. In particular, if η and η^{uni} have the same cost $c \in (0, 1)$, then $R_e((1 - \varepsilon)\eta^{\text{uni}} + \varepsilon\eta) - R_e(\eta^{\text{uni}}) = O(\varepsilon^2)$, which

means that the uniform strategy is a critical point for R_e among all the strategies with cost $c \in (0, 1)$. □

Proof of Proposition 5.4 We prove (i), and thus consider k constant degree and R_e convex. We first consider the case where k is irreducible. For any η , Lemma 5.13 and the convexity of R_e imply that

$$R_e(\eta^{\text{uni}}) + O(\varepsilon^2) = R_e((1 - \varepsilon)\eta^{\text{uni}} + \varepsilon\eta) \leq (1 - \varepsilon)R_e(\eta^{\text{uni}}) + \varepsilon R_e(\eta),$$

where η^{uni} the uniform strategy with the same cost as η . Letting ε go to 0, we get $R_e(\eta) \geq R_e(\eta^{\text{uni}})$, so R_e is minimal at η^{uni} .

Since $C(\eta^{\text{uni}}) = c$ and $R_e(\eta^{\text{uni}}) = (1 - c)R_0$, we deduce that $R_{e^*}(c) = (1 - c)R_0$ and thus, the Pareto frontier is a segment given by $\mathcal{F} = \{(c, (1 - c)R_0) : c \in [0, 1]\}$.

In what follows, we write $R_e[k]$ to stress that the reproduction function on Δ defined by (29) depends on the kernel k : $R_e[k](\eta) = \rho(T_{k\eta})$ for $\eta \in \Delta$. If k is not irreducible, then use the representation from Lemma 5.12 (or Delmas et al. 2021a, Lemma 5.3), to get that $R_e[k] = \max_{i \in I} R_e[k_i]$. Since the cost is affine, we deduce that a strategy η with $R_e[k](\eta) = \ell \in [0, R_0]$ is Pareto optimal if and only if, for all $i \in I$, the strategies $\eta_i = \eta \mathbb{1}_{\Omega_i}$ are Pareto optimal for the kernel k restricted to Ω_i and $R_e[k_i](\eta_i) = \ell$; see also (Delmas et al. 2022b, Proposition 5.7). Then the first step of the proof yields that $\eta_i = \ell \mathbb{1}_{\Omega_i}$ and thus the uniform strategy $\eta^{\text{uni}} = \ell \mathbb{1}_{\Omega}$ is Pareto optimal. This ends the proof of (i).

We now prove (ii). We first check that the kernel k is irreducible. Thanks to Lemma 5.12, the kernel k is completely reducible with a zero measure Ω_0 . However, (Delmas et al. 2021a, Lemma 5.10) also implies that it is *monatomic*, a notion introduced in (Delmas et al. 2021a, Section 5.2) which intuitively states that k has only one irreducible component. Together with complete reducibility, this implies that k is irreducible. The rest of the proof is then similar to the proof of (i) under the irreducibility assumption. □

6 Constant degree symmetric kernels of rank two

6.1 Motivation

Consider the integral operator T_k on L^2 associated to a kernel k with finite double norm on L^2 . According to (Conway 1990, p. 267), the operator T_k is an Hilbert-Schmidt integral operator. If furthermore the kernel k is symmetric, thanks to the spectral theorem for compact operators (Conway 1990, Theorem II.7.6), we have the following decomposition in $L^2(\Omega^2, \mu^{\otimes 2})$:

$$k(x, y) = \sum_{0 \leq n < N} \varepsilon_n \alpha_n(x) \alpha_n(y),$$

where $0 \leq N \leq +\infty$, $\varepsilon_n \in \{-, +\}$ and $(\alpha_n, 0 \leq n < N)$ is an orthogonal family of eigenvectors of T_k such that $\varepsilon_n \|\alpha_n\|^2$ is equal to the eigenvalue associated to α_n . In

particular, for constant degree symmetric kernel k and assuming that the rank of T_k is at least two ($N \geq 2$), α_0 is equal to $\mathbb{1}$ and the decomposition writes:

$$k(x, y) = R_0 + \sum_{1 \leq n < N} \varepsilon_n \alpha_n(x) \alpha_n(y),$$

where α_n for $1 \leq n < N$ is orthogonal to $\mathbb{1}$. The integral operator associated to the kernel $R_0 + \varepsilon_1 \alpha_1(x) \alpha_1(y)$ is the best $\|\cdot\|_{L^2}$ -approximation of T_k by an operator of rank two if $\|\alpha_1\| \geq \|\alpha_n\|$ for all $1 \leq n < N$.

Because it completes the study of the previous section but also because it can give some insights on the shape of the Pareto and anti-Pareto frontier for general symmetric constant degree kernels according to the stability results (Delmas et al. 2021b, Proposition 4.3 and Proposition 6.2), we will treat the case of symmetric constant degree kernels whose associated operator is of rank two, where one can explicitly minimize and maximize R_e among all strategies at a given cost.

6.2 Pareto and anti-Pareto frontiers

We suppose that $\Omega = [0, 1]$ is equipped with the Borel σ -field \mathcal{F} and a probability measure μ whose cumulative distribution function φ , defined by $\varphi(x) = \mu([0, x])$ for $x \in \Omega$, is continuous and increasing. We consider the following two kernels on Ω :

$$k^\varepsilon(x, y) = R_0 + \varepsilon \alpha(x) \alpha(y), \quad \text{with } \varepsilon \in \{-, +\}, \tag{48}$$

where $R_0 > 0$ and $\alpha \in L^2$ is strictly increasing and satisfies:

$$\sup_{\Omega} \alpha^2 \leq R_0 \quad \text{and} \quad \int_{\Omega} \alpha \, d\mu = 0. \tag{49}$$

Remark 6.1 (Generality) We note that this particular choice of Ω may be made without loss of generality, and that the strict monotonicity assumption on α is almost general: we refer the interested reader to Sect. 6.3 for further discussion on this point.

For $\varepsilon \in \{-, +\}$, the kernel k^ε is symmetric and constant degree. Furthermore, we have that R_0 and $\varepsilon \int_{\Omega} \alpha^2 \, d\mu$ are the only non-zero eigenvalues (and their multiplicity is one) of T_{k^ε} with corresponding eigen-vector $\mathbb{1}$ and α . Since $\alpha^2 \leq R_0$, we also get that R_0 is indeed the spectral radius of T_{k^ε} .

The Pareto (resp. anti-Pareto) frontier is already greedily parametrized by the uniform strategies for the kernel k^+ (resp. k^-), see Corollary 5.5. The following result restricts the choice of anti-Pareto (resp. Pareto) optimal strategies to two extreme strategies. Hence, in order to find the optima, it is enough to compute and compare the two values of R_e for each cost.

We recall the set of uniform strategies $\mathcal{S}^{\text{uni}} = \{t\mathbb{1} : t \in [0, 1]\}$ and consider the following sets of extremal strategies:

$$\mathcal{S}_0 = \{\mathbb{1}_{[0,t)} : t \in [0, 1]\} \quad \text{and} \quad \mathcal{S}_1 = \{\mathbb{1}_{[t,1)} : t \in [0, 1]\}$$

as well as the following set of strategies which contains \mathcal{S}^{uni} thanks to (49):

$$\mathcal{S}^{\perp\alpha} = \left\{ \eta \in \Delta : \int_{\Omega} \alpha \eta \, d\mu = 0 \right\}.$$

Recall that strategies are defined up to the a.s. equality. The proof of the next proposition is given in Sect. 6.4

Proposition 6.2 (Optima are uniform or on the sides) *Let $[0, 1]$ be endowed with a probability measure whose cumulative distribution function is increasing and continuous. Let k^ε be given by (48) with $R_0 > 0$ and α a strictly increasing function on $[0, 1]$ such that (49) holds.*

(i) **The kernel k^+ .** *A strategy is Pareto optimal if and only if it belongs to $\mathcal{S}^{\perp\alpha}$. In particular, for any $c \in [0, 1]$, the strategy $(1 - c)\mathbb{1}$ costs c and is Pareto optimal. The only possible anti-Pareto strategies of cost c are $\mathbb{1}_{[0, 1-c]}$ and $\mathbb{1}_{[c, 1]}$. In other words,*

$$\mathcal{P} = \mathcal{S}^{\perp\alpha} \quad \text{and} \quad \mathcal{P}^{\text{Anti}} \subset \mathcal{S}_0 \cup \mathcal{S}_1.$$

(ii) **The kernel k^- .** *A strategy is anti-Pareto optimal if and only if it belongs to $\mathcal{S}^{\perp\alpha}$. In particular, for any $c \in [0, 1]$, the strategy $(1 - c)\mathbb{1}$ costs c and is anti-Pareto optimal. The only possible Pareto strategies of cost c are $\mathbb{1}_{[0, 1-c]}$ and $\mathbb{1}_{[c, 1]}$. In other words,*

$$\mathcal{P} \subset \mathcal{S}_0 \cup \mathcal{S}_1 \quad \text{and} \quad \mathcal{P}^{\text{Anti}} = \mathcal{S}^{\perp\alpha}.$$

In both cases, we have $c^* = 0$ and $c_* = 1$.

Remark 6.3 Intuitively, the populations $\{\alpha < 0\}$ and $\{\alpha > 0\}$ behave in an assortative way for k^+ and in a disassortative way for k^- . As in Sect. 4, the uniform strategies are Pareto optimal in the ‘‘assortative’’ k^+ case and anti-Pareto optimal in the ‘‘disassortative’’ k^- case.

Remark 6.4 Under the assumptions of Proposition 6.2, if furthermore α is anti-symmetric with respect to $1/2$, that is $\alpha(x) = -\alpha(1 - x)$ for $x \in (0, 1)$, and μ is symmetric with respect to $1/2$, that is $\mu([0, x]) = \mu([1 - x, 1])$, then it is easy to check from the proof of Proposition 6.2 that the strategies from \mathcal{S}_0 and \mathcal{S}_1 are both optimal: $\mathcal{P}^{\text{Anti}} = \mathcal{S}_0 \cup \mathcal{S}_1$ for k^+ and $\mathcal{P} = \mathcal{S}_0 \cup \mathcal{S}_1$ for k^- . We plotted such an instance of k^+ and the corresponding Pareto and anti-Pareto frontiers in Fig. 9. We refer to Sect. 6.5 for an instance where α is not symmetric and $\mathcal{P} \neq \mathcal{S}_0 \cup \mathcal{S}_1$ for k^- .

6.3 On the choice of $\Omega = [0, 1]$ and on the monotonicity assumption

Using a reduction model technique from (Delmas et al. 2021b, Section 7), let us first see that there is no loss of generality by considering the kernel $k^\varepsilon = R_0 + \varepsilon\alpha \otimes \alpha$ on $\Omega = [0, 1]$ endowed with the Lebesgue measure μ and with α non-decreasing.

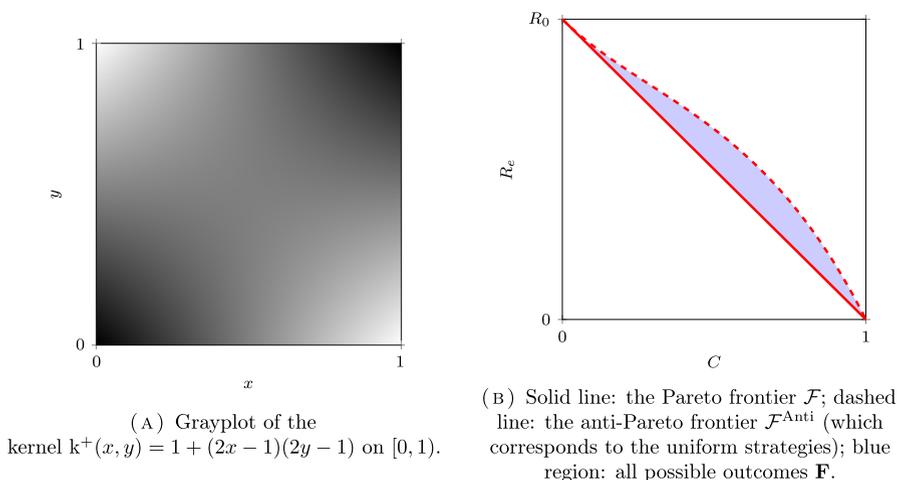


Fig. 9 An example of a constant degree kernel operator of rank two

Suppose that the function α in (48) is replaced by an \mathbb{R} -valued measurable function α_0 defined on a general probability space $(\Omega_0, \mathcal{F}_0, \mu_0)$ such that (49) holds. Thus, with obvious notations, for $\varepsilon \in \{-, +\}$, the kernel $R_0 + \varepsilon\alpha_0 \otimes \alpha_0$ is a kernel on Ω_0 . Denote by F the repartition function of α_0 (that is, $F(r) = \mu_0(\alpha_0 \leq r)$ for $r \in \mathbb{R}$) and take α as the quantile function of α_0 , that is, the right continuous inverse of F . Notice the function α is defined on the probability space $(\Omega, \mathcal{F}, \mu)$ is non-decreasing and satisfies (49). Consider the probability kernel $\kappa : \Omega_0 \times \mathcal{F} \rightarrow [0, 1]$ defined by $\kappa(x, \cdot) = \delta_{F(\alpha_0(x))}(\cdot)$, with δ the Dirac mass, if α is continuous at $\alpha_0(x)$ (that is, $F(\alpha_0(x)-) = F(\alpha_0(x))$) and the uniform probability measure on $[F(\alpha_0(x)-), F(\alpha_0(x))]$ otherwise. On the measurable space $(\Omega_0 \times \Omega, \mathcal{F}_0 \otimes \mathcal{F})$, we consider the probability measure $\nu(dx_1, dx_2) = \mu_0(dx_1)\kappa(x_1, dx_2)$, whose marginals are exactly μ_0 and μ . Then, for $\varepsilon \in \{-, +\}$, we have that :

$$R_0 + \varepsilon\alpha_0(x_1)\alpha_0(y_1) = R_0 + \varepsilon\alpha(x_2)\alpha(y_2) \quad \nu(dx_1, dx_2) \otimes \nu(dy_1, dy_2)\text{-a.s.}$$

According to (Delmas et al. 2021b, Section 7.3), see in particular Proposition 7.3 therein, the kernels $R_0 + \varepsilon\alpha_0 \otimes \alpha_0$ and $R_0 + \varepsilon\alpha \otimes \alpha$ are coupled and there is a correspondence between the corresponding (anti-)Pareto optimal strategies and their (anti-)Pareto frontiers are the same.

Hence, there is no loss in generality in assuming that the function α in (48) is indeed defined on $[0, 1)$ and is non-decreasing.

On the contrary, one cannot assume in full generality that α is *strictly* increasing, as when it is only non-decreasing, the situation is more complicated. Indeed, let us take the parameters $R_0 = 1$ and $\alpha = \mathbb{1}_{[0,0.5)} - \mathbb{1}_{[0.5,1)}$. Then, the kernel k^- is complete bipartite: $k^- = \mathbb{1}_{[0,0.5) \times [0.5,1)} + \mathbb{1}_{[0.5,1) \times [0,0.5)}$. Hence, according to Theorem 4.2 (iii), we have $c_* = 0.5$ for the kernel k^- . In a similar fashion, one can see that $k^+ = \mathbb{1}_{[0,0.5) \times [0,0.5)} + \mathbb{1}_{[0.5,1) \times [0.5,1)}$ is assortative and reducible; it is then easy to check that $c^* = 0.5$ for the kernel k^+ . However, it is still true that, for all costs c :

- $\mathbb{1}_{[0,1-c)}$ or $\mathbb{1}_{[c,1)}$ is solution of Problem (8) when the kernel k^+ is considered,
- $\mathbb{1}_{[0,1-c)}$ or $\mathbb{1}_{[c,1)}$ is solution of Problem (7) when the kernel k^- is considered.

From the proof of Proposition 6.2, we cannot expect to have strict inequalities in (59) if α is only non-decreasing, and thus one cannot expect $S_0 \cup S_1$ to contain $\mathcal{P}^{\text{Anti}}$ for the kernel k^+ or \mathcal{P} for the kernel k^- .

6.4 Proof of Proposition 6.2

We assume that $R_0 > 0$ and α is a strictly increasing function defined on $\Omega = [0, 1)$ such that (49) holds. Without loss of generality, we shall assume that $R_0 = 1$ unless otherwise specified. We write R_ϵ^ϵ for the effective reproduction function associated to the kernel k^ϵ . We shall also write ϵa for a if $\epsilon = +$ and $-\epsilon a$ if $\epsilon = -$. We first rewrite R_ϵ^ϵ in two different ways in Sect. 6.4.1. Then, we consider the kernel k^- in Sect. 6.4.2 and the kernel k^+ in Sect. 6.4.3.

6.4.1 Two expressions of the effective reproduction function

We provide an explicit formula for the function R_ϵ^ϵ , and an alternative variational formulation, both of which will be needed below.

Lemma 6.5 *Assume $R_0 = 1$ and α is a strictly increasing function defined on $\Omega = [0, 1)$ such that (49) holds. We have for $\epsilon \in \{+, -\}$ and $\eta \in \Delta$:*

$$2R_\epsilon^\epsilon(\eta) = \int \eta \, d\mu + \epsilon \int \alpha^2 \eta \, d\mu + \sqrt{\left(\int \eta \, d\mu - \epsilon \int \alpha^2 \eta \, d\mu\right)^2 + 4\epsilon \left(\int \alpha \eta \, d\mu\right)^2}. \tag{50}$$

Alternatively, $R_\epsilon^\epsilon(\eta)$ is the solution of the variational problem:

$$R_\epsilon^\epsilon(\eta) = \sup_{h \in B_+^\eta} \left(\int_0^1 h \eta \, d\mu \right)^2 + \epsilon \left(\int_0^1 h \alpha \eta \, d\mu \right)^2, \tag{51}$$

where

$$B_+^\eta = \left\{ h \in L_+^2 : \int_0^1 h^2 \eta \, d\mu = 1 \right\}.$$

The supremum in (51) is reached for the right Perron eigenfunction of $T_{k^\epsilon \eta}$ chosen in B_+^η .

Proof We first prove (50). For all $\eta \in \Delta$, the rank of the kernel operator $T_{k^\epsilon \eta}$ is smaller or equal to 2 and $\text{Im}(T_{k^\epsilon \eta}) \subset \text{Vect}(\mathbb{1}, \alpha)$. The matrix of $T_{k^\epsilon \eta}$ in the basis $(\mathbb{1}, \alpha)$ of the range of $T_{k^\epsilon \eta}$ is given by:

$$\begin{pmatrix} \int \eta \, d\mu & \int \alpha \eta \, d\mu \\ \epsilon \int \alpha \eta \, d\mu & \epsilon \int \alpha^2 \eta \, d\mu \end{pmatrix}. \tag{52}$$

An explicit computation of the spectrum of this matrix yields Equation (50) for its largest eigenvalue.

The variational formula (51) is a direct consequence of general Lemma 6.6 below. □

Lemma 6.6 (Variational formula for R_e when k is symmetric) *Suppose that k is a symmetric kernel on Ω with a finite double-norm in L^2 . Then, we have that for all $\eta \in \Delta$:*

$$R_e(\eta) = \sup_{h \in B_+^\eta} \int_{\Omega \times \Omega} h(x)\eta(x) k(x, y) h(y)\eta(y) \mu(dx)\mu(dy), \tag{53}$$

where

$$B_+^\eta = \left\{ h \in L_+^2 : \int_{\Omega} h^2 \eta \, d\mu = 1 \right\}.$$

The supremum in (53) is reached for the right Perron eigenfunction of $T_{k\eta}$ chosen in B_+^η .

Proof For a finite measure ν on (Ω, \mathcal{F}) , as usual, we denote by $L^2(\nu)$ the set of measurable real-valued functions f such that $\int_{\Omega} f^2 d\nu < +\infty$ endowed with the usual scalar product, so that $L^2(\nu)$ is an Hilbert space. Let $\eta \in \Delta$. We denote by $\mathcal{T}_{k\eta}$ the integral operator associated to the kernel $k\eta$ seen as an operator on the Hilbert space $L^2(\eta d\mu)$: for $g \in L^2(\eta d\mu)$ and $x \in \Omega$ we have $\mathcal{T}_{k\eta}(g)(x) = \int_{\Omega} k(x, y) \eta(y) g(y) \mu(dy)$. The operator $\mathcal{T}_{k\eta}$ is self-adjoint and compact since the double-norm of k in $L^2(\eta d\mu)$ is finite. It follows from the Krein-Rutman theorem and the Courant-Fischer-Weyl min-max principle that its spectral radius is given by the variational formula:

$$\rho(\mathcal{T}_{k\eta}) = \sup_{h \in B_+^\eta} \int_{\Omega \times \Omega} h(x) k(x, y) h(y) \eta(x) \mu(dx) \eta(y) \mu(dy).$$

Besides, the set $L^2(\mu)$ is densely and continuously embedded in $L^2(\eta d\mu)$ and the restriction of $\mathcal{T}_{k\eta}$ to $L^2(\mu)$ is equal to $T_{k\eta}$. Thanks to ((Delmas et al. 2021a, Lemmas 2.1 (iii) and 2.2), we deduce that $\rho(T_{k\eta})$ is equal to $\rho(\mathcal{T}_{k\eta})$, which gives (53).

Let h_0 be the right Perron eigenfunction of $T_{k\eta}$ chosen such that $h_0 \in B_+^\eta$. We get:

$$\int_{\Omega \times \Omega} \eta(x)h_0(x) k(x, y) \eta(y)h_0(y) \mu(dx)\mu(dy) = R_e(\eta) \int_{\Omega} \eta(x)h_0(x)^2 \mu(dx) = R_e(\eta).$$

Thus, the supremum in (53) is reached for $h = h_0$. □

6.4.2 The kernel k^-

Since α is increasing, we have $\mu(\alpha^2 = R_0) = 0$ and thus the symmetric kernel k^- is positive $\mu^{\otimes 2}$ -a.s. It follows from Remark 3.1 that $c^* = 0$ and $c_* = 1$, and the

strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost $c = 0$ (resp. $c = 1$). Since the kernel k^- is constant degree and symmetric, and the non-zero eigenvalues of T_{k^-} are given by $R_0 = 1$ and $-\int \alpha^2 d\mu$, the latter being negative, we deduce from Corollary 5.5 (ii) that $\mathcal{S}^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$. On the one hand, if η is anti-Pareto optimal with the same cost as $\eta^{\text{uni}} \in \mathcal{S}^{\text{uni}}$, one can use that $R_e^-(\eta) = \int \eta d\mu$ (as $R_e^-(\eta^{\text{uni}}) = \int \eta^{\text{uni}} d\mu$) and (50) to deduce that $\eta \in \mathcal{S}^{\perp\alpha}$. On the other hand, if η belongs to $\mathcal{S}^{\perp\alpha}$, we deduce from (50) that $R_e(\eta) = \int \eta d\mu$, and thus η is anti-Pareto optimal. In conclusion, we get $\mathcal{P}^{\text{Anti}} = \mathcal{S}^{\perp\alpha}$.

We now study the Pareto optimal strategies. We first introduce a notation inspired by the stochastic order of real valued random variables: we say that $\eta_1, \eta_2 \in \Delta$ with the same cost are in *stochastic order*, and we write $\eta_1 \leq_{\text{st}} \eta_2$ if:

$$\int_0^t \eta_1 d\mu \geq \int_0^t \eta_2 d\mu \quad \text{for all } t \in [0, 1]. \tag{54}$$

We also write $\eta_1 <_{\text{st}} \eta_2$ if the inequality in (54) is strict for at least one $t \in (0, 1)$. If $\eta_1 <_{\text{st}} \eta_2$ and h is an increasing bounded function defined on $[0, 1]$, then we have:

$$\int_{\Omega} h \eta_1 d\mu < \int_{\Omega} h \eta_2 d\mu. \tag{55}$$

Let $c \in (0, 1)$ be fixed. Define the vaccination strategies with cost c :

$$\eta_0 = \mathbb{1}_{[0, 1-c]} \quad \text{and} \quad \eta_1 = \mathbb{1}_{[c, 1]}. \tag{56}$$

In particular we have $\eta_0 <_{\text{st}} \eta_1$ as μ has no atom and Ω as full support. Let $\eta \notin \{\eta_0, \eta_1\}$ be a vaccination strategy with cost c ; necessarily

$$\eta_0 <_{\text{st}} \eta <_{\text{st}} \eta_1.$$

We now rewrite the function R_e^- in order to use the stochastic order on the vaccination strategies. We deduce from (50) that:

$$4R_e^-(\eta) = 4 \int \eta d\mu - H(\eta)^2 \quad \text{with} \quad H(\eta) = \sqrt{\int (1 + \alpha)^2 \eta d\mu} - \sqrt{\int (1 - \alpha)^2 \eta d\mu}. \tag{57}$$

Then, using that α is increasing and $[-1, 1]$ -valued, we deduce from (55) (with $h = (1 + \alpha)^2$ and $h = -(1 - \alpha)^2$) and the definition of H in (57) that:

$$H(\eta_0) < H(\eta) < H(\eta_1).$$

This readily implies that $R_e^-(\eta) > \min(R_e^-(\eta_0), R_e^-(\eta_1))$. Thus, among strategies of cost c , the only possible Pareto optimal ones are η_0 and η_1 . We deduce that $\mathcal{P} \subset \mathcal{S}_0 \cup \mathcal{S}_1$.

6.4.3 The kernel k^+

Arguing as for k^- , we get that $c^* = 0$ and $c_* = 1$, and the strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost $c = 0$ (resp. $c = 1$). Since the kernel k^+ is constant degree and symmetric, and the non-zero eigenvalues of T_{k^+} given by R_0 and $\int_{\Omega} \alpha^2 d\mu$ are positive, we deduce from Corollary 5.5 (i) that $\mathcal{S}^{\text{uni}} \subset \mathcal{P}$.

Arguing as in Sect. 6.4.2 for the identification of the anti-Pareto optima based on (50) (with $\varepsilon = +$ instead of $\varepsilon = -$) and using that $\mathcal{S}^{\text{uni}} \subset \mathcal{P}$ (instead of $\mathcal{S}^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$), we deduce that $\mathcal{P} = \mathcal{S}^{\perp\alpha}$.

We now consider the anti-Pareto optima. Let $c \in (0, 1)$. We first start with some comparison of integrals with respect to the vaccination strategies, with cost c , η_0 and η_1 defined by (56). Let η be a strategy of cost c not equal to η_0 or η_1 (recall that a strategy is defined up to the a.s. equality). Consider the monotone continuous non-negative functions defined on $[0, 1]$:

$$\phi_0 : x \mapsto \varphi^{-1} \left(\int_{[0,x)} \eta d\mu \right), \quad \text{and} \quad \phi_1 : x \mapsto \varphi^{-1} \left(1 - \int_{[x,1)} \eta d\mu \right).$$

Let $i \in \{0, 1\}$. Let ϕ_i^{-1} denote the generalized left-continuous inverse of ϕ_i . Note that $\eta(x)\mu(dx)$ -a.s., $\phi_i^{-1} \circ \phi_i(x) = x$. The measure $\eta_i d\mu$ is the push-forward of $\eta d\mu$ through ϕ_i , so that for h bounded measurable:

$$\int h \eta d\mu = \int h_i \eta_i d\mu \quad \text{with} \quad h_i = h \circ \phi_i^{-1}. \tag{58}$$

Since η is not equal to η_0 a.s., there exists $x_0 < 1 - c$ such that, $\phi_0(x) = x$ for $x \in [0, x_0]$ and $\phi_0(x) < x$ for $x \in (x_0, 1]$. Thus, we deduce that $\phi_0^{-1}(y) = y$ for all $y \in [0, x_0]$ and $\phi_0^{-1}(y) > y$ for all $y \in (x_0, 1 - c]$. Similarly, since η is not equal to η_1 almost surely, there exists $x_1 > c$ such that $\phi_1^{-1}(y) = y$ for all $y \in (x_1, 1]$ and $\phi_1^{-1}(y) < y$ for all $y \in [c, x_1]$. Since α is increasing and μ has no atom and full support in Ω , we deduce from (58), applied to $h\alpha$, that if h is a.s. positive bounded measurable, then:

$$\int h_0 \alpha \eta_0 d\mu < \int h \alpha \eta d\mu < \int h_1 \alpha \eta_1 d\mu. \tag{59}$$

Let h be the right Perron eigenfunction of $T_{k^+ \eta}$ chosen such that $h \in B_+^\eta$. Since k^+ is positive a.s. and thus irreducible with positive spectral radius, we have that h is positive a.s. Thanks to Lemma 6.5, we have:

$$R_e^+(\eta) = \left(\int h \eta d\mu \right)^2 + \left(\int h \alpha \eta d\mu \right)^2 \quad \text{and} \quad \int h^2 \eta d\mu = 1. \tag{60}$$

We deduce from (58) that for $i \in \{0, 1\}$:

$$\int h \eta \, d\mu = \int h_i \eta_i \, d\mu \quad \text{and} \quad 1 = \int h^2 \eta \, d\mu = \int h_i^2 \eta_i \, d\mu.$$

In particular h_i belongs to $B_+^{\eta_i}$. Using that $h > 0$ a.s., we then deduce from (60) and (59) that:

$$R_e^+(\eta) < \max_{i \in \{0,1\}} \left(\int h_i \eta_i \, d\mu \right)^2 + \left(\int h_i \alpha \eta_i \, d\mu \right)^2 \leq \max_{i \in \{0,1\}} R_e(\eta_i).$$

We conclude that only η_0 or η_1 can maximize R_e^+ among the strategies of cost $c \in (0, 1)$. We deduce that $\mathcal{P}^{\text{Anti}} \subset \mathcal{S}_0 \cup \mathcal{S}_1$.

6.5 An example where all parametrizations of the Pareto frontier have an infinite number of discontinuities

The purpose of this section is to give a particular example of kernel on a continuous model where we rigorously prove that the Pareto frontier cannot be greedily parametrized, that is, parametrized by a continuous path in Δ (as in the fully symmetric circle), and that all the parametrizations have an arbitrary large number of discontinuities (possibly countably infinite).

We keep the setting from Sect. 6.2. Without loss of generality, we assume that $R_0 = 1$, and we consider the kernel $k^- = 1 - \alpha \otimes \alpha$ on $\Omega = [0, 1)$ endowed with its Lebesgue measure. We know from the previous section that, for any cost, either η_0 or η_1 are Pareto optimal, and that all other strategies are non-optimal. The idea is then to build an instance of the function α in such a way that for some costs, one must vaccinate “on the left” and for other costs “on the right”.

Let $N \in \llbracket 2, +\infty \rrbracket$. Consider an increasing sequence $(x_n, n \in \llbracket 0, N \rrbracket)$ such that $x_0 = 1/2, x_N = 1$ and $\lim_{n \rightarrow \infty} x_n = 1$ if $N = \infty$. For $0 \leq n < N$, let $p_n = x_{n+1} - x_n$ and assume that $p_{n+1} < p_n$ for $n \in \llbracket 0, N \rrbracket$. For $n \geq 1$, let x_{-n} be the symmetric of x_n with respect to $1/2$, i.e., $x_{-n} = 1 - x_n$. The function α is increasing piecewise linear defined on $(0, 1)$ by:

$$\alpha(x) = \begin{cases} 2x - 1, & \text{for } x \in [x_{2m}, x_{2m+1}), \\ x - 1 + \frac{x_{2m-1} + x_{2m}}{2} & \text{for } x \in [x_{2m-1}, x_{2m}). \end{cases} \tag{61}$$

See Fig. 10A for an instance of the graph of α given in Example 6.9. Note that for all $n \in \llbracket 0, N \rrbracket$, we have:

$$\int_{x_n}^{x_{n+1}} \alpha \, d\mu = - \int_{x_{-n-1}}^{x_{-n}} \alpha \, d\mu. \tag{62}$$

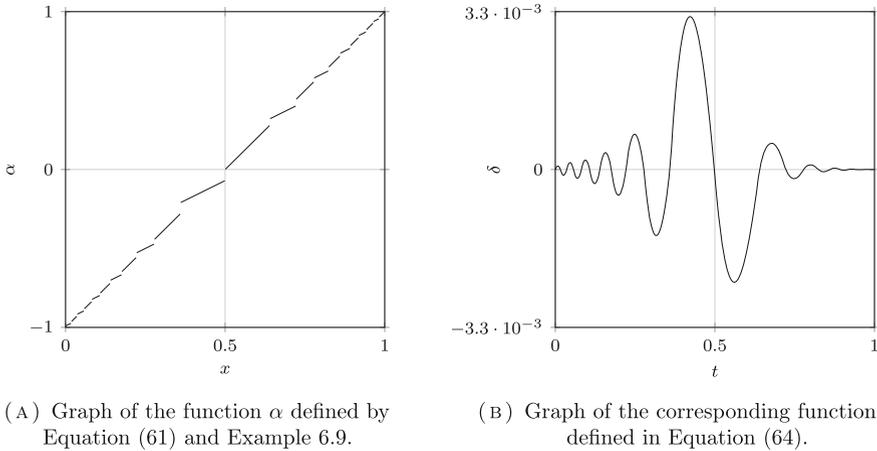


Fig. 10 Plots of the functions of interest in Sect. 6.5

This proves that the integral of α over $[0, 1)$ is equal to 0. Of course, $\sup_{[0,1)} \alpha^2 = 1 = R_0$. Hence, α satisfies Condition (49).

We recall that a function $\gamma : [0, c_\star] \mapsto \Delta$ is a parametrization of the Pareto frontier if for all $c \in [0, c_\star]$ the strategy $\gamma(c)$ is Pareto optimal with cost $C(\gamma(c)) = c$. Now we can prove there exists no greedy parametrization of the Pareto frontier of the kernel k^- and even impose an arbitrary large lower bound for the number of discontinuities.

Proposition 6.7 *Let $N \in \llbracket 2, +\infty \rrbracket$. Consider the kernel $k^- = 1 - \alpha \otimes \alpha$ from (48) on $\Omega = [0, 1)$ endowed with its Lebesgue measure, with α given by (61). Then, any parametrization of the Pareto frontier has at least $2N - 2$ and at most $20N - 2$ discontinuities.*

The proof is given at the end of this section, and relies on the following technical lemma based on the comparison of the following monotone paths γ_0 and γ_1 from $[0, 1]$ to Δ :

$$\gamma_0(t) = \mathbb{1}_{[0,t)}, \quad \text{and} \quad \gamma_1(t) = \mathbb{1}_{[1-t,1)}, \quad t \in [0, 1] \tag{63}$$

which parameterizes S_0 and S_1 as $\gamma_0([0, 1]) = S_0$ and $\gamma_1([0, 1]) = S_1$. Notice that strategies $\gamma_0(t)$ and $\gamma_1(t)$ have the same cost $1 - t$.

Consider the function $\delta : [0, 1] \rightarrow \mathbb{R}$ which, according to Proposition 6.2, measures the difference between the effective reproduction numbers at the extreme strategies:

$$\delta(t) = R_e(\gamma_0(t)) - R_e(\gamma_1(t)). \tag{64}$$

The function δ is continuous and $\delta(0) = \delta(1) = 0$; see for example Fig. 10B for its graph when α is taken from Example 6.9. We say that $t \in (0, 1)$ is a zero crossing of δ if $\delta(t) = 0$ and there exists $\varepsilon > 0$ such that $\delta(t+r)\delta(t-r) < 0$ for all $r \in (0, \varepsilon)$. The following result gives some information on the zeros of the function δ .

Lemma 6.8 *Under the assumptions of Proposition 6.7, the function δ defined in (64) has at least $2N - 2$ zero-crossings in $(0, 1)$ and at most $20N$ zeros in $[0, 1]$. Besides, if $N = \infty$, 0 and 1 are the only accumulation points of the set of zeros of δ .*

Proof Using the explicit representation of R_e^- from Lemma 6.5, see (50) with $\varepsilon = -$, we get the function δ can be expressed as:

$$2\delta(t) = V_1(t) - V_0(t) + \sqrt{V_0(t)^2 - M_0(t)^2} - \sqrt{V_1(t)^2 - M_1(t)^2}, \tag{65}$$

where, as $\int \alpha \, d\mu = 0$:

$$M_0(t) = 2 \int_0^t \alpha \, d\mu, \quad V_0(t) = t + \int_0^t \alpha^2 \, d\mu, \quad M_1(t) = M_0(1 - t) \quad \text{and} \quad V_1(t) = t + \int_{1-t}^1 \alpha^2 \, d\mu.$$

Elementary computations give that for all $n \in \llbracket 0, N \rrbracket$:

$$\int_{x_n}^{x_{n+1}} \alpha(x)^2 \, dx - \int_{x_{n-1}}^{x_n} \alpha(x)^2 \, dx = \frac{(-1)^n p_n^3}{4}, \tag{66}$$

where we recall that $p_n = x_{n+1} - x_n$. Hence, we obtain that for all $n \in \llbracket -N, N \rrbracket$:

$$V_1(x_n) - V_0(x_n) = \frac{1}{4} \sum_{i=|n|}^{\infty} (-1)^i p_i^3. \tag{67}$$

Since the sequence $(p_n, n \in \llbracket 0, N \rrbracket)$ is decreasing, we deduce that the sign of $V_1(x_n) - V_0(x_n)$ alternates depending on the parity of $n \in \llbracket -N, N \rrbracket$: it is positive for odd n and negative for even n . The same result holds for the numbers $\delta(x_n)$ since $M_0(x_n) = M_1(1 - x_n)$ for all $n \in \llbracket -N, N \rrbracket$ according to (62) (use that, with $b > 0$, the function $x \mapsto x - \sqrt{x^2 - b^2}$ is decreasing for $x \geq \sqrt{b}$ as its derivative is negative). This implies that δ has at least $2N - 2$ zero-crossings in $(0, 1)$.

We now prove that δ has at most $20N$ zeros in $[0, 1]$ and that 0 and 1 are the only possible accumulation points of the set of zeros of δ . It is enough to prove that δ has at most 10 zeros on $[x_n, x_{n+1}]$ for all finite $n \in \llbracket -N, N \rrbracket$. On such an interval $[x_n, x_{n+1}]$, the function α is a polynomial of degree one. Consider first n odd and non-negative, so that for $t \in [x_n, x_{n+1}]$, we get that with $a = 1 - (x_n + x_{n+1})/2$:

$$\begin{aligned} M_0(t) &= 2t^2 - 2t + b_1, & V_0(t) &= \frac{4}{3}t^3 - 2t^2 + 2t + b_2, \\ M_1(t) &= t^2 - 2at + b_3, & V_1(t) &= -\frac{1}{3}t^3 + at^2 + (1 - a^2)t + b_4, \end{aligned}$$

where b_i are constants. If t is a zero of δ , then it is also a zero of the polynomial P given by:

$$P = 4(V_1 - V_0) \left(V_0 M_1^2 - V_1 M_0^2 \right) - \left(M_0^2 - M_1^2 \right)^2.$$

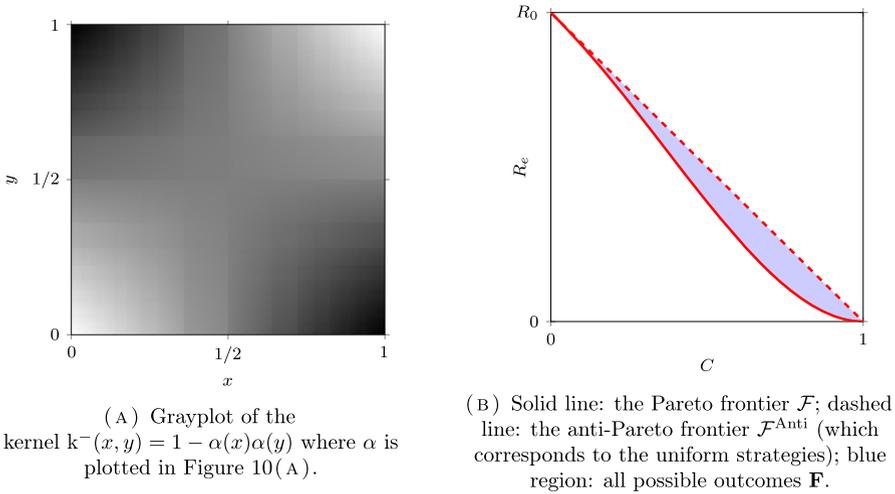


Fig. 11 An example of a constant degree kernel operator of rank two

Since the degree of P is exactly 10, it has at most 10 zeros. Thus δ has at most 10 zeros on $[x_n, x_{n+1}]$. This ends the proof. \square

Proof of Proposition 6.7 According to Proposition 6.2, the only possible Pareto strategies of cost $c = 1 - t \in [0, 1]$ are $\gamma_0(t)$ and $\gamma_1(t)$, and only one of them is optimal when $\delta \neq 0$. A zero crossing of the function δ on $(0, 1)$ therefore corresponds to a discontinuity of any parametrization of the Pareto frontier. We deduce from Lemma 6.8 that in $(0, 1)$ there are at least $2N - 2$ and at most $20N - 2$ zeros crossing and thus discontinuities of any parametrization of the Pareto frontier. \square

Example 6.9 In Fig. 10A, we have represented the function α defined by (61) where:

$$x_n = \frac{1}{2} \log_{12}(12(n + 1)), \quad 0 \leq n \leq N = 11.$$

Hence, the mesh $(x_n, -N \leq n \leq N)$ is composed by $2N + 1 = 23$ points. The graph of the corresponding function δ defined in (64) is drawn in Fig. 10B. The graph of the kernel $k^- = 1 - \alpha \otimes \alpha$ is given in Fig. 11A and the associated Pareto and anti-Pareto frontiers are plotted in Fig. 11B.

7 Geometric kernels on the sphere

A geometric random graph is an undirected graph constructed by assigning a random point in a latent metric space to each node and by connecting two nodes according to a certain probability that depends on the distance between their latent point. Because of its geometric structure, this model is appealing for a wide-range of applications such as wireless networks modelling (Hekmat and Van Mieghem 2003), social networks

(Hoff et al. 2002) and biological networks (Higham et al. 2008). A geometric random graph model can be represented as a symmetric kernel defined on the latent space (also called *graphon*) according to Lovász (2012).

In this section, we focus our study on the latent space given by the unit sphere. In Sect. 7.1 we present the mathematical model, and give in Sect. 7.2 sufficient conditions on the kernel for uniform strategies to be Pareto or anti-Pareto optimal. Section 7.3 is devoted to the explicit descriptions of the Pareto and anti-Pareto optimal vaccination strategies in the affine case.

7.1 The model

Let $d \geq 2$. Let $\Omega = \mathbb{S}^{d-1}$ be the unit sphere of the Euclidean d -dimensional space \mathbb{R}^d endowed with the usual Borel σ -field and the uniform probability measure μ . Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product on \mathbb{R}^d and let

$$\delta(x, y) = \arccos(\langle x, y \rangle)$$

denote the geodesic distance between $x, y \in \mathbb{S}^{d-1}$. By symmetry, the distribution on $[-1, 1]$ of the scalar product of two independent uniformly distributed random variables in \mathbb{S}^{d-1} is equal to the distribution of the first coordinate of a uniformly distributed unit vector: it is the probability measure on $[-1, 1]$ with density with respect to the Lebesgue measure proportional to the function w_d defined on $[-1, 1]$ by:

$$w_d(t) = (1 - t^2)^{(d-3)/2} \mathbb{1}_{(-1,1)}(t).$$

In particular, we deduce from the Funk-Heck formula (take $n = 0$ in (Dai and Xu 2013, Theorem 1.2.9) that for any non-negative measurable function h defined on $[-1, 1]$ and $x \in \mathbb{S}^{d-1}$, we have:

$$\int_{\mathbb{S}^{d-1}} h(\langle x, y \rangle) \mu(dy) = c_d \int_{-1}^1 h(t) w_d(t) dt \quad \text{with} \quad c_d = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})\sqrt{\pi}}. \tag{68}$$

We consider a symmetric kernel k on \mathbb{S}^{d-1} corresponding to a geometric random graph model on \mathbb{S}^{d-1} , given by:

$$k(x, y) = p(\langle x, y \rangle) = f \circ \delta(x, y), \quad x, y \in \mathbb{S}^{d-1}, \tag{69}$$

where $p : [-1, 1] \rightarrow \mathbb{R}_+$ is a measurable function and $f = p \circ \cos : [0, \pi] \rightarrow \mathbb{R}_+$. We assume that k has finite double-norm on L^2 ; thanks to (68), this is equivalent to:

$$\int_{-1}^1 p(t)^2 w_d(t) dt = \int_0^\pi f(\theta)^2 \sin(\theta)^{d-2} d\theta < \infty. \tag{70}$$

By symmetry, using that the scalar product and the measure μ are invariant by rotations, we deduce that the kernel k is a constant degree kernel. According to (45) and using (68), we get that the basic reproduction number is given by:

$$R_0 = c_d \int_{-1}^1 p(t) w_d(t) dt = c_d \int_0^\pi f(\theta) \sin(\theta)^{d-2} d\theta. \tag{71}$$

By (Dai and Xu 2013, Theorem 1.2.9), the eigenvectors of the symmetric operator T_k on $L^2(\mathbb{S}^{d-1})$ are the spherical harmonics, and in particular, they don't depend on the function p . We recall the linear subspace of spherical harmonics of degree n for $n \in \mathbb{N}$ has dimension d_n given by $d_0 = 1$ and for $n \in \mathbb{N}^*$:

$$d_n = \frac{2n + d - 2}{n + d - 2} \binom{n + d - 2}{n}.$$

The corresponding eigenvalues $(\lambda_n, n \in \mathbb{N})$ are real and given by:

$$\lambda_n = c_d \int_{-1}^1 p(t) \frac{G_n(t)}{G_n(1)} w_d(t) dt = c_d \int_0^\pi f(\theta) \frac{G_n(\cos(\theta))}{G_n(1)} \sin(\theta)^{d-2} d\theta, \tag{72}$$

where G_n is the Gegenbauer polynomial of degree n and parameter $(d - 2)/2$ (see (Dai and Xu 2013, Section B.2) with $G_n = C_n^{(d-1)/2}$). We simply recall that $G_0 = 1$ and that for $d = 2$, the Gegenbauer polynomials are, up to a multiplicative constant, the Chebyshev polynomials of the first kind:

$$G_n(\cos(\theta)) = \frac{2}{n} \cos(n\theta) \quad \text{for } \theta \in [0, \pi] \text{ and } n \in \mathbb{N}^*;$$

and that for $d \geq 3, r \in (-1, 1)$ and $\theta \in [0, \pi]$:

$$\sum_{n=0}^\infty r^n G_n(\cos(\theta)) = (1 + r^2 - 2r \cos(\theta))^{-(d-2)/2} \quad \text{and} \quad G_n(1) = \binom{n + d - 3}{n} \quad \text{for } n \in \mathbb{N}^*.$$

Thus, if $\lambda \neq 0$ is an eigenvalue of T_k , then its multiplicity is the sum of all the dimensions d_n such that $\lambda_n = \lambda$. The eigenvalue R_0 (associated to the eigenvector $\mathbb{1}$ which is the spherical harmonic of degree 0) is in fact simple according to the next Lemma.

Lemma 7.1 *Let k be a kernel on \mathbb{S}^{d-1} given by (69), with finite double-norm and such that $R_0 > 0$. Then the kernel k is constant degree and irreducible, and its eigenvalue R_0 is simple.*

Proof The kernel k is trivially a constant degree kernel. Since $d_0 = 1$, we only need to prove that $\lambda_n < \lambda_0 = R_0$ for all $n \in \mathbb{N}^*$ to get that R_0 is simple, and then use Lemma 5.12 to get that k is irreducible.

According to (Abramowitz and Stegun 1972, Equation 22.14.2) or (Atkinson and Han 2012, Section 3.7.1), we get that $|G_n(t)| \leq G_n(1)$ for $t \in [-1, 1]$. Since G_n is

a polynomial, the inequality is strict for a.e. $t \in [-1, 1]$. Using (72), we obtain that $\lambda_n < \lambda_0$ for all $n \in \mathbb{N}^*$. □

Example 7.2 (*The circle: $d = 2$*) In case $d = 2$, we identify the circle \mathbb{S}^1 with $\Omega = \mathbb{R}/2\pi\mathbb{Z}$ and the scalar product $\langle \theta, \theta' \rangle = \cos(\theta - \theta')$. The kernel k from (69) is the convolution kernel given by $k(\theta, \theta') = p(\cos(\theta - \theta')) = f(\theta - \theta')$, where f is symmetric non-negative and 2π periodic and its restriction to $[0, \pi]$ is square integrable. Then, we can consider the development in $L^2([0, \pi])$ of f as a Fourier series:

$$f(\theta) = \sum_{n=0}^{\infty} a_n(f) \cos(n\theta), \quad \theta \in [0, \pi], \tag{73}$$

where:

$$a_0(f) = \frac{1}{\pi} \int_0^\pi f(\theta) \, d\theta \quad \text{and} \quad a_n(f) = \frac{2}{\pi} \int_0^\pi \cos(n\theta) f(\theta) \, d\theta \quad \text{for } n \geq 1. \tag{74}$$

It follows from Equation (73) that the kernel has the following decomposition in $L^2([0, 2\pi]^2)$:

$$k(\theta, \theta') = a_0(f) + \sum_{n=1}^{\infty} a_n(f) (\cos(n\theta) \cos(n\theta') + \sin(n\theta) \sin(n\theta')), \quad \theta, \theta' \in [0, 2\pi). \tag{75}$$

Assume that $a_0(f) > 0$, that is, f is non-zero. Then, the spectral radius $R_0 = a_0(f)$ is an eigenvalue with multiplicity one associated to the eigenfunction $\mathbb{1}$ (and thus k is a constant degree kernel). The other eigenvalues are given by $\lambda_n = a_n(f)/2$ for all $n \geq 1$ and, when non zero and distinct, have multiplicity 2.

7.2 Sufficient condition for convexity or concavity

We would like to provide conditions on the function f or p that ensure that the eigenvalues $(\lambda_n, n \geq 1)$ given by (72) of the operator T_k with the kernel k defined by (69) are all non-negative or all non-positive so that R_e is convex or concave according to Corollary 5.5. Schoenberg’s theorem, see (Dai and Xu 2013, Theorem 14.3.3) or (Gneiting 2013, Theorem 1), characterizes the continuous function f such that the kernel k is positive semi-definite (and thus the eigenvalues $(\lambda_n, n \geq 1)$ are all non-negative) as those with non-negative Gegenbauer coefficients: $f = \sum_{n=0}^{\infty} a_n G_n$, where the convergence is uniform on $[-1, 1]$, with $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} a_n G_n(1)$ finite. When $d = 2$, this corresponds to the Böchner theorem. We refer to Gneiting (2013) and references therein for some characterization of functions f such that the kernel k from (69) is definite positive. We end this section with some examples.

Example 7.3 We give an elementary example in the setting of Example 7.2 when $d = 2$. Set

$$f_+(\theta) = (\pi - \theta)^2 \quad \text{and} \quad f_-(\theta) = \pi^2 - (\pi - \theta)^2 \quad \text{for } \theta \in [0, \pi].$$

We can compute the Fourier coefficients of f_+ and f_- as:

$$(\pi - \theta)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(n\theta), \quad \theta \in [0, \pi].$$

Using Corollary 5.5 and (Delmas et al. 2021a, Theorem 4.10), we deduce that the function R_e associated to the convolution kernel $k = f_+ \circ \delta$ is convex and $\mathcal{S}^{\text{uni}} \subset \mathcal{P}$; whereas the function R_e associated to the convolution kernel $k = f_- \circ \delta$ is concave and $\mathcal{S}^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$.

Example 7.4 (Kernel from a completely monotone function) Let φ be a continuous non-negative function defined on \mathbb{R}_+ , such that φ is completely monotone, that is, φ is infinitely differentiable on $(0, +\infty)$ and $(-1)^n \varphi^{(n)} \geq 0$ on $(0, +\infty)$ for all $n \geq 1$. Using (Gneiting 2013, Theorem 7), we get that the geometric kernel $k = f \circ \delta$ on \mathbb{S}^{d-1} , with $d = 2$, where $f = \varphi_{[0, \pi]}$ is positive definite (thus all the eigenvalues of T_k are non-negative). Thanks to Corollary 5.5 and (Delmas et al. 2021a, Theorem 4.10), we deduce that R_e is convex and the uniform strategies are Pareto optimal: $\mathcal{S}^{\text{uni}} \subset \mathcal{P}$.

Example 7.5 (Kernel from a Bernstein function) Let φ be a Bernstein function, that is a non-negative C^1 function defined on \mathbb{R}_+ such that $\varphi^{(1)}$ is completely monotone. Assume furthermore that $\sup_{\mathbb{R}_+} \varphi < \infty$. This gives that the function $t \mapsto (\sup_{\mathbb{R}_+} \varphi) - \varphi(t)$ defined on \mathbb{R}_+ is continuous non-negative and completely monotone. Consider the geometric kernel $k = f \circ \delta$ on \mathbb{S}^{d-1} , with $d = 2$, where $f = \varphi_{[0, \pi]}$. We deduce from (Gneiting 2013, Theorem 7), see also the previous example, that all the eigenvalues of the integral operator T_k , but for R_0 , are non-positive. Then, using Corollary 5.5 and (Delmas et al. 2021a, Theorem 4.10), we get that R_e is concave and the uniform strategies are anti-Pareto optimal: $\mathcal{S}^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$.

Example 7.6 (Kernel from a power function) Let $m \geq 1$ be an integer and $\theta_0 \geq \pi$ a real number. Using (Gneiting 2013, Lemma 4), we get that for $f(\theta) = (\theta_0 - \theta)^m$, R_e is convex and the uniform vaccination strategies are Pareto optimal; and that for $f(\theta) = \theta_0^m - (\theta_0 - \theta)^m$, R_e is concave and the uniform strategies are anti-Pareto optimal.

Example 7.7 (The function p is a power series) According to (Gneiting 2013, Theorem 1), if the function p can be written as $p(t) = \sum_{n \in \mathbb{N}} b_n t^n$ with b_n non-negative and $\sum_{n \in \mathbb{N}} b_n$ finite, then, for all $d \geq 2$, the kernel k defined by (69) on \mathbb{S}^{d-1} is semi-definite positive (and definite positive if the coefficients b_n are positive for infinitely many even and infinitely many odd integers n), and thus the function R_e is convex and the uniform vaccination strategies are Pareto optimal thanks to Corollary 5.5 and (Delmas et al. 2021a, Theorem 4.10).

Example 7.8 (The kernel is a power of the metric) Consider the function $p(t) = 2^{\nu/2}|1 - t|^{\nu/2}$, with $\nu > (1 - d)/2$, so that condition (70) holds. This corresponds to the kernel $k(x, y) = |x - y|^\nu$ which is a power of the distance between x and y . According to (Atkinson and Han 2012, Section 3.7.1) and Equation (3.74) therein, for $n \geq 1$, the eigenvalues λ_n have the same sign as $\prod_{k=0}^{n-1} (-\nu + 2k)$. So, we deduce that for $\nu \in ((1 - d)/2, 0)$ all the eigenvalues are positive and thus R_e is convex and the uniform vaccination strategies are Pareto optimal; and for $\nu \in (0, 2)$ all the eigenvalues (but $\lambda_0 = R_0 > 0$) are negative and thus R_e is concave and the uniform strategies are anti-Pareto optimal. The latter case is also a consequence of (Gneiting 2013, Theorem 1), whereas the former case is not a direct consequence of (Gneiting, 2013, Theorem 1) as $\sum_{n \in \mathbb{N}} b_n$ is not finite when ν is negative.

7.3 The affine model

Recall $\Omega = \mathbb{S}^{d-1} \subset \mathbb{R}^d$, with $d \geq 2$, is endowed with the uniform probability measure μ . In this section, we suppose that the model is affine, that is, the kernel k given by (69), i.e. $k(x, y) = p(\langle x, y \rangle)$, has a linear envelope:

$$p(t) = a + bt \quad \text{for } t \in [-1, 1].$$

The kernel k being non-negative non-constant with $R_0 > 0$ is equivalent to the condition $a \geq |b| > 0$ on the parameter (a, b) . This model corresponds to $f(\theta) = a + b \cos(\theta)$ for $\theta \in [0, \pi]$. Since the Gegenbauer polynomials $(G_n, n \in \mathbb{N})$ are orthogonal with respect to the measure $w_d(t) dt$, we easily deduce from (72) that the non-zero eigenvalues of the integral operator T_k are $R_0 = a$ (with multiplicity $d_0 = 1$) and $\lambda_1 = b/d$ (with multiplicity $d_1 = d$).

For $x \in \mathbb{S}^{d-1}$ and $t \in [-1, 1]$, we consider the following balls centered at x :

$$B(x, t) = \{y \in \mathbb{S}^{d-1} : \langle x, y \rangle \geq t\}.$$

Recall that strategies are defined up to the a.s. equality. We consider the following sets of extremal strategies, for $x \in \mathbb{S}^{d-1}$:

$$\mathcal{S}^{\text{balls}} = \left\{ \mathbb{1}_{B(x,t)} : x \in \mathbb{S}^{d-1}, t \in [-1, 1] \right\},$$

as well as the following set of strategies which contains the set of uniform strategies $\mathcal{S}^{\text{uni}} = \{t\mathbb{1} : t \in [0, 1]\}$:

$$\mathcal{S}^{\text{Lid}} = \left\{ \eta \in \Delta : \int_{\mathbb{S}^{d-1}} x \eta(x) \mu(dx) = 0 \right\}.$$

Proposition 7.9 Let $a \geq |b| > 0$ and the kernel k on \mathbb{S}^{d-1} , with $d \geq 2$, be given by:

$$k(x, y) = a + b\langle x, y \rangle.$$

(i) **The case $b > 0$.** A strategy is Pareto optimal if and only if it belongs to $\mathcal{S}^{\perp\text{id}}$. In particular, for any $c \in [0, 1]$, the strategy $(1 - c)\mathbb{1}$ costs c and is Pareto optimal. The anti-Pareto optimal strategies are $\mathbb{1}_{B(x,t)}$ for $x \in \mathbb{S}^{d-1}$ and $t \in [-1, 1]$. In other words:

$$\mathcal{P} = \mathcal{S}^{\perp\text{id}} \quad \text{and} \quad \mathcal{P}^{\text{Anti}} = \mathcal{S}^{\text{balls}}.$$

(ii) **The case $b < 0$.** A strategy is anti-Pareto optimal if and only if it belongs to $\mathcal{S}^{\perp\text{id}}$. In particular, for any $c \in [0, 1]$, the strategy $(1 - c)\mathbb{1}$ costs c and is anti-Pareto optimal. The Pareto optimal strategies are $\mathbb{1}_{B(x,t)}$ for $x \in \mathbb{S}^{d-1}$ and $t \in [-1, 1]$. In other words:

$$\mathcal{P}^{\text{Anti}} = \mathcal{S}^{\text{balls}} \quad \text{and} \quad \mathcal{S}^{\text{uni}} = \mathcal{S}^{\perp\text{id}}.$$

In both cases, we have $c_\star = 1$ and $c^\star = 0$.

Example 7.10 We consider the kernel $k = 1 + b\langle \cdot, \cdot \rangle$ on the sphere \mathbb{S}^{d-1} , with $d = 2$. This model has the same Pareto and anti-Pareto frontiers as the equivalent model given by $\Omega = [0, 1)$ endowed with the Lebesgue measure and the kernel $(x, y) \mapsto 1 + b \cos(\pi(x - y))$, where the equivalence holds in the sense of (Delmas et al. 2021b, Section 7), with an obvious deterministic coupling $\theta \mapsto \exp(2i\pi\theta)$. We provide the Pareto and anti-Pareto frontiers in Fig. 12 with $b = 1$ (top) and with $b = -1$ (bottom).

Proof The proof of Proposition 7.9 is decomposed in four steps. *Step 1:* $R_e(\eta)$ is the eigenvalue of a 2×2 matrix $M(\eta)$. Without loss of generality, we shall assume that $R_0 = a = 1$. Since k is positive a.s., we deduce that $c_\star = 1$ and $c^\star = 0$ thanks to Lemma 3.1; and the strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost 0 (resp. 1). So we shall only consider strategies $\eta \in \Delta$ such that $C(\eta) \in (0, 1)$.

Let $z_0 \in \mathbb{S}^{d-1}$. Write $b = \varepsilon\lambda^2$ with $\varepsilon \in \{-1, +1\}$ and $\lambda \in (0, 1]$, and define the function α on \mathbb{S}^{d-1} by:

$$\alpha = \lambda \langle \cdot, z_0 \rangle.$$

Let $\eta \in \Delta$ with cost $c \in (0, 1)$. As $c_\star = 1 > C(\eta)$, we get that $R_e(\eta) > 0$. We deduce from the special form of the kernel k that the eigenfunctions of $T_{k\eta}$ are of the form $\zeta + \beta\lambda\langle \cdot, y \rangle$ with $\zeta, \beta \in \mathbb{R}$ and $y \in \mathbb{S}^{d-1}$. Since $R_e(\eta) > 0$, the right Perron eigenfunction, say h_η , being non-negative, can be chosen such that $h_\eta = 1 + \beta_\eta\lambda\langle \cdot, y_\eta \rangle$ with $\beta_\eta \geq 0$ and $\beta_\eta\lambda \leq 1$. Up to a rotation on the vaccination strategy, we shall take $y_\eta = z_0$, that is:

$$h_\eta = 1 + \beta_\eta \alpha.$$

From the equality $R_e(\eta)h_\eta = T_{k\eta}h_\eta$, we deduce that:

$$R_e(\eta) = \int_{\mathbb{S}^{d-1}} \eta(y) \mu(dy) + \beta_\eta \lambda \int_{\mathbb{S}^{d-1}} \eta(y) \langle y, z_0 \rangle \mu(dy), \tag{76}$$

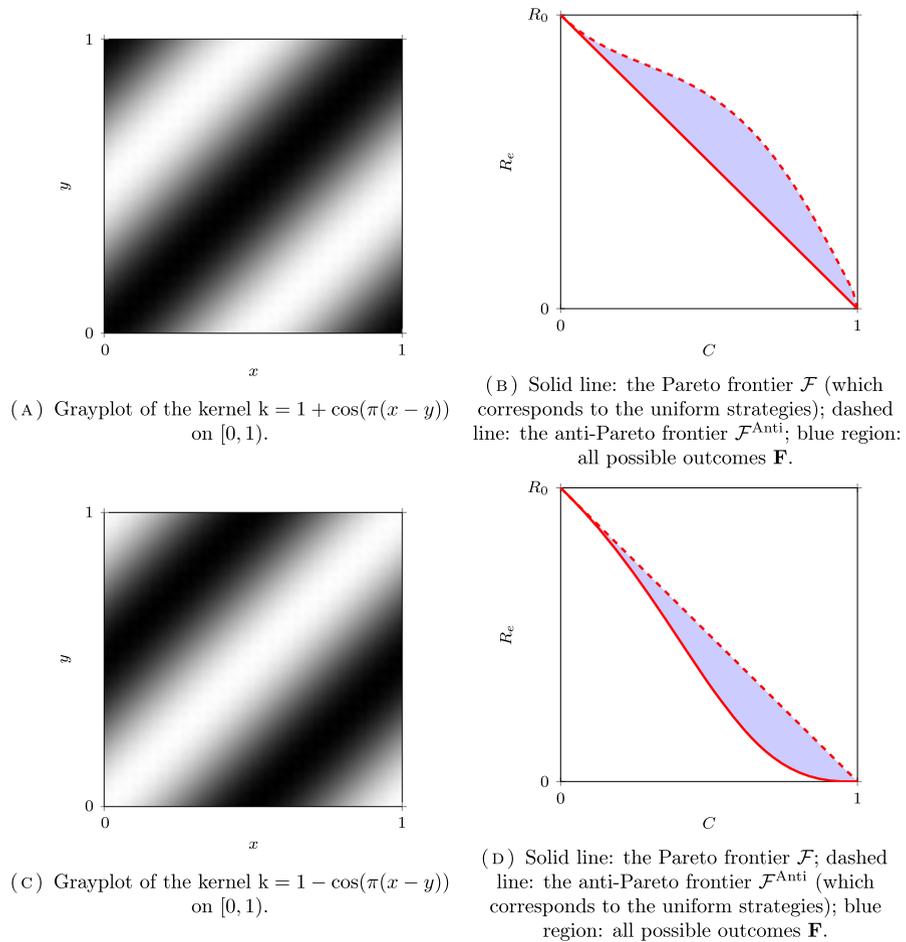


Fig. 12 Two examples of a geometric kernel on the circle $\mathbb{R} \setminus \mathbb{Z}$

$$\beta_\eta R_e(\eta)\langle \cdot, z_0 \rangle = \varepsilon \lambda \int_{\mathbb{S}^{d-1}} \eta(y) \langle \cdot, y \rangle \mu(dy) + \beta_\eta \varepsilon \lambda^2 \int_{\mathbb{S}^{d-1}} \eta(y) \langle \cdot, y \rangle \langle y, z_0 \rangle \mu(dy). \tag{77}$$

Evaluating the latter equality at $x = z_0$, we deduce that $R_e(\eta)$ is a positive eigenvalue of the matrix $M(\eta)$ associated to the eigenvector $(1, \beta_\eta)$, where:

$$M(\eta) = \begin{pmatrix} \int \eta \, d\mu & \int \alpha \eta \, d\mu \\ \varepsilon \int \alpha \eta \, d\mu & \varepsilon \int \alpha^2 \eta \, d\mu \end{pmatrix}. \tag{78}$$

We end this step by proving the following equivalence:

$$\beta_\eta = 0 \iff \int \alpha \eta \, d\mu = 0. \tag{79}$$

Indeed, if $\beta_\eta = 0$, then the vector $(1, 0)$ is an eigenvector of $M(\eta)$ associated to the eigenvalue $R_e(\eta)$. We deduce from (78) that $\int \alpha \eta \, d\mu = 0$. Conversely, if $\int \alpha \eta \, d\mu = 0$, then the matrix $M(\eta)$ is diagonal with eigenvalues $\int \eta \, d\mu$ and $\int \alpha^2 \eta \, d\mu$. As $\alpha^2 \leq 1$ with strict inequality on a set of positive μ -measure, we deduce that:

$$\int \eta \, d\mu > \int \alpha^2 \eta \, d\mu. \tag{80}$$

Since $(1, \beta_\eta)$ is an eigenvector of $M(\eta)$, this implies that $\beta_\eta = 0$. This proves (79).

Step 2: $R_e(\eta)$ is the spectral radius of the matrix $M(\eta)$, that is, $R_e(\eta) = \rho(M(\eta))$. We first consider the case $\varepsilon = -1$. Since α is non constant as $\lambda > 0$, we deduce from the Cauchy-Schwarz inequality, that the determinant of $M(\eta)$ is negative. As $c_\star = 1$ a.s., we deduce that $R_e(\eta) > 0$, and thus the other eigenvalue is negative. Since $\alpha^2 \leq 1$, the trace of $M(\eta)$ is non-negative, thus $R_e(\eta)$ is the spectral radius of the matrix $M(\eta)$.

We now consider the case $\varepsilon = +1$. Let η^{uni} be the uniform strategy with the same cost as η . Thanks to (76), we get $R_e(\eta^{\text{uni}}) = \int \eta^{\text{uni}} \, d\mu = \int \eta \, d\mu$. Since the non-zero eigenvalues of T_k , that is, 1 and λ^2/d , are positive, we deduce from Corollary 5.5 (i), that the uniform strategies are Pareto optimal ($\mathcal{S}^{\text{uni}} \subset \mathcal{P}$), so we have:

$$R_e(\eta) \geq R_e(\eta^{\text{uni}}) = \int \eta \, d\mu.$$

We deduce from (76) that $\beta_\eta \int \alpha \eta \, d\mu \geq 0$.

On the one hand, if $\beta_\eta \int \alpha \eta \, d\mu = 0$, then, by (79), the matrix $M(\eta)$ is diagonal. Using (80), we obtain that $R_e(\eta) = \rho(M(\eta))$. On the other hand, if $\beta_\eta \int \alpha \eta \, d\mu > 0$, then the matrix $M(\eta)$ has positive entries. Since the eigenvector $(1, \beta_\eta)$ also has positive entries, it is the right Perron eigenvector and the corresponding eigenvalue is the spectral radius of $M(\eta)$, that is, $R_e(\eta) = \rho(M(\eta))$. To conclude, the equality $R_e(\eta) = \rho(M(\eta))$ holds in all cases.

Step 3: $R_e(\eta) = \int \eta \, d\mu \iff \eta \in \mathcal{S}^{\perp \text{id}}$. Let $\eta \in \Delta$ such that $R_e(\eta) = \int \eta \, d\mu$. We deduce from (76) that $\beta_\eta \int \alpha \eta \, d\mu = 0$. Thanks to (79), this implies that $\beta_\eta = 0$. Using (77), we obtain that $\int y\eta(y) \, \mu(dy) = 0$ and thus $\eta \in \mathcal{S}^{\perp \text{id}}$. Clearly if $\eta \in \mathcal{S}^{\perp \text{id}}$, we deduce from (76) that $R_e(\eta) = \int \eta \, d\mu$.

As a consequence and since $\mathcal{S}^{\text{uni}} \subset \mathcal{S}^{\perp \text{id}}$, we deduce from Corollary 5.5 that if $\varepsilon = +1$, then $\mathcal{S}^{\text{uni}} \subset \mathcal{P}$ and thus $\mathcal{P} = \mathcal{S}^{\perp \text{id}}$; and that if $\varepsilon = -1$, then $\mathcal{S}^{\text{uni}} \subset \mathcal{P}^{\text{Anti}}$ and thus $\mathcal{P}^{\text{Anti}} = \mathcal{S}^{\perp \text{id}}$.

Step 4: A relation with the constant degree symmetric kernels of rank two from Sect. 6. This step is in the spirit of (Delmas et al. 2021b, Section 7) on coupled models. Let X be a uniform random variable on \mathbb{S}^{d-1} . Let $\Omega_0 = [-1, 1]$ endowed with the probability measure $\mu_0(dt) = c_d w_d(t) \, dt$, and set Δ_0 the set of $[0, 1]$ -valued measurable functions defined on Ω_0 . According to (Kallenberg 2021, Theorem 8.9), there exists $\eta_0 \in \Delta_0$ such that:

$$\eta_0(\langle X, z_0 \rangle) = \mathbb{E}[\eta(X) \mid \langle X, z_0 \rangle] \quad \text{a.s.} \tag{81}$$

Set $\alpha_0 = \lambda t$, and define the matrix:

$$M_0(\eta_0) = \begin{pmatrix} \int_{\Omega_0} \eta_0 \, d\mu_0 & \int_{\Omega_0} \alpha_0 \eta_0 \, d\mu_0 \\ \varepsilon \int_{\Omega_0} \alpha_0 \eta_0 \, d\mu_0 & \varepsilon \int_{\Omega_0} \alpha_0^2 \eta_0 \, d\mu_0 \end{pmatrix}.$$

By construction of η_0 , we have $M_0(\eta_0) = M(\eta)$. Thanks to Sect. 6, see Lemma 6.5 (but for the fact that Ω therein is replaced by $[-1, 1]$), we get that $M_0(\eta_0)$ is exactly the matrix in (52), and thus the spectral radius of $M_0(\eta_0)$ is the effective reproduction number of the model associated to the constant degree symmetric kernel of rank two $k_0^\varepsilon = 1 + \varepsilon \alpha_0 \otimes \alpha_0$ given in (48) (with Ω, μ, α replaced by Ω_0, μ_0 and α_0). We deduce that: if η is Pareto or anti-Pareto optimal for the model $(\mathbb{S}^{d-1}, \mu, k)$ then so is η_0 for the model $(\Omega_0, \mu_0, k_0^\varepsilon)$; and if η_0 is Pareto or anti-Pareto optimal for the model $(\Omega_0, \mu_0, k_0^\varepsilon)$, so is any strategy η such that (81) holds.

We first consider the case $\varepsilon = +1$. According to Proposition 6.2, we get that the anti-Pareto optimal strategies are $\eta_0 = \mathbb{1}_{[-1,t]}$ or $\eta_0 = \mathbb{1}_{[-t,1]}$ for a given cost c (with t uniquely characterized by c). Using that $0 \leq \eta \leq 1$, we deduce that the only possible choice for η such that (81) holds is to take $\eta = \mathbb{1}_{B(-z_0,t)}$ or $\eta = \mathbb{1}_{B(z_0,t)}$. Since z_0 was arbitrary, we get that the only possible anti-Pareto optimal strategies belong to $\mathcal{S}^{\text{balls}}$. Notice that anti-Pareto optimal strategies exist for all cost $c \in [0, 1]$ as $k > 0$ a.s., see Lemma 3.1 and (Delmas et al. 2021b, Section 5.4) for irreducible kernels, loss function R_e and uniform cost function C given by (28). Since the set of anti-Pareto optimal strategies is also invariant by rotation, we deduce that $\mathcal{P}^{\text{Anti}} = \mathcal{S}^{\text{balls}}$.

The case $\varepsilon = -1$ is similar and thus $\mathcal{P} = \mathcal{S}^{\text{balls}}$ in this case. (Note that the irreducibility of the kernel k is only used in (Delmas et al. 2021b, Lemma 5.13) for the study of anti-Pareto frontier.) □

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