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Sujet : Généalogie et Q-processus

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Généalogie et Q-processus

Résumé :

Cette thèse étudie le Q-processus de certains processus de branchement (superprocessus inhomogènes) ou de recombinaison (processus de Λ-Fleming-Viot) via une approche généalogique. Dans le premier cas, le Q-processus est défini comme le processus conditionné à la non-extinction, dans le second cas comme le processus conditionné à la non-absorption. Des constructions trajectorielles des Q-processus sont proposées dans les deux cas. Une nouvelle relation entre superprocessus homogènes et processus de Λ-Fleming-Viot est établie. Enfin, une étude du Q-processus est menée dans le cadre général des processus régénératifs.

Plus précisément, pour un superprocessus inhomogène, qui satisfait la propriété d'extinction presque sûre, le Q-processus est le processus conditionné à la non extinction en temps long. Nous le construisons grâce à une décomposition en épine dorsale appelée décomposition de Williams. Ceci forme le second chapitre, qui résulte d'un travail joint avec Jean-François Delmas.

Pour un processus de recombinaison, connu sous le nom de processus de Λ-Fleming-Viot, qui satisfait la propriété d'absorption presque sûre, nous définissons le Q-processus comme le processus conditionné à la non-absorption en temps long. Nous donnons une construction trajectorielle du processus conditionné via un système de particules look-down. Pour une classe de processus à valeurs mesure assez générale, comprenant à la fois superprocessus homogènes et processus de Λ-Fleming-Viot, et incluant en outre un déplacement spatial, nous représentons la h-transformée additive à l'aide d'un système de particules look-down. Ceci forme le troisième chapitre.

Pour un superprocessus de branchement homogène, basé sur un mécanisme de branchement α-stable et une immigration $(\alpha-1)$-stable, nous montrons que le processus du ratio convenablement changé de temps est encore un processus de Markov, un processus de Λ-Fleming-Viot pour $\Lambda = \text{Beta}(2-\alpha, \alpha-1)$ plus précisément. En outre, l'immigration peut être comprise comme résultant d'un conditionnement à la non extinction. Ceci forme le quatrième chapitre, et résulte d'un travail joint avec Clément Foucart.

Pour un processus régénératif, le Q-processus est défini comme le processus conditionné à ne pas admettre un certain type d'excursions (arbitraires). Le conditionnement diffère selon qu'il est effectué dans l'échelle du temps réel ou du temps local. Nous identifions les deux processus conditionnés comme deux éléments distincts d'une famille paramétrée de processus confinés, dont nous étudions les propriétés. Ce travail joint avec Stephan Gufler forme le cinquième et dernier chapitre.

Mots-clés :
Q-processus, généalogie, superprocessus, processus de Λ-Fleming-Viot, coalescent, processus régénératif, excursions.

Genealogy and Q-process

Abstract :

This work is concerned with the definition and study of the Q-process of some branching processes (inhomogeneous superprocesses) or recombination processes (Λ-Fleming-Viot process). In the first case, the Q-process is defined as the process conditioned on non-extinction, whereas in the second case, it is defined as the process conditioned on non-absorbtion. A pathwise construction of the Q-process is given in both cases. A link between a class of homogeneous superprocesses and Λ-Fleming-Viot processes is provided. Last, a study of the Q-process in the more general framework of regenerative processes is performed.

For an inhomogeneous superprocess, satisfying the almost sure extinction property, the Q-process is the process conditioned on non-extinction in remote time. We construct this process thanks to a spinal decomposition called the Williams decomposition. This corresponds to the second chapter, based on a joint work with Jean-François Delmas.

For a recombination process, called the Λ-Fleming-Viot process, satisfying the almost sure absorbtion property, we define the Q-process as the process conditioned on non-absorbtion in remote time. We give a pathwise representation of the conditioned process. For a class of measure valued processes including homogeneous superprocesses and Λ-Fleming-Viot processes, and allowing for a spatial motion, we construct a look-down particle system for the additive h-transform. This corresponds to the third chapter.

For an homogeneous superprocess, based on an α-stable branching mechanism and $(\alpha - 1)$-stable immigration, we show that the ratio process suitably time changed is still Markov, and yields a Beta$(2 - \alpha, \alpha - 1)$-Fleming-Viot process. Moreover, the immigration may be understood as resulting from conditioning on non-extinction. This builds the fourth chapter, and stems from a joint work with Clément Foucart.

For a regenerative process, we define the Q-process as the process without excursions in some prescribed set. The condititioning may be achieved in two different time scales, either the real time scale or the local time scale. The two conditioned processes are viewed as elements of a more general parametrized family of confined processes. This forms the fifth chapter, based on joint work with Stephan Gufler.

Keywords :
Q-process, genealogy, inhomogeneous superprocess, Λ-Fleming-Viot process, coalescent, regenerative process, excursions.

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Introduction générale

1.1 Résumé des travaux

Le premier article, écrit avec Jean-François Delmas,

A Williams Decomposition for Spatially Dependent Superprocesses, [31],

considère des superprocessus avec mécanisme de branchement inhomogène. Ces superprocessus constituent un modèle pour une grande population composée d'individus mobiles dans l'espace, se reproduisant aléatoirement, indépendamment mais pas identiquement : la loi de reproduction peut en effet être affectée par la position spatiale des individus. Sous l'hypothèse d'extinction presque sûre de la population, nous proposons une décomposition de la généalogie du superprocessus par rapport à la lignée ancestrale du "dernier" individu en vie, aussi appelée décomposition de Williams. Nous en déduisons une représentation du superprocessus conditionné à la non extinction en temps long, aussi appelé Q-processus dans la littérature.

Le second article,

Change of measure in the look-down particle system, [65],

considère le système de particules look-down de Donnelly et Kurtz [35] : pour une population d'individus identiques, qui ne vérifie pas nécessairement la propriété de branchement, on classe les individus en fonction de la pérennité de leur descendance. Ceci donne lieu à un système de particules dénombrable et échangeable, que l'on soumet à divers changements de mesures. Nous commençons par étudier un mod èle de population à taille constante appelé processus de Λ-Fleming-Viot. Nous supposons vérifiée la propriété d'absorption pr esque sûre, au sens où tous les membres de la population partagent le même type en temps fini. Dans ce cadre, nous définissons le Q-processus comme le processus conditionné à la non absorption en temps long. Le premier changement de mesure considéré permet de donner une construc tion trajectorielle du Q-processus basée sur la suppression de certains évènements de reproduction. Ceci est à comparer à la contructio n de Kesten d'arbres de Galton-Watson conditionnés à la non extinction, où des évènements de reproduction sont au contraire ajoutés (immi gration). En présence de mutations, nous définisson la h-transformée additive, qui correspond au second changement de mesure considéré. Nous proposons un système de particules pour la représenter. Ce système de particules confirme certains résultats d'Overbeck [103].

Le troisième article, écrit avec Clément Foucart, Stable continuous-state branching processes with immigration and Beta-Fleming-Viot processes with immigration, [55],

énonce un lien entre processus de Λ-Fleming-Viot et superprocessus en l'absence de mouvement spatial, et lorsque la reproduction est donnée par un mécanisme de branchement α stable avec immigration $(\alpha-1)$ stable. Nous montrons que le superprocessus normalisé en mesure de probabilité (processus du ratio) et convenablement changé de temps, donne lieu à un $Beta(2-\alpha, \alpha-1)$-processus de Fleming-Viot, ce qui complète les résultats de Birkner *et al.* [18] qui traitent du cas sans immigration. Nous notons que les CBI considérés ne sont autres que les CB considérés dans [18], une fois conditionnés à la non-extinction.

Le quatrième et dernier article, écrit avec Stephan Gufler,

A conditioned subordinator and the excursions of the Q-process, [61],

se place dans le cadre des processus régénératifs. Nous étudions alors le processus conditionné à ne pas admettre d'excursions dans un ensemble arbitraire. Nous observons, après Knight [76], qu'il n'est pas équivalent de conditionner le processus dans l'échelle du temps local ou du temps réel, et montrons que ces deux processus appartiennent à une famille paramétrée de processus confinés dont nous étudions les propriétés.

1.2 Les processus de branchement (CB)

1.2.1 Définition et premières propriétés

Les processus de branchement recensent le nombre d'individus au cours du temps dans une population composée d'individus identiques se reproduisant aléatoirement et indépendamment les uns des autres. Cette description permet de définir les processus de branchement dans un cadre discret : le nombre d'individus, à valeurs dans l'ensemble $\mathbb{N}$ des entiers naturels, évolue en fonction de la génération, également dans $\mathbb{N}$. L'introduction de tels processus de branchement remonte à Bienaymé [66] au milieu du 19-ième siècle, et indépendamment, à Galton et Watson [131] à la fin du 19-ième siècle, Pour de grandes populations, l'approximation continue est pertinente, et Jiřina [70] et Lamperti [87, 88] ont initié dans la seconde moitié du 20-ième siècle l'étude de processus de branchement dans un tel cadre : la taille de la population est désormais assimilée à un réel $\mathbb{R}^+$, et évolue en fonction du temps, dans $\mathbb{R}^+$ également. Les individus sont désormais de taille infinitésimale et la valeur du processus à l'instant t doit alors être comprise en rapport à la taille de la population initiale. Quelques monographies consacrées aux processus de branchement sont les suivantes : Harris [64], Athreya et Ney [5] et Jagers [69] dans un cadre discret, le chapitre 10 de Kyprianou [79] et Li [95] dans un cadre continu.

Formellement, un processus de branchement à temps et espace d'état continu (noté CB dans la suite) est un processus stochastique $X = (X_t, t \geq 0)$ à valeurs dans $[0, \infty]$ et à trajectoires càdlàg, dont la famille de lois $(\mathbb{P}_x, x \geq 0)$ (sous laquelle le processus est issu de x) satisfait la propriété de Markov par rapport à la filtration naturelle du processus ainsi que la propriété de branchement :

$$\mathbb{P}_{x+x'} = \mathbb{P}_x * \mathbb{P}_{x'}, x, x' \in \mathbb{R}^+. \tag{1.1}$$

Si l'on interprète $\mathbb{P}_{x+x'}$ comme la loi (de la taille) d'une population issue de $x+x'$, la propriété de branchement rend compte de l'indépendance des sous-populations issues de x et x' respectivement. Cette propriété modélise donc au niveau macroscopique une absence d'interactions entre individus.

La classe des processus qui satisfont à la propriété de branchement est caractérisée comme suit par Silverstein [125], voir aussi Le Gall [93], Théorème 1 du chapitre II (sous une hypothèse supplémentaire de moments).

Théorème. *Si $(X_t, t \geq 0)$ est un processus de Markov càdlàg à valeurs dans $[0,\infty]$ qui vérifie la propriété (1.1), alors*

$$\mathbb{E}_x(\mathrm{e}^{-\lambda X_t}) = \mathrm{e}^{-x u_t(\lambda)},\ \lambda \geq 0, \tag{1.2}$$

où $(u_t(\lambda), t \geq 0, \lambda \geq 0)$ est l'unique solution positive de l'équation intégrale :

$$u_t(\lambda) + \int_{(0,t)} ds\, \psi(u_s(\lambda)) = \lambda, \tag{1.3}$$

et ψ est une fonction de la forme :

$$\psi(\lambda) = \alpha\lambda^2 + \beta\lambda + \int_{(0,\infty)} \nu(dr)\,(\mathrm{e}^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r\leq 1\}}),\ \lambda \geq 0 \tag{1.4}$$

pour $\alpha \geq 0$, $\beta \in \mathbb{R}$, et ν une mesure de Radon sur $(0,\infty)$ qui satisfait $\int_{(0,\infty)}(1\wedge r^2)\nu(dr) < \infty$.

Un tel processus $(X_t, t \geq 0)$ est alors appelé processus de branchement à espace d'état continu, et noté CB(ψ).

La fonction ψ, appelée mécanisme de branchement, est convexe et infiniment différentiable sur $(0,\infty)$, et vérifie

$$\psi'(0+) = \beta - \int_{(1,\infty)} r\nu(dr) \in [-\infty, +\infty).$$

On notera que l'équation (1.3) peut encore s'écrire :

$$\int_{u_t(\lambda)}^{\lambda} \frac{ds}{\psi(s)} = t. \tag{1.5}$$

Une première remarque concerne le cas particulier des points 0 et ∞. Soit $\zeta \in \{0,\infty\}$: du fait de la propriété de branchement, $\mathbb{P}_\zeta$ est égal à la masse de Dirac en la trajectoire constante égale à ζ, puis, du fait de la propriété de Markov forte, le point ζ est un état absorbant pour X, c'est à dire que $X_s = \zeta$ pour un certain $s \geq 0$ implique $X_t = \zeta$ pour tout $t \geq s$.

De plus, partant d'une condition initiale finie, le point 0 est atteint en temps fini avec probabilité strictement positive lorsque la condition suivante, dite de Grey, est satisfaite :

$$\text{il existe } M > 0 \text{ tel que } \int_M^\infty \frac{ds}{\psi(s)} < \infty, \tag{1.6}$$

et avec probabilité 1 si de plus $\psi'(0+) \geq 0$. De même, partant d'une condition initiale non nulle, ∞ est atteint en temps fini avec probabilité strictement positive si et seulement si

$$\text{il existe } \varepsilon > 0 \text{ tel que } \int_0^\varepsilon \frac{ds}{|\psi(s)|} < \infty,$$

auquel cas le processus est dit non conservatif. Une seconde remarque concerne le calcul de l'espérance. On la calcule comme suit :

$$\begin{aligned}\mathbb{E}_x(X_t) &= \mathbb{E}_x(-\partial_{\lambda|\lambda=0}\left[\mathrm{e}^{-\lambda X_t}\right]) \\ &= -\partial_{\lambda|\lambda=0}\left[\mathrm{e}^{-xu_t(\lambda)}\right] \\ &= x\ \partial_{\lambda|\lambda=0}\left[u_t(\lambda)\right]\mathrm{e}^{-xu_t(0)}\,.\end{aligned}$$

Maintenant, si l'on pose $v(t) = \partial_{\lambda|\lambda=0}\left[u_t(\lambda)\right]$, alors

$$v(t) + \psi'(0+)\int_0^t v(s)ds = 1$$

découle de l'équation (1.3), et implique que :

$$\mathbb{E}_x(X_t) = x\ \mathrm{e}^{-t\psi'(0+)}\,. \tag{1.7}$$

Cette relation amène à classer les CB(ψ) en trois catégories distinctes, en fonction du signe de $\psi'(0+)$: si $\psi'(0+) < 0$, le processus est dit surcritique, si $\psi'(0+) = 0$, le processus est dit critique ; enfin, le processus est dit sous-critique lorsque $\psi'(0+) > 0$. Si le processus est critique ou sous-critique, X_t est une surmartingale positive et converge donc presque sûrement lorsque $t \to \infty$. Quelle est la valeur de la limite ? Le lemme de Fatou permet de conclure dans le cas sous-critique, puisque :

$$0 \le \mathbb{E}(X_\infty) = \mathbb{E}(\liminf_{t\to\infty} X_t) \le \liminf_{t\to\infty}\mathbb{E}(X_t) = x\,\mathrm{e}^{-t\psi'(0+)} \to 0 \text{ quand } t \to \infty,$$

implique $X_\infty = 0$ p.s. Mettons le cas critique en suspens pour quelques instants, le temps d'introduire la transformée de Lamperti.

1.2.2 La transformée de Lamperti

On se propose maintenant d'interpréter les différents paramètres intervenant dans l'expression du mécanisme de branchement (1.4). Pour cela, il est utile de reconnaître en ψ l'exposant de Laplace d'un processus de Lévy spectralement positif noté Y. On constate ensuite sans difficultés que le générateur infinitésimal $\mathcal{L}^X$ du CB(ψ) X et celui $\mathcal{L}^Y$ de Y agissent comme suit sur la famille des fonctions exponentielles $f_\lambda(x) = \mathrm{e}^{-\lambda x}$:

$$\mathcal{L}^X(f_\lambda)(x) = x\ \psi(\lambda)f_\lambda(x) = x\mathcal{L}^Y(f_\lambda)(x).$$

Ceci permet à Lamperti [87] d'établir, à l'aide du résultat de Volkonski [127] sur les changements de temps de processus de Markov, que les processus X et Y sont liés par un changement de temps, désormais appelé tranformation de Lamperti. Bien entendu, ce résultat ne vaut que si les deux processus sont issus du même point à l'instant initial. Les mots suivants, qui sont dûs à Lamperti, nous permettent d'appréhender la compréhension profonde qu'il avait des CB dès 1967 : "The examples [...] suggest that a CB function would be translation invariant, at least away from the absorbing state 0, if it were not for the fact that the 'local speed' of the process at x is not constant but proportional to x. It is therefore quite plausible to attempt removing this factor by means of a random time change." De nouvelles preuves de la transformation de Lamperti ont vu le jour récemment, voir Caballero, Lambert et Uribe Bravo [23]. Nous empruntons la formulation suivante à Kyprianou [79]. Nous notons $\tau_0(X) = \inf\{t > 0, X_t = 0\}$ et $\tau_0(Y) = \inf\{t > 0, Y_t = 0\}$ les premiers temps d'atteinte de 0 par X et Y respectivement.

Théorème. *Soit* $x > 0$.
– *Soit* X *un CB(*ψ*) issu de* x. *Posons* $\varphi(t) = \inf\{s \geq 0, \int_0^s ds\, X_s \geq t\}$. *Alors*

$$Y_t = X_{\varphi(t)} \text{ pour } 0 \leq t \leq \int_0^{\tau_0(X)} ds\, X_s$$

définit un processus de Lévy d'exposant de Laplace ψ*, issu de* x*, jusqu'à son temps d'atteinte de* 0.
– *Soit* Y *un processus de Lévy d'exposant de Laplace* ψ *issu de* x. *Si* $\theta(t) = \inf\{s > 0, \int_0^s \frac{ds}{Y_s} > t\}$, *avec la convention* $\inf\{\emptyset\} = \infty$*, alors :*

$$X_t = Y_{\theta(t) \wedge \tau_0(Y)} \text{ pour } t \geq 0$$

définit un CB(ψ*) issu de* x.

Notons que la donnée du processus Y jusqu'à son temps d'atteinte de 0 code bien toute la trajectoire de X puisque 0 est un état absorbant pour X. Notons également que $\tau_0(Y)$ est p.s. fini lorsque $\psi'(0+) \geq 0$, mais que le temps $\tau_0(X)$ est sous ces mêmes hypothèses :
– ou bien p.s. fini si la condition de Grey (1.6) est satisfaite.
– ou bien p.s. infini si la condition de Grey (1.6) n'est pas satisfaite.
Quelques applications de la transformée de Lamperti sont les suivantes : on obtient immédiatement que les sauts de X sont nécessairement positifs, ce qui n'est pas immédiat à la lecture de la propriété de branchement. Aussi, les trajectoires de X sont continues si et seulement si ν est identiquement nulle, auquel cas le mécanisme de branchement, et par extension le CB, sont dit quadratiques. Enfin, on peut maintenant conclure que les trajectoires d'un CB critique convergent presque sûrement vers 0 en $+\infty$.

Notons enfin qu'on rencontre deux transformations de Lamperti dans la littérature : la première du point de vue historique, que nous venons de présenter, lie CB et processus de Lévy spectralement positifs ; elle ne doit pas être confondue avec la seconde, qui lie les processus de Markov autosimilaires dans $\mathbb{R}^+$ et les processus de Lévy, voir Lamperti [89]. Un travail de Kyprianou et Pardo [80] explore le lien entre ces deux tranformations et en déduit des identités en loi pour les CB stables, qui sont sujets aux deux types de transformation.

1.2.3 La mesure canonique $\mathbb{N}$

La propriété de branchement (1.1) implique l'infinie divisibilité de la loi $\mathbb{P}_x$ du CB issu de x en tant que processus. Or, des variables aléatoires infiniment divisibles peuvent être représentées à l'aide d'une mesure de Poisson. Dans ce cas, l'intensité de la mesure de Poisson est appelée mesure canonique. Dans notre cas, cette mesure prend la forme d'une mesure sigma-finie sur l'espace des trajctoires càdlàg, et le processus de branchement est issu de 0 sous cette mesure. Nous renvoyons au cours de Perkins de Saint-Flour [104], section II.7, pour plus d'informations au sujet de la mesure canonique de variables aléatoires infiniment divisibles.

La façon la plus simple de faire apparaître la mesure canonique dans notre contexte est la suivante : de l'équation (1.2), on déduit le calcul suivant :

$$\frac{\mathbb{E}_x(1 - e^{-\lambda X_t})}{x} = \frac{1 - e^{-x u_t(\lambda)}}{x} \to u_t(\lambda) \text{ quand } x \to 0,$$

valable pour tout $\lambda \geq 0$. Ce calcul suggère l'existence d'une limite pour les mesures $\mathbb{P}_x/x$. Bien entendu, ces mesures sont de masse totale $1/x$, et nous ne sommes pas dans le cadre habituel de la convergence en loi de mesures de probabilités, comme décrite dans Billingsley [17] par exemple. La mesure limite en question est une mesure σ-finie, de masse totale infinie, traditionnellement notée $\mathbb{N}$ qui vérifie :

$$u_t(\lambda) = \mathbb{N}(1 - \mathrm{e}^{-\lambda X_t}). \tag{1.8}$$

C'est cette mesure qu'on appelle mesure canonique. La relation (1.2) correspond alors à la formule exponentielle pour les processus ponctuels de Poisson et se lit désormais comme suit : si $\sum_i \delta_{(x_i, X^i)}(dx, dX)$ est une mesure ponctuelle de Poisson sur $\mathbb{R}^+ \times D(\mathbb{R}^+, \mathbb{R}^+)$, avec $D(\mathbb{R}^+, \mathbb{R}^+)$ l'espace de Skorokhod des fonctions càdlàg de $\mathbb{R}^+$ dans $\mathbb{R}^+$, d'intensité $dx \times \mathbb{N}(dX)$, alors $(\sum_{i, x_i \leq x} X_t^i, t \geq 0)$ est de loi $\mathbb{P}_x$. Les relations (1.7) et (1.8) permettent de calculer l'espérance de X_t sous $\mathbb{N}$:

$$\mathbb{N}(X_t) = \mathrm{e}^{-\psi'(0+)t},\ t > 0. \tag{1.9}$$

Ensuite,

$$\mathrm{e}^{-xu_t(\lambda)} = \mathbb{E}_x(\mathrm{e}^{-\lambda X_t}) = \mathbb{E}_x(\mathbb{E}_{X_s}(\mathrm{e}^{-\lambda X_{t-s}})) = \mathbb{E}_x(\mathrm{e}^{-X_s u_{t-s}(\lambda)}) = \mathrm{e}^{-xu_s(u_{t-s}(\lambda))}$$

d'où l'on déduit que

$$u_t(\lambda) = u_s(u_{t-s}(\lambda)), \tag{1.10}$$

et enfin :

$$\mathbb{N}(1 - \mathrm{e}^{-\lambda X_t}) = \mathbb{N}(1 - \mathrm{e}^{-u_{t-s}(\lambda) X_s}) = \mathbb{N}(\mathbb{E}_{X_s}(1 - \mathrm{e}^{-\lambda X_{t-s}})).$$

Plus généralement, on peut montrer que le processus X sous $\mathbb{N}$ est markovien, avec les transitions du CB(ψ) en dehors de l'instant initial. Enfin, $(\mathbb{N}(X_t \in \cdot), t > 0)$ est une mesure d'entrée pour le semi-groupe du CB(ψ).

1.2.4 Les processus de branchement avec immigration (CBI)

Les processus de branchement avec immigration (CBI dans la suite) ont été introduits par Kawazu et Watanabe [72] comme limite d'échelle de processus de Galton-Watson avec immigration. Une introduction à ces processus pourra être trouvée au chapitre 12 de Kyprianou [79].

Un CBI(ψ, ϕ) est un processus de Markov fort X^∞ à trajectoires càdlàg de transformée de Laplace :

$$\mathbb{E}_x(\mathrm{e}^{-\lambda X_t^\infty}) = \mathrm{e}^{-xu_t(\lambda) - \int_0^t \phi(u_s(\lambda))ds},$$

lorsqu'il est issu de $x \geq 0$, avec u_t défini par l'équation (1.3), et

$$\phi(\lambda) = \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda r})\eta(dr),\ \lambda \geq 0 \tag{1.11}$$

l'exposant de Laplace d'un subordinateur, encore appelé mécanisme d'immigration. Ici, $\mathtt{d} \geq 0$ et η est une mesure de Radon sur $(0, \infty)$ telle que $\int_{(0,\infty)} (1 \wedge r)\eta(dr) < \infty$. Soit X un CB(ψ) issu de x et $\sum_i \delta_{(s_i, X^i)}$ un processus ponctuel de Poisson indépendant de X d'intensité

$$ds \times \left(\mathtt{d}\,\mathbb{N} + \int_{(0,\infty)} \eta(dr)\mathbb{P}_r \right)$$

sur $(0, \infty) \times D(\mathbb{R}^+, \mathbb{R}^+)$. Alors

$$(X_t + \sum_i X^i_{t-s_i} \mathbf{1}_{\{t \geq s_i\}}, t \geq 0) \tag{1.12}$$

a loi du CBI(ψ,ϕ).

Ainsi, du point de vue dynamique, un CBI(ψ, ϕ) se comporte comme un CB(ψ) avec un flux d'immigrants additionnel gouverné par un subordinateur d'exposant de Laplace ϕ. De plus les immigrants sont ensuite indistinguables des autres individus de la population originelle, dans le sens où ils partagent les mêmes caractéristiques reproductives, données par le mécanisme de branchement ψ. On notera que la loi du CBI(ψ, ϕ) satisfait l'analogue suivant de la propriété de branchement (1.1) :

$$\mathbb{P}_{x+x'}(X^\infty \in \cdot) = \mathbb{P}_x(X^\infty \in \cdot) * \mathbb{P}_{x'}(X \in \cdot), x, x' \in \mathbb{R}^+. \tag{1.13}$$

avec X un CB(ψ). Sans plus d'information sur la généalogie, la représentation (1.12) des CBI permet déjà d'identifier l'instant de naissance A_t de l'ancêtre commun le plus récent des individus en vie à l'instant t dans X^∞ selon la formule :

$$A_t = \inf\{s_i; X^i_{t-s_i} \neq 0\}\, \mathbf{1}_{\{X_t = 0\}}.$$

Cela permet ensuite d'étudier :
- le phénomène de bottleneck selon lequel la population est stochastiquement plus petite à l'instant A_t qu'en t, du moins en régime stationnaire, voir Chen et Delmas [26],
- la dynamique du processus $(A_t, t \geq 0)$, voir Evans et Ralph [51].

L'ensemble des zéros d'un CBI est un ensemble régénératif. Des conditions pour savoir si un ensemble régénératif est vide ont été établies par Fitzsimmons, Fristedt et Shepp [52]. Cependant, il n'est pas évident de transcrire ces conditions en terme du couple (ψ, ϕ). Nous faisons cet exercice dans un cas particulier dans l'article avec Clément Foucart [55], voir Proposition 4.3.1.

1.2.5 Le Q-processus des CB

Informellement, le Q-processus du CB X est défini comme le CB X conditionné par l'évènement $\{X_t \neq 0\}$ pour t grand. On dit encore que le Q-processus du CB est le CB conditionné à la non-extinction en temps long. Lamperti et Ney [90] ont les premiers travaillé sur ce sujet. Roelly et Rouault [112] et Evans [48] ont initié l'étude des Q-processus pour des superprocessus homogènes. Enfin, les travaux de Lambert [85] traitent le cas du Q-processus pour des CB généraux.

Soit X un CB(ψ) et $\tau_0(X) = \inf\{t > 0, X_t = 0\}$. On commence par écrire le calcul suivant :

$$\mathbb{P}_x(\tau_0(X) \leq t) = \mathbb{E}_x(\lim_{\lambda \to \infty} \mathrm{e}^{-\lambda X_t}) = \lim_{\lambda \to \infty} \mathrm{e}^{-x u_t(\lambda)} = \mathrm{e}^{-x u_t(\infty)},$$

où l'on utilise à la première égalité le caractère càdlàg des trajectoires, et où l'on note $u_t(\infty)$ la limite croissante de $u_t(\lambda)$ lorsque $\lambda \to \infty$. De l'équation (1.5), on déduit que $\{\tau_0(X) < \infty\}$ avec probabilité strictement positive lorsque la condition de Grey (1.6) est satisfaite, comme annoncé auparavant. Plus précisément, dans ce cas, $\lim_{t \to \infty} u_t(\infty)$ est égal à la plus grande racine de l'équation $\psi(\lambda) = 0$, qui vaut 0 lorsque le CB est critique ou sous-critique (on exclut le cas particulier $\psi(\lambda) = 0$ de la discussion).

Soit donc un CB sous-critique ou critique qui satisfait la condition de Grey, de sorte que l'évènement $\{\tau_0(X) = \infty\}$ a une probabilité $\mathbb{P}_x$ nulle et une mesure d'excursion $\mathbb{N}$ nulle. La loi du Q-processus notée $\mathbb{N}^\infty$ est définie comme suit, avec $(\mathcal{F}_t, t \geq 0)$ la filtration naturelle du processus X :

$$\mathbb{N}^\infty(A) = \lim_{s\to\infty} \mathbb{N}(A|\tau_0(X) \geq t+s), \;\; A \in \mathcal{F}_t, \tag{1.14}$$

sous réserve que cette limite ait un sens. Le théorème suivant est dû à Roelly et Rouault [112] et Evans [48] dans le cas d'un CB quadratique et Lambert [85] dans le cas général.

Proposition. *Soit X un CB(ψ) sous-critique ou critique qui satisfait la condition de Grey (1.6). La relation (1.14) définit une mesure de probabilité $\mathbb{N}^\infty$, qui peut être exprimée comme une h-transformée de Doob pour la fonction harmonique espace temps $h(t,x) = xe^{\psi'(0+)t}$, c'est-à-dire :*

$$\mathbb{N}^\infty(A) = \mathbb{N}(X_t\, e^{\psi'(0+)t}, A), \;\; A \in \mathcal{F}_t. \tag{1.15}$$

De plus, le processus X est sous $\mathbb{N}^\infty$ un processus de branchement avec immigration issu de 0 de mécanisme de branchement ψ et de mécanisme d'immigration $\phi(\lambda)$:

$$\phi(\lambda) = \psi'(\lambda) - \psi'(0+) = 2\alpha\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda r}) r\nu(dr). \tag{1.16}$$

On notera que le membre de droite de (1.15) définit encore une mesure de probabilité même si la condition de Grey n'est pas satisfaite.

Nous esquissons maintenant la démonstration de la Proposition 1.15. Le premier résultat tout d'abord, relatif à la h-transformée. On se donne $A \in \mathcal{F}_t$. Du fait de la propriété de Markov de $\mathbb{N}$, on a

$$\mathbb{N}(A|X_{t+s} > 0) = \frac{\mathbb{N}(\mathbb{P}_{X_t}(X_s > 0), A)}{\mathbb{N}(X_{t+s} > 0)} = \mathbb{N}\left(\frac{1 - e^{-X_t v_s}}{v_{t+s}}, A\right).$$

avec $v_s = \lim_{\lambda\to\infty} u_s(\lambda)$, caractérisé par

$$\int_{v_s}^\infty \frac{dr}{\psi(r)} = s. \tag{1.17}$$

On montre que v_s converge lorsque s tend vers l'infini vers la plus grande racine de ψ, égale à 0 puisque le CB est critique ou sous-critique. D'autre part, il est possible d'établir la convergence suivante, voir Lambert [85] :

$$\frac{v_s}{v_{t+s}} \xrightarrow[s\to\infty]{} e^{\psi'(0+)t}.$$

Une domination immédiate permet alors d'appliquer le théorème de convergence dominée et de conclure. Il reste à montrer que le Q-processus est un processus de branchement avec immigration. On commence par noter que, en tant que h-transformée d'un processus de Markov fort, X sous $\mathbb{N}^\infty$ est encore un processus de Markov fort. On calcule ensuite :

$$\begin{aligned} \mathbb{N}^\infty(\mathrm{e}^{-\lambda X_t}) &= \mathbb{N}(X_t\, \mathrm{e}^{\psi'(0+)t}\, \mathrm{e}^{-\lambda X_t}) \\ &= \mathrm{e}^{\psi'(0+)t}\, \partial_\lambda \mathbb{N}(1 - \mathrm{e}^{-\lambda X_t}) \\ &= \mathrm{e}^{\psi'(0+)t}\, \partial_\lambda u_t(\lambda) \\ &= \mathrm{e}^{\psi'(0+)t}\, \mathrm{e}^{-\int_0^t \psi'(u_s(\lambda))\, ds} \\ &= \mathrm{e}^{-\int_0^t (\psi'(u_s(\lambda)) - \psi'(0+))\, ds}, \end{aligned}$$

la troisième équation étant obtenue en différenciant l'équation intégrale (1.3) par rapport à λ. On a donc bien que X est sous $\mathbb{N}^\infty$ un processus de branchement avec immigration issu de 0 avec le mécanisme de branchement $\psi(\lambda)$ et de mécanisme d'immigration $\phi(\lambda) = \psi'(\lambda) - \psi'(0+)$.

On peut se demander si conditionnement à la non-extinction et transformation de Lamperti commutent : un tel résultat ne vaut que dans le cas d'un mécanisme de branchement critique, voir le Lemme 10.14 de la monographie de Kyprianou [79]. D'ailleurs, le conditionnement d'un processus de Lévy spectralement positif Y par $\{\tau_0(Y) \geq t\}$ peut donner lieu à une perte de masse à la limite $t \to \infty$, au sens où :

$$\lim_{s_0 \to \infty} \lim_{t \to \infty} \mathbb{P}(Y_t \leq s_0 | \tau_0(Y) \geq t) < 1,$$

voir le lemme 10.11 de la même référence [79], tandis que la Proposition 1.15 ci-dessus montre que l'on n'a pas de perte de masse lorsqu'on conditionne les CB associés.

1.3 Généalogie des CB : Les arbres réels

Les CB apparaissent naturellement comme limite d'échelle de processus de branchement discrets, appelés processus de Galton-Watson, convenablement renormalisés, voir Jiřina [70]. Les processus de Galton-Watson sont naturellement pourvus d'une généalogie. Il est moins facile de définir une généalogie pour les CB. Nous verrons deux méthodes complémentaires pour donner un sens à la généalogie d'un CB : les arbres continus en section 1.3.1, et le système de particules look-down en section 1.5.

1.3.1 Les arbres continus

Soit $g : [0, \infty) \to [0, \infty)$ une fonction continue à support compact telle que $g(0) = 0$. On définit alors :

$$d_g(s, t) = g(s) + g(t) - 2 \inf \{g(u), s \wedge t \leq u \leq s \vee t\}.$$

pour $s, t \in [0, \infty)$, et on a les deux propriétés suivantes :

$$d_g(s, t) = d_g(t, s) \text{ et } d_g(s, t) \leq d_g(s, u) + d_g(u, t).$$

La relation d'équivalence suivante :

$$s \sim t \text{ si et seulement si } d_g(s, t) = 0,$$

fait donc de d_g une distance sur l'espace quotient $\mathcal{T}_g = [0, \infty)/\sim$. Le couple $(\mathcal{T}_g, d_g)$ est alors un arbre réel, et on note $\mathbb{T}$ l'ensemble de ces arbres. Cela signifie, voir [49], que $(\mathcal{T}_g, d_g)$ est un espace métrique qui satisfait aux deux propriétés suivantes, pour tout $x, y \in \mathcal{T}_g$:

– Il existe une unique application isométrique $f_{x,y}$ de $[0, d_g(x, y)]$ dans $\mathcal{T}_g$ telle que :

$$f_{x,y}(0) = x \text{ et } f_{x,y}(d_g(x, y)) = y.$$

– Si ϕ est une application continue injective de $[0, 1]$ dans $\mathcal{T}_g$ telle que $\phi(0) = x$ et $\phi(1) = y$, alors :

$$\phi([0, 1]) = f_{x,y}([0, d(x, y)]).$$

Un arbre réel enraciné est un arbre réel avec un point distingué ρ. Nous enracinerons $\mathcal{T}_g$ en la classe d'équivalence de 0. L'image de $[0, d(x,y)]$ par l'application $f_{x,y}$ sera notée $[\![x,y]\!]$. On note $\tilde{\mathcal{L}}$ l'ensemble des feuilles de $\mathcal{T}_g$, défini comme l'ensemble des éléments x de $\mathcal{T}_g \setminus \{\rho\}$ tel que $\mathcal{T}_g$ privé de x reste connexe. On définit enfin le squelette $\mathcal{S}_g = \mathcal{T}_g \setminus \tilde{\mathcal{L}}$ comme le complémentaire de l'ensemble des feuilles dans $\mathcal{T}_g$. Deux mesures sont naturellement attachées à $\mathcal{T}_g$:

- la mesure de longueur ℓ_g, qui est une mesure sur le squelette définie par $\ell_g([\![x,y]\!]) = d_g(x,y)$,
- la mesure de masse m_g, définie comme la mesure image de la mesure de Lebesgue par la projection canonique p qui à un élément de $[0,\infty)$ associe son représentant dans $\mathcal{T}_g$.

Le plus récent ancêtre commun de x et y est $[\![\rho,x]\!] \cap [\![\rho,y]\!] \cap [\![x,y]\!]$, il sera noté $x \wedge y$.

Notons que l'ensemble des arbres réels compacts est polonais, voir Evans [49], ce qui va nous permettre de considérer des variables aléatoires à valeurs dans cet espace.

1.3.2 Les arbres de Lévy, et les décompositions de Bismut et de Williams.

Nous décrivons dans cette section une procédure pour donner un sens à la généalogie d'un CB(ψ) en terme d'arbre réel. Nous proposons ensuite une décomposition de cette généalogie par rapport à un point x de l'arbre :

- choisi selon la mesure de masse $m(dx)$ (Bismut).
- choisi de sorte que $d(\rho, x)$ soit maximal (Williams).

Nous précisons enfin comment ces généalogies permettent de définir la généalogie du Q-processus.

Soit un processus de Lévy $Y = (Y_t, t \geq 0)$ d'exposant de Laplace ψ donné par (1.4), dont les trajectoires sont à variation infinie, et qui ne dérive pas vers $+\infty$. Le Gall et Le Jan lui associent dans [94] un processus des hauteurs $(H_t, t \geq 0)$ défini comme suit. On note

$$\hat{Y}^t_s = Y_t - Y_{(t-s)-},\ 0 \leq s \leq t,$$

le processus retourné à l'instant t et

$$\hat{S}^t_s = \sup_{r \leq s} \hat{Y}^t_r$$

le processus du supremum. Le processus des hauteurs à l'instant t, H_t, est alors défini comme le temps local au niveau 0 et à l'instant t du processus $\hat{S}^t - \hat{Y}^t$. Le processus des hauteurs, qui n'est pas une semi-martingale, admet cependant un temps local, dont il existe une version conjointement mesurable notée $(L^t_s, t \geq 0, s \geq 0)$, continue et croissante en s.

On a la généralisation suivante du théorème de Ray-Knight, due à Duquesne et Le Gall [37].

Théorème (Théorème de Ray-Knight généralisé)**.** *Supposons que le CB(ψ) soit critique ou sous-critique, à trajectoires de variation infinie. Alors le processus* $(L^t_{\tau_{-r}}, t \geq 0)$ *est un CB(ψ) issu de* r*, avec* $\tau_{-r} = \tau_{-r}(Y)$*.*

Remarque 1.3.1. Lorsque $\psi(\lambda) = 2\lambda^2$, le processus des hauteurs a la loi d'un mouvement brownien standard réfléchi, et donc le théorème de Ray-Knight ci-dessus dit que la famille des temps locaux d'un mouvement brownien réfléchi stoppé lorsque le temps local en 0 excède r est un CB(ψ) issu de r, encore appelé diffusion de Feller. Ce résultat constitue le second théorème de Ray-Knight, dû indépendamment à Ray [109] et Knight [75].

Supposons que le CB(ψ) est critique ou sous-critique, à trajectoires de variation infinie, et vérifie la condition de Grey (1.6). Duquesne et Le Gall [37] prouvent alors que le processus des hauteurs est continu en la variable t. Dès lors, il est possible de considérer l'arbre réel associé $\mathcal{T}_g$ pour $g(s) = H_{s\wedge\tau_{-r}}$, où $\tau_{-r} = \tau_{-r}(Y)$. L'arbre réel $\mathcal{T}_g$ ainsi défini est alors compact. On notera $\mathbf{P}_r$ la loi de l'espace métrique $\mathcal{T}_g$ ainsi défini muni de d_g.

On peut encore définir l'arbre $\mathcal{T}^g$ sous la mesure d'excursion n du processus $Y - I$, où I désigne l'infimum de Y : $I_t = \inf_{0\leq s\leq t} Y_s$ pour $t \geq 0$. Soit $g(s) = H_s$ une excursion du processus des hauteurs sous n. Nous noterons $\mathbf{N}$ la "loi" de l'espace métrique égal à $\mathcal{T}_g$ muni de d_g. Le lien entre $\mathbb{N}$ et $\mathbf{N}$ s'énonce alors comme suit : si $\mathcal{T}$ est distribué selon $\mathbf{N}$, alors $(L^t_{\tau_0}, t \geq 0)$ est distribué selon $\mathbb{N}$.

Pour $t \geq 0$ fixé, la famille des temps locaux $(L^t_s, s \geq 0, t \geq 0)$ induit une mesure $d_s L^t_s$ sur $(0, \tau_0)$, et nous notons $\ell^t(du)$ la mesure image de la mesure $d_s L^t_s$ par l'application p. Ainsi, ℓ^t est une mesure sur l'arbre $\mathcal{T}_g$ qui vérifie, pour toute fonction φ mesurable bornée,

$$\int_{u\in\mathcal{T}_g} \ell^t(du)\varphi(u) = \int_0^{\tau_0} d_s L^t_s\, \varphi(p(s)). \tag{1.18}$$

On notera que $L^t_{\tau_0} = \ell^t(\mathbf{1})$, avec $\tau_0 = \inf\{t > 0, Y_t = 0\}$. Le lien avec la mesure de masse est le suivant :

$$m(du) = \int_{t\in\mathbb{R}^+} dt\, \ell^t(du). \tag{1.19}$$

On donne maintenant les décompositions de Bismut et de Williams sous la mesure d'excursion n de $Y - I$. Etant donné deux espaces métriques $(\mathcal{T}, d, \rho)$ et $(\mathcal{T}', d', \rho')$ et un élément x_0 de $\mathcal{T}$, on introduit une opération de greffe. Définissons

$$\tilde{\mathcal{T}} = \mathcal{T} \circledast (\mathcal{T}', x_0),$$

comme suit :
- $\tilde{\mathcal{T}} = \mathcal{T} \cup \mathcal{T}'$ avec x_0 et ρ' identifiés dans $\tilde{\mathcal{T}}$.
- la racine de $\tilde{\mathcal{T}}$ est la racine ρ de $\mathcal{T}$.
- $\tilde{d}$ est définie comme suit :

$$\tilde{d}(x,y) = \begin{cases} d(x,y) & \text{si } x, y \in \mathcal{T}, \\ d'(x,y) & \text{si } x, y \in \mathcal{T}', \\ d(x,x_0) + d'(\rho', y) & \text{si } x \in \mathcal{T}, y \in \mathcal{T}' \end{cases}$$

Soit $h \in [0, \infty]$. Considèrons l'espace métrique $\mathcal{T}^h = [0, h]$, muni de sa distance naturelle. On pose maintenant :

$$\mu_B(dT) = 2\alpha\mathbf{N}(dT) + \int_{(0,\infty)} \nu(dr) r\, \mathbf{P}_r(dT).$$

On se donne ensuite une mesure ponctuelle de Poisson $\sum_{i\in\mathcal{I}} \delta_{(s_i,\mathcal{T}_i)}(ds, dT)$ sur $\mathcal{T}^h \times \mathbb{T}$ d'intensité $\mathbf{1}_{[0,h]}(s)ds\ \mu_B(dT)$. Alors

$$\mathcal{T}^{B,h} = \mathcal{T}^h \underset{i\in\mathcal{I}}{\circledast} (\mathcal{T}_i, s_i)$$

définit encore un arbre réel. Noter que cette construction, dans laquelle nous greffons un nombre infini d'arbres, est bien licite. Nous renvoyons à Le Gall Le Jan [94] et Duquesne Le Gall [37] pour la démonstration de la décomposition de Bismut suivante.

Théorème (Décomposition de Bismut)**.** *Supposons $\mathcal{T}$ construit à partir d'un CB(ψ) critique ou sous-critique, à trajectoires de variation infinie, et qui satisfait la condition de Grey. Nous avons alors la relation :*

$$\mathbf{N}\left(\int_{x\in\mathcal{T}} m(dx)F(d(\rho,x),\mathcal{T})\right) = \int_{h\in\mathbb{R}^+} \mathrm{e}^{-\psi'(0+)h}\, dh\, \mathbb{E}(F(h,\mathcal{T}^{B,h})).$$

Nous choisissons $(\mathcal{T}, x)$ distribué selon $\mathbf{N}(dT)m(dx)$: cela signifie que $\mathcal{T}$ est distribué selon $\mathbf{N}(m(\mathcal{T}), dT)$ puis, conditionnellement à $\mathcal{T}$, que x est de loi $m(dx)/m(\mathcal{T})$. Alors $d(\rho, x)$ a pour densité $\mathrm{e}^{-\psi'(0+)h}$ par rapport à la mesure de Lebesgue dh sur $\mathbb{R}^+$, et conditionnellement à $d(\rho, x) = h$, $\mathcal{T}$ est distribué comme $\mathcal{T}^{B,h}$.

Remarque 1.3.2. Lorsque $\psi(\lambda) = 2\lambda^2$, le processus des hauteurs est distribué selon la mesure d'excursion d'Itô, et on retrouve la décomposition originale de l'excursion brownienne due à Bismut [19].

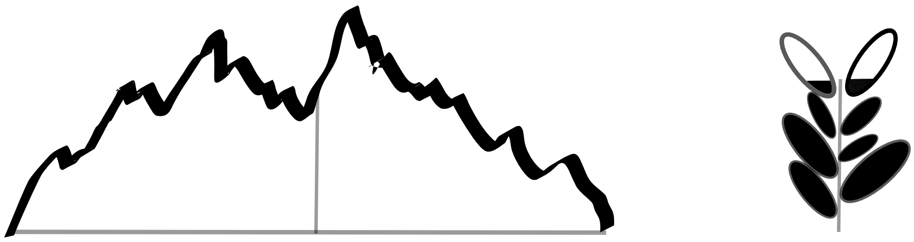

Figure 1.1: A gauche : une excursion du processus des hauteurs sous n et un point uniforme sur celle-ci, associé au point x dans l'arbre. A droite : l'arbre réel associé, de loi $\mathcal{T}^{B,h}$, décomposé le long du chemin de ρ à x, de longueur h.

Soit $h < \infty$, et soit A un ensemble qui ne dépend que de la restriction de l'arbre réel à l'ensemble $\{x \in \mathcal{T}, d(\rho, x) \leq h\}$. On a par construction la propriété de compatibilité suivante :

$$\mathbb{P}(\mathcal{T}^{B,h} \in A) = \mathbb{P}(\mathcal{T}^{B,\infty} \in A).$$

De plus, on peut déduire de (1.19) et du théorème précédent que :

$$\mathbb{P}(\mathcal{T}^{B,h} \in A) = \mathbf{N}(\mathrm{e}^{\psi'(0+)h} L^h_{\tau_0}, \mathcal{T} \in A).$$

On rappelle de plus que $(L^h_{\tau_0}, h \geq 0)$ a la loi d'un CB(ψ) pris sous sa mesure d'excursion. Maintenant, la martingale $(L^h_{\tau_0}\, \mathrm{e}^{\psi'(0+)h}, h \geq 0)$ est encore la dérivée de Radon-Nikodym du Q-processus par rapport au processus de branchement original dans sa filtration naturelle $(\mathcal{F}_h, h \geq 0)$, selon (1.15). Ainsi $\mathcal{T}^{B,\infty}$ donne une généalogie au Q-processus. Ce même résultat vaut encore dans le cas discret : voir les preuves conceptuelles du théorème de Kesten et Stigum par Lyons, Pemantle et Peres [97].

On exprime maintenant la décomposition de Williams en terme d'arbre réel. On note

$$H_{\max} = \sup\{H_t, t \geq 0\} = \sup\{d(\rho, x), x \in \mathcal{T}\}$$

la hauteur de l'arbre réel $\mathcal{T}$ associé à H. On définit :

$$\mu_{W,h}(dT) = 2\alpha\mathbf{N}(dT) + \int_{(0,\infty)} \nu(dr) r\, \mathrm{e}^{-rv_h}\, \mathbf{P}_r(dT).$$

Soit $h \geq 0$ fixé. On considère à nouveau l'espace métrique $\mathcal{T}^h = [0, h]$ muni de sa distance naturelle. Soit une mesure ponctuelle de Poisson $\sum_i \delta_{(s_i, \mathcal{T}_i)}(ds, dT)$ sur le produit de $\mathcal{T}^\infty \times \mathbb{T}$ d'intensité

$$\mathbf{1}_{[0,h)}(s) ds\ \mu_{W,h-s}(dT).$$

Alors $\mathcal{T}^{W,h} = \mathcal{T}^h \underset{i \in \mathcal{I}^h}{\circledast} (\mathcal{T}_i, s_i)$, où $\mathcal{I}^h = \{i \in \mathcal{I}; s_i + H_{\max}(\mathcal{T}_i) \leq h\}$, définit encore un arbre réel. Abraham et Delmas [1] prouvent alors le Théorème suivant.

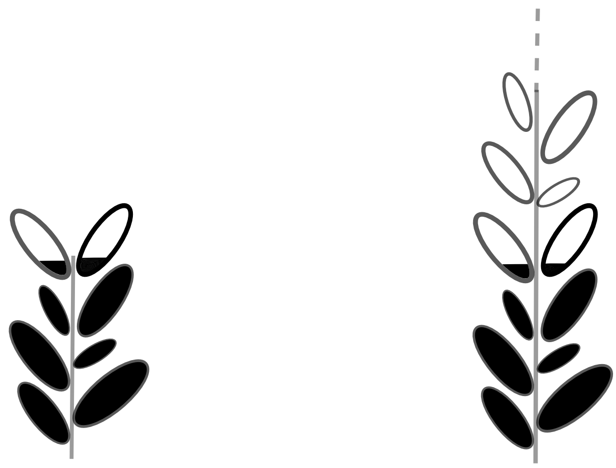

Figure 1.2: A gauche : l'arbre réel $\mathcal{T}^{B,h}$. A droite : l'arbre réel $\mathcal{T}^{B,\infty}$. Les deux arbres coïncident à distance inférieure à h de la racine

Théorème (Décomposition de Williams)**.** *Supposons $\mathcal{T}$ construit à partir d'un CB(ψ) critique ou sous-critique, à trajectoires de variation infinie, et qui satisfait la condition de Grey. On a :*
– $\mathbf{N}(H_{max} \geq h) = v_h$ *pour tout* $h > 0$.
– *Conditionnellement à* $\{H_{max} = h\}$, $\mathcal{T}$ *a même loi que* $\mathcal{T}^{W,h}$.
Ainsi, on a la relation :

$$\mathbb{N}(F(\mathcal{T})) = \int_{h \in \mathbb{R}^+} dh \, |\partial_h v_h| \, \mathbb{E}(F(\mathcal{T}^{W,h})).$$

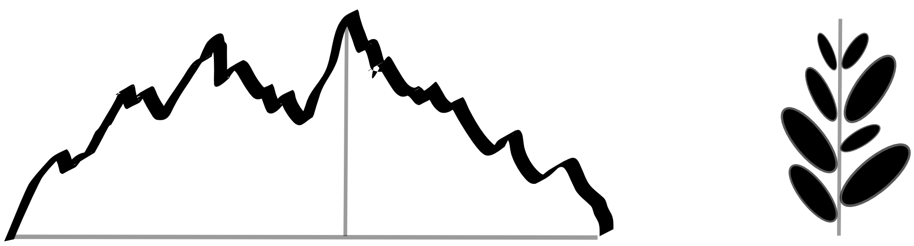

Figure 1.3: A gauche : une excursion du processus des hauteurs sous n décomposée selon son supremum atteint en x. A droite : l'arbre réel associé, décomposé le long du chemin de ρ à x, de longueur h, de même loi que $\mathcal{T}^{W,h}$. On notera qu'aucun des sous-arbres greffés le long de $\mathcal{T}^h$ ne dépasse la hauteur h.

Remarque 1.3.3. Lorsque $\psi(\lambda) = 2\lambda^2$, le processus des hauteurs est distribué selon la mesure d'excursion d'Itô, et on retrouve la décomposition de l'excursion brownienne par rapport à son supremum due à Williams, voir Rogers et Williams [114], Théorème 55.11.

La décomposition de Williams fournit une seconde approche de la généalogie du Q-processus, puisqu'il est possible de montrer que les conditionnements par les évènements $\{H_{\max} > h\}$ et $\{H_{\max} = h\}$ donnent le même processus a la limite $h \to \infty$, voir le lemme 2.5.1. Ainsi, $\mathcal{T}^{W,\infty}$ fournit à nouveau la généalogie du Q-processus. On notera enfin que $\mathcal{T}^{W,\infty} \overset{(\mathcal{L})}{=} \mathcal{T}^{B,\infty}$.

En résumé, nous avons vu deux méthodes pour donner une généalogie au Q-processus des CB :

– ou bien commencer par prouver l'existence du Q-processus du CB en montrant que c'est une h-transformée puis représenter cette h-transformée en terme d'arbre réel : c'est la méthode explicitée dans la section 1.2.5 et poursuivie avec la décomposition de Bismut.
– ou bien obtenir une représentation en terme d'arbre réel du processus conditionné à $\{H_{\max} = h\}$ puis passer à la limite en h sur cette représentation, comme nous venons de le voir avec la décomposition de Williams.

1.3.3 Une construction des superprocessus homogènes

Cette section présente la définition du serpent brownien introduit par Le Gall dans [92], qui permet de construire un superprocessus homogène à partir de sa généalogie donnée par un arbre réel.

Soit un processus de Markov Y sur un espace polonais E sans discontinuités fixes. On note P_x la loi du processus issu de x dans E. Soit $\mathcal{T}$ un arbre réel, supposé déterministe pour le moment, enraciné en ρ. Il existe un processus $(W_v, v \in \mathcal{T})$ indicé par $\mathcal{T}$ tel que :
– Pour chaque $u \in \mathcal{T}$, le processus $(W_{f_{(\rho,u)}(r)}, 0 \leq r \leq d(\rho, u))$ est de loi P_x.
– Conditionnellement à la valeur $W_{u\wedge v}$, $(W_z, v \in [\![u \wedge v, u]\!])$ et $(W_z, v \in [\![u \wedge v, z]\!])$ sont indépendants.

Soit un mécanisme de branchement ψ sous-critique ou critique qui vérifie la condition de Grey (1.6). On construit à partir du processus de Lévy spectralement positif d'exposant de Laplace ψ l'arbre réel $\mathcal{T}$ de “loi” $\mathbf{N}$ comme en section 1.3.2. On définit alors le superprocessus homogène $(Z_t, t \geq 0)$ à valeurs dans l'ensemble $\mathcal{M}_f(E)$ des mesures finies sur E par :

$$Z_t(dx) = \int_{u\in\mathcal{T}} \ell^t(du)\delta_{W_u}(dx). \tag{1.20}$$

On notera encore $\mathbb{N}_x$, par abus de notation, la “loi” de la mesure aléatoire Z_t et pour une mesure finie $\mu \in \mathcal{M}_f(E)$ et f positive et mesurable, $\mu(f)$ la quantité définie par $\mu(f) = \int_E f(x)\mu(dx)$. La quantité $u_t(f, x) = \mathbb{N}_x(1 - \mathrm{e}^{-Z_t(f)})$ satisfait à l'équation intégrale :

$$u_t(f, x) + \mathrm{E}_x \left(\int_0^t ds\ \psi(u_{t-s}(f, Y_s)) \right) = \mathrm{E}_x(f(Y_t)), \tag{1.21}$$

voir le Théorème 2.1 de [92] pour le cas où $\psi(\lambda) = 2\lambda^2$ et le Théorème 4.2.2 de Duquesne et Le Gall [37] pour le cas général. Le processus $(Z_t, t \geq 0)$ est un superprocessus homogène, caractérisé par la donnée du mouvement spatial de loi P et du mécanisme de branchement ψ. On doit l'introduction des superprocessus à Dawson [29, 30] et Watanabe [129], d'où le nom qui leur est parfois donné de processus de Dawson-Watanabe. La notation $u_t(f, x) = \mathbb{N}_x(1 - \mathrm{e}^{-Z_t(f)})$ est similaire à la notation $u_t(\lambda) = \mathbb{N}(1 - \mathrm{e}^{-\lambda X_t})$ utilisée auparavant, avec la fonction f dans le rôle de la condition initiale $\lambda \in \mathbb{R}^+$. La donnée de x est une donnée spatiale supplémentaire. On notera que, pour le choix de $f = \lambda \mathbf{1}_E$,

$$u_t(f, x) = \mathbb{N}_x(1 - \mathrm{e}^{-\lambda Z_t(E)}) = u_t(\lambda),$$

car la masse totale d'un superprocessus homogène est par construction un CB(ψ). Notons enfin que des variations de cette approche permettent de construire le superprocessus associé à des mécanismes de branchement surcritiques, voir Duquesne et Winkel [38] et Abraham et Delmas [2].

1.3.4 Les CB multitypes

Nous proposons dans cette section une généralisation des CB quadratiques, appelée processus de branchement multitypes. On peut retracer l'origine de ces processus à Watanabe, voir [130] pour un exemple à 2 types. On consultera Champagnat et Roelly [25] pour une étude plus récente. Nous observons que la définition de leur généalogie pose problème.

Désormais X_t n'est plus un scalaire mais un vecteur de $(\mathbb{R}^+)^n$. Notons (x, y) le produit scalaire de deux éléments de $\mathbb{R}^n$. Soit $\lambda \in (\mathbb{R}^+)^n$. Il existe un unique processus de Markov X_t à trajectoires càdlàg dont la transformée de Laplace s'écrit :

$$\mathbb{E}_x(\mathrm{e}^{-(\lambda, X_t)}) = \mathrm{e}^{-(x, u_t(\lambda, \cdot))},$$

où le vecteur $u_t(\lambda, \cdot) = (u_t(\lambda, i), 1 \le i \le n)$ est l'unique solution du système suivant :

$$\begin{cases} \partial_t u_t(\lambda, i) = \sum_{1 \le j \le K} \beta_{ij} u_t(\lambda, j) - \alpha u_t(\lambda, i)^2 \\ u_0(\lambda, i) = \lambda_i. \end{cases}$$

Notons $\beta_i = -\sum_j \beta_{i,j}$. Soit Y une chaîne de Markov à valeurs dans $\{1, \ldots, n\}$ de taux de transition infinitésimal β_{ij} de i vers j pour $j \neq i$. Notons P_i la loi de Y issue de i. Alors le système précédent peut se mettre sous la forme suivante :

$$u_t(\lambda, i) + \mathrm{E}_i\left(\int_0^t ds\ \psi(Y_s, u_{t-s}(\lambda, Y_s))\right) = \mathrm{E}_i(\lambda_{Y_t}), \tag{1.22}$$

avec $\psi(i, z) = \alpha z^2 + \beta_i z$. Dans le cas où tous les β_i sont constants, égaux à β, $\psi(i, z)$ est fonction de z seulement, il s'agit d'un mécanisme de branchement homogène, l'équation (1.22) est en fait une équation du type (1.21), et la construction (1.20) de X via sa généalogie est valable. Si les β_i ne sont pas constants en revanche, le mécanisme de branchement $\psi(i, z) = \alpha z^2 + \beta_i z$ est qualifié d'inhomogène, et l'équation (1.22) ne peut se réduire à une équation de type (1.21). Heuristiquement, les individus de la population se reproduisent encore indépendamment mais plus identiquement, dans le sens où la loi de reproduction est fonction de la position spatiale, qui elle-même évolue selon un processus de Markov à espace d'état discret. Une question d'intérêt est alors la compréhension de l'interaction entre la généalogie et la composante spatiale. On ne peut plus en effet choisir d'abord la structure généalogique puis, indépendamment, le mouvement spatial, comme dans (1.20). Cette question a donné naissance à l'article [31] avec Jean-François Delmas.

On notera que ces CB multitypes apparaissent comme limite d'échelle de processus de Galton-Watson multitype avec mutations rares, voir Champagnat et Roelly [25] pour la définition de ces processus. Dans le cas où les mutations ne sont pas rares, c'est à dire si la matrice de transition de la chaîne de Markov est indépendante du changement d'échelle, Miermont [98] montre qu'un superprocessus homogène apparaît à la limite.

Les processus de branchement multitypes apparaissent comme un cas particulier de superprocessus plus généraux, les superprocessus inhomogènes, comme noté par Dynkin à l'exemple 2 p. 10 de [41]. Nous allons dorénavant nous placer dans ce cadre.

1.4 Les superprocessus inhomogènes : généalogie, décomposition de Williams et Q-processus

On établit dans cette section une décomposition de Williams pour des processus de branchement intègrant une composante spatiale qui interagit avec le branchement, puis on en déduit une construction du Q-processus. Cette section correspond au travail [31] avec Jean-François Delmas, dont les résultats principaux sont exposés dans les sections 1.4.2 et 1.4.3.

1.4.1 Définition

On propose maintenant un cadre d'étude adapté à notre problème, le cadre des superprocessus inhomogènes. Après l'introduction des superprocessus par Dawson et Watanabe, Dynkin [40, 41] a considérablement développé ce champ d'étude, qui continue à faire l'objet d'une grande attention de la part de la communauté probabiliste, comme en témoignent les monographies suivantes, classées par ordre chronologique : Perkins [104], Etheridge [46], Duquesne et Le Gall [37] et Li [95].

On se donne un processus de Markov fort sur un espace polonais, de loi P_x lorsqu'il est issu de $x \in E$, et de générateur infinitésimal noté $\mathcal{L}$. Soit un mécanisme de branchement inhomogène de la forme :

$$\psi(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2, \tag{1.23}$$

pour des fonctions β et α continues bornées sur E à valeurs dans $\mathbb{R}$ et $\mathbb{R}^+$ respectivement. Lorsque les fonctions β et α sont constantes, le mécanisme de branchement est dit homogène.

Soit f positive, mesurable et bornée. D'après le théorème II.5.11 de Perkins [104], il existe une unique solution $u_t(f, x)$ mesurable et bornée sur les ensembles $[0, T] \times E$ pour tout $T > 0$ de l'équation intégrale :

$$u_t(f, x) + \mathrm{E}_x \left(\int_0^t ds \; \psi(Y_s, u_{t-s}(f, Y_s)) \right) = \mathrm{E}_x(f(Y_t)). \tag{1.24}$$

On peut maintenant définir le superprocessus inhomogène. Le Théorème d'existence suivant est une conséquence des Théorèmes II.5.1 et II.5.11 de Perkins [104].

Définition. *Il existe un unique processus de Markov $(Z_t, t \geq 0)$ à trajectoires continues dans l'espace $\mathcal{M}_f(E)$ des mesures finies sur E muni de la topologie de la convergence étroite, qui satisfait, pour toute fonction f positive, mesurable et bornée, et pour tout $x \in E$,*

$$\mathbb{N}_x(1 - \mathrm{e}^{-Z_t(f)}) = u_t(f, x), \tag{1.25}$$

avec $u_t(f, \cdot)$ la solution de (1.24) bornée sur les ensembles $[0, T] \times E$ pour tout $T \geq 0$.

On dira que Z est le superprocessus de mouvement spatial de loi P (ou de générateur infinitésimal $\mathcal{L}$) et de mécanisme de branchement (1.23). On peut encore définir la loi $\mathbb{P}_\nu$ de ce même superprocessus issu de $\nu \in \mathcal{M}_f(E)$ par :

$$\mathbb{P}_\nu(\mathrm{e}^{-Z_t(f)}) = \mathrm{e}^{-\nu(u_t(f, \cdot))}.$$

On le construit comme suit : Si $\sum_i \delta_{Z^i}(dZ)$ est une mesure de Poisson d'intensité $\int_E \nu(dx)\mathbb{N}_x$, alors $\sum_i Z^i$ a pour loi $\mathbb{P}_\nu$. Le superprocessus $Z = (Z_t, t \geq 0)$ est appelé inhomogène : il s'agit d'une inhomogénéité en espace qui traduit le fait que la reproduction est fonction de la position spatiale. Le paramètre $-\beta(x)$ est un paramètre malthusien qui gouverne la croissance exponentielle de la population en x, et $\alpha(x)$ gouverne l'intensité de la reproduction en x. Noter que Z est, en tant que processus de Markov, un processus homogène en temps au sens usuel où ses noyaux de transition de l'instant s à un instant ultérieur t ne dépendent que de la différence $t-s$. Noter aussi que la masse totale du superprocessus $Z(\mathbf{1})$, qui était un CB(ψ) dans le cas homogène, n'est plus désormais un processus de Markov en général.

1.4.2 Une généalogie via les transformées de Pinsky et de Dawson-Girsanov

Avant de présenter nos résultats, et afin de développer une meilleure intuition des superprocessus inhomogènes, on se propose de présenter brièvement deux techniques utiles : la h-transformée au sens de Pinsky, et la transformation de Dawson-Girsanov.

On définit la h-transformée au sens de Pinsky du superprocessus Z par la relation :

$$Z^h(dx) = h(x)Z(dx). \tag{1.26}$$

On vérifie alors que, sous certaines conditions sur h explicitées au lemme 2.3.5, $Z^h(dx)$ est encore un superprocessus, de mouvement spatial P^h_x, défini par :

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^h_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{h(Y_t)}{h(x)}\,\mathrm{e}^{-\int_0^t ds\ (\mathcal{L}h/h)(Y_s)}\,. \tag{1.27}$$

et de mécanisme de branchement :

$$\psi(x,z) = \frac{(-\mathcal{L}+\beta)h}{h}z + \alpha(x)h(x)z^2,$$

avec la nouvelle condition initiale $Z^h_0(dx) = h(x)Z_0(dx)$. S'il existe une fonction ϕ_0 positive qui satisfait à $(-\mathcal{L}+\beta)\phi_0 = \lambda_0\phi_0$ (nous reviendrons en section 1.4.4 sur l'existence d'une telle fonction), alors $Z^{\phi_0}(dx)$ est encore un superprocessus, de mouvement spatial $\mathrm{P}^{\phi_0}_x$, et de mécanisme de branchement :

$$\lambda_0 z + \alpha(x)\phi_0(x)z^2.$$

On aimerait alors éliminer la dépendance en x du coefficient en z^2 du mécanisme de branchement. On peut penser, à la suite de Dhersin et Serlet [33], à utiliser un changement de temps aléatoire. Cependant, ce changement de temps impacte également le coefficient en z, annulant l'effet de la tranformation précédente. En outre, ce changement de temps affecte les longueurs de l'arbre généalogique et en particulier le temps d'extinction $H_{\max} = \inf\{t>0, Z_t = 0\}$ du superprocessus Z. Cette approche ne semble donc pas adaptée pour prouver une décomposition de Williams, qui est une décomposition de la loi du superprocessus par rapport à $H_{\max}$.

Nous avons donc choisi une autre méthode pour réduire le superprocessus inhomogène à un superprocessus homogène. Précisément, nous nous donnons un superprocessus inhomogène Z qui satisfait la propriété d'extinction presque sûre au sens où $\mathbb{N}_x(H_{\max} = \infty) = 0$ pour tout $x \in E$. Nous supposons de plus que les coefficients α et β du mécanisme de branchement satisfont aux propriétés de régularité $(H2)$ et $(H3)$ explicitées dans l'article, ce qui nous permet de justifier l'emploi des transformées suivantes. Nous considérons alors le superprocessus $Z^h(dx)$ avec $h = 1/\alpha$, de mécanisme de branchement :

$$\tilde{\psi}(x,z) = \tilde{\beta}(x)z + z^2, \text{ avec } \tilde{\beta}(x) = \alpha(-\mathcal{L}+\beta)(1/\alpha)(x),$$

de mouvement spatial $\mathrm{P}^{\frac{1}{\alpha}}$, dont nous notons le générateur infinitésimal $\tilde{\mathcal{L}}$. Nous notons $\tilde{\mathbb{P}}_\nu$ la loi du superprocessus $Z^{\frac{1}{\alpha}}(dx)$ lorsqu'il est issu de $\nu \in \mathcal{M}_f(E)$. Il nous reste à éliminer la dépendance en x du coefficient en z dans $\tilde{\psi}$. Nous utilisons une transformation de Dawson-Girsanov de la loi du superprocessus, voir le cours de Saint-Flour de Dawson [30]. Précisément, nous posons :

$$\beta_0 = \sup_{x\in E} \max\left(\tilde{\beta}(x), \sqrt{(\tilde{\beta}^2(x) - 2\tilde{\mathcal{L}}(\tilde{\beta})(x))_+}\right) \quad \text{et} \quad q(x) = \frac{\beta_0 - \tilde{\beta}(x)}{2},$$

ainsi que :

$$\varphi(x) = \tilde{\psi}(x, q(x)) - \tilde{\mathcal{L}}(q)(x), \quad x \in E.$$

Alors, la relation :

$$\frac{d\mathbb{P}^0_\nu}{d\tilde{\mathbb{P}}_\nu} = \mathrm{e}^{Z_0(q) - \int_0^{+\infty} ds\; Z_s(\varphi)}\, \mathbf{1}_{\{H_{\max} < +\infty\}} \tag{1.28}$$

définit la loi $\mathbb{P}^0_\nu$ d'un superprocessus homogène de mouvement spatial encore donné par le générateur $\tilde{\mathcal{L}}$ et de mécanisme de branchement :

$$\psi^0(\lambda) = \beta_0\lambda + \lambda^2.$$

Nous sommes donc arrivés à nos fins. Réciproquement, on peut construire l'arbre réel $\mathcal{T}$ associé au mécanisme de branchement homogène ψ^0, puis $(W_u, u \in \mathcal{T})$ le processus (indicé par l'arbre) de mouvement spatial donné par le générateur infinitésimal $\tilde{\mathcal{L}}$, selon la construction explicitée en section 1.3.3. On note $\mathbf{N}^0$ sa distribution. On définit alors $\tilde{\mathbf{N}}$ par absolue continuité comme suit :

$$\frac{d\tilde{\mathbf{N}}_x}{d\mathbf{N}^0_x}(W) = \mathrm{e}^{\int_0^{+\infty} ds\; Z_s(\varphi)},$$

et on rappelle que Z est une fonction du processus W. On obtient finalement la construction suivante de la généalogie du superprocessus inhomogène.

Proposition. [31] *Si Z satisfait la propriété d'extinction presque sûre, et α et β satisfont à $(H2)$ et $(H3)$, alors le processus à valeurs mesures $(Z_t, t \geq 0)$ défini par :*

$$Z_t(dx) = \int_{u\in\mathcal{T}} \ell^t(du)\; \alpha(W_s)\delta_{W_s}(dx), \quad t \geq 0.$$

avec W sous $\tilde{\mathbf{N}}_x$ est distribué selon $\mathbb{N}_x$.

Un mot de terminologie pour finir : on notera que la h-transformée de Pinsky est une transformation trajectorielle qui consiste en une repondération du superprocessus. En particulier, il ne s'agit pas, comme la h-transformée au sens de Doob, d'une transformation absolument continue de la loi du processus. Au contraire, la loi de la h-transformée de Pinsky est en général étrangère à la loi du processus initial.

1.4.3 La décomposition de Williams

Du fait de l'inhomogénéité du mécanisme de branchement $\psi(x,z)$ défini en (1.23), la décomposition de Williams pour un superprocessus inhomogène ne peut se réduire, comme dans le cas homogène, à la décomposition de Williams du CB associé. En particulier, on peut se demander quel est le mouvement spatial des plus longues lignées ancestrales.

Nous énoncerons nos résultats sous la mesure canonique $\mathbb{N}$, les résultats analogues sous $\mathbb{P}$ peuvent être consultés dans l'article. Un rôle clef dans l'analyse est joué par la fonction

$$v_h(x) = \mathbb{N}_x\left[Z_h \neq 0\right],$$

qui généralise la fonction v_h définie en (1.17) dans le cadre homogène. La fonction $v_h(x)$ est constante en la variable d'espace x dans le cas d'un mécanisme de branchement homogène, c'est à dire lorsque β et α sont constantes dans (1.22). La relation suivante définit de manière licite une mesure de probabilité :

$$\forall\ 0 \leq t < h,\quad \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\partial_h v_{h-t}(Y_t)}{\partial_h v_h(x)}\ \mathrm{e}^{-\int_0^t ds\ \partial_\lambda \psi(Y_s, v_{h-s}(Y_s))}, \tag{1.29}$$

avec $\mathcal{D}_t = \sigma(Y_s, 0 \leq s \leq t)$ la filtration naturelle de Y. Notons que $\mathrm{P}^{(h)}$ et P coïncident sur $\mathcal{D}_h$ pour un mécanisme de branchement homogène. On présente maintenant un second théorème intitulé décomposition de Williams, cette fois-ci valable pour des superprocessus inhomogènes. L'intérêt principal de cette décomposition vient de l'identification de la loi P^h de la lignée ancestrale la plus pérenne.

Théorème (Décomposition de Williams, [31])**.** *Supposons que le superprocessus Z satisfait la propriété d'extinction presque sûre, et que α et β sont continues bornées, et satisfont aux hypothèses $(H2)$ et $(H3)$. Soit $x \in E$ et $Y_{[0,h_0)}$ de loi $\mathrm{P}_x^{(h_0)}$. On se donne une mesure de Poisson $\sum_{j\in J} \delta_{(s_j, Z^j)}$ d'intensité :*

$$2\ \mathbf{1}_{[0,h_0)}(s)ds\ \mathbf{1}_{\{H_{max}<h_0-s\}}\alpha(Y_s)\ \mathbb{N}_{Y_s}[dZ]. \tag{1.30}$$

Conditionnellement à $\{H_{max} = h_0\}$, le superprocessus Z sous $\mathbb{N}_x$ a même loi que $Z^{(h_0)} = (Z_t^{(h_0)}, t \geq 0)$ défini par :

$$Z_t^{(h_0)} = \sum_{j\in J,\, s_j<t} Z^j_{t-s_j},\ t \geq 0.$$

Il s'agit d'une décomposition de la loi du superprocessus par rapport à son temps d'extinction $H_{\max}$. Conditionnellement à $\{H_{\max} = h_0\}$, les superprocessus greffés à la hauteur s ne peuvent excéder la hauteur h_0 du fait de la restriction à $\{H_{\max} \leq h_0 - s\}$ de la mesure de Poisson définie dans (1.30).

1.4.4 Le Q-processus d'un superprocessus inhomogène

L'objectif est maintenant de passer à la limite en h dans le théorème précédent. Ceci ne pose pas de difficultés en ce qui concerne la loi des superprocessus que l'on greffe : la restriction à $\{H_{\max} \leq h_0 - s\}$ dans l'énoncé du théorème s'en trouve simplement levée. La difficulté concerne la convergence du mouvement spatial $\mathrm{P}^{(h)}$. Pour cela, on a besoin de la notion de valeur propre généralisée, et ceci nous amène à nous restreindre aux deux cas où nous savons que cette valeur propre est bien définie, à savoir les cas où Y est une chaîne de Markov à espace d'état fini ou une diffusion dans $\mathbb{R}^K$. Notons $\mathcal{D}$ le domaine du générateur $\mathcal{L}$. La valeur propre généralisée λ_0 de l'opérateur $\beta - \mathcal{L}$ est définie dans ces deux cas par :

$$\lambda_0 = \sup \{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ telle que } (\beta - \mathcal{L})u = \ell\, u\}\cdot$$

Une interprétation plus probabiliste de cette quantité est fournie par la relation suivante :

$$\lambda_0 = - \sup_{A \subset E} \lim_{t \to \infty} \frac{1}{t} \log \mathrm{E}_x \big[\mathrm{e}^{-\int_0^t ds\, \beta(Y_s)} \mathbf{1}_{\{\tau_{A^c} > t\}} \big],$$

où le supremum est pris sur les sous-ensembles compacts A de E, et A^c désigne le complémentaire dans E de l'ensemble A. On renvoie au livre de Pinsky [107] lorsque Y est une diffusion sur $\mathbb{R}^d$, et au livre de Seneta [121] lorsque Y est une chaîne de Markov à espace d'état fini. Dans ce cas, la valeur propre généralisée correspond à la valeur propre de Perron Frobenius.

Nous définissons maintenant l'hypothèse de "product-criticality" d'après Pinsky [107]. L'opérateur $(\beta - \lambda_0) - \mathcal{L}$ est dit critique lorsque l'espace des fonctions harmoniques positives associées est non vide, mais la fonction de Green est infinie. Dans ce cas, l'espace vectoriel des fonctions harmoniques positives pour l'opérateur $(\beta - \lambda_0) - \mathcal{L}$ est de dimension 1, engendré par une fonction notée ϕ_0 appelée vecteur propre généralisé. De plus, l'ensemble des fonctions harmoniques positives de l'opérateur adjoint de $(\beta - \lambda_0) - \mathcal{L}$ est de dimension 1, engendré par une fonction notée $\tilde{\phi}_0$. Si en outre $\int_S dx\, \phi_0(x)\, \tilde{\phi}_0(x) < \infty$, l'opérateur $(\beta - \lambda_0) - \mathcal{L}$ est dit "product-critical", et la mesure de probabilité P^{ϕ_0}, donnée par :

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\phi_0(Y_t)}{\phi_0(Y_0)}\, \mathrm{e}^{-\int_0^t ds\ (\beta(Y_s) - \lambda_0)},$$

définit alors un processus de Markov récurrent au sens où il existe une mesure de probabilité ν sur E telle que :

$$\sup_{f \in b\mathcal{E}, \|f\|_\infty \leq 1} |\mathrm{E}^{\phi_0}_x[f(Y_t)] - \nu(f)| \xrightarrow[t \to +\infty]{} 0,$$

avec $b\mathcal{E}$ les fonctions mesurables de E dans $\mathbb{R}$ et $\|f\|_\infty = \sup_{x \in E} |f(x)|$ la norme infinie de f, voir le Théorème 9.9 p. 192 de Pinsky [107]. Ainsi, la notion d'opérateur "product-critical" est étroitement liée à la notion de récurrence positive. Nous énonçons maintenant un résultat de convergence des superprocessus $Z^{(h_0)}$, sous cette hypothèse de "product-criticality".

Théorème. [31]. *Supposons que la valeur propre généralisée est positive ou nulle, $\lambda_0 \geq 0$, que le vecteur propre généralisé ϕ_0 est minoré et majoré par des constantes strictement positives, que l'opérateur $(\beta - \lambda_0) - \mathcal{L}$ est "product critical". Supposons enfin α et β continues bornées et $\alpha \in \mathcal{C}^4$ bornée inférieurement par une constante positive dans le cas de la superdiffusion.*

Soit Y de loi $\mathrm{P}^{\phi_0}_x$, et, conditionnellement à Y, soit $\sum_{j \in \mathcal{I}} \delta_{(s_j, Z^j)}(ds, dZ)$ une mesure de Poisson d'intensité :

$$2\, \mathbf{1}_{\mathbb{R}^+}(s) ds\ \alpha(Y_s) \mathbb{N}_{Y_s}[dZ].$$

On considère le processus $Z^{(\infty)} = (Z^{(\infty)}_t, t \geq 0)$, défini par :

$$Z^{(\infty)}_t = \sum_{j \in J,\, s_j < t} Z^j_{t - s_j},\ t \geq 0,$$

et on note $\mathbb{N}^{(\infty)}_x$ sa distribution. Alors le processus $(Z^{(h_0)}_s, s \in [0, t])$ converge en loi vers $(Z^{(\infty)}_s, s \in [0, t])$ lorsque h_0 tend vers $+\infty$.

On précise, pour faire le lien entre les hypothèses des deux théorèmes précédents, que le caractère positif ou nul de λ_0 associé au fait que ϕ_0 soit minorée et majorée par deux constantes positives implique la propriété d'extinction presque sûre, d'après un argument de couplage explicité au lemme 2.6.1. En outre, ce théorème énonce un résultat sur le Q-processus, puisqu'on vérifie au lemme 2.5.1 que $\mathbb{N}_x^{(\infty)}$ correspond encore à la limite en loi de $\mathbb{N}_x^{(\geq h_0)} = \mathbb{N}_x\left[\cdot\ |H_{\max} \geq h_0\right]$ lorsque h_0 tend vers $+\infty$, ce qui constitue la définition du Q-processus. Ce théorème s'inscrit dans la littérature comme suit. Il permet de préciser la remarque 2.8 de Champagnat et Roelly [25], dans laquelle, après avoir défini le Q-processus en terme de h-transformée, les auteurs précisent qu'une construction de celui-ci en terme d'une "immigration interactive" est envisageable. Un processus similaire à $Z^{(\infty)}$ avait été défini auparavant dans Engländer et Kyprianou [43], et, à la discussion 2.2, les auteurs suggéraient que ce processus devait coïncider avec le Q-processus. Notre Théorème confirme donc cette suggestion.

Nous l'avons expliqué, notre intérêt dans ce travail réside principalement dans la façon dont le mouvement de la lignée généalogique la plus longue se trouve affecté par le caractère inhomogène du mécanisme de branchement. Engländer et Pinsky [44] s'intéressent à des superprocessus avec des mécanismes de branchement inhomogènes qui ne vérifient pas la propriété d'extinction presque sûre. Ils montrent que les lignées généalogiques infinies forment un arbre de Galton-Watson à temps continu, et que la loi de ces lignées infinies est P^w définie par (1.27) avec w une fonction positive telle que $\mathcal{L}(w) - \psi(w) = 0$. Ainsi, la loi P^w de ces lignées infinies dépend de $\mathcal{L}$, β et α, alors que la loi P^{ϕ_0} de l'unique lignée infinie du Q-processus ne dépend que de $\mathcal{L}$ et β.

Un dernier résultat concerne le superprocessus Z sous $\mathbb{N}_x^{(h)}$ vu depuis l'instant d'extinction h. Son énoncé nécessite l'introduction de $\mathrm{P}^{(-h)}$ la loi de Y sous $\mathrm{P}^{(h)}$ translatée de h :

$$\mathrm{P}^{(-h)}\left((Y_s, s \in [-h,0]) \in \bullet\right) = \mathrm{P}^{(h)}\left((Y_{h+s}, s \in [-h,0]) \in \bullet\right).$$

L'hypothèse de "product criticality" associée à $\lambda_0 > 0$ implique alors l'existence d'une mesure de probabilité $\mathrm{P}^{(-\infty)}$ telle que pour tout $x \in E$, $t \geq 0$:

$$\mathrm{P}_x^{(-h)}\left((Y_s, s \in [-t,0]) \in \bullet\right) \xrightarrow[h\to+\infty]{} \mathrm{P}^{(-\infty)}\left((Y_s, s \in [-t,0]) \in \bullet\right).$$

Théorème. [31]. *On suppose $\lambda_0 > 0$, ϕ_0 minorée et majorée par deux constantes strictement positives, et $(\beta - \lambda_0) - \mathcal{L}$ de type " product critical". De plus, on suppose α et β continues bornées et $\alpha \in \mathcal{C}^4$ bornée inférieurement par une constante positive dans le cas où Y est une diffusion.*

Soit Y de loi $\mathrm{P}^{(-\infty)}$, et, conditionnellement à Y, soit $\sum_{j\in J} \delta_{(s_j, Z^j)}$ une mesure ponctuelle de Poisson d'intensité :

$$2\ \mathbf{1}_{\{s<0\}}\alpha(Y_s)\, ds\ \mathbf{1}_{\{H_{max}(X)<-s\}}\ \mathbb{N}_{Y_s}[dZ].$$

On considère le processus $(Z_s^{(-\infty)}, s \leq 0)$, défini pour $s \leq 0$ par :

$$Z_s^{(-\infty)} = \sum_{j\in J,\ s_j<s} Z^j_{s-s_j}.$$

Alors le processus $(Z^{(h_0)}_{h_0+s}, s \in [-t,0])$ converge en loi vers $(Z_s^{(-\infty)}, s \in [-t,0])$ lorsque h_0 tend vers $+\infty$.

Du fait de l'indépendance entre structure généalogique et mouvement spatial dans la construction du superprocessus homogène explicitée en section 1.3.3, le Q-processus d'un superprocessus homogène peut être défini à partir du Q-processus du CB, comme nous l'avons déjà remarqué. La condition de récurrence que nous imposons sur le mouvement spatial pour obtenir le Q-processus peut donc sembler superflue dans ce cas. En revanche, cette condition est naturelle pour obtenir la convergence depuis le "sommet" énoncée dans le dernier Théorème.

On définit enfin la mesure de probabilité $\mathrm{P}_x^{(B,t)}$ suivante :

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\right]}.$$

Cette mesure de probabilité peut être vue comme la loi de la lignée ancestrale d'un individu choisi au hasard dans la population à l'instant t, voir la formule (2.44). Il s'agit également d'une pénalisation de "Feynman Kac" du mouvement spatial P_x, selon la terminologie de Roynette et Yor [115]. On prouve dans [31] que si ϕ_0 est minorée et majorée par deux constantes strictement positives, et si l'opérateur $(\beta - \lambda_0) - \mathcal{L}$ est "product -critical", alors $\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_s}$ converge en loi vers $\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_s}$ pour $s \geq 0$ fixé lorsque $t \to \infty$. Ceci peut aussi être interprété comme un état globulaire dans un modèle de polymère aléatoire, voir Cranston, Koralov and Molchanov [27].

1.4.5 Ouverture

En conclusion du travail [31] avec Jean-François Delmas :

– une extension possible : La limitation aux mécanismes de branchement quadratiques du type (1.23) tient au fait que nous voulions définir une généalogie. Cette limitation nous a permis de prouver nos résultats par "transport" à partir du superprocessus homogène comme expliqué en section 1.4.2. Cependant, pour ce qui est des résultats présentés dans cette introduction, formulés en terme de processus à valeurs mesures, on pourrait envisager une démonstration classique par l'analyse des équations différentielles partielles vérifiées par les cumulants $u_t(f)$, comme fait dans [43]. Cette approche permettrait d'envisager une généralisation à des mécanismes de branchement plus généraux du type :

$$\psi(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda u} - 1 + \lambda u \mathbf{1}_{u\leq 1})\nu(x, du).$$

  Dans ce cas, les formules exprimées en terme de ψ, comme (1.29), seraient inchangées.

– une ouverture : Les superprocessus inhomogènes modélisent de (grandes) populations dans lesquelles les individus se reproduisent encore indépendamment, mais plus identiquement : la loi de reproduction dépend en effet, comme on l'a vu, d'une coordonnée spatiale. On aurait pu imaginer des individus se reproduisant identiquement, mais plus indépendamment : par exemple, dans le modèle logistique, la loi de reproduction subit une rétroaction négative de la part de la population totale. Ce modèle, qui rend compte d'une population avec ressources limitées, a été étudié par Lambert [84], l'existence du Q-processus est un cas particulier du travail de Cattiaux *et al.* [24], et une généalogie a été définie par Pardoux et Wakolbinger [91]. On renvoie aussi au cours [104] de Saint-Flour de Perkins pour des modèles avec interactions plus générales. On peut encore se poser la question de la généalogie du Q-processus dans ces cas.

1.5 Généalogie : le système de particules look-down

Cette section fournit une seconde approche de la généalogie, valable pour des populations échangeables qui ne satisfont pas nécessairement la propriété de branchement, via un système de particules appelé look-down. Les principales applications que nous en avons tirées sont détaillées dans les deux sections suivantes, numérotées 1.6 et 1.7.

1.5.1 Construction du système de particules

Les processus de branchement apparaissent comme limite d'échelle de processus de Galton-Watson convenablement renormalisés, comme l'a établi Jiřina [70]. Pour établir la convergence jointe de la généalogie, une possibilité consiste à définir le processus de contour des processus de Galton-Watson, ce qui mène à la définition de la généalogie via les arbres réels. Une autre possibilité consiste à classer les individus dans le modèle discret en fonction de la persistence de leur descendance. A tout instant $t \geq 0$, on donne un *niveau* (dans $\mathbb{N}$) à chaque individu : 1 pour celui qui à la descendance la plus pérenne, 2 pour le suivant, etc... Donnelly et Kurtz appliquent cette idée dans [35] et prouvent ainsi la convergence de l'arbre généalogique restreint aux n premiers niveaux pour des modèles de populations neutres (c'est-à-dire composée d'individus identiques) très généraux. Ces modèles comprennent les processus de branchement, mais ne sont pas restreints à ces processus. De plus, les systèmes de particules limites sont compatibles quand n croît : on obtient alors un système de particules échangeable discret apte à décrire une population continue. Ce système de particules a été nommé look-down par Donnelly et Kurtz, en référence à un premier modèle [34]. Nous ne détaillerons pas les questions de convergence, et présentons maintenant la construction du système de particules limite.

On se donne deux processus càdlàg à valeurs dans $\mathbb{R}^+$: $X = (X_t, t \geq 0)$ le processus de masse totale, $\tau_0(X) = \inf\{t > 0, X_t = 0\}$, et $U = (U_t, t \geq 0)$ le processus de "rééchantillonage". On suppose $U_0 = 0$ et U croissant, et on décompose U comme suit :

$$U_t = U_t^k + \sum_{s \leq t} \Delta U_s,\ U^k \text{continu},\ \Delta U_s = U_s - U_{s-}.$$

Le point 0 sera supposé absorbant pour X. Enfin, pour pouvoir définir le modèle, on a besoin que $\Delta U_t \leq X_t^2$ pour tout $t \geq 0$. Conditionnellement à U et X, on définit alors deux mesures ponctuelles de Poisson N^ρ et N^k sur $[0, \tau_0(X)) \times \mathcal{P}_\infty$, où $\mathcal{P}_\infty$ représente l'ensemble des partitions de $\mathbb{N}$:

- $N^\rho = \sum_{0 \leq t < \tau_0(X), \Delta U_t \neq 0} \delta_{(t,\pi)}(dt, d\pi)$ où les partitions échangeables π sont i.i.d. selon la loi $\rho_x(d\pi)$ pour $x = \sqrt{\Delta U_t}/X_t$. Pour $0 \leq x \leq 1$, ρ_x désigne la loi de la partition aléatoire échangeable π qui comprend un unique bloc non trivial de fréquence asymptotique x. On la construit simplement comme suit : chaque entier fait partie du bloc non trivial de π avec probabilité x, indépendamment des autres entiers.
- $N^k = \sum_{0 \leq t < \tau_0(X)} \delta_{(t,\pi)}(dt, d\pi)$ est une mesure de Poisson, indépendante de N^ρ, d'intensité $(dU_t^k/(X_t)^2) \times \mu^k$, où μ^k est la mesure sur $\mathcal{P}_\infty$ qui donne une masse 1 à toute partition avec un unique bloc non réduit à un singleton composé de deux entiers, et une masse nulle aux autres.

Conditionnellement à (X, U, N^k, N^ρ), on définit ensuite un système de particules

$$\xi = (\xi_t(n), 0 \leq t < \tau_0(X), n \in \mathbb{N}),$$

avec $\xi_t(n)$ le type de l'individu au niveau n à l'instant t. La construction de ce système suit les règles suivantes :

- $(\xi_0(n), n \in \mathbb{N})$ est une suite i.i.d. de loi uniforme sur $[0,1]$.
- A chaque atome (t,π) de la mesure $N := N^k + N^\rho$, on associe un évènement de reproduction : soit $j_1 < j_2 < \ldots$ les élèments de l'unique bloc de π qui n'est pas un singleton. C'est ou bien une paire si (t,π) est un atome de N^k, auquel cas on conviendra de poser $j_k = \infty$ pour $j \geq 3$, ou bien un ensemble infini si (t,π) est un atome de N^ρ. Les individus aux niveaux $j_1 < j_2 < \ldots$ à l'instant t sont considérés comme étant les enfants de l'individu au niveau j_1 à l'instant $t-$, qui joue le rôle du père. Celui-ci leur transmet son type $\xi_{t-}(j_1)$, tandis que les autres types sont distribués aux individus restants comme suit : pour tout entier ℓ, $\xi_t(j_\ell) = \xi_{t-}(j_1)$ et pour tout $k \notin \{j_\ell, \ell \in \mathbb{N}\}$, $\xi_t(k) = \xi_{t-}(k - \#J_k)$ avec $\#J_k$ le cardinal de l'ensemble $J_k := \{\ell > 1, j_\ell \leq k\}$.

Ceci définit un système de particules ξ sur $[0, \tau_0(X))$. On pose ensuite $\xi_t(j) = \lim_{s \to \tau_0(X)} \xi_s(j)$ pour $t \geq \tau_0(X)$. Avec cette définition, conditionnellement à (X, U), la suite $(\xi_t(j), j \in \mathbb{N})$ est échangeable pour tout $t \geq 0$, voir la Proposition 3.1 de [35], et l'on note R_t sa mesure de de Finetti :

$$R_t(dx) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi_t(n)}(dx).$$

Une version càdlàg de ce processus à valeurs mesures de probabilité existe d'après [35], Théorème 3.2. Ce processus est encore appelé processus du ratio.

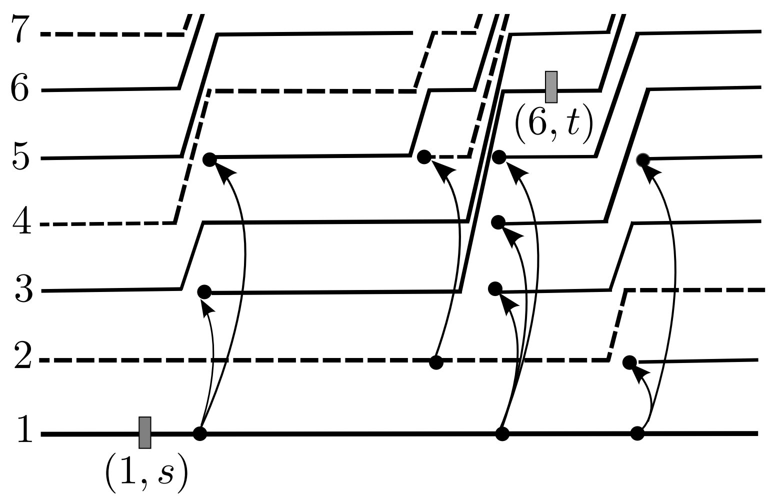

Figure 1.4: Les 7 premiers niveaux d'un système de particules look-down à deux types, symbolisés par les traits pleins ou les traits pointillés. On observera la façon dont sont transmis les types au moment d'un évènement de reproduction. L'individu 1 à l'instant s est un ancêtre de l'individu 6 à l'instant ultérieur t : une flèche permet en effet de passer du niveau 1 au niveau 3, puis un autre évènement de reproduction peu avant l'instant t envoie le niveau 3 au niveau 6.

Le processus d'intérêt, noté $Z(dx) = (Z_t(dx), t \geq 0)$, est alors défini comme suit :

$$(Z_t(dx), t \geq 0) = (X_t \, R_t(dx), t \geq 0). \tag{1.31}$$

La mesure finie $Z_t(dx)$ sur $[0,1]$ représente donc une population dont la masse totale est donnée par X_t et dont les types sont rééchantillonnés selon le processus U. L'approche look-down consiste en une factorisation polaire (1.31) de la mesure $Z(dx)$: les deux facteurs sont la taille totale X de la population d'une part et le ratio R d'autre part, qui comprend l'information de la généalogie. L'idée d'une telle factorisation remonte au moins à Shiga [124].

1.5.2 Superprocessus homogènes et processus de Λ-Fleming-Viot

Superprocessus homogènes

Lorsque X est un CB(ψ) pour ψ de la forme (1.4) conservatif, et $U = [X]$ est sa variation quadratique, le processus $Z(dx)$ est le superprocessus homogène, ou processus de Dawson-Watanabe, sans déplacement spatial sur $[0,1]$, déjà introduit en section 1.3.3. Ce résultat non trivial est prouvé dans [35], Section 4, voir également Birkner *et al.* [18]. Les mesures N^ρ et N^k prennent alors la forme suivante :

- $N^\rho(ds, d\pi)$ est une mesure ponctuelle de Poisson de compensateur prévisible :

$$X_{s-}\ ds \int_{(0,1)} \varphi^\star_{X_{s-}}(\nu)(dr)\ \rho_r(d\pi),$$

  avec $\varphi_y : x \to x/(x+y)$ et $\varphi^\star_y(\nu)(dr)$ la mesure image de ν par φ_y, qui est donc une mesure sur $(0,1)$.
- $N^k(ds, d\pi)$ est une mesure ponctuelle de Poisson, indépendante de N^ρ, de compensateur prévisible :

$$\frac{\alpha}{X_{s-}}\ ds\ \mu^k(d\pi)$$

  pour $0 \le s < \tau_0(X)$.

On notera que le drift β qui apparaît dans (1.4) ne joue de rôle qu'au travers de X_{s-} dans les expressions des deux compensateurs.

On notera également que si les évènements de reproduction sont clairement identifiables dans le modèle look-down, les évènements de mort sont possibles, et se font par poussée du niveau à l'infini. Dans le cas du processus de Dawson-Watanabe, les évènements de morts surviennent si et seulement si la condition de Grey est vérifiée.

Mentionnons qu'il est difficile de trouver la généralisation du système de particules look-dwon apte à représenter des superprocessus inhomogènes, voir néanmoins [78].

Processus de Λ-Fleming-Viot

Lorsque $X = 1$ et U est un subordinateur, le processus $Z(dx) = R(dx)$ est appelé processus de Λ-Fleming-Viot, (ou plus simplement Λ-Fleming-Viot) sans déplacement spatial sur $[0,1]$, pour Λ une mesure finie sur $[0,1]$ liée à l'exposant de Laplace $c\lambda + \int_{(0,1]}(1 - \mathrm{e}^{-\lambda x})\nu^U(dx)$ du subordinateur U comme suit :

$$\int_{[0,1]} g(x)\Lambda(dx) = cg(0) + \int_{(0,1]} g(\sqrt{x})x\nu^U(dx), \tag{1.32}$$

voir la section 3.1.4 de [35]. Lorsque $\Lambda(dx) = \delta_0(dx)$, on parle simplement de processus de Fleming-Viot. Posons $\nu(dx) = x^{-2}\Lambda_{|(0,1]}(dx)$. Les mesures N^ρ et N^k prennent alors la forme particulièrement simple suivante :

– $N^\rho(ds, d\pi)$ est une mesure ponctuelle de Poisson d'intensité $ds \ \int_{(0,1]} \nu(dx) x^{-2} \rho_x(d\pi)$.
– $N^k(ds, d\pi)$ est une mesure ponctuelle de Poisson d'intensité $c\ ds\ \mu^k(d\pi)$, indépendante de N^ρ.
Lorsque $c = 0$, la dynamique du Λ-Fleming Viot $Z(dx) = R(dx)$ est particulièrement simple à décrire. On se donne une mesure de Poisson d'intensité $dt\, x^{-2}\ \nu(dx)$, puis pour chaque atome (t, x) de cette mesure, on ajourne comme suit la valeur du processus :

$$R_t = (1 - x)\ R_{t-} + x\ \delta_U \text{ avec } U \text{ de loi } R_{t-}.$$

où les différents choix de U sont indépendants les uns des autres, et sont indépendants de la mesure de Poisson.

Le système de particules donne un accès direct aux propriétés dynamiques des processus de Fleming-Viot. On consultera Pfaffelhuber et Wakolbinger [106] et Delmas, Dhersin et Siri-Jégousse [32] pour des travaux sur le MRCA dans un cadre à population constante, à comparer avec ceux mentionnés en Section 1.2.4 pour des populations branchantes.

On peut se demander quel est le lien entre la notion de généalogie définie par le système de particules look-down et celle définie par les arbres continus dans les cas où les deux constructions ont du sens, c'est-à-dire pour des processus de Dawson-Watanabe homogènes construits à partir de CB ou de CBI. Ou encore quel est le lien avec la notion de généalogie définie par les flots de ponts (dans le cas des processus de Λ-Fleming-Viot) ou par les flots de subordinateurs (dans le cas des CB), ainsi que les ont définies Bertoin et Le Gall [14, 15]. Berestycki, Berestycki et Schweinsberg [8] et, plus récemment, Labbé [81, 82] font le lien entre ces différentes approches.

Notre contribution consiste en deux nouvelles applications [55, 65]. Notons cependant que l'article [55] avec Clément Foucart a finalement fait l'objet d'une rédaction qui ne fait pas appel à ce système de particules. La fin de cette section détaille quelques applications immédiates du système de particules look-down.

1.5.3 M-Fleming-Viot

Donnelly et Kurtz considèrent une population d'individus tous identiques (au sens où ils ont la même loi de reproduction), et classent ces individus en fonction de la perennité de leur descendance (en "regardant dans le futur"). L'ajout du niveau différencie implicitement les individus : en effet, l'individu de niveau 1 doit heuristiquement mieux se reproduire que l'individu de niveau 2, pour assurer le fait que sa descendance est plus pérenne ; de même, l'individu de niveau 2 doit mieux se reproduire que l'individu de niveau 3 et ainsi de suite.

Un calcul simple permet de confirmer ce fait rigoureusement dans le cadre d'un Λ-Fleming-Viot. Supposons pour simplifier que $\Lambda\{0\} = 0$. On rappelle que $\nu(dx) = x^{-2}\Lambda_{|(0,1]}(dx)$. Soit $i \geq 1$ un entier. Un évènement de reproduction qui génère une fraction x de la population advient à taux $\nu(dx)$, et, conditionnellement à la donnée d'un tel évènement, l'individu de niveau i est choisi comme père avec probabilité $(1 - x)^{i-1}x$. L'individu de niveau i engendre donc une fraction x de la population à taux

$$x(1 - x)^{i-1}\nu(dx) = x^{-1}(1 - x)^{i-1}\Lambda(dx).$$

Ainsi, le taux de reproduction $\nu(dx)$ d'un individu "moyen" se trouve biaisé par $x(1 - x)^{i-1}$, qui est une fonction décroissante de i pour tout $x \in [0, 1]$.

Nous profitons de l'occasion pour introduire la notion de processus de M-Fleming-Viot, définie par Clément Foucart dans [54], et que nous étudions plus avant dans l'article [55]. Si l'on conçoit les immigrants comme étant les fils d'un individu immortel *membre* de la population, étant donnée la construction du modèle look-down, cet individu doit être l'individu au niveau 1 (sous réserve qu'il existe un unique individu immortel dans la population). Soit un couple de mesures $M = (\Lambda_0, \Lambda_1)$, avec Λ_0 et Λ_1 deux mesures finies sur $[0, 1]$. Supposons pour simplifier que Λ_0 et Λ_1 n'ont pas d'atome en 0. Le M-Fleming-Viot peut être défini à partir d'une modification du modèle lookdown usuel :

– L'individu de niveau 1 se comporte comme dans le système de particules look-down d'un Λ_0-Fleming-Viot usuel, c'est-à-dire qu'il génère une fraction x de la population avec intensité
$$x^{-1}\Lambda_0(dx).$$
– Les individus aux niveaux supérieurs $\{2, 3, \dots\}$ se comportent comme dans le système de particules look-down d'un Λ_1-Fleming-Viot dont les niveaux commenceraient à 2, et non à 1. Ainsi, l'individu de niveau $i \geq 2$ génère une fraction x de la population avec intensité
$$x^{-1}(1-x)^{i-2}\Lambda_1(dx) = x^{-1}(1-x)^{i-1}\frac{\Lambda_1(dx)}{1-x}.$$

En comparant les taux de reproduction, un Λ-Fleming-Viot apparaît comme un cas particulier de M-Fleming-Viot avec $M = (\Lambda, (1-x)\Lambda)$. Ceci fait l'objet du lemme 4.4.1 dans le cas général. On notera que le modèle des M-Fleming-Viot ne donne plus lieu en général à un système de particules échangeable.

1.5.4 Superprocessus homogènes et δ_0-Fleming-Viot : un premier lien

Le modèle look-down consiste en une description de la généalogie conditionnellement à la taille totale de la population X et au processus de rééchantillonage U. Dès lors, il est très facile de comprendre l'effet sur la généalogie du conditionnement par une des deux quantités X et U.

Un tel conditionnement est dû à Etheridge et March [45], et donne un lien entre :
– les processus de Dawson-Watanabe, de masse totale des CB(ψ).
– les Λ-Fleming-Viot, de processus de rééchantillonage un subordinateur U.
Soit un processus de Dawson-Watanabe $Z(dx)$ de masse totale X un CB(ψ) avec $\psi(\lambda) = 2\lambda^2$, que nous conditionnons par l'évènement $\{X_s = 1, s \geq 0\}$. Bien entendu, cet évènement est de probabilité nulle, et il faut donc définir une façon de l'approcher. On peut par exemple fixer ϵ et $t \geq 0$ et conditionner par l'évènement $\{|X_s - 1| \leq \epsilon, 0 \leq s \leq t\}$. Lorsque ϵ tend vers 0, la loi du processus $((X_s, U_s), 0 \leq s \leq t)$ conditionné converge vers celle de $((1, 4s), 0 \leq s \leq t)$. Ensuite, par construction du modèle look-down, le processus Z construit à partir de
$$((X_s, U_s), 0 \leq s \leq t) = ((1, 4s), 0 \leq s \leq t)$$
est un $(4\delta_0)$-Fleming-Viot restreint à $[0, t]$. Nous verrons un second lien entre superprocessus homogènes et Λ-Fleming-Viot en section 1.7.

1.6 Changement de mesure dans le système de particules look-down

Cette section se rapporte au travail [65]. Elle fournit de nouvelles applications du système de particules look-down. Nous définissons notamment le Q-processus pour les Λ-Fleming-Viot en section 1.6.2, dont nous fournissons une construction trajectorielle par effacement de certains évènements de reproduction.

1.6.1 Le Q-processus d'un superprocessus homogène

On considère un processus de Dawson-Watanabe $Z(dx)$ construit comme expliqué en section 1.5.2 à partir de X un CB(ψ) sous-critique ou critique qui satisfait la condition de Grey (1.6). On sait de la section 1.2.5 que le CB X conditionné à la non extinction, noté X^∞ dans la suite, est un CBI(ψ,$\psi' - \psi'(0+)$). Le Q-processus associé au processus à valeurs mesures $Z(dx)$ est le processus à valeurs mesures $Z^\infty(dx)$, construit à partir de X^∞ et $U = [X^\infty]$ comme expliqué en section 1.5.2. Une question naturelle, sachant que X^∞ est un processus de branchement avec immigration, consiste à se demander d'où provient l'immigration *dans le système de particules* look-down associé à Z^∞. On suppose pour simplifier que $\alpha = 0$ dans l'expression (1.4) du mécanisme de branchement de ψ. De la construction poissonienne (1.12) de X^∞, on déduit que $\sum_{0\le s\le t} \delta_{(s,\Delta X_s^\infty)}(ds, du)$ admet pour compensateur prévisible

$$ds\ (X_{s-}^\infty\ \nu(du) + u\nu(du)).$$

Dans cette expression, le terme $ds\ X_{s-}^\infty\ \nu(du)$ correspond au terme de branchement, et la multiplication par X_{s-}^∞ à la transformation de Lamperti expliquée en section 1.2.2. Le terme $ds\ u\nu(du)$, indépendant de la taille X_{s-} de la population, correspond au terme d'immigration. Maintenant, conditionnellement à la valeur du saut $\Delta X_s^\infty = u$, l'évènement $\{j_1(s) = 1\}$ a pour probabilité

$$\frac{u}{X_s^\infty} = \frac{u}{X_{s-}^\infty + u}.$$

La mesure ponctuelle

$$\sum_{0\le s\le t} \delta_{(s,\Delta X_s^\infty)}(ds, du)\mathbf{1}_{\{j_1(s)=1\}}$$

admet donc pour compensateur prévisible :

$$ds\ \left(\frac{u}{X_{s-}^\infty + u}\right)(X_{s-}^\infty\nu(du) + u\nu(du)) = ds\ u\nu(du).$$

On en déduit le résultat suivant :

Proposition. [65]. *Le processus* $\left(\sum_{0\le s\le t} \Delta X_s^\infty\ \mathbf{1}_{\{j_1(s)=1\}}, t \ge 0\right)$ *est un processus de Lévy à sauts purs, de mesure de Lévy* $u\nu(du)$*, c'est-à-dire un subordinateur d'exposant de Laplace* $\psi'(\lambda) - \psi'(0+)$.

En d'autres termes : l'immigration provient de l'individu au niveau 1, et on retrouve le fait que cet individu génère une fraction x de la population selon la mesure de Lévy biaisée $u\nu(du)$. (Attention à ne pas confondre ce résultat avec la loi de reproduction de la particule de niveau 1 dans le modèle du Λ-Fleming-Viot vue en section 1.5.3.) Le même résultat avec l'ajout d'une composante brownienne ($\alpha \neq 0$) est plus délicat à obtenir, et fait l'objet d'une remarque dans l'article [65].

1.6.2 Le Q-processus d'un Λ-Fleming-Viot

Nous avons conditionné en section 1.2.5 un CB vérifiant $\{\tau_0(X) < \infty\}$ p.s. à l'évènement $\{\tau_0(X) = \infty\}$ par le biais d'un passage à la limite, et avons appelé le processus résultant Q-processus. Sous une condition explicite (que nous donnons en (1.39)), le Λ-Fleming-Viot vérifie la propriété d'absorption p.s., au sens où R_t est, pour t assez grand, p.s. réduit à une masse de Dirac en le type aléatoire $\xi_0(1)$. Ceci signifie que le seul type $\xi_0(1)$ subsiste au bout d'un temps suffisamment long. Notre objectif est maintenant de conditionner tout Λ-Fleming-Viot qui vérifie la propriété d'absorption p.s. à la non absorption. Cet objectif avait déjà été menée à bien à l'aide de techniques analytiques dans le cas de la diffusion de Wright-Fisher, on pourra consulter Lambert [86], qui pointe lui-même vers les travaux de Kimura [73]. Nous proposons une approche nouvelle, basée sur le système de particules look-down, qui permet de calculer le générateur du Q-processus, et permet en outre d'expliquer la forme de ce générateur.

Nous supposons donc dans toute cette section que $Z(dx) = R(dx)$ est un Λ-Fleming-Viot, et on rappelle que l'exposant de Laplace du subordinateur U est lié à la mesure Λ par l'équation (1.32). On choisit de travailler avec l'espace des types $E = \{1, \dots, K'\}$ plutôt que $[0, 1]$ par commodité.

Un outil important dans cette étude est le changement de filtration. On définit à cet effet les deux filtrations d'intérêt :
– $(\mathcal{F}_t = \sigma\{(\xi_s(n), n \in \mathbb{N}), 0 \le s \le t\})$ la filtration associé au système de particules.
– $(\mathcal{G}_t = \sigma\{R_s, 0 \le s \le t\})$ la filtration associée au processus R.
On considère le processus R conditionné à la coexistence des K premiers types à l'instant t, pour K entier fixé compris entre 1 et K' :

$$\forall A \in \mathcal{G}_t,\ \mathbb{P}(R^{(\ge t)} \in A) = \mathbb{P}(R \in A | \prod_{i=1}^{K} R_t\{i\} \neq 0),$$

Il n'est pas aisé de définir la structure probabiliste du processus $R^{(\ge t)}$ à t fixé. Néanmoins, pour t grand, on peut en s'aidant du système de particules look-down prouver que $R^{(\ge t)}$ restreint à une fenêtre de temps $[0, s]$ près de l'origine, admet une structure simple, qui peut être décrite en fonction d'un nouveau système de particules ξ^∞. Un élément clef dans l'analyse est l'introduction de $L(t)$ le premier niveau auquel les K premiers types sont apparus dans le système de particules original ξ :

$$L(t) = \inf\{i \ge K, \{1, \dots, K\} \subset \{\xi_t(1), \dots, \xi_t(i)\}\}.$$

On définit le système de particules ξ^∞, semblable au système de particules original ξ, avec deux différences majeures : les K premiers types sont forcés d'occuper les K premiers niveaux, puis les évènements de reproduction impliquant au moins 2 des K premiers niveaux sont interdits, de sorte que les K premiers types restent aux K premiers niveaux à tout instant ultérieur. La définition précise est la suivante :
(i) La suite finie $(\xi_0^\infty(j), 1 \le j \le K)$ est une permutation uniforme de $\{1, \dots, K\}$, et la suite $(\xi_0^\infty(j), j \ge K + 1)$ est une suite aléatoire indépendante échangeable, de fréquence asymptotique R_0^∞ de loi :

$$\mathbb{P}(R_0^\infty \in A) = \mathbb{E}\left(\mathbf{1}_A(R_0)\frac{\prod_{i=1}^K R_0\{i\}}{\mathbb{E}(\prod_{i=1}^K R_0\{i\})}\right).$$

(ii) Les évènements de reproduction sont gouvernés par la restriction de la mesure de Poisson N (définie dans à la section 1.5.1 lors de la construction de R) à l'ensemble

$$V := \left\{(s, \pi), \pi_{|[K]} = \{\{1\}, \{2\}, \dots, \{K\}\}\right\},$$

où $\pi_{|[K]}$ désigne la restriction de la partition π à $\{1, \dots, K\}$.
Par un argument d'absolue continuité, on montre que le système de particules ξ^∞ ainsi construit est bien défini, et admet une mesure de de Finetti notée R_t^∞ :

$$R_t^\infty(dx) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi_t^\infty(n)}(dx),$$

Notons $\mathbb{P}_K$ la loi du processus de Markov $(L(t), t \geq 0)$ issu de K. Nous pouvons alors énoncer le Théorème suivant.

Théorème. [65]. *Soit $s \geq 0$ fixé. Si :*

$$\lim_{t\to\infty} \frac{\mathbb{P}_{K+1}(L(t) < \infty)}{\mathbb{P}_K(L(t) < \infty)} = 0, \tag{1.33}$$

alors la famille de processus $(R_u^{(\geq t)}, 0 \leq u \leq s)$ converge en loi vers le processus $(R_u^\infty, 0 \leq u \leq s)$ quand t tend vers ∞.

Noter que la condition (1.33) implique l'absorption p.s. En effet, on a $\mathbb{P}_{K+1}(L(t) < \infty) \to 0$ de (1.33), et donc L est infini p.s. en temps fini. Des conditions suffisantes sous lesquelles la condition (1.33) vaut sont explicitées dans l'article. On introduit, pour $i \geq 1$, les quantités :

$$r_i = \frac{i(i-1)}{2} c + \int_{(0,1]} \nu(dx) \left(1 - (1-x)^i - ix(1-x)^{i-1}\right).$$

Elles représentent le paramètre de la variable aléatoire exponentielle qui gouverne le temps passé par une particule au niveau i dans le sytème de particules look-down original ξ. On a alors l'égalité suivante :

Proposition. [65]. *Soit $t \geq 0$, et $A \in \mathcal{G}_t$:*

$$\mathbb{P}(R^\infty \in A) = \mathbb{E}\left(\mathbf{1}_A(R) \frac{\prod_{i=1}^K R_t\{i\}}{\mathbb{E}(\prod_{i=1}^K R_0\{i\})} \mathrm{e}^{r_K t}\right). \tag{1.34}$$

Noter que cette Proposition vaut sans faire l'hypothèse d'absorption p.s..

Plaçons nous désormais dans le cas $K = K' = 2$. La population compte donc deux types, et nous conditionnons le processus à la survie de ces types en temps long. Dans ce cadre simple, le Λ-Fleming-Viot est maintenant assimilable au processus $(R_t\{1\}, t \geq 0)$ à valeurs dans $[0,1]$, et Bertoin et Le Gall donnent dans [15] son générateur infinitésimal :

$$Gf(x) = \frac{1}{2}cx(1-x)f''(x) + x\int_{(0,1]} \nu(dy)[f(x(1-y)+y) - f(x)] + (1-x)\int_{(0,1]} \nu(dy)[f(x(1-y)) - f(x)],$$

de domaine $f \in \mathcal{C}^2([0,1])$. On obtient alors le théorème suivant, et son interprétation à suivre :

Théorème. [65]. *Supposons $K = K' = 2$. Définissons pour $f \in \mathcal{C}^2([0,1])$, et $x \in [0,1]$:*

$$G^0 f(x) = c(1-2x)f'(x) + \int_{(0,1]} y(1-y)\nu(dy)[f(x(1-y)+y) - f(x)] + \int_{(0,1]} y(1-y)\nu(dy)[f(x(1-y)) - f(x)],$$

and

$$G^1 f(x) = \frac{1}{2} cx(1-x)f''(x) + x \int_{(0,1]} (1-y)^2 \nu(dy)[f(x(1-y)+y)) - f(x)] \\ + (1-x) \int_{(0,1]} (1-y)^2 \nu(dy)[f(x(1-y)) - f(x)].$$

Alors l'opérateur $G^0 + G^1$ est le générateur du processus $(R_t^\infty, t \geq 0)$.

Une preuve directe dans le cadre d'un Λ-Fleming-Viot à sauts purs, c'est à dire pour lequel $\Lambda\{0\} = 0$, est la suivante. On écrit :

$$\nu(dy) = 2y(1-y)\nu(dy) + (1-y)^2\nu(dy) + y^2\nu(dy),$$

puis on interprète comme suit les différents termes de la somme :

1. Le premier terme est la somme des deux mesures $y(1-y)\nu(dy)$ qui apparaissent dans l'expression de G^0. Chacune de ces deux mesures correspond à l'intensité des évènements de reproduction qui concernent le niveau 1 mais pas le niveau 2, ou le niveau 2 mais pas le niveau 1, puisque ces évènements ont pour probabilité $y(1-y)$ lorsque l'évènement de reproduction implique une fraction y de la population. Nous interprétons ces évènements comme des évènements d'immigration.
2. Le second terme correspond à la mesure $(1-y)^2\nu(dy)$qui apparaît dans le générateur G^1 et correspond à l'intensité des évènements de reproduction qui n'impliquent ni 1 ni 2 : à nouveau, cet évènement a pour probabilité $(1-y)^2$ lorsque l'évènement de reproduction implique une fraction y de la population. Le père est alors de type 1 avec probabilité x la fréquence asymptotique des particules de type 1, et de type 0 avec probabilité $1-x$. Nous interprétons ces évènements comme des évènements de reproduction.
3. Le troisième terme de la somme n'apparaît ni dans l'expression de G^0, ni dans celle de G^1 : il correspond à l'intensité des évènements de reproduction qui impliquent à la fois les niveaux 1 et 2, mais ces évènements ont été effacés dans notre construction de R^∞.

Nous donnons dans l'article une version plus générale, incluant les cas où $\Lambda\{0\} \neq 0$, dont la preuve est basée sur l'expression (1.34) de R^∞ comme h-transformée de R. Il est intéressant de comparer le Q-processus du Λ-Fleming-Viot avec le Q-processus des processus de branchement : alors qu'il est nécessaire d'ajouter des évènements de reproduction lorsqu'on conditionne un processus de branchement à la non-extinction (voir la construction de Kesten de l'arbre de Galton-Watson conditionné à la non extinction par exemple), il faut effacer des évènements de reproduction lorsqu'on conditionne un Λ-Fleming-Viot à la non-absorption. On peut certes interpréter ξ^∞ comme un système de particules où les K premiers niveaux agissent comme K sources d'immigration, mais alors les niveaux suivants se reproduisent selon une mesure biaisée égale à $(1-y)^K\Lambda(dy)$, et cette mesure est dominée par $\Lambda(dy)$ au sens de l'ordre stochastique.

1.6.3 Une relation d'entrelacement

Nous concluons la présentation des résultats de [65] par une décomposition trajectorielle de la diffusion de Wright-Fisher, conséquence directe de la construction du système de particules look-down.

Lorsque $Z = R$ est un $(c\delta_0)$-Fleming-Viot, le processus $(R_t([0,x]), t \geq 0)$ est appelée diffusion de Wright-Fisher issue de x, de générateur infinitésimal :

$$Gf(x) = \frac{1}{2}cx(1-x)f''(x)$$

pour $f \in \mathcal{C}^2([0,1])$. Plutôt que de choisir les types initiaux $(\xi_0(n), n \geq 1)$ uniformes dans $[0,1]$, on prend des variables de Bernoulli indépendantes dans $\{0,1\}$, égales à 1 avec probabilité x, de sorte que $(R_t\{1\}, t \geq 0)$ a la loi d'une diffusion de Wright-Fisher issue de x. Ceci nous permet d'être en accord avec les notations de [65]. On considère alors le premier niveau occupé par une particule de type 1 :

$$L^1(t) = \inf\{i \geq 1, 1 \in \{\xi_t(1), \ldots, \xi_t(i)\}\}$$

à valeurs dans $\mathbb{N} \cup \{\infty\}$. Par construction du système de particules look-down, le processus L saute de ℓ à $\ell+1$ à taux $c\ell(\ell-1)/2$. Ensuite, conditionnellement à $\{L^1 = \ell\}$, le processus $(R_t\{1\}, t \geq 0)$ est une diffusion de Wright-Fisher avec $\ell - 1$ sources d'immigration de type 0 et une source d'immigration de type 1, et on peut alors montrer que le générateur d'une telle diffusion vaut

$$G^\ell f(x) = \frac{1}{2}cx(1-x)\partial_{xx}f(x,\ell) + c\left[(1-x) - (\ell-1)x\right]\partial_x f(x,\ell).$$

On en déduit que l'opérateur $\hat{G}$ défini par

$$\hat{G}f(x,\ell) = G^\ell f(x) + c\frac{\ell(\ell-1)}{2}\left[f(x,\ell+1) - f(x,\ell)\right],$$

pour les fonctions f qui, en tant que fonction de x, appartiennent à $\mathcal{C}^2([0,1])$, est le générateur infinitésimal du processus $(R\{1\}, L^1)$.

On a donc interprété la trajectoire d'une diffusion de Wright-Fisher comme la concaténation de trajectoires de diffusions de Wright-Fisher avec immigration. Une question naturelle est alors la suivante : Les temps de sauts du processus L sont-ils identifiables à la lecture du processus de Wright-Fisher ? C'est-à-dire : sont-ils mesurables par rapport à la filtration $\mathcal{G}$?

En outre, cette décomposition trajectorielle donne lieu à une relation analytique. L'espérance conditionnelle de L sachant $R\{1\}$ est donnée par le noyau K défini comme suit :

$$K(x,\ell) = (1-x)^{\ell-1}x, x \in (0,1], \ell \in \mathbb{N} \quad \text{et} \quad K(0,\infty) = 1.$$

Ainsi, le processus L peut être vu comme une fonction aléatoire de $R\{1\}$, le noyau K jouant le rôle de fonction aléatoire. De plus, L est un processus markovien. Les fonctions (aléatoires) markoviennes de processus markoviens ont été étudiées par Rogers et Pitman [113] (nous rappelons leur résultat principal au Théorème 3.2.14). Leurs générateurs vérifient des relations dites d'entrelacemenent, dont nous donnons maintenant un exemple.

Proposition. [65]. *Soit f dans le domaine de $\hat{G}$ et $x \in (0,1]$. On a*

$$\hat{K}\hat{G}(f)(x) = G\hat{K}(f)(x).$$

où $\hat{K}$ agit comme suit sur les fonctions $f(x,\ell)$: $\hat{K}f(x) = \sum_{\ell \geq 1} K(x,\ell)f(x,\ell)$.

On notera enfin qu'une autre relation d'entrelacement pour les diffusions de Wright-Fisher a été établie par Swart [126], qui lie la diffusion de Wright-Fisher et son Q-processus.

1.6.4 Ouverture

En conclusion du travail [65], deux questions :

– Comment décrire le Q-processus d'un processus de Fleming-Viot avec sélection ? Un système de particules look-down de mesure de de Finetti le processus de Fleming-Viot avec sélection a été proposé dans [34] et, plus récemment, dans Bah, Pardoux et Sow [6]. Il est naturel de se demander si cette représentation est encore adaptée à la dérivation du Q-processus dans ce nouveau contexte.
– La condition (1.33) dans l'énoncé du Théorème 3.2.6 est-elle équivalente à l'hypothèse d'absorption presque sûre ? S'il est possible de montrer que (1.33) implique l'absorption presque sûre, nous n'avons pas réussi à établir la réciproque. Nous pensons néanmoins qu'elle est vraie.

1.7 Superprocessus homogènes et Λ-Fleming-Viot : un second lien

Nous avons vu un premier lien entre superprocessus homogènes et processus de Fleming-Viot à la section 1.5.4 : un superprocessus homogène conditionné à avoir une masse totale constante est un processus de Fleming-Viot. Peut-on retrouver un processus de Fleming-Viot sans contraindre la masse totale ? On savait depuis Birkner *et al.* [18] que cela est effectivement possible dans certains cas particuliers de superprocessus, liés à des mécanismes de branchement stables. Nous donnons de nouveaux exemples dans l'article [55] avec Clément Foucart en ajoutant une immigration supplémentaire.

1.7.1 Le processus du ratio de la diffusion de Wright Fisher

Soit Z le superprocessus homogène sans déplacement spatial construit à partir de X un CBI et $U = [X]$ sa variation quadratique. Notons $X_t(x) = Z_t([0,x])$ pour $t \geq 0$ et pour $x \geq 0$. Notons qu'une construction directe du superprocessus $(X_t(x), t \geq 0, x \geq 0)$ est fournie par le théorème de Daniell-Kolmogorov. Fixons $x \in [0,1]$. Peut on, moyennant un changement de temps, rendre le processus du ratio $R_t(x) = X_t(x)/X_t(1)$ Markovien ?

La réponse est aisée dans le cas du CB(ψ) avec $\psi(\lambda) = 2\lambda^2$. On sait en effet que ce CB(ψ) est sous $\mathbb{P}_x$ l'unique solution en loi de l'équation différentielle stochastique d'inconnue X :

$$X_t(x) = x + 2\int_0^t \sqrt{X_s(x)}dB_s,$$

avec B un mouvement brownien standard. Une application de la formule d'Itô assure alors que le ratio $R_t(x) = X_t(x)/X_t(1)$ pour $0 \leq x \leq 1$ défini pour $0 \leq t < \tau_0(X(1))$, satisfait l'équation différentielle stochastique suivante :

$$R_t(x) = x + 2\int_0^t \sqrt{\frac{R_s(x)(1 - R_s(x))}{X_s(1)}}dB_s, \quad 0 \leq t < \tau_0(X(1)),$$

avec B un mouvement brownien standard indépendant de $X(1)$. On prendra soin de décomposer $X_t(1)$ en la somme des deux processus $X_t(x)$ et $X_t(1) - X_t(x)$, indépendants d'après la propriété de branchement, au moment d'appliquer la formule d'Itô. Posons $C(t) = \int_0^t ds/X_s(1)$. Alors $C(\tau_0(X(1)))$ est infini p.s., d'après l'argument de scaling utilisé dans la preuve de la Proposition

4.3.2 par exemple. Le processus $\tilde{R} = R \circ C^{-1}$, défini sur $\mathbb{R}^+$, est indépendant de $(X_s(1), s \geq 0)$ et satisfait l'équation différentielle stochastique suivante :

$$\tilde{R}_t(x) = x + 2\int_0^t \sqrt{\tilde{R}_s(x)(1 - \tilde{R}_s(x))}dB_s. \tag{1.35}$$

avec B un mouvement brownien standard. Cette équation admet une unique solution en loi, qui est une diffusion de Wright-Fisher : en particulier, la propriété de Markov est vérifiée. De plus, le processus $\tilde{R}(x)$ est indépendant de $X(1)$. Ces résultats peuvent être vus comme une version simplifiée du théorème de Perkins, voir [105].

Il est également possible de considérer le cas du CBI(ψ, ϕ) avec $\psi(\lambda) = 2\lambda^2$ et $\phi(\lambda) = 4\lambda$, noté X^∞, qui est le Q-processus du CB(ψ). A nouveau, on pose, pour t et $x \geq 0$, $X_t^\infty(x) = Z_t^\infty([0, x])$ pour Z^∞ construit à partir de X^∞ un CBI et U sa variation quadratique. Alternativement, on peut encore construire $(X_t^\infty(x), t \geq 0, x \geq 0)$ à partir du théorème de Daniell-Kolmogorov. On considère alors $R_t^\infty(x) = X_t^\infty(x)/X_t^\infty(1)$ pour $0 \leq x \leq 1$. Le processus X^∞ est l'unique solution de l'équation différentielle stochastique :

$$X_t^\infty(x) = x + 2\int_0^t \sqrt{X_s^\infty}dB_s + 2ds.$$

Posons $C^\infty(t) = \int_0^t ds/X_s^\infty(1)$. Alors $\tau_0(X^\infty(1)) = \infty$ et $C^\infty(\tau_0(X^\infty(1))) = \infty$, voir à nouveau la preuve de la Proposition 4.3.2. Avec $\tilde{R}^\infty = R^\infty \circ (C^\infty)^{-1}$, le même raisonnement que précédemment donne maintenant :

$$\tilde{R}_t^\infty(x) = x + 2\int_0^t \sqrt{\tilde{R}_s^\infty(1 - \tilde{R}_s^\infty)}dB_s + 2(1 - \tilde{R}_s^\infty)ds. \tag{1.36}$$

Cette équation admet une unique solution en loi, qui est une diffusion appelée diffusion de Wright-Fisher avec immigration : en particulier, la propriété de Markov est à nouveau vérifiée. De plus, le processus $\tilde{R}^\infty(x)$ est indépendant de $X^\infty(1)$. Ces résultats sont mentionnés en introduction de Warren et Yor [128]. On notera enfin que $(\tilde{R}^\infty, t \geq 0)$ correspond au processus $(\tilde{R}_t, t \geq 0)$ conditionné par l'évènement $\{\tau_0(\tilde{R}) = \infty\}$.

1.7.2 Le processus du ratio des CBIs stables

Nous étudions dans cette section quand le processus du ratio $R(dx)$ est, à un changement de temps près, Markovien, pour des CBI plus généraux. Soit X un CBI(ψ, ϕ). Nous supposons pour simplifier que $\psi(\lambda)$ et $\phi(\lambda)$ vérifient $\beta = 0$ dans (1.4) et $\mathtt{d} = 0$ dans (1.11). À nouveau, la construction poissonnienne (1.12) des CBI permet d'écrire le compensateur prévisible de la mesure ponctuelle $\sum_{0 \leq s \leq t} \delta_{(s, \Delta X_s)}(ds, du)$ comme :

$$ds\ (X_{s-}\nu(du) + \eta(du)).$$

On rappelle la définition de l'application $\varphi_x : y \to y/(x+y)$. Le compensateur prévisible de la mesure $\sum_{0 \leq s \leq t} \delta_{(s, \Delta X_s / X_s)}(ds, dr)$ s'écrit comme suit :

$$ds\ (\varphi^\star_{X_{s-}}(X_{s-}\nu + \eta))(dr) = ds\left(X_{s-}\varphi^\star_{X_{s-}}(\nu)(dr) + \varphi^\star_{X_{s-}}(\eta)(dr)\right), \tag{1.37}$$

avec $\varphi_x^\star(\eta)$ la mesure image de η par φ_x.

Lemma 1.7.1 (Lemme 3.5 de Birkner *et al*, [18]). *Les mesures images* $(\varphi_x^\star(\eta), x > 0)$ *de la mesure* $\eta(du)$ *par l'application* φ_x *sont toutes proportionelles à une même mesure, i.e. satisfont*

$$(\varphi_x^\star)(\eta)(dr) = \lambda_x \eta^0(dr)$$

pour une application λ *et une mesure* $\eta^0(dr)$ *sur* $(0,1)$ *si et seulement si*

$$\eta(du) = cu^{-1-\alpha}du \text{ pour un certain } \alpha \in (0,2),$$

auquel cas on a également $\eta^0(dr) = cr^{-2}Beta(2-\alpha,\alpha)(dr)$ *et* $\lambda_x = x^{-\alpha}$.

Ainsi, le second membre de (1.37) peut être factorisé :
– lorsque

$$\nu(du) = cu^{-1-\alpha}du \text{ pour un certain } \alpha \in (0,2), \text{ et } \eta(du) = 0,$$

auquel cas le second membre de (1.37) s'écrit comme suit :

$$ds\, X_{s-}\varphi^\star_{X_{s-}}(\nu)(dr) = X_{s-}^{1-\alpha} ds\, cr^{-2}\, \mathrm{Beta}(2-\alpha,\alpha)(dr).$$

Ceci permet à Birkner *et al.* de prouver le théorème 1.1 de [18], que l'on peut résumer ainsi : Les processus de Dawson-Watanabe Z construits à partir d'un CB(ψ) X (comme expliqué en 1.5.2) dont le ratio R est, à changement de temps près, Markovien ont pour mécanisme de branchement :

$$\psi(\lambda) = \begin{cases} \tilde\beta\lambda + d\lambda^\alpha & \text{pour } \alpha \in (1,2), \\ \tilde\beta\lambda + d\lambda\log(\lambda) & \text{pour } \alpha = 1, \\ \tilde\beta\lambda - d\lambda^\alpha & \text{pour } \alpha \in (0,1), \end{cases}$$

et $\tilde\beta \in \mathbb{R}$. Dans ce cas, le ratio changé de temps est un Beta$(2-\alpha,\alpha)$-Fleming-Viot.
– lorsque

$$\nu(du) = cu^{-1-\alpha}du \text{ et } \eta(du) = cu^{-\alpha}du \text{ pour un certain } \alpha \in (1,2),$$

auquel cas le second membre de (1.37) s'écrit comme suit :

$$\begin{aligned} & ds\, \left(X_{s-}\varphi^\star_{X_{s-}}(\nu)(dr) + \varphi^\star_{X_{s-}}(\eta)(dr)\right), \\ &= ds\, \left(X_{s-}X_{s-}^{-\alpha}cr^{-2}\, \mathrm{Beta}(2-\alpha,\alpha)(dr) + X_{s-}^{1-\alpha}cr^{-2}\, \mathrm{Beta}(3-\alpha,\alpha-1)(dr)\right) \\ &= X_{s-}^{1-\alpha}\, ds\, cr^{-2}\, \left(\mathrm{Beta}(2-\alpha,\alpha)(dr) + \mathrm{Beta}(3-\alpha,\alpha-1)(dr)\right) \\ &= X_{s-}^{1-\alpha}\, ds\, cr^{-2}\, \mathrm{Beta}(2-\alpha,\alpha-1)(dr) \end{aligned}$$

Ceci nous permet, avec Clément Foucart, d'établir la la deuxième partie du théorème 4.3.3, que l'on énonce maintenant sous une forme compacte.

Soit X^∞ un CBI(ψ,ϕ), et $\tau(X^\infty) = \inf\{t > 0, X_t^\infty = 0\}$. Pour $0 \le t \le \tau(X^\infty)$, et $1 < \alpha < 2$, on introduit le changement de temps :

$$C(t) = \int_0^t (X_s^\infty)^{1-\alpha} ds.$$

Théorème. [55]. *Soit* $Z^\infty(dx)$ *le processus de Dawson-Watanabe construit à partir de* X^∞ *un CBI*(ψ,ϕ) *et* $U = [X^\infty]$ *sa variation quadratique.*

Si

$$\psi(\lambda) = d\lambda^\alpha \text{ et } \phi(\lambda) = d\alpha\lambda^{\alpha-1} \text{ pour } 1 < \alpha < 2,$$

alors le processus $(R^{\infty}_{C^{-1}(t)}, t \geq 0)$ *est un processus de* Λ*-Fleming-Viot avec :*

$$\Lambda(dr) = d\,\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\mathrm{Beta}(2-\alpha,\alpha-1)(dr),$$

où l'on convient que $\mathrm{Beta}(2-\alpha,\alpha-1)(dr) = \delta_0(dr)$ *si* $\alpha = 2$.

On discute maintenant de la divergence des changements de temps. Nous avons prouvé que le changement de temps divergeait, c'est-à-dire que $C(\tau(X^{\infty})) = \infty$ p.s. Ceci découle du caractère autosimilaire de X^{∞}, prouvé par Kyprianou et Pardo [80], qui s'écrit ainsi :

$$(xX^{\infty}_{x^{1-\alpha}t}(1), t \geq 0) \overset{loi}{=} (X^{\infty}_t(x), t \geq 0)\,,$$

avec $X^{\infty}_t(x)$ la valeur à l'instant t de X^{∞} issu de x à l'instant 0.

En fait, nous avons prouvé une version plus générale de ce Théorème, qui permet de prendre une valeur de la constante d distincte pour $\psi(\lambda)$ et $\phi(\lambda)$, auquel cas le ratio changé de temps est donnée par un M-Fleming-Viot. Ces processus, introduits par Clément Foucart [54], permettent de distinguer l'individu de niveau 1 dans le système de particules lookdown, en lui affectant un taux de reproduction propre. Le point remarquable du Théorème ci-dessus est que, pour ce choix particulier de $\psi(\lambda)$ et $\phi(\lambda)$, nous retrouvons un Λ-Fleming-Viot usuel. Ceci est lié au fait que les $\mathrm{CBI}(\psi,\phi)$ que nous étudions correspondent aux Q-processus des $\mathrm{CB}(\psi)$ obtenus par Birkner *et al.* [18]. Nous approfondissons ce point dans la section 1.7.4 consacrée à la généalogie en temps décroissant du ratio changé de temps. Au préalable, nous devons définir les Λ-coalescents.

1.7.3 Définition des Λ-coalescents

Les Λ-coalescents ont été introduits en 1999 par Pitman [108] et Sagitov [116], indépendamment. Pitman en donne la définition suivante. On note $\Pi_{|[n]}$ la restriction à l'ensemble $\{1,\ldots,n\}$ de la partition $\Pi \in \mathcal{P}_{\infty}$.

Théorème. *Soit* $(\Lambda_{b,k}, 2 \leq k \leq n)$ *des nombres réels. Il existe pour tout* $\pi \in \mathcal{P}_{\infty}$ *un processus* $(\Pi = \Pi_t, t \geq 0)$ *à valeurs dans* $\mathcal{P}_{\infty}$*, issu de* $\Pi_0 = \pi$*, dans lequel* k *blocs coalescent avec taux* $\Lambda_{b,k}$ *lorsque* $\Pi_{|[n]}$ *a* b *blocs, si et seulement si il existe une mesure finie* Λ *sur* $[0,1]$ *telle que :*

$$\Lambda_{b,k} = \int_{[0,1]} x^{k-2}(1-x)^{b-k}\Lambda(dx). \tag{1.38}$$

Dans ce cas, le processus Π *est appelé* Λ*-coalescent. Le coalescent est dit standard lorsqu'il est issu de la partition de* $\mathbb{N}$ *en singletons, c'est-à-dire* $\Pi_0 = \{\{1\},\{2\},\{3\},\ldots\}$.

Lorsque k blocs parmi b coalescent, alors considérant un $b+1$-ième bloc, ou bien il a pris part à l'évènement de coalescence, ou bien il n'y a pas pris part, d'où la relation

$$\Lambda_{b,k} = \Lambda_{b+1,k} + \Lambda_{b+1,k+1},$$

qui implique que les taux de sauts s'écrivent sous la forme (1.38).

Le coalescent de Kingman pour lequel $\Lambda(dx) = \delta_0(dx)$, n'autorise que les évènements de coalescence binaire $\Lambda_{b,k} = \mathbf{1}_{\{k=2\}}$. On le construit très simplement comme suit : toute paire de blocs coalesce indépendamment à taux constant égal à 1. Le Λ-coalescent associé à une mesure Λ telle que $\Lambda\{0\} = 0$ peut quant à lui être construit comme suit : on se donne N une mesure de Poisson sur $\mathbb{R}^+ \times \mathcal{P}_\infty$ d'intensité $dt \times \int_0^1 \Lambda(dx)x^{-2}\rho_x(d\pi)$. Si (t, π) est un atome de N, et l'unique bloc non réduit à un singleton de π est $(j_1, j_2, \ldots)$, alors les blocs numérotés (par ordre de leur plus petit élément) j_1, j_2, ... coalescent à l'instant t. Enfin, la construction du Λ-coalescent dans le cas où $\Lambda\{0\}$ et $\Lambda(0, 1]$ sont tous deux non nuls s'obtient simplement par superposition, puisque les taux de sauts sont additifs en Λ.

Une question naturelle consiste à se demander si le nombre de blocs à tout instant positif est fini ou non dans un coalescent standard (dont le nombre de blocs à l'instant initial est infini par définition). Pitman montre qu'une loi du $0 - 1$ prévaut : dès lors que $\Lambda\{1\} \neq 0$, ou bien p.s. le nombre de blocs est fini à tout instant strictement positif : card $\Pi_t < \infty$ pour tout $t > 0$, auquel cas on dit que le coalescent descend de l'infini ; ou bien p.s. le nombre de blocs reste infini à tout instant positif : card $\Pi_t = \infty$ pour tout $t \geq 0$. Dans le premier cas, tous les évènements de coalescence n'impliquent qu'un nombre fini de blocs et dans le second cas, tous les évènements de coalescence impliquent un nombre infini de blocs (ce qui exclut de fait les coalescences binaires). Supposons $\Lambda\{1\} = 0$. Schweinsberg définit dans [120] le taux de décroissance moyen γ_b du nombre de blocs en présence de b blocs :

$$\gamma_b = \sum_{k=2}^{b} (k-1) \binom{b}{k} \Lambda_{b,k}$$

puis obtient la condition nécessaire et suffisante suivante pour la descente de l'infini du coalescent :

$$\sum_{b \geq 2} \frac{1}{\gamma_b} < \infty, \tag{1.39}$$

Bertoin et Le Gall observent ensuite que cette condition équivaut à la condition de Grey (1.6) pour le mécanisme de branchement :

$$\psi(\lambda) = \Lambda(\{0\})\lambda^2 + \int_{(0,1)} (\mathrm{e}^{-\lambda x} - 1 - \lambda x)x^{-2}\Lambda(dx).$$

Détaillons maintenant quelques exemples de Λ-coalescent : Pour $\alpha, \beta > 0$, le choix de

$$\Lambda(dx) = \frac{1}{\mathrm{Beta}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)dx$$

définit le Beta(α, β)-coalescent. Le cas $\alpha = \beta = 1$ correspond à $\Lambda(dx) = \mathbf{1}_{[0,1]}(dx)$, et donne lieu aux transitions : $\Lambda_{b,k} = (k-2)!(b-k)!/(b-1)!$. Ce coalescent, appelé coalescent de Bolthausen-Sznitman [21], est remarquable à de nombreux égards. Il est en lien avec les arbres récursifs, voir Goldschmidt et Martin [58]. Il décrit aussi la généalogie du ratio du CB(ψ) pour $\psi(\lambda) = \lambda \log(\lambda)$, appelé CB de Neveu, sans besoin de changement de temps qui plus est, voir Bertoin et Le Gall [14].

1.7.4 Le Beta$(2 - \alpha, \alpha - 1)$-coalescent dans le processus du ratio des CBIs stables.

Le théorème suivant permet de donner un sens à la généalogie de certains processus de branchement avec immigration.

On considère le sytème de particules look-down associé à Z^∞ construit à partir d'un CBI(ψ,ϕ) et $U=[X^\infty]$ sa variation quadratique. On définit un processus $(\hat{\Pi}^t_s, 0\le s\le t)$ à valeurs dans l'ensemble $\mathcal{P}_\infty$ des partitions de l'ensemble $\mathbb{N}$ des entiers naturels comme suit. Etant donné $0\le s\le t$ et $i,j\in\mathbb{N}$, on dira que i et j appartiennent au même bloc de $\hat{\Pi}^t_s$ lorsque les individus aux niveaux i et j à l'instant t partagent le même ancêtre à l'instant s dans le graphe look-down associé à Z^∞. On se reportera à la figure 1.4 pour une définition de la notion d'ancêtre. On notera que $\hat{\Pi}^t_t$ est la partition de $\mathbb{N}$ en singletons.

On rappelle la définition de $C(t)=\int_0^t (X^\infty_s)^{1-\alpha}ds$ pour $0\le t<\tau_0(X^\infty)$.

Théorème. [55]. *Posons* $d=\Gamma(2-\alpha)/\alpha(\alpha-1)$. *Si*

$$\psi(\lambda)=d\lambda^\alpha \text{ et } \phi(\lambda)=d\alpha\lambda^{\alpha-1} \text{ pour } 1<\alpha<2,$$

alors le processus $(\hat{\Pi}^{C^{-1}(t)}_{C^{-1}(t-s)}, 0\le s\le t)$ *est la restriction d'un* Beta$(2-\alpha,\alpha-1)$*-coalescent, où l'on convient à nouveau que* Beta$(2-\alpha,\alpha-1)(dr)=\delta_0(dr)$ *lorsque* $\alpha=2$.

Appelons coalescence de fréquence x une coalescence dans laquelle chaque bloc participe avec probabilité x. Par construction du Beta$(2-\alpha,\alpha-1)$-coalescent, l'intensité des coalescences de fréquence x est $x^{1-\alpha}(1-x)^{\alpha-2}$. Or on a :

$$x^{1-\alpha}(1-x)^{\alpha-2}=x^{2-\alpha}(1-x)^{\alpha-2}+x^{1-\alpha}(1-x)^{\alpha-1}.$$

Le premier terme de la somme peut être compris comme l'intensité des coalescences de fréquence x qui impliquent le bloc 1 et le second terme comme l'intensité des coalescences de fréquence x qui n'impliquent pas le bloc 1. Cela découle en effet de la construction poissonienne du Λ-coalescent. Ainsi, on peut voir le Beta$(2-\alpha,\alpha-1)$-coalescent comme la superposition de deux coalescents :
- Le premier est un Beta$(3-\alpha,\alpha-1)$-coalescent sur les blocs numérotés $\{2,3,\ldots\}$, et le bloc qui contient 1 est associé à chaque coalescence.
- Le second est un Beta$(2-\alpha,\alpha)$ coalescent usuel sur les blocs numérotés $\{2,3,\ldots\}$.

Cela s'accorde à la description du CBI(ψ,ϕ) comme une superposition de CB(ψ) greffés selon une mesure ponctuelle de Poisson : lues en temps décroissant, les instants de greffe (ou encore d'immigration) correspondent aux instants de coalescence du Beta$(3-\alpha,\alpha-1)$-coalescent, et les évènements de reproduction des CB greffés aux évènements de coalescence dans le Beta$(2-\alpha,\alpha)$-coalescent.

1.7.5 Ouverture

Une question ouverte pour clore la présentation du travail [55] avec Clément Foucart : La généalogie du CB stable comme celle du CBI stable ne sont déterminées qu'à compter de l'instant aléatoire $C^{-1}(t)$, comme l'ont observé Berestycki et Berestycki [7] dans le cas du CB stable. Hormis dans le cas $\alpha=2$, on ne connaît pas la généalogie (en terme de coalescent) des CB stables à compter d'un instant fixé t. A cet effet, il peut être utile de noter que Duquesne et Le Gall déterminent au chapitre 4.6 de [37] la structure de l'arbre de Lévy associé au CB stable conditionnellement à la non extinction en t. Plus précisément, il n'est pas difficile de déduire de leur Théorème 4.6.2 une description de cet arbre.

1.8 Un subordinateur conditionné et les excursions du Q-processus

L'objectif de cette dernière section, basée sur le travail [61] avec Stephan Gufler, est de décrire le Q-processus de certains processus régénératifs.

1.8.1 L'exemple d'une chaîne de Markov à espace d'état fini

Le développement suivant s'inspire de l'appendice M du livre d'Aldous [3]. Il s'agit d'un exemple simple, celui d'une chaîne de Markov à espace d'état fini, qui montre l'approche classique du Q-processus. Soit une chaîne de Markov $(X_n, n \geq 0)$ à espace d'état fini I de matrice de transition $P_{ij} = \mathbb{P}(X_1 = j|X_0 = i)$ pour i et j dans I. Cette matrice $P = (P_{ij}, i, j \in I)$ a, par définition, tous ses élements compris entre 0 et 1 et de plus, la somme de chacune de ses lignes vaut 1 : on l'appelle matrice stochastique. Le Q-processus $X^\infty = (X_n^\infty, n \geq 0)$ de X est défini comme le processus X confiné dans $I \setminus J$ au sens suivant :

$$\mathbb{P}_i(X^\infty \in A) = \lim_{n\to\infty} \mathbb{P}_i(X \in A|\, T_J > n),\ A \in \mathcal{F}_m = \sigma(X_1, \ldots, X_m)$$

avec $T_J = \inf\{n \geq 0, X_n \in J\}$ le temps d'atteinte de J. L'idée est alors d'appliquer le théorème de Perron-Frobenius à la matrice sous-stochastique $\hat{P} = (\hat{P}_{ij}, i, j \in I \setminus J)$ définie simplement par restriction comme suit : $\hat{P}_{ij} = P_{ij}$ pour $i, j \in I \setminus J$. Supposant que (la chaîne tuée associée à) la matrice $\hat{P}$ est irréductible et apériodique, on obtient l'équivalent suivant pour les coefficients de $\hat{P}$ élevée à la puissance n :

$$(\hat{P}^n)_{ij} = \mathbb{P}(X_n = j|X_0 = i) \sim \theta^n \beta_i \alpha_j \text{ lorsque } n \to \infty,\ \text{pour } i, j \in I \setminus J,$$

où :
- θ est la valeur propre de $\hat{P}$ pour laquelle $|\theta|$ est maximal.
- θ est réelle, égale à 1 si $\hat{P}$ est stochastique, et $0 < \theta < 1$ sinon.
- α et β sont des vecteurs propres à gauche et à droite respectivement, $\alpha\hat{P} = \theta\alpha$, $\hat{P}\beta = \theta\beta$, et ont des coefficients positifs. En outre, nous pouvons choisir α normalisé de sorte que $\sum_{i\in I\setminus J} \alpha_i = 1$

En conséquence, le Q-processus X^∞ est une h-transformée de X qui s'exprime à l'aide du vecteur propre à droite β et de la valeur propre θ selon :

$$\mathbb{P}_i(X^\infty \in A) = \mathbb{E}_i\left(\theta^{-m}\frac{\beta_{X_m}}{\beta_i}, X \in A|T_j > m\right),\ A \in \mathcal{F}_m.$$

En particulier, c'est encore un processus de Markov, et sa matrice de transition, désormais stochastique, s'écrit :

$$\mathbb{P}(X_1^\infty = j|X_0^\infty = i) = P_{ij}\, \theta^{-1}\, \frac{\beta_j}{\beta_i},\ i, j \in I \setminus J.$$

On en déduit encore que $\mathbb{P}(T_J > n) \sim c\theta^n$ quand n tend vers l'infini, c'est-à-dire que T_J a une queue de nature géométrique. De plus, pour toute distribution initiale, on a la convergence suivante :

$$\lim_{n\to\infty} \mathbb{P}(X_n = i|T_J > n) = \alpha_i,$$

qui exprime le fait que α est la limite de Yaglom. On notera enfin que si X_0 a la distribution α, alors T_J a une distribution géométrique :

$$\mathbb{P}(T_J = n) = (1-\theta)\theta^{n-1}, n \geq 1.$$

Voilà pour l'approche classique du Q-processus. Noter que la représentation du Q-processus comme une h-transformée à l'aide d'un vecteur propre à droite positif, nul sur les bords du domaine où on cherche à confiner le processus, est un fait général : la formule (1.14) dans le cas du CB et la formule (1.34) dans le cas du Λ-Fleming Viot en donnent deux nouveaux exemples.

Notre approche, qui repose sur une étude de la longueur des excursions, fait quant à elle intervenir de façon naturelle un subordinateur conditionné, introduit en section 1.8.2. Elle nous permet d'étudier en section 1.8.4 un nouveau processus confiné par ses excursions, dont le lien avec le Q-processus usuel est partiellement étudié dans l'article.

1.8.2 Un subordinateur conditionné à être grand à un instant aléatoire

Soit $\mathbb{P}$ la loi du subordinateur σ d'exposant de Laplace

$$\phi(\lambda) = \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\eta(dx),$$

pour $\mathtt{d} \geq 0$ et η une mesure de Radon sur $(0,\infty)$ telle que $\int_{(0,\infty)}(1\wedge x)\eta(dx) < \infty$. On autorisera les arguments négatifs pour ϕ, qui sera donc vue comme une application de $(-\infty,+\infty)$ dans $[-\infty,+\infty)$. On définit $\mathbb{P}^s$ la loi du subordinateur σ conditionné à atteindre le niveau s avant un instant $\mathbf{e}$ indépendant distribué selon une loi exponentielle de paramètre $\kappa > 0$:

$$\mathbb{P}^s(\sigma \in A) := \mathbb{P}(\sigma \in A | \sigma_{\mathbf{e}} > s),\ A \in \mathcal{G}$$

où $\mathcal{G} = \cup_{\ell \geq 0} \mathcal{G}_\ell$ et $(\mathcal{G}_\ell, \ell \geq 0)$ est la filtration naturelle du processus σ. Notre objectif est de caractériser la limite en loi de la famille de mesures de probabilité $\mathbb{P}^s$ quand $s \to \infty$ sur $\mathcal{G}_\ell$ pour $\ell \geq 0$ fixé. On introduit :

$$\rho = \sup\{\lambda \geq 0, \mathbb{E}(\mathrm{e}^{\lambda\sigma_1}) \leq \mathrm{e}^{\kappa}\} = -\inf\{\lambda \leq 0, \phi(\lambda) \geq -\kappa\}.$$

La quantité ρ donne le taux de décroissance exponentielle de $f(s) = \mathbb{P}(\sigma_{\mathbf{e}} > s)$, au sens où :

$$\lim_{s\to\infty} \frac{-\log f(s)}{s} = \rho.$$

Ceci découle de la sous-additivité de la fonction $s \to -\log f(s)$, qui est elle-même une conséquence de la propriété de Markov. On montre que :

$$\mathbb{P}^\infty(A) = \mathbb{E}(\mathrm{e}^{\rho\sigma_\ell + \phi(-\rho)\ell}, A),\ A \in \mathcal{G}_\ell,$$

définit bien une mesure de probabilité sur $\mathcal{G}$. Il s'agit d'une transformée d'Esscher de $\mathbb{P}$. On notera que σ sous $\mathbb{P}^\infty$ est encore un subordinateur, d'exposant de Laplace l'exposant translaté :

$$\phi(\lambda - \rho) - \phi(-\rho).$$

On définit alors les deux hypothèses suivantes :

(C) Le nombre $\rho \geq 0$ satisfait $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{\kappa}$ et $\mathbb{E}(\sigma_1\,\mathrm{e}^{\rho\sigma_1}) < \infty$.

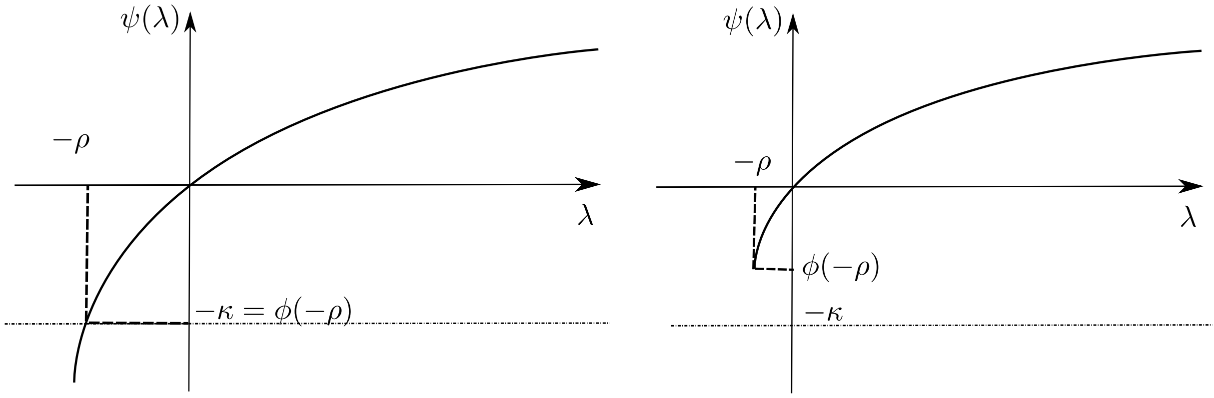

Figure 1.5: A gauche : un exemple d'exposant de Laplace pour lequel $\phi(-\rho) = -\kappa$. A droite : un exemple d'exposant de Laplace pour lequel $\phi(-\rho) > -\kappa$.

(R) $f(s)/f(s-t) \to \mathrm{e}^{-\rho t}$ quand $s \to \infty$.

Nous montrons que (C) est une condition suffisante pour vérifier (R), et donnons d'autres conditions suffisantes pour que (R) soit satisfaite quand (C) ne l'est pas nécessairement. On a le résultat suivant :

Théorème. [61] *Nous avons :*
– *Sous l'hypothèse* (C), *étant donné* $\ell \geq 0$ *:*

$$\lim_{s\to\infty} \mathbb{P}^s(A) = \mathbb{P}^\infty(A), \ A \in \mathcal{G}_\ell.$$

– *Sous l'hypothèse* (R), *étant donné* $\ell \geq 0$ *et* $s_0 \geq 0$ *:*

$$\lim_{s\to\infty} \mathbb{P}^s(A, \sigma_\ell \leq s_0) = \mathrm{e}^{-(\kappa+\phi(-\rho))\ell} \ \mathbb{P}^\infty(A, \sigma_\ell \leq s_0), \ A \in \mathcal{G}_\ell.$$

On peut interpréter comme suit ce théorème : On demande au subordinateur de prendre une grande valeur à un instant aléatoire indépendant. Il existe deux possibilités : faire en sorte que le subordinateur soit plus grand, en lui adjoignant de grands sauts, ce qui est le cas sous $\mathbb{P}^\infty$, ou augmenter la valeur de l'instant aléatoire. En général, ces deux phénomènes ont lieu simultanément, et $\mathbb{P}^s$ converge en loi vers $\mathbb{P}^\infty$ tué à un instant exponentiel indépendent de paramètre $\kappa + \phi(-\rho)$. Puisque $\kappa + \phi(-\rho) \leq \kappa$, la variable aléatoire exponentielle de paramètre $\kappa + \phi(-\rho)$ domine stochastiquement celle de paramètre κ. On notera que sous (C), on a nécessairement $\kappa+\phi(-\rho) = 0$, et donc le processus limite n'est pas tué. Pour des travaux proches dans le cadre des processus de Lévy, on consultera Griffin [59].

1.8.3 Les processus régénératifs

Cette section introduit la classe des processus régénératifs, suivant l'exposition du chapitre 22 de Kallenberg [71].

Soit $X = (X_t, t \geq 0)$ un processus càdlàg à valeurs dans un espace polonais E, et a un point de E. On note $\mathbb{P}_x$ la loi du processus issu de x pour tout $x \in E$, et simplement $\mathbb{P}$ dans le cas où $x = a$, et $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ la filtration naturelle du processus. Le processus X est dit régénératif en a, lorsque pour tout temps d'arrêt T relatif à la filtration $\mathcal{F}$:

$$\mathbb{P}_{x\ |\mathcal{F}_T} \circ \theta_T^{-1} = \mathbb{P}_a \text{ p.s. sur } \{T < \infty, X_T = a\},$$

avec θ_t l'opérateur de shift défini par $\theta_t(\omega) = \omega(t + \cdot)$. L'ensemble $\mathcal{Z} = \{t \geq 0, X_t = a\}$ est appelé l'ensemble régénératif. L'intérieur de $\mathbb{R}^+ \setminus \mathcal{Z}$ peut s'écrire comme réunion d'intervalles ouverts maximaux, et $\mathbb{R}^+ \setminus \mathcal{Z}$ comme une réunion d'intervalles maximaux ouverts à droite car le processus X est continu à droite. Chacun de ces intervalles, de la forme (u, v) ou $[u, v)$, est associé à une trajectoire e comme suit :

$$e = (X_{(u+s)\wedge v}, s \geq 0).$$

Ces trajectoires appartiennent à l'ensemble $\Omega^{\mathbf{e}}$ des excursions, défini comme le sous-ensemble suivant de l'ensemble Ω des trajectoires càdlàg à valeurs dans E :

$$\Omega^{\mathbf{e}} = \{\omega \in \Omega, (\omega(s) = a \text{ et } s > 0) \Rightarrow \omega(t) = a \text{ pour tout } t \geq s\}.$$

On a la dichotomie suivante : ou bien, p.s., les points de $\mathcal{Z}$ sont tous isolés, ou bien, p.s., aucun d'eux ne l'est.

Plaçons nous dans le cas où p.s. aucun des points de $\mathcal{Z}$ n'est isolé. On construit un processus L sur $\mathbb{R}^+$ croissant, continu et adapté, de support $\bar{\mathcal{Z}}$ p.s., appelé temps local. On note σ l'inverse continu à droite de L, $\sigma_\ell = \inf\{s > 0, L_s > \ell\}$. Le processus σ, appelé temps local inverse, est un subordinateur, issu de 0 sous $\mathbb{P}$. Puisque le processus croissant L a $\bar{\mathcal{Z}}$ comme support, il est constant sur chacun des intervalles d'excursion et on peut poser, pour chaque instant de saut ℓ de σ :

$$e_\ell(s) := X_{(\sigma_{\ell-}+s)\wedge\sigma_\ell}. \tag{1.40}$$

Enfin, on construit une mesure sigma-finie n sur $\Omega^{\mathbf{e}}$ telle que, si N est une mesure ponctuelle de Poisson sur $\mathbb{R}^+ \times \Omega^{\mathbf{e}}$ d'intensité $d\ell \times n(de)$, alors :

$$\sum_{\ell\geq 0, \sigma_{\ell-}<\sigma_\ell} \delta_{(\ell, e_\ell)}(d\ell, de) \text{ est la restriction de } N \text{ sur } [0, L_\infty] \times \Omega^{\mathbf{e}}. \tag{1.41}$$

De plus, le produit $n.L$ est p.s. unique. Posant $T_a(\omega) = \inf\{t > 0, \omega(t) = a\},\ \omega \in \Omega$, on peut écrire à l'aide de n l'exposant de Laplace de σ :

$$k + \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x}) n(T_a \in dx) \tag{1.42}$$

avec $k = n(T_a = \infty)$ le taux de mort, et $\mathtt{d} \geq 0$ qui satisfait :

$$\text{p.s.,} \int_{[0,t]} \mathbf{1}_{\mathcal{Z}}(s) ds = \mathtt{d} L_t,\ t \geq 0.$$

Noter que $\mathtt{d} = 0$ dans le cas du mouvement brownien.

Plaçons nous maintenant dans le cas où tous les éléments de $\mathcal{Z}$ sont isolés. Il existe alors une première excursion, et le temps local peut être défini simplement comme suit : soit $(L_i, i \geq 1)$ une suite de variables aléatoires exponentielles de paramètre strictement positif arbitraire et fixé. A la i-ème excursion, on associe alors la valeur du temps local suivante : $\sum_{1\leq j\leq i} L_j$. Le temps local inverse est alors par construction un subordinateur. Son exposant de Laplace peut encore s'écrire sous la forme (1.42), avec nécessairement dans ce cas $\mathtt{d} = 0$, et n une mesure finie. On définit e_ℓ comme dans (1.40) et on a encore (1.41) avec le produit $n.L$ p.s. unique.

1.8.4 Les excursions du processus confiné

On considère Ω^0 un sous-ensemble mesurable de l'ensemble $\Omega^{\mathbf{e}}$ des excursions qui vérifie de plus $0 < n(\Omega^0) < \infty$, et $n(T_a = \infty, \Omega^1) = 0$ avec Ω^1 le complémentaire dans $\Omega^{\mathbf{e}}$ de Ω^0. On note

$$T^0 = \inf\{\sigma_{\ell-}, e_\ell \in \Omega^0\}$$

l'instant où commence la première excursion dans Ω^0. La figure 1.6 illustre la définition de T_0 dans un cas particulier. On s'intéresse à

$$\mathbb{P}^{(t)}(A) = \mathbb{P}(A|T^0 > t),\ A \in \mathcal{F},$$

où $\mathcal{F} = \cup_{t\geq 0}\mathcal{F}_t$ et $(\mathcal{F}_t, t \geq 0)$ est la filtration naturelle du processus régénératif X. La famille décroissante $\{T^0 > t\}$ est une approximation possible de l'évènement $\{T^0 = \infty\}$, mais d'autres approximations sont possibles comme par exemple $\{T^0 > \sigma_\ell\}$ qui donne lieu à la famille de mesures de probabilités $\mathbb{P}^{(loc)}$:

$$\mathbb{P}^{(loc,\ell)}(A) = \mathbb{P}(A|T^0 > \sigma_\ell),\ A \in \mathcal{F}.$$

Notre intérêt réside dans la comparaison des deux approximations du conditionnement par l'évènement $\{T^0 = \infty\}$, de probabilité nulle sous nos hypothèses sur Ω^0. On commence par traiter le cas de $\mathbb{P}^{(loc,\ell)}$, plus facile. On pose $\mathcal{G}_\ell = \mathcal{F}_{\sigma_\ell}$. Les propriétés des mesures de Poisson assurent que le processus $(\mathrm{e}^{n(\Omega^0)\ell}\,\mathbf{1}_{\{T^0>\sigma_\ell\}}, \ell \geq 0)$ est une $\mathcal{G}$-martingale. On en déduit qu'il existe une unique mesure $\mathbb{P}^{(loc)}$ sur $\mathcal{G}$ telle que :

$$\mathbb{P}^{(loc)}(A) = \mathbb{E}(\mathrm{e}^{n(\Omega^0)\ell}\,\mathbf{1}_{\{T^0>\sigma_\ell\}}, A),\ A \in \mathcal{G}_\ell,$$

et on a pour chaque $\ell \geq 0$,

$$\mathbb{P}^{(loc)}(A) = \mathbb{P}^{(loc,\ell)}(A),\ A \in \mathcal{G}_\ell.$$

En ce sens, les conditionnements par $\{T^0 > \sigma_\ell\}$ sont consistants. On prouve que le temps local inverse sous $\mathbb{P}^{(loc)}$ est encore un subordinateur d'exposant de Laplace :

$$\mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})n(T_a \in dx, \Omega^1). \tag{1.43}$$

Dès lors, le lien avec la section 1.8.2 s'établit comme suit :

$$T^0 \text{ sous } \mathbb{P} \text{ a même loi que } \sigma_{\mathbf{e}} \text{ sous } \mathbb{P}^{(loc)}$$

où $\mathbf{e}$ est une variable aléatoire exponentielle de paramètre $\kappa = n(\Omega^0)$, indépendante de σ.

La quantité ρ est redéfinie comme suit :

$$\rho = \sup\{\lambda \geq 0, \mathbb{E}^{(loc)}(\mathrm{e}^{\lambda\sigma_1}) \leq \mathrm{e}^{n(\Omega^0)}\} = \sup\{\lambda \geq 0, \mathbb{E}(\mathrm{e}^{\lambda\sigma_1}, T_0 > \sigma_1) \leq 1\}.$$

On dira donc dorénavant que (C) est satisfaite quand :

$$\mathbb{E}^{(loc)}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{n(\Omega^0)} \text{ et } \mathbb{E}^{(loc)}(\sigma_1\,\mathrm{e}^{\rho\sigma_1}) < \infty,$$

soit encore, de façon équivalente,

$$\mathbb{E}(\mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) = 1 \text{ et } \mathbb{E}(\sigma_1 \mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) < \infty.$$

On introduit alors $\mathbb{P}^{(\infty)}$:

$$\mathbb{P}^{(\infty)}(A) = \mathrm{e}^{\ell(n(\Omega^0)+\phi(-\rho))}\, \mathbb{E}(\mathrm{e}^{\rho\sigma_\ell}, A, T^0 \geq \sigma_\ell), \;\; A \in \mathcal{G}_\ell. \tag{1.44}$$

et on vérifie que $\mathbb{P}^{(\infty)}$ définit une mesure de probabilité. On dira que (R) est satisfaite quand

$$\mathbb{P}^{(loc)}(\sigma_{\mathbf{e}} > s)/\mathbb{P}^{(loc)}(\sigma_{\mathbf{e}} > s - t) \to \mathrm{e}^{\rho t} \;\text{ quand } s \to \infty,$$

pour $\mathbf{e}$ une variable aléatoire exponentielle de paramètre $n(\Omega^0)$, indépendante de σ sous $\mathbb{P}^{(loc)}$.

Théorème. [61] *Nous avons :*
– *Sous l'hypothèse* (C)*, étant donné* $\ell \geq 0$ *:*

$$\lim_{t\to\infty} \mathbb{P}^{(t)}(A) = \mathbb{P}^{(\infty)}(A), \;\; A \in \mathcal{G}_\ell.$$

– *Sous l'hypothèse* (R)*, étant donné* $\ell \geq 0$ *et* $s_0 \geq 0$ *:*

$$\lim_{t\to\infty} \mathbb{P}^{(t)}(A, \sigma_\ell \leq s_0) = \mathrm{e}^{-\ell(n(\Omega^0)+\phi(-\rho))}\, \mathbb{P}^{(\infty)}(A, \sigma_\ell \leq s_0), \;\; A \in \mathcal{G}_\ell. \tag{1.45}$$

On notera que le préfacteur du membre de droite de (1.45) vaut 1 lorsque $n(\Omega^0) + \phi(-\rho) = 0$, ce qui est le cas si $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) = 1$. On a alors convergence vers une mesure de probabilité.

Enfin, dans le cas où

(H) Il existe $E_0 \subset E$ tel que $\Omega^0 = \{e \in \Omega^{\mathbf{e}}, e(s) \in E_0 \text{ pour un certain } s \geq 0\}$,

nous définissons le temps d'atteinte de E_0 par $\bar{T}^0 = \inf\{s > 0, X_s \in E_0\}$. Il est naturel de se demander quel est le lien entre le conditionnement par $\{\bar{T}^0 \geq t\}$ et celui par $\{T^0 \geq t\}$. Nous faisons le lien entre ces deux conditionnements dans l'article [61], pour t grand.

1.8.5 Ouverture

En conclusion, quelques questions suscitées par ce dernier travail [61] :

– Comment décrire l'excursion infinie qui apparaît dans le cas où $\phi(-\rho) > -\kappa$ sous $\mathbb{P}^{(\infty)}$? Une telle excursion apparaît par exemple lorsqu'on conditionne un mouvement brownien ou un processus de Lévy oscillant à rester positif. On peut alors changer de point de vue et considérer les excursions sous le supremum. Ceci nous amène naturellement à poser la question de domaines $\Omega^0(\ell)$ fonctions du temps local ℓ.
– Comment décrire la frontière de Martin ? La frontière de Martin d'un processus correspond à la description de toutes les fonctions harmoniques positives extrêmales, voir par exemple l'introduction au sujet par Sawyer [119] ; à chacune de ces fonctions harmoniques positives extrêmales est associé un processus qui converge vers un point de la frontière de Martin, et l'ensemble de ces processus décrit en un certain sens l'ensemble des possibilités pour un processus transient de quitter son espace d'état, voir Revuz [110]. La description des excursions des éléments de la frontière de Martin peut se révéler éclairante. On pourra relire à cette aune Ney et Spitzer [100] et Ignatiouk-Robert et Lorée [68] pour des processus en dimension 2, ou encore Salminen [118] et Overbeck [101] pour les processus de branchement.

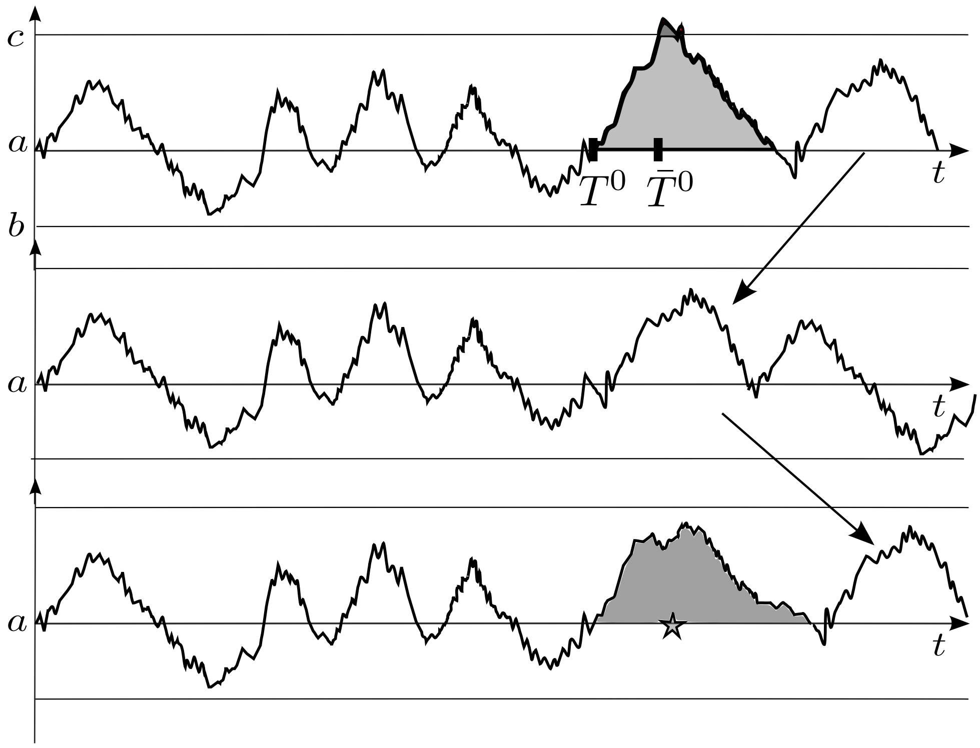

Figure 1.6: Ici, Ω^0 correspond aux excursions depuis le point a qui visitent $\mathbb{R} \setminus (b, c)$. En haut : une trajectoire sous $\mathbb{P}$. Au milieu : une trajectoire sous $\mathbb{P}^{(loc)}$, l'excursion qui atteint $\mathbb{R} \setminus (b, c)$ a été effacée par rapport à la trajectoire sous $\mathbb{P}$. En bas : une trajectoire sous $\mathbb{P}^{(\infty)}$, une nouvelle excursion biaisée par la taille est ajoutée par rapport à la trajectoire sous $\mathbb{P}^{(loc)}$. En outre, si on supprime les excursions de $\mathbb{P}^{(\infty)}$ marquées par une étoile, on retrouve la trajectoire sous $\mathbb{P}^{(loc)}$. Les étoiles sont placées aux instants de sauts d'un processus ponctuel de Poisson de paramètre ρ.

– Comment généraliser le propos à d'autres pénalisations ? Dans le cas (en partie traité dans l'article) où Ω^0 est de la forme (H), le conditionnement par $\{\bar{T}^0 \geq t\}$ est une forme de pénalisation particulière, par la fonction $\mathcal{F}_t$ mesurable $\mathbf{1}_{\{\bar{T}^0 \geq t\}}/\mathbb{P}(\bar{T}^0 \geq t)$. Pourrait-on étudier à l'aide des excursions d'autres formes de pénalisations ? La pénalisation de Feynman-Kac, évoquée dans cette introduction à la fin de la section 1.4.3, consiste à repondérer les trajectoires par

$$\frac{\mathrm{e}^{-\int_0^t ds\, \beta(X_s)}}{\mathbb{E}(\mathrm{e}^{-\int_0^t ds\, \beta(X_s)})}$$

pour une certaine fonction β positive. Cette pénalisation correspond à un “soft killing“, par opposition au “hard killing“ associé au conditionnement par $\{\bar{T}^0 \geq t\}$, qui correspond formellement à $\beta(x) = \infty \cdot \mathbf{1}_{x \in E_0} + 0 \cdot \mathbf{1}_{x \notin E_0}$. Une approche par les excursions semble encore adaptée à cette pénalisation, voir les travaux de Najnudel, Roynette et Yor [99]. On notera qu'une telle pénalisation a été étudiée en fin de section 1.4.3, voir la définition de $\mathrm{P}^{(B,t)}$.

2

A Williams decomposition for spatially dependent superprocessus

2.1 Introduction

Even if superprocesses with very general branching mechanisms are known, most of the works devoted to the study of their genealogy are concerned with homogeneous branching mechanisms, that is, populations with identical individuals. Four distinct approaches have been proposed for describing these genealogies. When there is no spatial motion, superprocesses are reduced to continuous state branching processes, whose genealogy can be understood by a flow of subordinators, see Bertoin and Le Gall [14], or by growing discrete trees, see Duquesne and Winkel [38]. With a spatial motion, the description of the genealogy can be done using the lookdown process of Donnelly and Kurtz [35] or the snake process of Le Gall [92]. Some works generalize both constructions to non-homogeneous branching mechanisms: Kurtz and Rodriguez [78] recently extended the lookdown process in this direction whereas Dhersin and Serlet proposed in [33] modifications of the snake.

Let X be a non-homogeneous superprocess. It models the evolution of a large population, where the location of the individuals is allowed to affect their reproduction law. We assume the extinction time $H_{\max}$ of this population is finite. We are interested in the two following conditionings on the genealogical structure of X:

1. The distribution $X^{(h_0)}$ of X conditioned on $H_{\max} = h_0$: we derive it using a spinal decomposition involving the ancestral lineage of the last individual alive (Williams' decomposition).
2. The convergence of the distribution of $X^{(h_0)}$ as h_0 goes to ∞. This convergence is studied from two viewpoints. On the one hand, we obtain a convergence result for $(X_s^{(h_0)}, s \in [0,t])$ towards the Q-process. On the other hand, we find a convergence result for the backward process, namely $(X_{h_0+s}^{(h_0)}, s \in [-t,0])$. We reduce both convergences to the convergence of the ancestral lineage of the last individual alive thanks to Williams' decomposition.

Concerning the first conditioning, we stress on the following difference between superprocesses with homogeneous and non-homogeneous branching mechanisms, which explains our interest in the latter model. For homogeneous branching mechanisms, the spatial motion is independent of the genealogical structure. As a consequence, the law of the ancestral lineage of the last individual alive does not distinguish. Therefore, in this setting, the description of $X^{(h_0)}$ may be deduced from Abraham and Delmas [1] where no spatial motion is taken into account. For non-homogeneous branching mechanisms on the contrary, the law of the ancestral lineage of the last individual alive should depend on the distance to the extinction time h_0. This fact will be precised by the second conditioning.

A few lines about the terminology "Williams' decompositions" are in order: Williams [132] decomposed the Brownian excursion with respect to its maximum. After Aldous recognized in [4] the genealogy of a branching process in this excursion, this name also designates decompositions of branching processes with respect to their height.

Our second conditioning exemplifies the interest of Williams' decomposition for investigating the process conditioned on extinction in remote time. The convergence of the superprocess conditioned on extinction in remote time essentially reduces to the convergence of the ancestral lineage of the last individual alive thanks to Williams' decomposition. Also, we may consider the limit either on a fixed time window $[0,t]$ to get the corresponding Q-process, either on an evolving time window attached to the extinction time $[h_0 - t, h_0]$. For non-homogeneous branching mechanisms, we expect a different behaviour on these two time windows for the ancestral lineage of the last individual: far away from the extinction time, it should favorize the fittest types; near the extinction time, it should select the weakest types. Once again, this conditioning is simpler for homogeneous branching mechanisms: in that setting, it goes back to Serlet [122] for quadratic branching mechanism; for more general branching mechanisms, it reduces to the corresponding decomposition for continuous state branching process, see Chen and Delmas [26] and references therein. For non-homogeneous branching a first construction of the Q-process is given in Champagnat and Roelly [25] in the particular case of a multitype Feller diffusion and without genealogical description.

Last, we stress a rigourous analysis of the ancestral lineage of the last individual alive necessitates the introduction of a genealogy for the superprocess, since the ancestral lineages are not immediately identifiable in the context of measure-valued processes. We found out that the previous genealogies defined for non-homogeneous branching mechanisms, see [78] and [33], were not suited to our need. In particular, the description in [33] preserves neither the extinction time nor the last individual alive. We thus provide a new description of the genealogy through a modification of the Brownian snake, which codes for the genealogy of superprocesses with homogeneous branching mechanisms. More precisely, starting with non-homogeneous branching mechanism, we go back to an homogeneous one via two transformations:

- The first transformation relies on the non-linear h transform, or reweighting of superprocesses, introduced in Engländer and Pinsky [44].
- The second transformation is based on a Girsanov change of measure on the law of superprocesses, as described in Chapter IV of Perkins [104].

Reversing the procedure, we define the genealogy associated to a non-homogeneous branching mechanism from the one associated to an homogeneous one. We then obtain our two conditionings by "transfer", using the previous knowledge for homogeneous branching mechanisms. We are aware of the drawback of this approach: namely, we restrict ourselves to quadratic branching mechanisms with bounded and smooth parameters.

Outline. We give some background on superprocesses with a non-homogeneous branching mechanism in Section 2.2. Section 2.3 begins with the definition of the h-transform in the sense of Engländer and Pinsky, Definition 2.3.4, goes on with a Girsanov Theorem, Proposition 2.3.7, and ends up with the definition of the genealogy, Proposition 2.3.12, by combining both tools. Section 2.4 is mainly devoted to the proof of Williams' decomposition, Theorem 2.4.12. By the way, we give a decomposition with respect to a randomly chosen individual, also known as a

Bismut decomposition, in Proposition 2.4.2. Section 2.5 gives some applications of Williams' decomposition. We first prove in Lemma 2.5.1 that the limit of the superprocesses conditioned to extinct *at* a remote time coincide with the Q-process (the superprocess conditioned to extinct *after* a remote time) and actually show in Theorem 2.5.5 that such a limit exists. We also look at the convergence of the process seen from the top (so, backward from the extinction time), see Theorem 2.5.9. All previous results are provided with a set of assumptions: we then give in Section 2.6 sufficient conditions for these assumptions to be valid in term of the generalized eigenvector and eigenvalue, and then check that they hold in Section 2.7 in two examples: the finite state space superprocess (with mass process the multitype Feller diffusion) and the superdiffusion.

2.2 Notations and definitions

This section, based on the lecture notes of Perkins [104], provides us with basic material about superprocesses, relying on their characterization via the Log Laplace equation.

We first introduce some definitions:

- (E, δ) is a Polish space, $\mathcal{B}$ its Borel sigma-field.
- $\mathcal{E}$ is the set of real valued measurable functions and $b\mathcal{E} \subset \mathcal{E}$ the subset of bounded functions.
- $\mathcal{C}(E, \mathbb{R})$, or simply $\mathcal{C}$, is the set of continuous real valued functions on E, $\mathcal{C}_b \subset \mathcal{C}$ the subset of continuous bounded functions.
- $D(\mathbb{R}^+, E)$, or simply D, is the set of càdlàg paths of E equipped with the Skorokhod topology, $\mathcal{D}$ is the Borel sigma field on D, and $\mathcal{D}_t$ the canonical right continuous filtration on D.
- For each set of functions, the superscript $.^+$ will denote the subset of the nonnegative functions: For instance, $b\mathcal{E}^+$ stands for the subset of non negative functions of $b\mathcal{E}$.
- $\mathcal{M}_f(E)$ is the space of finite measures on E. The standard inner product notation will be used: for $g \in \mathcal{E}$ integrable with respect to $M \in \mathcal{M}_f(E)$, $M(g) = \int_E M(dx)g(x)$.

We can now introduce the two main ingredients which enter in the definition of a superprocess, the spatial motion and the branching mechanism:

- Assume $Y = (D, \mathcal{D}, \mathcal{D}_t, Y_t, \mathrm{P}_x)$ is a Borel strong Markov process. "Borel" means that $x \to \mathrm{P}_x(A)$ is $\mathcal{B}$ measurable for all $A \in \mathcal{B}$. Let E_x denote the expectation operator, and $(P_t, t \geq 0)$ the semigroup defined by: $P_t(f)(x) = \mathrm{E}_x[f(Y_t)]$. We impose the additional assumption that $P_t : \mathcal{C}_b \to \mathcal{C}_b$. In particular the process Y has no fixed discontinuities. The generator associated to the semigroup will be denoted $\mathcal{L}$. Remember f belongs to the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ if $f \in \mathcal{C}_b$ and for some $g \in \mathcal{C}_b$,
$$f(Y_t) - f(x) - \int_0^t ds\ g(Y_s) \text{ is a } P_x \text{ martingale for all } x \text{ in } E, \tag{2.1}$$
in which case $g = \mathcal{L}(f)$.
- The functions α and β being elements of $\mathcal{C}_b$, with α bounded from below by a positive constant, the non-homogeneous quadratic branching mechanism $\psi^{\beta,\alpha}$ is defined by:
$$\psi^{\beta,\alpha}(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2, \tag{2.2}$$
for all $x \in E$ and $\lambda \in \mathbb{R}$. We will just write ψ for $\psi^{\beta,\alpha}$ when there is no possible confusion. If α and β are constant functions, we will call the branching mechanism (and by extension, the corresponding superprocess) homogeneous.

The mild form of the Log Laplace equation is given by the integral equation, for $\phi, f \in b\mathcal{E}^+$, $t \geq 0$, $x \in E$:

$$u_t(x) + \mathrm{E}_x\left[\int_0^t ds\ \psi(Y_s, u_{t-s}(Y_s))\right] = \mathrm{E}_x\left[f(Y_t) + \int_0^t ds\ \phi(Y_s)\right]\cdot \tag{2.3}$$

Theorem 2.2.1. *([104], Theorem II.5.11) Let $\phi, f \in b\mathcal{E}^+$. There is a unique jointly (in t and x) Borel measurable solution $u_t^{f,\phi}(x)$ of equation (2.3) such that $u_t^{f,\phi}$ is bounded on $[0,T]\times E$ for all $T>0$. Moreover, $u_t^{f,\phi}\geq 0$ for all $t \geq 0$.*

We shall write u^f for $u^{f,0}$ when ϕ is null.

We introduce the canonical space of continuous applications from $[0,\infty)$ to $\mathcal{M}_f(E)$, denoted by $\Omega := \mathcal{C}(\mathbb{R}^+, \mathcal{M}_f(E))$, endowed with its Borel sigma field $\mathcal{F}$, and the canonical right continuous filtration $\mathcal{F}_t$. Notice that $\mathcal{F} = \mathcal{F}_\infty$.

Theorem 2.2.2. *([104], Theorem II.5.11) Let $u_t^{f,\phi}(x)$ denote the unique jointly Borel measurable solution of equation (2.3) such that $u_t^{f,\phi}$ is bounded on $[0,T]\times E$ for all $T>0$. There exists a unique Markov process $Z = (\Omega, \mathcal{F}, \mathcal{F}_t, Z_t, (\mathbb{P}_\nu^{(\mathcal{L},\beta,\alpha)}, \nu \in \mathcal{M}_f(E)))$ such that:*

$$\forall \phi, f \in b\mathcal{E}^+, \qquad \mathbb{E}_\nu^{(\mathcal{L},\beta,\alpha)}\left[\mathrm{e}^{-Z_t(f) - \int_0^t ds\ Z_s(\phi)}\right] = \mathrm{e}^{-\nu(u_t^{f,\phi})}\,. \tag{2.4}$$

Z is called the $(\mathcal{L},\beta,\alpha)$-superprocess.

We now state the existence theorem of the canonical measures:

Theorem 2.2.3. *([104], Theorem II.7.3) There exists a measurable family of σ-finite measures $(\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}, x \in E)$ on $(\Omega, \mathcal{F})$ which satisfies the following properties: If $\sum_{j\in\mathcal{J}} \delta_{(x^j, Z^j)}$ is a Poisson point measure on $E \times \Omega$ with intensity $\nu(dx)\ \mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}$, then $\sum_{j\in\mathcal{J}} Z^j$ is an $(\mathcal{L},\beta,\alpha)$-superprocess started at ν.*

We will often abuse notation by denoting $\mathbb{P}_\nu$ (resp. $\mathbb{N}_x$) instead of $\mathbb{P}_\nu^{(\mathcal{L},\beta,\alpha)}$ (resp. $\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}$), and $\mathbb{P}_x$ instead of $\mathbb{P}_{\delta_x}$ when starting from δ_x the Dirac mass at point x.

Let Z be a $(\mathcal{L},\beta,\alpha)$-superprocess. The exponential formula for Poisson point measures yields the following equality:

$$\forall f \in b\mathcal{E}^+,\ \mathbb{N}_{x_0}\left[1 - \mathrm{e}^{-Z_t(f)}\right] = -\log \mathbb{E}_{x_0}\left[\mathrm{e}^{-Z_t(f)}\right] = u_t^f(x_0), \tag{2.5}$$

where u_t^f is (uniquely) defined by equation (2.4).

Denote $H_{\max}$ the extinction time of Z:

$$H_{\max} = \inf\{t > 0;\ Z_t = 0\}. \tag{2.6}$$

Definition 2.2.4 (Global extinction)**.** *The superprocess Z satisfies global extinction if $\mathbb{P}_\nu(H_{max} < \infty) = 1$ for all $\nu \in \mathcal{M}_f(E)$.*

We will need the the following assumption:

$(H1)$ **The $(\mathcal{L},\beta,\alpha)$-superprocess satisfies the global extinction property.**

We shall be interested in the function

$$v_t(x) = \mathbb{N}_x[H_{\max} > t]. \tag{2.7}$$

We set $v_\infty(x) = \lim_{t\to\infty} \downarrow v_t(x)$. The global extinction property is easily stated using v_∞.

Lemma 2.2.5. *The global extinction property holds if and only if $v_\infty = 0$.*

See also Lemma 2.4.9 for other properties of the function v.

Proof. The exponential formula for Poisson point measures yields:

$$\mathbb{P}_\nu(H_{\max} \le t) = \mathrm{e}^{-\nu(v_t)} .$$

To conclude, let t go to ∞ in the previous equality to get:

$$\mathbb{P}_\nu(H_{\max} < \infty) = \mathrm{e}^{-\nu(v_\infty)} .$$

□

For homogeneous superprocesses (α and β constant), the function v is easy to compute and the global extinction holds if and only β is nonnegative. Then, using a stochastic domination argument, one gets that a $(\mathcal{L}, \beta, \alpha)$-superprocess, with β nonnegative, exhibits global extinction (see [43] p.80 for details).

2.3 A genealogy for the non-homogeneous superprocesses

We first recall (Section 2.3.1) the h-transform for superprocess introduced in [44] and then (Section 2.3.2) a Girsanov theorem previously introduced in [104] for interactive superprocesses. Those two transformations allow us to give the Radon-Nikodym derivative of the distribution of a superprocess with non-homogeneous branching mechanism with respect to the distribution of a superprocess with an homogeneous branching mechanism. The genealogy of the superprocess with an homogeneous branching mechanism can be described using a Brownian snake, see [37]. Then, in Section 2.3.3, we use the Radon-Nikodym derivative to transport this genealogy and get a genealogy for the superprocess with non-homogeneous branching mechanism.

2.3.1 *h*-transform for superprocesses

We first introduce a new probability measure on $(D, \mathcal{D})$ using the next Lemma.

Lemma 2.3.1. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then, the process $\big(\frac{g(Y_t)}{g(x)}\,\mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}, t \ge 0\big)$ is a positive martingale under P_x.*

We set $\mathcal{D}_g(\mathcal{L}) = \{v \in \mathcal{C}_b,\ gv \in \mathcal{D}(\mathcal{L})\}$.

Proof. Let g be as in Lemma 2.3.1 and $f \in \mathcal{D}_g(\mathcal{L})$. The process:

$$\left((fg)(Y_t) - (fg)(x) - \int_0^t ds\ \mathcal{L}(fg)(Y_s),\ t \geq 0\right)$$

is a P_x martingale by definition of the generator $\mathcal{L}$. Thus, the process:

$$\left(\frac{(fg)(Y_t)}{g(x)} - f(x) - \int_0^t ds\ \frac{\mathcal{L}(fg)(Y_s)}{g(x)},\ t \geq 0\right)$$

is a P_x martingale. We set:

$$\begin{aligned} M_t^{f,g} &= \mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}\ \frac{(fg)(Y_t)}{g(x)} - f(x) \\ &\quad - \int_0^t ds\ \mathrm{e}^{-\int_0^s dr\ (\mathcal{L}g/g)(Y_r)} \left[\frac{\mathcal{L}(fg)(Y_s)}{g(x)} - \frac{\mathcal{L}(g)(Y_s)}{g(Y_s)}\frac{(fg)(Y_s)}{g(x)}\right]. \end{aligned} \tag{2.8}$$

Itô's lemma then yields that the process $(M_t^{f,g}, t \geq 0)$ is another P_x martingale. Notice this is a true martingale since it is bounded from our assumptions on f and g. Remark also that the constant function equal to 1 is in $\mathcal{D}_g(\mathcal{L})$. This choice of f yields to the result. □

Let P_x^g denote the probability measure on $(D, \mathcal{D})$ defined by:

$$\forall t \geq 0, \quad \frac{dP^g_{x\ |\mathcal{D}_t}}{dP_{x\ |\mathcal{D}_t}} = \frac{g(Y_t)}{g(x)}\ \mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}. \tag{2.9}$$

Note that in the case where g is harmonic for the linear operator $\mathcal{L}$ (that is $\mathcal{L}g = 0$), the probability distribution P^g is the usual Doob h-transform of P for $h = g$.

We also introduce the generator $\mathcal{L}^g$ of the canonical process Y under P^g and the expectation operator E^g associated to P^g.

Lemma 2.3.2. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then, we have $\mathcal{D}_g(\mathcal{L}) \subset \mathcal{D}(\mathcal{L}^g)$ and*

$$\forall u \in \mathcal{D}_g(\mathcal{L}), \quad \mathcal{L}^g(u) = \frac{\mathcal{L}(gu) - \mathcal{L}(g)u}{g}\cdot$$

Proof. As, for $f \in \mathcal{D}_g(\mathcal{L})$, the process $(M_t^{f,g}, t \geq 0)$ defined by (2.8) is a martingale under P_x, we get that the process:

$$f(Y_t) - f(x) - \int_0^t ds\ \left(\frac{\mathcal{L}(fg)(Y_s) - \mathcal{L}(g)(Y_s)f(Y_s)}{g(Y_s)}\right), \quad t \geq 0$$

is a P_x^g martingale. This gives the result. □

Remark 2.3.3. Let $((t,x) \to g(t,x))$ be a function bounded from below by a positive constant, differentiable in t, such that $g(t,.) \in \mathcal{D}(\mathcal{L})$ for each t and $((t,x) \to \partial_t g(t,x))$ is bounded from above. By considering the process (t, Y_t) instead of Y_t, we have the immediate counterpart of Lemma 2.3.1 for time dependent function $g(t,.)$. In particular, we may define the following probability measure on $(D, \mathcal{D})$ (still denoted P_x^g by a small abuse of notations):

$$\forall t \geq 0, \quad \frac{dP^g_{x\ |\mathcal{D}_t}}{dP_{x\ |\mathcal{D}_t}} = \frac{g(t, Y_t)}{g(0,x)}\ \mathrm{e}^{-\int_0^t ds\ \frac{\mathcal{L}g + \partial_t g}{g}(s, Y_s)}, \tag{2.10}$$

where $\mathcal{L}$ acts on g as a function of x.

We now define the h-transform for superprocesses, as introduced in [44] (notice this does not correspond to the Doob h-transform for superprocesses).

Definition 2.3.4. *Let $Z = (Z_t, t \geq 0)$ be an $(\mathcal{L}, \beta, \alpha)$ superprocess. For $g \in b\mathcal{E}^+$, we define the h-transform of Z (with $h = g$) as $Z^g = (Z_t^g, t \geq 0)$ the measure-valued process given for all $t \geq 0$ by:*

$$Z_t^g(dx) = g(x) Z_t(dx). \tag{2.11}$$

Note that (2.11) holds pointwise, and that the law of the h-transform of a superprocess may be singular with respect to the law of the initial superprocess.

We first give an easy generalization of a result in section 2 of [44] for a general spatial motion.

Proposition 2.3.5. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then the process Z^g is a $\left(\mathcal{L}^g, \frac{(-\mathcal{L}+\beta)g}{g}, \alpha g\right)$-superprocess.*

Proof. The Markov property of Z^g is clear. We compute, for $f \in b\mathcal{E}^+$:

$$\mathbb{E}_x[\mathrm{e}^{-Z_t^g(f)}] = \mathbb{E}_{\delta_x/g(x)}[\mathrm{e}^{-Z_t(fg)}] = \mathrm{e}^{-u_t(x)/g(x)},$$

where, by Theorem 2.2.2, u satisfies:

$$u_t(x) + \mathrm{E}_x\left[\int_0^t dr\ \psi(Y_r, u_{t-r}(Y_r))\right] = \mathrm{E}_x\left[(fg)(Y_t)\right], \tag{2.12}$$

which can also be written:

$$u_t(x) + \mathrm{E}_x\left[\int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))\right] + \mathrm{E}_x\left[\int_s^t dr\ \psi(Y_r, u_{t-r}(Y_r))\right] = \mathrm{E}_x\left[(fg)(Y_t)\right].$$

But (2.12) written at time $t-s$ gives:

$$u_{t-s}(x) + \mathrm{E}_x\left[\int_0^{t-s} dr\ \psi(Y_r, u_{t-s-r}(Y_r))\right] = \mathrm{E}_x\left[(fg)(Y_{t-s})\right].$$

By comparing the two previous equations, we get:

$$u_t(x) + \mathrm{E}_x\left[\int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))\right] = \mathrm{E}_x\left[u_{t-s}(Y_s)\right],$$

and the Markov property now implies that the process:

$$u_{t-s}(Y_s) - \int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))$$

with $s \in [0, t]$ is a P_x martingale. Itô's lemma now yields that the process:

$$u_{t-s}(Y_s)\, \mathrm{e}^{-\int_0^s dr (\mathcal{L}g/g)(Y_r)} - \int_0^s dr\ \ \mathrm{e}^{-\int_0^r du\ (\mathcal{L}g/g)(Y_u)} \left(\psi(Y_r, u_{t-r}(Y_r)) - (\mathcal{L}g/g)(Y_r)\ u_{t-r}(Y_r)\right)$$

with $s \in [0, t]$ is another P_x martingale (the integrability comes from the assumption $\mathcal{L}g \in \mathcal{C}_b$ and $1/g \in \mathcal{C}_b$). Taking expectations at time $s = 0$ and at time $s = t$, we have:

$$u_t(x) + \mathrm{E}_x\Big[\int_0^t ds\ \mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)}\ (\psi(Y_s, u_{t-s}(Y_s)) - (\mathcal{L}g/g)(Y_s)u_{t-s}(Y_s))\Big] = \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t dr(\mathcal{L}g/g)(Y_r)}(fg)(Y_t)\Big].$$

We divide both sides by $g(x)$ and expand ψ according to its definition:

$$(\frac{u_t}{g})(x) + \mathrm{E}_x\Big[\int_0^t ds\ \frac{g(Y_s)}{g(x)}\ \mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)}\ \Big((\alpha g)(Y_s)(\frac{u_{t-s}}{g})^2(Y_s) + (\beta - \frac{\mathcal{L}g}{g})(Y_s)(\frac{u_{t-s}}{g})(Y_s)\Big)\Big] = \mathrm{E}_x\Big[\frac{g(Y_t)}{g(x)}\ \mathrm{e}^{-\int_0^t dr(\mathcal{L}g/g)(Y_r)}\ f(Y_t)\Big].$$

By definition of P^g_x from (2.9), we get that:

$$(\frac{u_t}{g})(x) + \mathrm{E}^g_x\Big[\int_0^t ds\ \Big((\alpha g)(Y_s)(\frac{u_{t-s}}{g})^2(Y_s) + (\beta - \frac{\mathcal{L}g}{g})(Y_s)(\frac{u_{t-s}}{g})(Y_s)\Big]\Big) = \mathrm{E}^g_x[f(Y_t)].$$

We conclude from Theorem 2.2.2 that Z^g is a $(\mathcal{L}^g, \frac{(-\mathcal{L}+\beta)g}{g}, \alpha g)$-superprocess. □

In order to perform the h-transform of interest, we shall consider the following assumption.

$(H2)$ $1/\alpha$ **belongs to** $\mathcal{D}(\mathcal{L})$**.**

Notice that $(H2)$ implies that $\alpha\mathcal{L}(1/\alpha) \in \mathcal{C}_b$. Proposition 2.3.5 and Lemma 2.3.1 then yield the following Corollary.

Corollary 2.3.6. *Let Z be an $(\mathcal{L}, \beta, \alpha)$-superprocess. Assume $(H2)$. The process $Z^{1/\alpha}$ is an $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$-superprocess with:*

$$\tilde{\mathcal{L}} = \mathcal{L}^{1/\alpha} \quad \textit{and} \quad \tilde{\beta} = \beta - \alpha\mathcal{L}(1/\alpha). \tag{2.13}$$

Moreover, for all $t \geq 0$, the law $\tilde{\mathrm{P}}_x$ of the process Y with generator $\tilde{\mathcal{L}}$ is absolutely continuous on $\mathcal{D}_t$ with respect to P_x and its Radon-Nikodym derivative is given by:

$$\frac{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\alpha(x)}{\alpha(Y_t)}\ \mathrm{e}^{\int_0^t ds\ (\tilde{\beta}-\beta)(Y_s)}. \tag{2.14}$$

We will note $\tilde{\mathbb{P}}$ for the law of $Z^{1/\alpha}$ on the canonical space (that is $\tilde{\mathbb{P}} = \mathbb{P}^{(\tilde{\mathcal{L}},\tilde{\beta},1)}$) and $\tilde{\mathbb{N}}$ for its canonical measure. Observe that the branching mechanism of Z under $\tilde{\mathbb{P}}$, which we shall write $\tilde{\psi}$, is given by:

$$\tilde{\psi}(x, \lambda) = \tilde{\beta}(x)\ \lambda + \lambda^2, \tag{2.15}$$

and the quadratic coefficient is no more dependent on x. Notice that $\mathbb{P}_{\alpha\nu}(Z \in \cdot) = \tilde{\mathbb{P}}_\nu(\alpha Z \in \cdot)$. This implies the following relationship on the canonical measures (use Theorem 2.2.3 to check it):

$$\alpha(x)\mathbb{N}_x[Z \in \cdot] = \tilde{\mathbb{N}}_x[\alpha Z \in \cdot]. \tag{2.16}$$

Recall that $v_t(x) = \mathbb{N}_x[H_{\max} > t] = \mathbb{N}_x[Z_t \neq 0]$. We set $\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[Z_t \neq 0]$. As α is positive, equality (2.16) implies in particular that, for all $t > 0$ and $x \in E$:

$$\alpha(x)v_t(x) = \tilde{v}_t(x). \tag{2.17}$$

2.3.2 A Girsanov type theorem

The following assumption will be used to perform the Girsanov change of measure.

$(H3)$ **Assumption** $(H2)$ **holds and the function** $\tilde{\beta}$ **defined in (2.13) is in** $\mathcal{D}(\tilde{\mathcal{L}})$**, with** $\tilde{\mathcal{L}}$ **defined in (2.13).**

For $z \in \mathbb{R}$, we set $z_+ = \max(z, 0)$. Under $(H2)$ and $(H3)$, we define:

$$\beta_0 = \sup_{x \in E} \max\left(\tilde{\beta}(x), \sqrt{(\tilde{\beta}^2(x) - 2\tilde{\mathcal{L}}(\tilde{\beta})(x))_+}\right) \quad \text{and} \quad q(x) = \frac{\beta_0 - \tilde{\beta}(x)}{2}\cdot \tag{2.18}$$

Notice that $q \geq 0$.

We shall consider the distribution of the homogeneous $(\tilde{\mathcal{L}}, \beta_0, 1)$-superprocess, which we will denote by $\mathbb{P}^0$ ($\mathbb{P}^0 = \mathbb{P}^{(\tilde{\mathcal{L}}, \beta_0, 1)}$) and its canonical measure $\mathbb{N}^0$. Note that the branching mechanism of Z under $\mathbb{P}^0$ is homogeneous (the branching mechanism does not depend on x). We set ψ^0 for $\psi^{\beta_0, 1}$. Since ψ^0 does not depend anymore on x we shall also write $\psi^0(\lambda)$ for $\psi^0(x, \lambda)$:

$$\psi^0(\lambda) = \beta_0 \lambda + \lambda^2. \tag{2.19}$$

Proposition 2.3.7 below is a Girsanov's type theorem which allows us to finally reduce the distribution $\tilde{\mathbb{P}}$ to the homogeneous distribution $\mathbb{P}^0$. We introduce the process $M = (M_t, t \geq 0)$ defined by:

$$M_t = \exp\left(Z_0(q) - Z_t(q) - \int_0^t ds\ Z_s(\varphi)\right), \tag{2.20}$$

where the function φ is defined by:

$$\varphi(x) = \tilde{\psi}(x, q(x)) - \tilde{\mathcal{L}}(q)(x), \quad x \in E. \tag{2.21}$$

Proposition 2.3.7. *A Girsanov's type theorem. Assume $(H2)$ and $(H3)$ hold. Let Z be a $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$-superprocess.*
(i) The process M is a bounded $\mathcal{F}$-martingale under $\tilde{\mathbb{P}}_\nu$ which converges a.s. to

$$M_\infty = \mathrm{e}^{Z_0(q) - \int_0^{+\infty} ds\ Z_s(\varphi)}\, \mathbf{1}_{\{H_{max} < +\infty\}}.$$

(ii) We have:

$$\frac{d\mathbb{P}^0_\nu}{d\tilde{\mathbb{P}}_\nu} = M_\infty.$$

(iii) If moreover $(H1)$ holds, then $\mathbb{P}^0_\nu$-a.s. we have $M_\infty > 0$, the probability measure $\tilde{\mathbb{P}}_\nu$ is absolutely continuous with respect to $\mathbb{P}^0_\nu$ on $\mathcal{F}$:

$$\frac{d\tilde{\mathbb{P}}_\nu}{d\mathbb{P}^0_\nu} = \frac{1}{M_\infty}, \quad \text{and} \quad \frac{d\tilde{\mathbb{N}}_x}{d\mathbb{N}^0_x} = \mathrm{e}^{\int_0^{+\infty} ds\ Z_s(\varphi)}.$$

We also have:

$$q(x) = \mathbb{N}^0_x\left[\mathrm{e}^{\int_0^{+\infty} ds\ Z_s(\varphi)} - 1\right]. \tag{2.22}$$

The two first points are a particular case of Theorem IV.1.6 p.252 in [104] on interactive drift. For the sake of completeness, we give a proof based on the mild form of the Log Laplace equation (2.3) introduced in Section 2.2. Notice that:

$$\psi^0(\lambda) = \tilde{\psi}(x, \lambda + q(x)) - \tilde{\psi}(x, q(x)). \tag{2.23}$$

Thus, Proposition 2.3.7 appears as a non-homogeneous generalization of Corollary 4.4 in [2]. We first give an elementary Lemma.

Lemma 2.3.8. *Assume* $(H2)$ *and* $(H3)$ *hold. The function* φ *defined by (2.21) is nonnegative.*

Proof. The following computation:

$$\begin{aligned}\varphi(x) = \tilde{\psi}(x, q(x)) - \tilde{\mathcal{L}}(q)(x) &= q(x)^2 + \tilde{\beta} q(x) - \tilde{\mathcal{L}}(q)(x)\\ &= \left(\frac{\beta_0 - \tilde{\beta}(x)}{2}\right)^2 + \tilde{\beta}(x)\ \frac{\beta_0 - \tilde{\beta}(x)}{2} - \tilde{\mathcal{L}}(q)(x)\\ &= \frac{\beta_0^2 - \tilde{\beta}^2(x) + 2\tilde{\mathcal{L}}(\tilde{\beta})(x)}{4}\end{aligned}$$

and the definition (2.18) of β_0 ensure that the function φ is nonnegative. □

Proof of Proposition 2.3.7. First observe that M is $\mathcal{F}$-adapted. As the function q also is nonnegative, we deduce from Lemma 2.3.8 that the process M is bounded by $\mathrm{e}^{Z_0(q)}$.

Let $f \in b\mathcal{E}^+$. On the one hand, we have:

$$\tilde{\mathbb{E}}_x[M_t\, \mathrm{e}^{-Z_t(f)}] = \tilde{\mathbb{E}}_x[\mathrm{e}^{q(x) - Z_t(q+f) - \int_0^t ds\ Z_s(\varphi)}] = \mathrm{e}^{q(x) - r_t(x)},$$

where, according to Theorem 2.2.2, $r_t(x)$ is bounded on $[0,T] \times E$ for all $T > 0$ and satisfies:

$$\begin{aligned} r_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, r_s(Y_{t-s}))\right]& \\ = \tilde{\mathrm{E}}_x\left[\int_0^t ds\ (\tilde{\psi}(Y_{t-s}, q(Y_{t-s})) - \tilde{\mathcal{L}}(q)(Y_{t-s})) + (q+f)(Y_t)\right].& \end{aligned} \tag{2.24}$$

On the other hand, we have

$$\mathbb{E}^0_x[\mathrm{e}^{-Z_t(f)}] = \mathrm{e}^{-w_t(x)},$$

where $w_t(x)$ is bounded on $[0,T] \times E$ for all $T > 0$ and satisfies:

$$w_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \psi^0(Y_{t-s}, w_s(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x[f(Y_t)].$$

Using (2.23), rewrite the previous equation under the form:

$$w_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, (w_s + q)(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, q(Y_{t-s})) + f(Y_t)\right]. \tag{2.25}$$

We now make use of the Dynkin's formula with $(H3)$:

$$q(x) = -\tilde{\mathrm{E}}_x\Big[\int_0^t \tilde{\mathcal{L}}(q)(Y_s)\Big] + \tilde{\mathrm{E}}_x[q(Y_t)], \tag{2.26}$$

and sum the equations (2.25) and (2.26) term by term to get:

$$\begin{aligned}(w_t+q)(x) + \tilde{\mathrm{E}}_x\Big[\int_0^t ds\ \tilde{\psi}(Y_{t-s},(w_s+q)(Y_{t-s}))\Big] \\ = \tilde{\mathrm{E}}_x\Big[\int_0^t ds\ \big(\tilde{\psi}(Y_{t-s},q(Y_{t-s})) - \tilde{\mathcal{L}}(q)(Y_{t-s})\big) + (q+f)(Y_t)\Big].\end{aligned} \tag{2.27}$$

The functions $r_t(x)$ and $w_t(x)+q(x)$ are bounded on $[0,T]\times E$ for all $T>0$ and satisfy the same equation, see equations (2.24) and (2.27). By uniqueness, see Theorem 2.2.1, we finally get that $w_t+q=r_t$. This gives:

$$\tilde{\mathbb{E}}_x[M_t\,\mathrm{e}^{-Z_t(f)}] = \mathbb{E}^0_x[\mathrm{e}^{-Z_t(f)}]. \tag{2.28}$$

The Poissonian decomposition of the superprocesses, see Theorem 2.2.3, and the exponential formula enable us to extend this relation to arbitrary initial measures ν:

$$\tilde{\mathbb{E}}_\nu[M_t\,\mathrm{e}^{-Z_t(f)}] = \mathbb{E}^0_\nu[\mathrm{e}^{-Z_t(f)}]. \tag{2.29}$$

This equality with $f=0$ and the Markov property of Z proves the first part of item (i).

Now, a direct induction based on the Markov property yields that, for all positive integer n, and $f_1,\ldots,f_n \in b\mathcal{E}^+$, $0\leq s_1\leq\ldots\leq s_n\leq t$:

$$\tilde{\mathbb{E}}_\nu[M_t\,\mathrm{e}^{-\sum_{1\leq i\leq n} Z_{s_i}(f_i)}] = \mathbb{E}^0_\nu[\mathrm{e}^{-\sum_{1\leq i\leq n} Z_{s_i}(f_i)}]. \tag{2.30}$$

And we conclude with an application of the monotone class theorem that, for all nonnegative $\mathcal{F}_t$-measurable random variable A:

$$\tilde{\mathbb{E}}_\nu[M_t A] = \mathbb{E}^0_\nu[A].$$

The martingale M is bounded and thus converges a.s. to a limit M_∞. We deduce that for all nonnegative $\mathcal{F}_t$-measurable random variable A:

$$\tilde{\mathbb{E}}_\nu[M_\infty A] = \mathbb{E}^0_\nu[A]. \tag{2.31}$$

This also holds for any nonnegative $\mathcal{F}_\infty$-measurable random variable A. This gives the second item (ii).

On $\{H_{\max}<+\infty\}$, then clearly M_t converges to $\mathrm{e}^{Z_0(q)-\int_0^{+\infty} ds\ Z_s(\varphi)}$. Notice that $\mathbb{P}^0_\nu(H_{\max}=+\infty)=0$. We deduce from (2.31) with $A=\mathbf{1}_{\{H_{\max}=+\infty\}}$ that $\tilde{\mathbb{P}}_\nu$-a.s. on $\{H_{\max}=+\infty\}$, $M_\infty=0$. This gives the first part of item (i).

Now, we prove the third item (iii). Notice that (2.31) implies that $\mathbb{P}^0_\nu$-a.s. $M_\infty>0$. Thanks to $(H1)$, we also have that $\tilde{\mathbb{P}}_\nu$-a.s. $M_\infty>0$. Let A be a nonnegative $\mathcal{F}_\infty$-measurable random variable. Applying (2.31) with A replaced by $\mathbf{1}_{\{M_\infty>0\}}A/M_\infty$, we get:

$$\tilde{\mathbb{E}}_\nu[A] = \tilde{\mathbb{E}}_\nu\Big[M_\infty\mathbf{1}_{\{M_\infty>0\}}\frac{A}{M_\infty}\Big] = \mathbb{E}^0_\nu\Big[\frac{A}{M_\infty}\mathbf{1}_{\{M_\infty>0\}}\Big] = \mathbb{E}^0_\nu\Big[\frac{A}{M_\infty}\Big].$$

This gives the first part of item (iii).

Notice that for all positive integer n, and $f_1,\dots,f_n \in b\mathcal{E}^+$, $0 \le s_1 \le \ldots \le s_n$, we have

$$\begin{aligned}\tilde{\mathbb{N}}_x\left[1-\mathrm{e}^{\sum_{1\le i\le n} Z_{s_i}(f_i)}\right] &= -\log\left(\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\sum_{1\le i\le n} Z_{s_i}(f_i)}\right]\right)\\ &= -\log\left(\mathrm{E}^0_x\left[\mathrm{e}^{\sum_{1\le i\le n} Z_{s_i}(f_i)+\int_0^{+\infty} ds\; Z_s(\varphi)}\right]\right)+q(x)\\ &= \mathbb{N}^0_x\left[1-\mathrm{e}^{\sum_{1\le i\le n} Z_{s_i}(f_i)+\int_0^{+\infty} ds\; Z_s(\varphi)}\right]+q(x).\end{aligned}$$

Taking $f_i = 0$ for all i gives (2.22). This implies:

$$\tilde{\mathbb{N}}_x\left[1-\mathrm{e}^{\sum_{1\le i\le n} Z_{s_i}(f_i)}\right] = \mathbb{N}^0_x\left[\mathrm{e}^{\int_0^{+\infty} ds\; Z_s(\varphi)}\left(1-\mathrm{e}^{\sum_{1\le i\le n} Z_{s_i}(f_i)}\right)\right].$$

The monotone class theorem gives then the last part of item (iii). □

2.3.3 Genealogy for superprocesses

We now recall the genealogy of Z under $\mathbb{P}^0$ given by the Brownian snake from [37]. We assume $(H2)$ and $(H3)$ hold.

Let $\mathcal{W}$ denote the set of all càdlàg killed paths in E. An element $w \in \mathcal{W}$ is a càdlàg path: $w : [0, \eta(w)) \to E$, with $\eta(w)$ the lifetime of the path w. By convention the trivial path $\{x\}$, with $x \in E$, is a killed path with lifetime 0 and it belongs to $\mathcal{W}$. The space $\mathcal{W}$ is Polish for the distance:

$$d(w,w') = \delta(w(0), w(0)') + |\eta(w) - \eta(w')| + \int_0^{\eta(w)\wedge\eta(w')} ds\; d_s(w_{[0,s]}, w'_{[0,s]}),$$

where d_s refers to the Skorokhod metric on the space $D([0,s],E)$, and w_I is the restriction of w on the interval I. Denote $\mathcal{W}_x$ the set of stopped paths w such that $w(0) = x$. We work on the canonical space of continuous applications from $[0,\infty)$ to $\mathcal{W}$, denoted by $\bar{\Omega} := \mathcal{C}(\mathbb{R}^+, \mathcal{W})$, endowed with the Borel sigma field $\bar{\mathcal{G}}$ for the distance d, and the canonical right continuous filtration $\bar{\mathcal{G}}_t = \sigma\{W_s, s \le t\}$, where $(W_s, s \in \mathbb{R}^+)$ is the canonical coordinate process. Notice $\bar{\mathcal{G}} = \bar{\mathcal{G}}_\infty$ by construction. We set $H_s = \eta(W_s)$ the lifetime of W_s.

Definition 2.3.9 (Proposition 4.1.1 and Theorem 4.1.2 of [37])**.** *Fix $W_0 \in \mathcal{W}_x$. There exists a unique $\mathcal{W}_x$-valued Markov process $W = (\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathcal{G}}_t, W_t, \mathbf{P}^0_{W_0})$, called the Brownian snake, starting at W_0 and satisfying the two properties:*
(i) The lifetime process $H = (H_s, s \ge 0)$ is a reflecting Brownian motion with non-positive drift $-\beta_0$, starting from $H_0 = \eta(W_0)$.
(ii) Conditionally given the lifetime process H, the process $(W_s, s \ge 0)$ is distributed as a non-homogeneous Markov process, with transition kernel specified by the two following prescriptions, for $0 \le s \le s'$:
– $W_{s'}(t) = W_s(t)$ for all $t < H_{[s,s']}$, with $H_{[s,s']} = \inf_{s\le r\le s'} H_r$.
– Conditionally on $W_s(H_{[s,s']}-)$, the path $\left(W_{s'}(H_{[s,s']}+t), 0 \le t < H_{s'} - H_{[s,s']}\right)$ is independent of W_s and is distributed as $Y_{[0,H_{s'}-H_{[s,s']})}$ under $\tilde{\mathrm{P}}_{W_s(H_{[s,s']}-)}$.
This process will be called the β_0-snake started at W_0, and its law denoted by $\mathbf{P}^0_{W_0}$.

We will just write $\mathbf{P}_x^0$ for the law of the snake started at the trivial path $\{x\}$. The corresponding excursion measure $\mathbf{N}_x^0$ of W is given as follows: the lifetime process H is distributed according to the Itô measure of the positive excursion of a reflecting Brownian motion with non-positive drift $-\beta_0$, and conditionally given the lifetime process H, the process $(W_s, s \geq 0)$ is distributed according to (ii) of Definition 2.3.9. Let

$$\sigma = \inf\{s > 0; H_s = 0\}$$

denote the length of the excursion under $\mathbf{N}_x^0$.

Let $(l_s^r, r \geq 0, s \geq 0)$ be the bicontinuous version of the local time process of H; where l_s^r refers to the local time at level r at time s. We also set $\hat{w} = w(\eta(w)-)$ for the left end position of the path w. We consider the measure-valued process $Z(W) = (Z_t(W), t \geq 0)$ defined under $\mathbf{N}_x^0$ by:

$$Z_t(W)(dx) = \int_0^\sigma d_s l_s^t \, \delta_{\hat{W}_s}(dx). \tag{2.32}$$

The β_0-snake gives the genealogy of the $(\tilde{\mathcal{L}}, \beta_0, 1)$ superprocess in the following sense.

Proposition 2.3.10 ([37], Theorem 4.2.1)**.** *We have:*
– *The process $Z(W)$ is under $\mathbf{N}_x^0$ distributed as Z under $\mathbb{N}_x^0$.*
– *Let $\sum_{j\in\mathcal{J}} \delta_{(x^j,W^j)}$ be a Poisson point measure on $E \times \bar{\Omega}$ with intensity $\nu(dx)\, \mathbf{N}_x^0[dW]$. Then $\sum_{j\in\mathcal{J}} Z(W^j)$ is an $(\tilde{\mathcal{L}}, \beta_0, 1)$-superprocess started at ν.*

Notice that, under $\mathbf{N}_x^0$, the extinction time of $Z(W)$ is defined by

$$\inf\{t; Z_t(W) = 0\} = \sup_{s\in[0,\sigma]} H_s,$$

and we shall write this quantity $H_{\max}$ or $H_{\max}(W)$ if we need to stress the dependence in W. This notation is coherent with (2.6).

We now transport the genealogy of Z under $\mathbb{N}^0$ to a genealogy of Z under $\tilde{\mathbb{N}}$. In order to simplify notations, we shall write Z for $Z(W)$ when there is no confusion.

Definition 2.3.11. *Under (H1)-(H3), we define a measure $\tilde{\mathbf{N}}_x$ on $(\bar{\Omega}, \bar{\mathcal{G}})$ by:*

$$\forall W \in \bar{\Omega}, \quad \frac{d\tilde{\mathbf{N}}_x}{d\mathbf{N}_x^0}(W) = \frac{d\tilde{\mathbb{N}}_x}{d\mathbb{N}_x^0}(Z(W)) = \frac{1}{M_\infty} = \mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)}.$$

Notice the second equality in the previous definition is the third item of Proposition 2.3.7.

At this point, the genealogy defined for Z under $\tilde{\mathbb{N}}_x$ will give the genealogy of Z under $\mathbb{N}$ up to a ponderation. We set

$$\mathbf{N}_x = \frac{1}{\alpha(x)} \tilde{\mathbf{N}}_x. \tag{2.33}$$

Proposition 2.3.12. *We have:*
(i) $Z(W)$ under $\tilde{\mathbf{N}}_x$ is distributed as Z under $\tilde{\mathbb{N}}_x$.
(ii) The weighted process $Z^{weight} = (Z_t^{weight}, t \geq 0)$ with

$$Z_t^{weight}(dx) = \int_0^\sigma d_s l_s^t \; \alpha(\hat{W}_s)\delta_{\hat{W}_s}(dx), \quad t \geq 0, \tag{2.34}$$

is under $\mathbf{N}_x$ distributed as Z under $\mathbb{N}_x$.

We may write $Z^{\text{weight}}(W)$ for Z^{weight} to emphazise the dependence in the snake W.

Proof. This is a direct consequence of Definition 2.3.11 and (2.16). □

We shall say that W under $\mathbf{N}_x$ provides through (2.34) a genealogy for Z under $\mathbb{N}_x$.

2.4 A Williams decomposition

In Section 2.4.1, we give a decomposition of the genealogy of the superprocesses $(\mathcal{L}, \beta, \alpha)$ and $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$ with respect to a randomly chosen individual. In Section 2.4.2, we give a Williams decomposition of the genealogy of the superprocesses $(\mathcal{L}, \beta, \alpha)$ and $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$ with respect to the last individual alive.

2.4.1 Bismut's decomposition

A decomposition of the genealogy of the homogeneous superprocess with respect to a randomly chosen individual is well known in the homogeneous case, even for a general branching mechanism (see lemmas 4.2.5 and 4.6.1 in [37]).

We now explain how to decompose the snake process under the excursion measure ($\tilde{\mathbf{N}}_x$ or $\mathbf{N}_x^0$) with respect to its value at a given time. Recall $\sigma = \inf\{s > 0, H_s = 0\}$ denote the length of the excursion. Fix a real number $t \in [0, \sigma]$. We consider the process $H^{(g)}$ (on the left of t) defined on $[0, t]$ by $H_s^{(g)} = H_{t-s} - H_t$ for all $s \in [0, t]$. The excursion intervals above 0 of the process $(H_s^{(g)} - \inf_{0 \leq s' \leq s} H_{s'}^{(g)}, 0 \leq s \leq t)$ are denoted $\bigcup_{j \in J^{(g)}}(c_j, d_j)$. We also consider the process $H^{(d)}$ (on the right of t) defined on $[0, \sigma - t]$ by $H_s^{(d)} = H_{t+s} - H_t$. The excursion intervals above 0 of the process $(H_s^{(d)} - \inf_{0 \leq s' \leq s} H_{s'}^{(d)}, 0 \leq s \leq \sigma - t)$ are denoted $\bigcup_{j \in J^{(d)}}(c_j, d_j)$. We define the level of the excursion j as $s_j = H_{t-c_j}$ if $j \in J^{(g)}$ and $s_j = H_{t+c_j}$ if $j \in J^{(d)}$. We also define for the excursion j the corresponding excursion of the snake: $W^j = (W_s^j, s \geq 0)$ as

$$W_s^j(.) = W_{t-(c_j+s)\wedge d_j}(. + s_j) \quad \text{if } j \in J^{(g)}, \text{ and } \quad W_s^j(.) = W_{t+(c_j+s)\wedge d_j}(. + s_j) \quad \text{if } j \in J^{(d)}.$$

We consider the following two point measures on $\mathbb{R}^+ \times \bar{\Omega}$: for $\varepsilon \in \{g, d\}$,

$$R_t^\varepsilon = \sum_{j \in J^{(\varepsilon)}} \delta_{(s_j, W^j)}. \tag{2.35}$$

Notice that under $\mathbf{N}_x^0$ (and under $\tilde{\mathbf{N}}_x$ if $(H1)$ holds), the process W can be reconstructed from the triplet (W_t, R_t^g, R_t^d) as follows. If $s \in [0, t]$, then there exists $j \in J^{(g)}$ such that $(t-s) \in [c_j, d_j]$ and we have:

$$W_s(u) = W^j_{(t-s)-c_j}(u - s_j) \text{ if } u > s_j \text{ and } W_s(u) = W_t(u) \text{ if } u \leq s_j.$$

If $s \in [t, \sigma]$, then there exists $j \in J^{(d)}$ such that $(s-t) \in [c_j, d_j]$ and we have:

$$W_s(u) = W^j_{(s-t)-c_j}(u - s_j) \text{ if } u > s_j \text{ and } W_s(u) = W_t(u) \text{ if } u \leq s_j.$$

We are interested in the probabilistic structure of this triplet, when t is chosen according to the Lebesgue measure on the excursion time interval of the snake. Under $\mathbf{N}^0$, this result is as a consequence of Lemmas 4.2.4 and 4.2.5 from [37]. We recall this result in the next Proposition.

For a point measure $R = \sum_{j\in J} \delta_{(s_j,x_j)}$ on a space $\mathbb{R}\times\mathcal{X}$ and $A\subset\mathbb{R}$, we shall consider the restriction of R to $A\times\mathcal{X}$ given by $R_A = \sum_{j\in J} \mathbf{1}_A(s_j)\delta_{(s_j,x_j)}$.

Proposition 2.4.1 ([37], Lemmas 4.2.4 and 4.2.5)**.** *For every measurable nonnegative function F, the following formulas hold:*

$$\mathbf{N}^0_x\Big[\int_0^\sigma ds\ F(W_s, R^g_s, R^d_s)\Big] = \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,r)}, \hat{R}^{B,g}_{[0,r)}, \hat{R}^{B,d}_{[0,r)})\Big], \tag{2.36}$$

$$\mathbf{N}^0_x\Big[\int_0^\sigma d_s l^t_s\ F(W_s, R^g_s, R^d_s)\Big] = \mathrm{e}^{-\beta_0 t}\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,t)}, \hat{R}^{B,g}_{[0,t)}, \hat{R}^{B,d}_{[0,t)})\Big], \quad t>0, \tag{2.37}$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on Y, $\hat{R}^{B,g}$ and $\hat{R}^{B,d}$ are two independent Poisson point measures with intensity $\hat{\nu}^B(ds, dW) = ds\ \mathbf{N}^0_{Y_s}[dW]$.

The next Proposition gives a similar result in the non-homogeneous case.

Proposition 2.4.2. *Under $(H1)$-$(H3)$, for every measurable nonnegative function F, the two formulas hold:*

$$\tilde{\mathbf{N}}_x\Big[\int_0^\sigma ds\ F(W_s, R^g_s, R^d_s)\Big] = \int_0^\infty dr\ \tilde{\mathrm{E}}_x\Big[\mathrm{e}^{-\int_0^r ds\ \tilde{\beta}(Y_s)}\, F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\Big], \tag{2.38}$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on Y, $R^{B,g}_{[0,r)}$ and $R^{B,d}_{[0,r)}$ are two independent Poisson point measures with intensity

$$\nu^B(ds, dW) = ds\ \tilde{\mathbf{N}}_{Y_s}[dW] = ds\ \alpha(Y_s)\mathbf{N}_{Y_s}[dW]; \tag{2.39}$$

and

$$\mathbf{N}_x\Big[\int_0^\sigma ds\ \alpha(\hat{W}_s) F(W_s, R^g_s, R^d_s)\Big] = \int_0^\infty dr\ \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^r ds\ \beta(Y_s)}\, F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\Big], \tag{2.40}$$

where under E_x and conditionally on Y, $R^{B,g}_{[0,r)}$ and $R^{B,d}_{[0,r)}$ are two independent Poisson point measures with intensity ν^B.

Observe there is a weight $\alpha(\hat{W}_s)$ in (2.40) (see also (2.34) where this weight appears) which modifies the law of the individual picked at random, changing the modified diffusion $\tilde{\mathrm{P}}_x$ in (2.38) into the original one P_x.

We shall use the following elementary Lemma on Poisson point measures.

Lemma 2.4.3. *Let R be a Poisson point measure on a Polish space with intensity ν. Let f be a nonnegative measurable function f such that $\nu(\mathrm{e}^f - 1) < +\infty$. Then for any nonnegative measurable function F, we have:*

$$\mathbb{E}\left[F(R)\,\mathrm{e}^{R(f)}\right] = \mathbb{E}\left[F(\tilde{R})\right]\mathrm{e}^{\nu(\mathrm{e}^f - 1)}, \tag{2.41}$$

where $\tilde{R}$ is a Poisson point measure with intensity $\tilde{\nu}(dx) = \mathrm{e}^{f(x)}\,\nu(dx)$.

Proof of Proposition 2.4.2. We keep notations introduced in Propositions 2.4.1 and 2.4.2. We have:

$$
\begin{aligned}
\tilde{\mathbf{N}}_x\Big[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\Big] \\
&= \mathbf{N}_x^0\Big[\mathrm{e}^{\int_0^{+\infty} ds\ Z_s(\varphi)} \int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\Big] \\
&= \mathbf{N}_x^0\Big[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\, \mathrm{e}^{(R_s^g+R_s^d)(f)}\Big] \\
&= \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,r)}, \hat{R}^{B,g}_{[0,r)}, \hat{R}^{B,d}_{[0,r)})\, \mathrm{e}^{(\hat{R}^{B,g}_{[0,r)}+\hat{R}^{B,d}_{[0,r)})(f)}\Big] \\
&= \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\, \mathrm{e}^{2\int_0^r ds\ \mathbf{N}^0_{Y_s}[\mathrm{e}^{\int_0^{+\infty} Z_r(W)(\varphi)}-1]}\Big] \\
&= \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\, \mathrm{e}^{2\int_0^r ds\ q(Y_s)}\Big] \\
&= \int_0^\infty dr\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\, \mathrm{e}^{-\int_0^r ds\ \tilde{\beta}(Y_s)}\Big],
\end{aligned}
$$

where the first equality comes from $(H1)$ and item (iii) of Proposition 2.3.7, we set $f(s, W) = \int_0^{+\infty} Z_r(W)(\varphi)$ for the second equality, we use Proposition 2.4.1 for the third equality, we use Lemma 2.4.3 for the fourth, we use (2.22) for the fifth, and the definition (2.18) of q in the last. This proves (2.38).

Then replace $F(W_s, R_s^g, R_s^d)$ by $\alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)$ in (2.38) and use (2.14) as well as (2.33) to get (2.40). □

The proof of the following Proposition is similar to the proof of Proposition 2.4.2 and is not reproduced here.

Proposition 2.4.4. *Under* $(H1)$-$(H3)$*, for every measurable nonnegative function* F*, the two formulas hold: for fixed* $t > 0$*,*

$$
\tilde{\mathbf{N}}_x\Big[\int_0^\sigma d_s l_s^t\ F(W_s, R_s^g, R_s^d)\Big] = \tilde{\mathrm{E}}_x\Big[\mathrm{e}^{-\int_0^t ds\ \tilde{\beta}(Y_s)}\, F(Y_{[0,t)}, R^{B,g}_{[0,t)}, R^{B,d}_{[0,t)})\Big], \tag{2.42}
$$

where under $\tilde{\mathrm{E}}_x$ *and conditionally on* Y*,* $R^{B,g}$ *and* $R^{B,d}$ *are two independent Poisson point measures with intensity* ν^B *defined in (2.39), and*

$$
\mathbf{N}_x\Big[\int_0^\sigma d_s l_s^t\ \alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)\Big] = \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\, F(Y_{[0,t)}, R^{B,g}_{[0,t)}, R^{B,d}_{[0,t)})\Big], \tag{2.43}
$$

where under E_x *and conditionally on* Y*,* $R^{B,g}$ *and* $R^{B,d}$ *are two independent Poisson point measures with intensity* ν^B*.*

As an example of application of this Proposition, we can recover easily the following well known result.

Corollary 2.4.5. *Under* $(H1)$-$(H3)$*, for every measurable nonnegative functions* f *and* g *on* E*, we have:*

$$
\mathbb{N}_x\Big[Z_t(f)\, \mathrm{e}^{-Z_t(g)}\Big] = \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \partial_\lambda \psi\left(Y_s,\ \mathbb{N}_{Y_s}\left[1-\mathrm{e}^{Z_{t-s}(g)}\right]\right)}\, f(Y_t)\Big].
$$

In particular, we recover the so-called "many-to-one" formula (with $g = 0$ in Corollary 2.4.5):

$$\mathbb{N}_x[Z_t(f)] = \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\, f(Y_t)\Big]. \tag{2.44}$$

Proof. We set for $w \in \mathcal{W}$ with $\eta(w) = t$ and r_1, r_2 two point measures on $\mathbb{R}^+ \times \bar{\Omega}$

$$F(w, r_1, r_2) = f(\hat{w})\, \mathrm{e}^{h(r_1)+h(r_2)},$$

where $h(\sum_{i\in I} \delta_{(s_i,W^i)}) = \sum_{s_i<t} Z^{\text{weight}}(W^i)_{t-s_i}(g)$. We have:

$$\begin{aligned}
\mathbb{N}_x\Big[Z_t(f)\,\mathrm{e}^{-Z_t(g)}\Big] &= \mathbf{N}_x\Big[\int_0^\sigma d_s l_s^t\ \alpha(\hat{W}_s) F(W_s, R_s^g, R_s^d)\Big]\\
&= \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\, f(Y_t)\,\mathrm{e}^{h(R^{B,g}_{[0,r)})+h(R^{B,d}_{[0,r)})}\Big]\\
&= \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\, f(Y_t)\,\mathrm{e}^{-\int_0^t 2\alpha(Y_s)\mathbf{N}_{Y_s}[1-\mathrm{e}^{Z^{\text{weight}}_{t-s}(g)}]}\Big]\\
&= \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \partial_\lambda\psi\left(Y_s,\ \mathbb{N}_{Y_s}\left[1-\mathrm{e}^{Z_{t-s}(g)}\right]\right)}\, f(Y_t)\Big],
\end{aligned}$$

where we used item (ii) of Proposition 2.3.12 for the first and last equality, (2.43) with F previously defined for the second, formula for exponentials of Poisson point measure and (2.33) for the third. □

Remark 2.4.6. Equation (2.44) justifies the introduction of the following family of probability measures indexed by $t \geq 0$:

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}}{\mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\Big]}, \tag{2.45}$$

which can be understood as the law of the ancestral lineage of an individual sampled at random at height t under the excursion measure $\mathbb{N}_x$, and also correspond to Feynman Kac penalisation of the original spatial motion P_x (see [115]). Notice that this law does not depend on the parameter α. These probability measures are not compatible as t varies but will be shown in Lemma 2.6.13 to converge as $t \to \infty$ in restriction to $\mathcal{D}_s$, s fixed, $s \leq t$, under some ergodic assumption (see $(H9)$ in Section 2.6).

2.4.2 Williams decomposition

We first recall Williams decomposition for the Brownian snake (see [132] for Brownian excursions, [122] for Brownian snake or [1] for general homogeneous branching mechanism without spatial motion).

Under the excursion measures $\mathbf{N}^0_x$, $\tilde{\mathbf{N}}_x$ and $\mathbf{N}_x$, recall that $H_{\max} = \sup_{[0,\sigma]} H_s$. Because of the continuity of H, we can define $T_{\max} = \inf\{s > 0, H_s = H_{\max}\}$. Notice the properties of the Brownian excursions implies that a.e. $H_s = H_{\max}$ only if $s = T_{\max}$. We set $v^0_t(x) = \mathbb{N}^0_x[H_{\max} > t]$ and recall this function does not depend on x. Thus, we shall write v^0_t for $v^0_t(x)$. Standard computations give:

$$v_t^0 = \frac{\beta_0}{\mathrm{e}^{\beta_0 t} - 1}.$$

The next result is a straightforward adaptation from Theorem 3.3 of [1] and gives the distribution of $(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$ under $\mathbf{N}_x^0$.

Proposition 2.4.7 (Williams decomposition under $\mathbf{N}_x^0$)**.** *We have:*
(i) The distribution of H_{max} under $\mathbf{N}_x^0$ is characterized by: $\mathbf{N}_x^0[H_{max} > h] = v_h^0$.
(ii) Conditionally on $\{H_{max} = h_0\}$, $W_{T_{max}}$ under $\mathbf{N}_x^0$ is distributed as $Y_{[0,h_0)}$ under $\tilde{\mathrm{P}}_x$.
(iii) Conditionally on $\{H_{max} = h_0\}$ and $W_{T_{max}}$, $R^g_{T_{max}}$ and $R^d_{T_{max}}$ are under $\mathbf{N}_x^0$ independent Poisson point measures on $\mathbb{R}^+ \times \bar{\Omega}$ with intensity:

$$\mathbf{1}_{[0,h_0)}(s) ds\ \mathbf{1}_{\{H_{max}(W) < h_0 - s\}}\ \mathbf{N}^0_{W_{T_{max}}(s)}[dW].$$

In other words, for any nonnegative measurable function F, we have

$$\mathbf{N}_x^0\Big[F(H_{max}, W_{T_{max}}, R^g_{T_{max}}, R^d_{T_{max}})\Big] = -\int_0^\infty \partial_h v_h^0\ dh\ \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, \hat{R}^{W,(h),g}, \hat{R}^{W,(h),d})\Big],$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on $Y_{[0,h)}$, $\hat{R}^{W,(h),g}$ and $\hat{R}^{W,(h),d}$ are two independent Poisson point measures with intensity $\hat{\nu}^{W,(h)}(ds, dW) = \mathbf{1}_{[0,h)}(s) ds\ \mathbf{1}_{\{H_{max}(W) < h - s\}}\ \mathbf{N}^0_{Y_s}[dW]$.

Notice that items (ii) and (iii) in the previous Proposition implies the existence of a measurable family $(\mathbf{N}_x^{0,(h)}, h > 0)$ of probabilities on $(\bar{\Omega}, \bar{\mathcal{G}})$ such that $\mathbf{N}_x^{0,(h)}$ is the distribution of W (more precisely of $(W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$) under $\mathbf{N}_x^0$ conditionally on $\{H_{\max} = h\}$.

Remark 2.4.8. In Klebaner & al [74], the Esty time reversal "is obtained by conditioning a [discrete time] Galton Watson process in negative time upon entering state 0 (extinction) at time 0 when starting at state 1 at time $-n$ and letting n tend to infinity". The authors then observe that in the linear fractional case (modified geometric offspring distribution) the Esty time reversal has the law of the same Galton Watson process conditioned on non-extinction. Notice that in our continuous setting, the process $(H_s, 0 \le s \le T_{\max})$ is under $\mathbf{N}_x^{0,(h)}$ a Bessel process up to its first hitting time of h, and thus is reversible: $(H_s, 0 \le s \le T_{\max})$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(h - H_{T_{\max}-s}, 0 \le s \le T_{\max})$ under $\mathbf{N}_x^{0,(h)}$. It is also well known (see Corollary 3.1.6 of [37]) that $(H_{\sigma-s}, 0 \le s \le \sigma - T_{\max})$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(H_s, 0 \le s \le T_{\max})$ under $\mathbf{N}_x^{0,(h)}$. We deduce from these two points that $(Z_s(1), 0 \le s \le h)$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(Z_{h-s}(1), 0 \le s \le h)$ under $\mathbf{N}_x^{0,(h)}$. This result, which holds at fixed h, gives a prelimiting version in continuous time of the Esty time reversal. Passing to the limit as $h \to \infty$, see Section 2.5.2, we then get the equivalent of the Esty time reversal in a continuous setting.

Before stating Williams decomposition, Theorem 2.4.12, let us prove some properties for the functions $v_t(x) = \mathbb{N}_x[H_{\max} > t] = \mathbb{N}_x[Z_t \neq 0]$ and $\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[H_{\max} > t]$ which will play a significant rôle in the next Section. Recall (2.17) states that

$$\alpha v_t = \tilde{v}_t.$$

Notice also that (2.18) implies that q is bounded from above by $(\beta_0 + \|\tilde{\beta}\|_\infty)/2$.

Lemma 2.4.9. *Assume* $(H1)$-$(H3)$*. We have:*

$$q(x) + v_t^0 \ge \tilde{v}_t(x) \ge v_t^0. \tag{2.46}$$

Furthermore for fixed $x \in E$, $\tilde{v}_t(x)$ is of class $\mathcal{C}^1$ in t and we have:

$$\partial_t \tilde{v}_t(x) = \tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^t \Sigma_r(Y_{t-r})\, dr} \right] \partial_t v_t^0, \tag{2.47}$$

where the function Σ defined by:

$$\Sigma_t(x) = 2(v_t^0 + q(x) - \tilde{v}_t(x)) = \partial_\lambda \psi^0(v_t^0) - \partial_\lambda \tilde{\psi}(x, \tilde{v}_t(x)) \tag{2.48}$$

satisfies:

$$0 \leq \Sigma_t(x) \leq 2q(x) \leq \beta_0 + \| \tilde{\beta} \|_\infty . \tag{2.49}$$

Proof. We deduce from item (iii) of Proposition 2.3.7 that, as $\varphi \geq 0$ (see Lemma 2.3.8),

$$\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[Z_t \neq 0] = \mathbb{N}_x^0 \left[\mathbf{1}_{\{Z_t \neq 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right] \geq \mathbb{N}_x^0[Z_t \neq 0] = v_t^0.$$

We also have

$$\begin{aligned}
\tilde{v}_t(x) &= \mathbb{N}_x^0 \left[\mathbf{1}_{\{Z_t \neq 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right] \\
&= \mathbb{N}_x^0 \left[\mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} - 1 \right] + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{Z_t = 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right] \\
&= q(x) + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{Z_t = 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right] \\
&\leq q(x) + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{Z_t = 0\}} \right] \\
&= q(x) + v_t^0,
\end{aligned}$$

where we used (2.22) for the third equality. This proves (2.46).
Using Williams decomposition under $\mathbf{N}_x^0$, we get:

$$\tilde{v}_t(x) = -\int_t^{+\infty} \partial_r v_r^0 \, dr \, \mathbf{N}_x^{0,(r)} \left[\mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right].$$

Using again Williams decomposition under $\mathbf{N}_x^0$, we have

$$\begin{aligned}
\mathbf{N}_x^{0,(r)} \left[\mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right] &= \tilde{\mathrm{E}}_x \left[\mathrm{e}^{2 \int_0^r ds\, \mathbf{N}^0_{Y_{r-s}} \left[(\mathrm{e}^{\int_0^{+\infty} dt\, Z_t(\varphi)} - 1) \mathbf{1}_{\{Z_s = 0\}} \right]} \right] \\
&= \tilde{\mathrm{E}}_x \left[\mathrm{e}^{2 \int_0^r ds\, \mathbf{N}^0_{Y_s} \left[(\mathrm{e}^{\int_0^{+\infty} dt\, Z_t(\varphi)} - 1) \mathbf{1}_{\{Z_{r-s} = 0\}} \right]} \right].
\end{aligned} \tag{2.50}$$

We deduce that, for fixed x, $r \mapsto \mathbf{N}_x^{0,(r)} \left[\mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right]$ is non-decreasing and continuous as $\mathbf{N}_y^0[H_{\max} = t] = 0$ for $t > 0$. Therefore, we deduce that for fixed x, $\tilde{v}_t(x)$ is of class $\mathcal{C}^1$ in t:

$$\partial_t \tilde{v}_t(x) = \mathbf{N}_x^{0,(t)} \left[\mathrm{e}^{\int_0^{+\infty} ds\, Z_s(\varphi)} \right] \partial_t v_t^0.$$

We have thanks to item (iii) from Proposition 2.3.7:

$$\begin{aligned}\mathbf{N}_y^0&\left[(\mathrm{e}^{\int_0^{+\infty} dt\ Z_t(\varphi)}-1)\mathbf{1}_{\{Z_s=0\}}\right]\\&=\mathbf{N}_y^0\left[Z_s\neq 0\right]+\mathbf{N}_y^0\left[\mathrm{e}^{\int_0^{+\infty} dt\ Z_t(\varphi)}-1\right]-\mathbf{N}_y^0\left[\mathrm{e}^{\int_0^{+\infty} dt\ Z_t(\varphi)}\mathbf{1}_{\{Z_s\neq 0\}}\right]\\&=v_s^0+q(y)-\tilde{v}_s(y)\\&=\frac{1}{2}\Big[\partial_\lambda\psi^0(v_s^0)-\partial_\lambda\tilde{\psi}(y,\tilde{v}_s(y))\Big],\end{aligned}\tag{2.51}$$

where the last equality follows from (2.15), (2.18) and (2.19). Thus, with $\Sigma_s(y)=\partial_\lambda\psi^0(v_s^0)-\partial_\lambda\tilde{\psi}(y,\tilde{v}_s(y))$, we deduce that:

$$\mathbf{N}_x^{0,(t)}\left[\mathrm{e}^{\int_0^{+\infty} ds\ Z_s(\varphi)}\right]=\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^t ds\ \Sigma_s(Y_{t-s})}\right].$$

This implies (2.47). Notice that, thanks to (2.46), Σ is nonnegative and bounded from above by $2q$. □

Fix $h>0$. We define the probability measures $\mathrm{P}^{(h)}$ absolutely continuous with respect to P and $\tilde{\mathrm{P}}$ on $\mathcal{D}_h$ with Radon-Nikodym derivative:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_h}}{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_h}}=\frac{\mathrm{e}^{\int_0^h \Sigma_{h-r}(Y_r)\ dr}}{\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\ dr}\right]}.\tag{2.52}$$

Notice this Radon-Nikodym derivative is 1 if the branching mechanism ψ is homogeneous. We deduce from (2.47) and (2.48) that:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_h}}{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_h}}=\frac{\partial_h v_h^0}{\partial_h\tilde{v}_h(x)}\ \mathrm{e}^{-\int_0^h dr\ \left(\partial_\lambda\tilde{\psi}(Y_r,\tilde{v}_{h-r}(Y_r))-\partial_\lambda\psi^0(v^0_{h-r})\right)}$$

and, using (2.14):

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_h}}{d\mathrm{P}_{x\ |\mathcal{D}_h}}=\frac{1}{\alpha(Y_h)}\ \frac{\partial_h v_h^0}{\partial_h v_h(x)}\ \mathrm{e}^{-\int_0^h dr\ \left(\partial_\lambda\psi(Y_r,v_{h-r}(Y_r))-\partial_\lambda\psi^0(v^0_{h-r})\right)}.\tag{2.53}$$

In the next Lemma, we give an intrinsic representation of the Radon-Nikodym derivatives (2.52) and (2.53), which does not involve β_0 or v^0.

Lemma 2.4.10. *Assume* $(H1)$-$(H3)$. *Fix* $h>0$. *The processes* $M^{(h)}=(M_t^{(h)},t\in[0,h))$ *and* $\tilde{M}^{(h)}=(\tilde{M}_t^{(h)},t\in[0,h))$, *with:*

$$M_t^{(h)}=\frac{\partial_h v_{h-t}(Y_t)}{\partial_h v_h(x)}\ \mathrm{e}^{-\int_0^t ds\ \partial_\lambda\psi(Y_s,v_{h-s}(Y_s))}\quad \text{and}\quad \tilde{M}_t^{(h)}=\frac{\partial_h\tilde{v}_{h-t}(Y_t)}{\partial_h\tilde{v}_h(x)}\ \mathrm{e}^{-\int_0^t ds\ \partial_\lambda\tilde{\psi}(Y_s,\tilde{v}_{h-s}(Y_s))},$$

are nonnegative bounded $\mathcal{D}_t$*-martingales respectively under* P_x *and* $\tilde{\mathrm{P}}_x$. *Furthermore, we have for* $0\leq t<h$:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}}=M_t^{(h)}\quad \text{and}\quad \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_t}}=\tilde{M}_t^{(h)}.\tag{2.54}$$

Notice the limit $M_h^{(h)}$ of $M^{(h)}$ and the limit $\tilde{M}_h^{(h)}$ of $\tilde{M}^{(h)}$ are respectively given by the right-handside of (2.53) and (2.52).

Remark 2.4.11. Comparing (2.10) and (2.54), we have that $\mathrm{P}_x^{(h)} = \mathrm{P}_x^g$ with $g(t,x) = \partial_h v_{h-t}(x)$, if g satisfies the assumptions of Remark 2.3.3.

Proof. First of all, the process $\tilde{M}^{(h)}$ is clearly $\mathcal{D}_t$-adapted. Using (2.47), we get:

$$\tilde{\mathrm{E}}_y\left[\mathrm{e}^{\int_0^{h-t} \Sigma_{h-t-r}(Y_r)\, dr}\right] = \frac{\partial_h \tilde{v}_{h-t}(y)}{\partial_h v^0_{h-t}}.$$

We set:

$$\tilde{M}_h^{(h)} = \frac{\mathrm{e}^{\int_0^h \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr}\right]}.$$

We have:

$$\begin{aligned}
\tilde{\mathrm{E}}_x[\tilde{M}_h^{(h)}|\mathcal{D}_t] &= \frac{\mathrm{e}^{\int_0^t \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr}\right]} \tilde{\mathrm{E}}_{Y_t}\left[\mathrm{e}^{\int_0^{h-t} \Sigma_{h-t-r}(Y_r)\, dr}\right] \\
&= \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \frac{\partial_h v^0_h}{\partial_h v^0_{h-t}} \mathrm{e}^{\int_0^t \Sigma_{h-r}(Y_r)\, dr} \\
&= \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \mathrm{e}^{-\int_0^t \partial_\lambda \tilde{\psi}\left(Y_s, \tilde{v}_{h-s}(Y_s)\right)\, ds} \frac{\partial_h v^0_h}{\partial_h v^0_{h-t}} \mathrm{e}^{\int_0^t \partial_\lambda \psi^0\left(v^0_{h-s}\right)\, ds}
\end{aligned}$$

In the homogeneous setting, v^0 simply solves the ordinary differential equation:

$$\partial_h v^0_h = -\psi^0(v^0_h).$$

This implies that

$$\partial_h \log(\partial_h v^0_h) = \frac{\partial_h^2 v^0_h}{\partial_h v^0_h} = -\partial_\lambda \psi^0(v^0_h)$$

and thus

$$\frac{\partial_h v^0_h}{\partial_h v^0_{h-t}} \mathrm{e}^{\int_0^t \partial_\lambda \psi^0\left(v^0_{h-s}\right)\, ds} = 1. \tag{2.55}$$

We deduce that

$$\tilde{\mathrm{E}}_x[\tilde{M}_h^{(h)}|\mathcal{D}_t] = \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \mathrm{e}^{-\int_0^t dr\, \partial_\lambda \tilde{\psi}(Y_r, \tilde{v}_{h-r}(Y_r))} = \tilde{M}_t^{(h)}.$$

Therefore, $\tilde{M}^{(h)}$ is a $\mathcal{D}_t$-martingale under $\tilde{\mathrm{P}}_x$ and the second part of (2.54) is a consequence of (2.52). Then, use (2.14) to get that $M^{(h)}$ is a $\mathcal{D}_t$-martingale under P_x and the first part of (2.54). □

We now give the Williams' decomposition: the distribution of $(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$ under $\mathbf{N}_x$ or equivalently under $\tilde{\mathbf{N}}_x/\alpha(x)$. Recall the distribution $\mathrm{P}_x^{(h)}$ defined in (2.52) or (2.53).

Theorem 2.4.12 (Williams' decomposition under $\mathbf{N}_x$). *Assume* $(H1)$-$(H3)$. *We have:*
(i) The distribution of H_{max} *under* $\mathbf{N}_x$ *is characterized by:* $\mathbf{N}_x[H_{max} > h] = v_h(x)$.

(ii) Conditionally on $\{H_{max} = h_0\}$*, the law of* $W_{T_{max}}$ *under* $\mathbf{N}_x$ *is distributed as* $Y_{[0,h_0)}$ *under* $\mathrm{P}_x^{(h_0)}$*.*

(iii) Conditionally on $\{H_{max} = h_0\}$ *and* $W_{T_{max}}$*,* $R^g_{T_{max}}$ *and* $R^d_{T_{max}}$ *are under* $\mathbf{N}_x$ *independent Poisson point measures on* $\mathbb{R}^+ \times \bar{\Omega}$ *with intensity:*

$$\mathbf{1}_{[0,h_0)}(s)ds\ \mathbf{1}_{\{H_{max}(W')<h_0-s\}}\alpha(W_{T_{max}}(s))\ \mathbf{N}_{W_{T_{max}}(s)}[dW'].$$

In other words, for any nonnegative measurable function F*, we have*

$$\mathbf{N}_x\Big[F(H_{max}, W_{T_{max}}, R^g_{T_{max}}, R^d_{T_{max}})\Big] = -\int_0^\infty \partial_h v_h(x)\ dh\ \mathrm{E}_x^{(h)}\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\Big],$$

where under $\mathrm{E}_x^{(h)}$ *and conditionally on* $Y_{[0,h)}$*,* $R^{W,(h),g}$ *and* $R^{W,(h),d}$ *are two independent Poisson point measures with intensity:*

$$\nu^{W,(h)}(ds, dW) = \mathbf{1}_{[0,h)}(s)ds\ \mathbf{1}_{\{H_{max}(W)<h-s\}}\alpha(Y_s)\ \mathbf{N}_{Y_s}[dW]. \tag{2.56}$$

Notice that items (ii) and (iii) in the previous Proposition imply the existence of a measurable familly $(\mathbf{N}_x^{(h)}, h > 0)$ of probabilities on $(\bar{\Omega}, \bar{\mathcal{G}})$ such that $\mathbf{N}_x^{(h)}$ is the distribution of W (more precisely of $(W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$) under $\mathbf{N}_x$ conditionally on $\{H_{\max} = h\}$.

Proof. We keep notations introduced in Proposition 2.4.7 and Theorem 2.4.12. We have:

$$\begin{aligned}
&\tilde{\mathbf{N}}_x\Big[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\Big]\\
&= \mathbf{N}_x^0\Big[\mathrm{e}^{\int_0^{+\infty} ds\ Z_s(\varphi)}\ F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\Big]\\
&= \mathbf{N}_x^0\Big[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\ \mathrm{e}^{(R^g_{T_{\max}}+R^d_{T_{\max}})(f)}\Big]\\
&= -\int_0^\infty \partial_h v_h^0\ dh\ \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, \hat{R}^{W,(h),g}, \hat{R}^{W,(h),d})\ \mathrm{e}^{(\hat{R}^{W,(h),g}+\hat{R}^{W,(h),d})(f)}\Big]\\
&= -\int_0^\infty \partial_h v_h^0\ dh\ \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\ \mathrm{e}^{2\int_0^h ds\ \mathbf{N}^0_{Y_s}\Big[(\mathrm{e}^{\int_0^{+\infty} dt\ Z_t(\varphi)}-1)\mathbf{1}_{\{Z_{r-s}=0\}}\Big]}\Big]\\
&= -\int_0^\infty \partial_h v_h^0\ dh\ \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\ \mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\ ds}\Big]\\
&= -\int_0^\infty \partial_h v_h^0\ dh\ \tilde{\mathrm{E}}_x\Big[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\ ds}\Big]\mathrm{E}_x^{(h)}\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\Big]\\
&= -\int_0^\infty \partial_h \tilde{v}_h(x)\ dh\ \mathrm{E}_x^{(h)}\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\Big],
\end{aligned}$$

where the first equality comes from $(H1)$ and item (iii) of Proposition 2.3.7; we set $f(s, W) = \int_0^{+\infty} Z_r(W)(\varphi)$ for the second equality; we use Proposition 2.4.7 for the third equality; we use Lemma 2.4.3 for the fourth with $R^{W,(h),g}$ and $R^{W,(h),d}$ which under $\tilde{\mathrm{E}}_x^{(h)}$ and conditionally on $Y_{[0,h)}$ are two independent Poisson point measures with intensity $\nu^{W,(h)}$; we use (2.51) for the fifth, definition (2.52) of $\mathrm{E}_x^{(h)}$ for the sixth, and (2.47) for the seventh. Then use (2.33) and (2.17) to conclude. □

The definition of $\mathbf{N}_x^{(h)}$ gives in turn sense to the conditional law $\mathbb{N}_x^{(h)} = \mathbb{N}_x(.|H_{\max} = h)$ of the $(\mathcal{L}, \beta, \alpha)$ superprocess conditioned to die at time h, for all $h > 0$. The next Corollary is then a straightforward consequence of Theorem 2.4.12.

Corollary 2.4.13. *Assume* $(H1)$-$(H3)$*. Let* $h > 0$*. Let* $x \in E$ *and* $Y_{[0,h)}$ *be distributed according to* $\mathrm{P}_x^{(h)}$*. Consider the Poisson point measure* $\mathcal{N} = \sum_{j\in J} \delta_{(s_j, Z^j)}$ *on* $[0,h) \times \Omega$ *with intensity:*

$$2\mathbf{1}_{[0,h)}(s)ds \; \mathbf{1}_{\{H_{max}(Z)<h-s\}}\alpha(Y_s)\; \mathbb{N}_{Y_s}[dZ].$$

The process $Z^{(h)} = (Z_t^{(h)}, t \geq 0)$*, which is defined for all* $t \geq 0$ *by:*

$$Z_t^{(h)} = \sum_{j\in J,\, s_j<t} Z_{t-s_j}^j,$$

is distributed according to $\mathbb{N}_x^{(h)}$*.*

We now give the superprocess counterpart of Theorem 2.4.12.

Corollary 2.4.14 (Williams' decomposition under $\mathbb{P}_\nu$)**.** *Assume* $(H1)$-$(H3)$*.*
(i) The distribution of H_{max} *under* $\mathbb{P}_\nu$ *is:* $\mathbb{P}_\nu(H_{max} \leq h) = \mathrm{e}^{-\nu(v_h)}$*.*
(ii) Conditionally on $\{H_{max} = h_0\}$*, the distribution of* Z *under* $\mathbb{P}_\nu$ *is that of the sum of* $Z'+Z^{(h_0)}$*, where:*
(iii) $Z^{(h_0)}$ *has distribution:*

$$\frac{\partial_h v_{h_0}(x)}{\nu(\partial_h v_{h_0})}\nu(dx)\; \mathbb{N}_{x_0}^{(h_0)}.$$

(iv) Z' *is independent of* $Z^{(h_0)}$ *and has distribution* $\mathbb{P}_\nu(.|H_{max} < h_0)$*.*
Then the measure-valued process $Z' + Z^{(h_0)}$ *has distribution* $\mathbb{P}_\nu$*.*

In particular the distribution of $Z' + Z^{(h_0)}$ conditionally on h_0 (which is given by (ii)-(iv) from Corollary 2.4.14) is a regular version of the distribution of the $(\mathcal{L}, \beta, \alpha)$ superprocess conditioned to die at a fixed time h_0, which we shall write $\mathbb{P}_\nu^{(h_0)}$.

Proof. Let μ be a finite measure on $\mathbb{R}^+$ and f a nonnegative measurable function defined on $\mathbb{R}^+ \times E$. For a measure-valued process $A = (A_t, t \geq 0)$ on E, we set $A(f\mu) = \int f(t,x)\; A_t(dx)\mu(dt)$. We also write $f_s(t,x) = f(s+t,x)$.

Let Z' and $Z^{(h_0)}$ be defined as in Corollary 2.4.14. In order to characterized the distribution of the process $Z' + Z^{(h_0)}$, we shall compute

$$A = \mathbb{E}[\mathrm{e}^{-Z'(f\mu) - Z^{(h_0)}(f\mu)}].$$

We shall use notations from Corollary 2.4.13. We have:

$$\begin{aligned} A &= -\int_0^{+\infty} \nu(\partial_h v_h)\, \mathrm{e}^{-\nu(h)}\, dh \int_E \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)}\, \nu(dx) \\ &\qquad \mathrm{E}_x^{(h)}\left[\mathbb{E}[\mathrm{e}^{-\sum_{j\in J} Z^j(f_{s_j}\mu)}\, |Y_{[0,h)}]\right] \mathbb{E}_\nu\left[\mathrm{e}^{-Z(f\mu)}\, |H_{\max} < h\right] \\ &= -\int_E \nu(dx) \int_0^{+\infty} \partial_h v_h(x)\, dh \\ &\qquad \mathrm{E}_x^{(h)}\left[\mathbb{E}[\mathrm{e}^{-\sum_{j\in J} Z^j(f_{s_j}\mu)}\, |Y_{[0,h)}]\right] \mathbb{E}_\nu\left[\mathrm{e}^{-Z(f\mu)}\, \mathbf{1}_{\{H_{\max}<h\}}\right], \end{aligned}$$

where we used the definition of Z' and $\mathcal{N}$ for the first equality, and the equality $\mathbb{P}_\nu(H_{\max} < h) = \mathbb{P}_\nu(H_{\max} \leq h) = \mathrm{e}^{-\nu(h)}$ for the second. Recall notations from Theorem 2.4.12. We set:

$$G\left(\sum_{i\in I} \delta_{(s_i, W^i)}, \sum_{i'\in I'} \delta_{(s_{i'}, W^{i'})}\right) = \mathrm{e}^{-\sum_{j\in I\cup I'} Z^j(W^j)(f_{s_j}\mu)}$$

and $g(h) = \mathbb{E}_\nu\left[\mathrm{e}^{-Z(f\mu)}\,\mathbf{1}_{\{H_{\max}<h\}}\right]$. We have:

$$\begin{aligned}
A &= -\int_E \nu(dx) \int_0^{+\infty} \partial_h v_h(x)\, dh\; \mathrm{E}_x^{(h)}\left[G(R^{W,(h),g}, R^{W,(h),d}) g(h)\right] \\
&= \int_E \nu(dx)\, \mathbf{N}_x\left[G(R_g^{T_{\max}}, R_d^{T_{\max}}) g(H_{\max})\right] \\
&= \int_E \nu(dx)\, \mathbb{N}_x\left[\mathrm{e}^{-Z(f\mu)}\, \mathbb{E}_\nu\left[\mathrm{e}^{-Z(f\mu)}\,\mathbf{1}_{\{H_{\max}<h\}}\right]_{|h=H_{\max}}\right] \\
&= \mathbb{E}\left[\sum_{i\in I} \mathrm{e}^{-Z^i(f\mu)} \prod_{j\in I;\, j\neq i} \mathrm{e}^{-Z^j(f\mu)}\,\mathbf{1}_{\{H^j_{\max}<H^i_{\max}\}}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-\sum_{i\in I} Z^i(f\mu)}\right] \\
&= \mathbb{E}_\nu\left[\mathrm{e}^{-Z(f\mu)}\right],
\end{aligned}$$

where we used the definition of G and g for the first and third equalities, Theorem 2.4.12 for the second equality, the master formula for Poisson point measure $\sum_{i\in I} \delta_{Z^i}$ with intensity $\nu(dx)\, \mathbb{N}_x[dZ]$ for the fourth equality (and the obvious notation $H^i_{\max} = \inf\{t \geq 0; Z^i_t = 0\}$) and Theorem 2.2.3 for the last equality. Thus we get:

$$\mathbb{E}[\mathrm{e}^{-Z'(f\mu)-Z^{(h_0)}(f\mu)}] = \mathbb{E}_\nu\left[\mathrm{e}^{-Z(f\mu)}\right].$$

This readily implies that the process $Z' + Z^{(h_0)}$ is distributed as Z under $\mathbb{P}_\nu$. □

2.5 Some applications

2.5.1 The law of the Q-process

Recall $\mathbb{P}_\nu^{(h)}$ defined after Corollary 2.4.14 is the distribution of the $(\mathcal{L}, \beta, \alpha)$-superprocess started at $\nu \in \mathcal{M}_f(E)$ conditionally on $\{H_{\max} = h\}$. We consider also $\mathbb{P}_\nu^{(\geq h)} = \mathbb{P}_\nu(\cdot | H_{\max} \geq h)$ the distribution of the $(\mathcal{L}, \beta, \alpha)$-superprocess started at $\nu \in \mathcal{M}_f(E)$ conditionally on $\{H_{\max} \geq h\}$.

The distribution of the Q-process, when it exists, is defined as the weak limit of $\mathbb{P}_\nu^{(\geq h)}$ when h goes to infinity. The next Lemma insures that if $\mathbb{P}_\nu^{(h)}$ weakly converges to a limit $\mathbb{P}_\nu^{(\infty)}$, then this limit is also the distribution of the Q-process.

Lemma 2.5.1. *Fix $t > 0$. If $\mathbb{P}_\nu^{(h)}$ converges weakly to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$, then $\mathbb{P}_\nu^{(\geq h)}$ converges weakly to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.*

Proof. Let $A = \mathbf{1}_A$ with $A \in \mathcal{F}_t$ such that $\mathbb{P}_\nu^{(\infty)}(\partial A) = 0$. Using the Williams' decomposition under $\mathbb{P}_\nu$ given by Corollary 2.4.14, we have for $h > t$:

$$\mathbb{E}_\nu^{(\geq h)}[A] = \mathrm{e}^{\nu(v_h)} \int_h^\infty \mathbb{E}_\nu^{(h')}[A]\, f(h')dh',$$

where $f(h) = -\nu(\partial_h v_h) \exp(-\nu(v_h))$. We write down the difference:

$$\mathbb{E}_\nu^{(\geq h)}[A] - \mathbb{E}_\nu^{(\infty)}[A] = \mathrm{e}^{\nu(v_h)} \int_h^\infty (\mathbb{E}_\nu^{(h')}[A] - \mathbb{E}_\nu^{(\infty)}[A])\, f(h')dh'.$$

Since $\mathbb{P}_\nu^{(h')}$ weakly converges to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$ and $\mathbb{P}_\nu^{(\infty)}(\partial A) = 0$, we get that $\lim_{h' \to +\infty} \mathbb{E}_\nu^{(h')}[A] - \mathbb{E}_\nu^{(\infty)}[A] = 0$. We conclude that $\lim_{h \to +\infty} \mathbb{E}_\nu^{(\geq h)}[A] - \mathbb{E}_\nu^{(\infty)}[A] = 0$, which gives the result. □

We now address the question of convergence of the family of probability measures $(\mathbb{P}_x^{(h)}, h \geq 0)$. Recall from (2.54) that for all $0 \leq t < h$:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = M_t^{(h)}.$$

We shall consider the following assumption on the convergence in law of the spine.

$(H4)$ **For all $t \geq 0$, P_x-a.s. $(M_t^{(h)}, h > t)$ converges to a limit say $M_t^{(\infty)}$, and $\mathrm{E}_x[M_t^{(\infty)}] = 1$.**

Note that Scheffé's lemma implies that the convergence also holds in $\mathrm{L}^1(\mathrm{P}_x)$. Furthermore, since $(M_t^{(h)}, t \in [0, h))$ is a nonnegative martingale, there exists a version of $(M_t^{(\infty)}, t \geq 0)$ which is a nonnegative martingale.

Remark 2.5.2. We provide in Section 2.7 sufficient conditions for $(H1)$-$(H4)$ to hold in the case of the multitype Feller diffusion and the superdiffusion. These conditions are stated in term of the generalized eigenvalue λ_0 defined by

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L})u = \ell\, u\}, \tag{2.57}$$

and its associated eigenfunction.

Remark 2.5.3. The family $(\mathrm{P}_x^{(h)}, h \geq 0)$ and the family $(\mathrm{P}_x^{(B,h)}, h \geq 0)$ defined in Remark 2.4.6 will be shown in Lemma 2.6.13 to converge to the same limiting probability measure.

Under $(H4)$, we define the probability measure $\mathrm{P}_x^{(\infty)}$ on $(D, \mathcal{D})$ by its Radon Nikodym derivative, for all $t \geq 0$:

$$\frac{d\mathrm{P}^{(\infty)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = M_t^{(\infty)}. \tag{2.58}$$

By construction, the probability measure $\mathrm{P}_x^{(h)}$ converges weakly to $\mathrm{P}_x^{(\infty)}$ on $\mathcal{D}_t$, for all $t \geq 0$.

Let $\nu \in \mathcal{M}_f(E)$. We shall consider the following assumption:

$(H5)_\nu$ **There exists a measurable function ρ such that the following convergence holds in $\mathrm{L}^1(\nu)$:**

$$\frac{\partial_h v_h}{\nu(\partial_h v_h)} \xrightarrow[h\to+\infty]{} \rho.$$

In particular, we have $\nu(\rho) = 1$. Let $\nu \in \mathcal{M}_f(E)$. Under $(H4)$ and $(H5)_\nu$, we set:

$$\mathrm{P}_\nu^{(\infty)}(dY) = \int_E \nu(dx)\rho(x)\,\mathrm{P}_x^{(\infty)}(dY).$$

Notice then that $\int_E \nu(dx)\,\frac{\partial_h v_h(x)}{\nu(\partial_h v_h)}\mathrm{P}_x^{(h)}(dY)$ converges weakly to $\mathrm{P}_\nu^{(\infty)}(dY)$ on $\mathcal{D}_t$, for all $t \geq 0$.

Remark 2.5.4. If ν a constant times the Dirac mass δ_x, for some $x \in E$, then $(H5)_\nu$ holds if $(H4)$ holds and in this case we have $\mathrm{P}_\nu^{(\infty)} = \mathrm{P}_x^{(\infty)}$.

We can now state the result on the convergence of $\mathbf{N}_x^{(h)}$.

Theorem 2.5.5. *Assume* $(H1)$*-*$(H4)$*. Let* $t \geq 0$*. The triplet* $((W_{T_{max}})_{[0,t]}, (R^g_{T_{max}})_{[0,t]}, (R^d_{T_{max}})_{[0,t]})$ *under* $\mathbf{N}_x^{(h)}$ *converges weakly to the distribution of the triplet* $(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})$ *where* Y *has distribution* $\mathrm{P}_x^{(\infty)}$ *and conditionally on* Y*,* $R^{B,g}$ *and* $R^{B,d}$ *are two independent Poisson point measures with intensity* ν^B *given by (2.39). We even have the slightly stronger result: for any bounded measurable function* F*,*

$$\mathbf{N}_x^{(h)}\Big[F((W_{T_{max}})_{[0,t]}, (R^g_{T_{max}})_{[0,t]}, (R^d_{T_{max}})_{[0,t]})\Big] \xrightarrow[h\to+\infty]{} \mathrm{E}_x^{(\infty)}\Big[F(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})\Big]. \tag{2.59}$$

Proof. Let $h > t$. We use notations from Theorem 2.4.12. Let F be a bounded measurable function on $\mathcal{W} \times (\mathbb{R}^+ \times \bar{\Omega})^2$. From the Williams' decomposition, Theorem 2.4.12, we have:

$$\begin{aligned}\mathbf{N}_x^{(h)}\Big[F((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})\Big] &= \mathrm{E}_x^{(h)}\Big[F(Y_{[0,t]}, R^{W,g,(h)}_{[0,t]}, R^{W,d,(h)}_{[0,t]})\Big] \\ &= \mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big],\end{aligned}$$

where φ^h is defined by:

$$\varphi^h(y_{[0,t]}) = \mathrm{E}_x^{(h)}\Big[F(y_{[0,t]}, R^{W,g,(h)}_{[0,t]}, R^{W,d,(h)}_{[0,t]})\Big| Y = y\Big].$$

We also set:

$$\varphi^\infty(y_{[0,t]}) = \mathrm{E}_x^{(\infty)}\Big[F(y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})\Big| Y = y\Big].$$

We want to control:

$$\varDelta_h = \mathbf{N}_x^{(h)}\Big[F((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})\Big] - \mathrm{E}_x^{(\infty)}\Big[F(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})\Big].$$

Notice that:

$$\begin{aligned}\varDelta_h &= \mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^\infty(Y_{[0,t]})\big] \\ &= \big(\mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^h(Y_{[0,t]})\big]\big) + \mathrm{E}_x^{(\infty)}\big[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\big].\end{aligned} \tag{2.60}$$

We prove the first term of the right hand-side of (2.60) converges to 0. We have:

$$\mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^h(Y_{[0,t]})\big] = \mathrm{E}_x\big[(M_t^{(h)} - M_t^{(\infty)})\, \varphi^h(Y_{[0,t]})\big].$$

Then use that φ^h is bounded by $\| F \|_\infty$ and the convergence of $(M_t^{(h)}, h > t)$ towards $M_t^{(\infty)}$ in $\mathrm{L}^1(\mathrm{P}_x)$ to get:

$$\lim_{h\to\infty} \mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^h(Y_{[0,t]})\big] = 0. \tag{2.61}$$

We then prove the second term of the right hand-side of (2.60) converges to 0. Notice that conditionally on Y, $R_{[0,t]}^{W,g,(h)}$ and $R_{[0,t]}^{W,d,(h)}$ (resp. $R_{[0,t]}^{B,g}$ and $R_{[0,t]}^{B,d}$) are independent Poisson point measures with intensity $\mathbf{1}_{[0,t]}(s)\, \nu^{W,(h)}(ds, dW)$ where $\nu^{W,(h)}$ is given by (2.56) (resp. $\mathbf{1}_{[0,t]}(s)\, \nu^B(ds, dW)$ where ν^B is given by (2.39)). And we have:

$$\mathbf{1}_{[0,t]}(s)\, \nu^{W,(h)}(ds, dW) = \mathbf{1}_{\{H_{\max}(W)<h-s\}} \mathbf{1}_{[0,t]}(s)\, \nu^B(ds, dW).$$

Thanks to (2.17) and (2.46), we get that:

$$\int \mathbf{1}_{\{H_{\max}(W)\geq h-s\}} \mathbf{1}_{[0,t]}(s)\, \nu^B(ds, dW) = \int_0^t ds\; \alpha(y_s) \mathbb{N}_{y_s}[H_{\max} \geq h-s] = \int_0^t ds\; v_{h-s}(y_s) < +\infty.$$

The proof of the next Lemma is postponed to the end of this Section.

Lemma 2.5.6. *Let R and $\tilde{R}$ be two Poisson point measures on a Polish space with respective intensity ν and $\tilde{\nu}$. Assume that $\tilde{\nu}(dx) = \mathbf{1}_A(x)\nu(dx)$, where A is measurable and $\nu(A^c) < +\infty$. Then for any bounded measurable function F, we have:*

$$\Big|\mathbb{E}[F(R)] - \mathbb{E}[F(\tilde{R})]\Big| \leq 2 \| F \|_\infty \nu(A^c).$$

Using this Lemma with ν given by $\mathbf{1}_{[0,t]}(s)\, \nu^B(ds, dW)$ and A given by $\{H_{\max}(W) < h - s\}$, we deduce that:

$$\Big|(\varphi^h - \varphi^\infty)(y_{[0,t]})\Big| \leq 4 \| F \|_\infty \int_0^t ds\; v_{h-s}(y_s).$$

We deduce that:

$$\Big|\mathrm{E}_x^{(\infty)}\big[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\big]\Big| \leq 4 \| F \|_\infty \mathrm{E}_x^{(\infty)}\left[\int_0^t ds\; v_{h-s}(Y_s)\right].$$

Recall that $(H1)$ implies that $v_{h-s}(x)$ converges to 0 as h goes to infinity. Since v is bounded (use (2.17) and (2.46)), by dominated convergence, we get:

$$\lim_{h\to\infty} \mathrm{E}_x^{(\infty)}\big[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\big] = 0. \tag{2.62}$$

Therefore, we deduce from (2.60) that $\lim_{h\to+\infty} \Delta_h = 0$, which gives (2.59). □

We now define a superprocess with spine distribution $\mathrm{P}_\nu^{(\infty)}$.

Definition 2.5.7. *Let $\nu \in \mathcal{M}_f(E)$. Assume $\mathrm{P}_\nu^{(\infty)}$ is well defined. Let Y be distributed according to $\mathrm{P}_\nu^{(\infty)}$, and, conditionally on Y, let $\mathcal{N} = \sum_{j\in J} \delta_{(s_j, Z^j)}$ be a Poisson point measure with intensity:*

$$2\mathbf{1}_{\mathbb{R}^+}(s) ds\; \alpha(Y_s) \mathbb{N}_{Y_s}[dZ].$$

Consider the process $Z^{(\infty)} = (Z_t^{(\infty)}, t \geq 0)$, which is defined for all $t \geq 0$ by:

$$Z_t^{(\infty)} = \sum_{j\in J,\, s_j<t} Z_{t-s_j}^j.$$

(i) Let Z' be independent of $Z^{(\infty)}$ and be distributed according to $\mathbb{P}_\nu$. Then, we write $\mathbb{P}_\nu^{(\infty)}$ for the distribution of $Z' + Z^{(\infty)}$.
(ii) If ν is the Dirac mass at x, we write $\mathbb{N}_x^{(\infty)}$ for the distribution of $Z^{(\infty)}$.

As a consequence of Theorem 2.5.5, we get the convergence of $\mathbb{P}_\nu^{(h)}$. We shall write $\mathbb{P}_x^{(h)}$ when ν is the Dirac mass at x.

Corollary 2.5.8. *Under $(H1)$-$(H4)$, we have that, for all $t \geq 0$:*
(i) The distribution $\mathbb{N}_x^{(h)}$ converges weakly to $\mathbb{N}_x^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.
(ii) The distribution $\mathbb{P}_x^{(h)}$ converges weakly to $\mathbb{P}_x^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.
(iii) Let $\nu \in \mathcal{M}_f(E)$. If furthermore $(H5)_\nu$ holds, then the distribution $\mathbb{P}_\nu^{(h)}$ converges weakly to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.

Proof. Point (i) is a direct consequence of Theorem 2.5.5, Definition 2.5.7 and Proposition 2.3.12.

Point (ii) is a direct consequence of point (i), Corollary 2.4.14 and the weak convergence of $\mathbb{P}_x^{(\leq h)}$ to $\mathbb{P}_x$ as h goes to infinity.

According to Corollary 2.4.14, under $\mathbb{P}_\nu^{(h)}$, Z is distributed according to $Z' + Z^{(h)}$ where Z' and $Z^{(h)}$ are independent, Z' is distributed according to $\mathbb{P}_\nu^{(\leq h)}$ and $Z^{(h)}$ is distributed according to

$$\int_E \nu(dx) \, \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)} \, \mathbb{N}_x^{(h)}[dZ].$$

Assumption $(H5)_\nu$ implies this distribution converges weakly to:

$$\int_E \nu(dx) \; \rho(x) \; \mathbb{N}_x^{(\infty)}[dZ]$$

(because of the convergence of the densities in $\mathrm{L}^1(\nu)$) on $(\Omega, \mathcal{F}_t)$ as h goes to infinity. This and the weak convergence of $\mathbb{P}_\nu^{(\leq h)}$ to $\mathbb{P}_\nu$ as h goes to infinity gives point (iii). □

Proof of Lemma 2.5.6. Similarly to Lemma 2.4.3 (formally take $f = -\infty \mathbf{1}_{A^c}$), we have:

$$\mathbb{E}\left[F(R)\mathbf{1}_{\{R(A^c)=0\}}\right] = \mathbb{E}\left[F(\tilde{R})\right] \mathrm{e}^{-\nu(A^c)} .$$

We deduce that:

$$\begin{aligned}
\left|\mathbb{E}[F(R)] - \mathbb{E}[F(\tilde{R})]\right| &= \left|\mathbb{E}[F(R)] - \mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}]\,\mathrm{e}^{\nu(A^c)}\right| \\
&\leq \left|\mathbb{E}[F(R)] - \mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}]\right| + \left|\mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}](1 - \mathrm{e}^{\nu(A^c)})\right| \\
&\leq \|F\|_\infty (1 - \mathbb{P}(R(A^c) = 0)) + \|F\|_\infty \, \mathbb{P}(R(A^c) = 0)(\mathrm{e}^{\nu(A^c)} - 1) \\
&= 2\,\|F\|_\infty (1 - \mathrm{e}^{-\nu(A^c)}) \\
&\leq 2\,\|F\|_\infty \, \nu(A^c).
\end{aligned}$$

This gives the result. □

2.5.2 Backward from the extinction time

We shall work in this section with the space $D^- = D(\mathbb{R}^-, E)$ equipped with the Skorokhod topology. We also consider the σ-fields $\mathcal{D}_I = \sigma(Y_r, r \in I)$ for I an interval on $(-\infty, 0]$.

Let us denote by θ the translation operator, which maps any process R to the shifted process $\theta_h(R)$ defined by:

$$\theta_h(R)_\cdot = R_{\cdot+h} \ .$$

The process R may be a path, a killed path or a point measure, in which case we set, for $R = \sum_{j\in J} \delta_{(s_j,x_j)}$, $\theta_h(R) = \sum_{j\in J} \delta_{(h+s_j,x_j)}$. We also denote $\mathrm{P}^{(-h)}$ the push forward probability measure of $\mathrm{P}^{(h)}$ by θ_h, defined on $\mathcal{D}_{[-h,0]}$ by:

$$\mathrm{P}^{(-h)}(Y \in \bullet) = \mathrm{P}^{(h)}(\theta_h(Y) \in \bullet) = \mathrm{P}^{(h)}((Y_{h+s}, s \in [-h,0]) \in \bullet). \tag{2.63}$$

We introduce the following assumptions.

$(H6)$ **There exists a probability measure on** $(D^-, \mathcal{D}_{(-\infty,0]})$ **denoted** $\mathrm{P}^{(-\infty)}$ **such that for all** $x \in E$**,** $t \geq 0$**, and** f **bounded and** $\mathcal{D}_{[-t,0]}$ **measurable:**

$$\mathrm{E}_x^{(-h)}\big[f(Y_{[-t,0]})\big] \xrightarrow[h\to+\infty]{} \mathrm{E}^{(-\infty)}\big[f(Y_{[-t,0]})\big].$$

$(H7)$ **For all** $t > 0$**, there exists a non negative function** q **such that for all** $x \in E$**, for all** $h > 0$**:**

$$v_h(x) - v_{h+t}(x) \leq q(h) \quad \text{and} \quad \int_1^\infty dr\ q(r) < \infty.$$

Note that the probability measure $\mathrm{P}^{(-\infty)}$ in $(H6)$ does not depend on the starting point x.

We can now state the result on the convergence of the superprocess backward from the extinction time.

Theorem 2.5.9. *Under* $(H1)$-$(H4)$ *and* $(H6)$.

(i) The distribution of the triplet $\big(\theta_h(W_{T_{max}})_{[-t,0]}, \theta_h(R_g^{T_{max}})_{[-t,0]}, \theta_h(R_d^{T_{max}})_{[-t,0]}\big)$ *under* $\mathbf{N}_x^{(h)}$ *converges weakly to the distribution of the triplet* $(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]})$ *where* Y *has distribution* $\mathrm{P}^{(-\infty)}$ *and conditionally on* Y*,* $R^{W,g}$ *and* $R^{W,d}$ *are two independent Poisson point measures with intensity:*

$$\mathbf{1}_{\{s<0\}}\alpha(Y_s)\, ds\ \mathbf{1}_{\{H_{max}(W)<-s\}}\ \mathbf{N}_{Y_s}[dW].$$

We even have the slightly stronger result: for any bounded measurable function F*,*

$$\mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{max}})_{[-t,0]}, \theta_h(R_g^{T_{max}})_{[-t,0]}, \theta_h(R_d^{T_{max}})_{[-t,0]}\big)\Big] \xrightarrow[h\to+\infty]{} \mathrm{E}^{(-\infty)}\Big[F\big(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big]. \tag{2.64}$$

(ii) If furthermore $(H7)$ *holds, then the process* $\theta_h(Z)_{[-t,0]} = (Z_{h+s}, s \in [-t,0])$ *under* $\mathbb{N}_x^{(h)}$ *weakly converges towards* $Z^{(-\infty)}_{[-t,0]}$*, where for* $s \leq 0$*:*

$$Z_s^{(-\infty)} = \sum_{j\in J,\, s_j<s} Z^j_{s-s_j},$$

and conditionally on Y *with distribution* $\mathrm{P}^{(-\infty)}$, $\sum_{j\in J} \delta_{(s_j,Z^j)}$ *is a Poisson point measure with intensity:*

$$2\, \mathbf{1}_{\{s<0\}} \alpha(Y_s)\, ds\, \mathbf{1}_{\{H_{max}(Z)<-s\}}\, \mathbb{N}_{Y_s}[dZ].$$

Remark 2.5.10. We provide in Lemmas 2.7.3 and 2.7.6 sufficient conditions for $(H6)$ and $(H7)$ to hold in the case of the multitype Feller diffusion and the superdiffusion. These conditions are stated in term of the generalized eigenvalue λ_0 defined in (2.57) and its associated eigenfunction.

Proof. Let $0 < t < h$. We use notations from Theorems 2.4.12, 2.5.5. Let F be a bounded measurable function on $\mathcal{W}^- \times (\mathbb{R}^- \times \bar{\Omega})^2$ with $\mathcal{W}^-$ the set of killed paths indexed by negative times. We want to control δ_h defined by:

$$\delta_h = \mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R^g_{T_{\max}})_{[-t,0]}, \theta_h(R^d_{T_{\max}})_{[-t,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[F\big(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big].$$

We set:

$$\Upsilon(y_{[-t,0]}) = \mathrm{E}^{(-\infty)}\Big[F\big(y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big| Y = y\Big].$$

We deduce from Williams' decomposition, Theorem 2.4.12, and the definition of $R^{W,g}$ and $R^{W,d}$, that:

$$\mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R^g_{T_{\max}})_{[-t,0]}, \theta_h(R^d_{T_{\max}})_{[-t,0]}\big)\Big] = \mathrm{E}_x^{(-h)}\left[\Upsilon(Y_{[-t,0]})\right].$$

We thus can rewrite δ_h as:

$$\delta_h = \mathrm{E}_x^{(-h)}\left[\Upsilon(Y_{[-t,0]})\right] - \mathrm{E}^{(-\infty)}\left[\Upsilon(Y_{[-t,0]})\right].$$

The function Υ being bounded by $\|F\|_\infty$ and measurable, we may conclude under assumption $(H6)$ that $\lim_{h\to+\infty} \delta_h = 0$. This proves point (i).

We now prove point (ii). Let $t > 0$ and $\varepsilon > 0$ be fixed. Let F be a bounded measurable function on the space of continuous measure-valued applications indexed by negative times. For a point measure on $\mathbb{R}^- \times \bar{\Omega}$, $M = \sum_{i\in\mathcal{I}} \delta_{(s_i,W_i)}$, we set:

$$\tilde{F}(M) = F\left(\big(\sum_{i\in\mathcal{I}} \theta_{s_i}(Z(W_i))\big)_{[-t,0]}\right).$$

For $h > t$, we want a control of $\bar{\delta}_h$ defined by:

$$\bar{\delta}_h = \mathbb{N}_x^{(h)}\Big[F\big(\theta_h(Z)_{[-t,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[\tilde{F}\big(R^{W,g} + R^{W,d}\big)\Big].$$

By Corollary 2.4.13, we have:

$$\mathbb{N}_x^{(h)}\Big[F\big(\theta_h(Z)_{[-t,0]}\big)\Big] = \mathbf{N}_x^{(h)}\Big[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\Big].$$

Thus, we get:

$$\bar{\delta}_h = \mathbf{N}_x^{(h)}\Big[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\Big] - \mathrm{E}^{(-\infty)}\Big[\tilde{F}\big(R^{W,g} + R^{W,d}\big)\Big]. \tag{2.65}$$

For $a > s$ fixed, we introduce $\bar{\delta}_h^a$, for $h > a$, defined by:

$$\bar{\delta}_h^a = \mathbf{N}_x^{(h)}\Big[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-a,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[\tilde{F}\big((R^{W,g} + R^{W,d})_{[-a,0]}\big)\Big]. \tag{2.66}$$

Notice the restriction of the point measures to $[-a, 0]$. Point (i) directly yields that $\lim_{h\to+\infty} \bar{\delta}_h^a = 0$. Thus, there exists $h_a > 0$ such that for all $h \geq h_a$,

$$\bar{\delta}_h^a \leq \varepsilon/2.$$

We now consider the difference $\bar{\delta}_h - \bar{\delta}_h^a$. We associate to the point measures M introduced above the most recent common ancestor of the population alive at time $-t$:

$$A(M) = \sup\{s > 0; \sum_{i\in\mathcal{I}} \mathbf{1}_{\{s_i<-s\}} \mathbf{1}_{\{H_{\max}(W_i)>-t-s_i\}} \neq 0\}.$$

Let us observe that:

$$\mathbf{N}_x^{(h)} \text{ a.s.,}\;\; \tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\mathbf{1}_{\{A\leq a\}} = \tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-a,0]}\big)\mathbf{1}_{\{A\leq a\}}, \tag{2.67}$$

with $A = A(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-h,0]})$ in the left and in the right hand side. Similarly, we have:

$$\mathrm{P}^{(-\infty)} \text{ a.s.,}\;\; \tilde{F}\big(R^{W,g} + R^{W,d}\big)\mathbf{1}_{\{A\leq a\}} = \tilde{F}\big((R^{W,g} + R^{W,d})_{[-a,0]}\big)\mathbf{1}_{\{A\leq a\}}, \tag{2.68}$$

with $A = A\big(R^{W,g} + R^{W,d}\big)$ in the left and in the right hand side. We thus deduce the following bound on $\bar{\delta}_h - \bar{\delta}_h^a$:

$$\begin{aligned}
|\bar{\delta}_h - \bar{\delta}_h^a| &\leq 2\|F\|_\infty \Big[\mathbf{N}_x^{(h)}[A > a] + \mathrm{P}^{(-\infty)}[A > a]\Big] \\
&= 2\|F\|_\infty \Big[\mathrm{E}_x^{(-h)}[1 - \mathrm{e}^{-\int_a^h dr\;\, 2\alpha(Y_{-r})(v_{r-t}-v_r)(Y_{-r})}] + \mathrm{E}^{(-\infty)}[1 - \mathrm{e}^{-\int_a^\infty dr\;\, 2\alpha(Y_{-r})(v_{r-t}-v_r)(Y_{-r})}]\Big] \\
&\leq 8\|F\|_\infty \|\alpha\|_\infty \int_{a-t}^\infty dr\; g(r),
\end{aligned}$$

where we used (2.65), (2.66), (2.67) and (2.68) for the first inequality, the definition of A for the first equality, as well as $(H7)$ and the fact that $1 - \mathrm{e}^{-x} \leq x$ if $x \geq 0$ for the last inequality. From $(H7)$, we can choose a large enough such that: $|\bar{\delta}_h - \bar{\delta}_h^a| \leq \varepsilon/2$. We deduce that for all $h \geq \max(a, h_a)$: $|\bar{\delta}_h| \leq |\bar{\delta}_h - \bar{\delta}_h^a| + |\bar{\delta}_h^a| \leq \varepsilon$. This proves point (ii). □

2.6 The assumptions $(\boldsymbol{H}4)$, $(\boldsymbol{H}5)_\nu$ and $(\boldsymbol{H}6)$

We assume in all this section that P **is the distribution of a diffusion in** $\mathbb{R}^K$ **for** K **integer or the law of a finite state space Markov Chain,** see Section 2.7 and the references therein. In particular, the generalized eigenvalue λ_0 of $(\beta - \mathcal{L})$ (see (2.86) or (2.88)) is known to exist. We will denote by ϕ_0 the associated right eigenvector. We shall consider the assumption:

$(H8)$ **There exist two positive constants** C_1 **and** C_2 **such that** $\forall x \in E,\ C_1 \leq \phi_0(x) \leq C_2$**; and** $\phi_0 \in \mathcal{D}(\mathcal{L})$**.**

Under $(H8)$, let $\mathrm{P}_x^{\phi_0}$ be the probability measure on $(D, \mathcal{D})$ defined by (2.9) with g replaced by ϕ_0:

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\phi_0(Y_t)}{\phi_0(Y_0)}\,\mathrm{e}^{-\int_0^t ds\ (\beta(Y_s)-\lambda_0)}. \tag{2.69}$$

We shall also consider the assumption:

$(H9)$ **The probability measure** P^{ϕ_0} **admits a stationary measure** π**, and we have:**

$$\sup_{f\in b\mathcal{E}, \|f\|_\infty \leq 1} |\mathrm{E}_x^{\phi_0}[f(Y_t)] - \pi(f)| \xrightarrow[t\to+\infty]{} 0. \tag{2.70}$$

Notice the two hypotheses $(H8)$ and $(H9)$ hold for the examples of Section 2.7, see Lemmas 2.7.1 and 2.7.5.

Let us mention at this point that we will check that $\mathrm{P}_x^{\phi_0} = \mathrm{P}_x^{(\infty)}$ with $\mathrm{P}_x^{(\infty)}$ defined by (2.58), see Proposition 2.6.8.

2.6.1 Proof of $(\boldsymbol{H}4)$-$(\boldsymbol{H}6)$

Notice $(H9)$ implies that the probability measure $\mathrm{P}_\pi^{\phi_0}$ admits a stationary version on $D(\mathbb{R}, E)$, which we still denote by $\mathrm{P}_\pi^{\phi_0}$.

We introduce a specific h-transform of the superprocess. From Proposition 2.3.5 and the definition of the generalized eigenvalue (2.86) and (2.88), we have that the h-transform given by Definition 2.3.4 with $g = \phi_0$ of the $(\mathcal{L}, \beta, \alpha)$ superprocess is the $(\mathcal{L}^{\phi_0}, \lambda_0, \alpha\phi_0)$ superprocess. We define v^{ϕ_0} for all $t > 0$ and $x \in E$ by:

$$v_t^{\phi_0}(x) = \mathbb{N}_x^{(\mathcal{L}^{\phi_0}, \lambda_0, \alpha\phi_0)}[H_{\max} > t]. \tag{2.71}$$

Observe that, as in (2.17), the following normalization holds between v^{ϕ_0} and v:

$$v_t^{\phi_0}(x) = \frac{v_t(x)}{\phi_0(x)}\cdot \tag{2.72}$$

Our first task is to give precise bounds on the decay of $v_t^{\phi_0}$ as t goes to ∞.

We first offer bounds for the case $\lambda_0 = 0$ in Lemma 2.6.1, relying on a coupling argument. This in turn gives sufficient condition under which $(H1)$ holds in Lemma 2.6.2 We then enounce

Feynman-Kac representation formulae, Lemma 2.6.3, which yield exponential bounds in the case $\lambda_0 > 0$, see Lemma 2.6.4. We finally strenghten in Lemma 2.6.6 the bound of Lemma 2.6.4 by proving the exponential behaviour of $v_t^{\phi_0}$ in the case $\lambda_0 > 0$. The proofs of Lemmas 2.6.1, 2.6.2, 2.6.3, 2.6.4 and 2.6.6 are given in Section 2.6.2.

We first give a bound in the case $\lambda_0 = 0$. The proof relies on a coupling argument on the construction from Dhersin and Serlet [33]. It yields bounds from below and from above for the extinction time $H_{\max}$.

Lemma 2.6.1. *Assume $\lambda_0 = 0$, $(H2)$ and $(H8)$. Then for all $t > 0$:*

$$\alpha\phi_0(x)\frac{1}{\|\alpha\phi_0\|_\infty^2} \leq t\, v_t^{\phi_0}(x) \leq \alpha\phi_0(x)\left\|\frac{1}{\alpha\phi_0}\right\|_\infty^2.$$

We deduce from this lemma that assumption $(H1)$ holds:

Lemma 2.6.2. *Assume $\lambda_0 \geq 0$, $(H2)$ and $(H8)$. Then $(H1)$ holds.*

We give a Feynman-Kac's formula for v_0^ϕ and ∂v_0^ϕ.

Lemma 2.6.3. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$. Let $\varepsilon > 0$. We have:*

$$v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\, v_\varepsilon^{\phi_0}(Y_h)\right], \tag{2.73}$$

$$\partial_h v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\, \partial_h v_\varepsilon^{\phi_0}(Y_h)\right]. \tag{2.74}$$

We give exponential bounds for v_0^ϕ and $\partial_t v_0^\phi$ in the subcritical case.

Lemma 2.6.4. *Assume $\lambda_0 > 0$, $(H2)$-$(H3)$ and $(H8)$. Fix $t_0 > 0$. There exists C_3 and C_4 two positive constants such that, for all $x \in E$, $t > t_0$:*

$$C_3 \leq v_t^{\phi_0}(x)\,\mathrm{e}^{\lambda_0 t} \leq C_4. \tag{2.75}$$

There exists C_5 and C_6 two positive constants such that, for all $x \in E$, $t > t_0$:

$$C_5 \leq |\partial_t v_t^{\phi_0}(x)|\,\mathrm{e}^{\lambda_0 t} \leq C_6. \tag{2.76}$$

As a direct consequence of (2.75), we get the following Lemma.

Lemma 2.6.5. *Assume $\lambda_0 > 0$, $(H2)$-$(H3)$ and $(H8)$. Then $(H7)$ holds.*

In what follows, the notation $o_h(1)$ refers to any function F_h such that $\lim_{h\to+\infty} \| F_h \|_\infty = 0$. We now improve on Lemma 2.6.4, by using the ergodic formula (2.70).

Lemma 2.6.6. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$ hold. Then for all $\varepsilon > 0$, we have:*

$$\partial_t v_{h+\varepsilon}^{\phi_0}(x)\ \mathrm{e}^{\lambda_0 h} = \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha\,\phi_0\,v_{s+\varepsilon}^{\phi_0}(Y_{-s})}\,\partial_t v_\varepsilon^{\phi_0}(Y_0)\right](1 + o_h(1)). \tag{2.77}$$

In addition, for $\lambda_0 > 0$, we have that:

$$\mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_0^\infty ds\ \alpha\,\phi_0\,v_{s+\varepsilon}^{\phi_0}(Y_{-s})}\,\partial_t v_\varepsilon^{\phi_0}(Y_0)\right]$$

is finite (notice the integration is up to $+\infty$) and:

$$\partial_t v_{h+\varepsilon}^{\phi_0}(x)\ \mathrm{e}^{\lambda_0 h} = \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_0^\infty ds\ \alpha\,\phi_0\,v_{s+\varepsilon}^{\phi_0}(Y_{-s})}\,\partial_t v_\varepsilon^{\phi_0}(Y_0)\right] + o_h(1). \tag{2.78}$$

Our next goal is to prove $(H4)$ from $(H8)$-$(H9)$, see Proposition 2.6.8.

Fix $x \in E$. We observe from (2.54) and (2.69) that $\mathrm{P}_x^{(h)}$ is absolutely continuous with respect to $\mathrm{P}_x^{\phi_0}$ on $\mathcal{D}_{[0,t]}$ for $0 \leq t < h$. We define $M_t^{(h),\phi_0}$ the corresponding Radon-Nikodym derivative:

$$M_t^{(h),\phi_0} = \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}.$$

Using (2.54), (2.69) and the normalization $v(x) = v^{\phi_0}(x)\ \phi_0(x)$, we get:

$$\begin{aligned} M_t^{(h),\phi_0} &= \frac{\partial_t v_{h-t}(Y_t)\, \mathrm{e}^{-\lambda_0 t}}{\partial_t v_h(Y_0)} \frac{\phi_0(Y_0)}{\phi_0(Y_t)}\, \mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\ v_{h-s}(Y_s)} \\ &= \frac{\partial_t v^{\phi_0}_{h-t}(Y_t)\, \mathrm{e}^{-\lambda_0 t}}{\partial_t v^{\phi_0}_h(Y_0)}\, \mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{h-s}(Y_s)}. \end{aligned} \tag{2.79}$$

We have the following result on the convergence of $M_t^{(h),\phi_0}$.

Lemma 2.6.7. *Assume* $(H2)$-$(H3)$ *and* $(H8)$-$(H9)$*. For* $\lambda_0 \geq 0$*, we have:*

$$M_t^{(h),\phi_0} \xrightarrow[h\to+\infty]{} 1 \quad \mathrm{P}_x^{\phi_0}\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x^{\phi_0}),$$

and for $\lambda_0 > 0$*, we have:*

$$M_{h/2}^{(h),\phi_0} \xrightarrow[h\to+\infty]{} 1 \quad \mathrm{P}_x^{\phi_0}\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x^{\phi_0}).$$

Proof. We compute:

$$\begin{aligned} M_t^{(h),\phi_0} &= \frac{\partial_t v^{\phi_0}_{h-t}(Y_t)\, \mathrm{e}^{\lambda_0(h-t)}}{\partial_t v^{\phi_0}_h(Y_0)\, \mathrm{e}^{\lambda_0 h}}\, \mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\phi_0(Y_s)v^{\phi_0}_{h-s}(Y_s)} \\ &= \frac{\mathrm{E}^{\phi_0}_\pi\big[\,\mathrm{e}^{-2\int_{-(h-t-\varepsilon)}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\big]\,(1+o_h(1))}{\mathrm{E}^{\phi_0}_\pi\big[\,\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\big]\,(1+o_h(1))}(1+o_h(1)) \\ &= \frac{\mathrm{E}^{\phi_0}_\pi\big[\,\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\big]}{\mathrm{E}^{\phi_0}_\pi\big[\,\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\big]}(1+o_h(1)) \\ &= 1+o_h(1), \end{aligned}$$

where we used (2.79) for the first equality, (2.77) twice and the boundedness of α and ϕ_0 as well as the convergence of v_h to 0 for the second, and Lemma 2.6.4 (if $\lambda_0 > 0$) or Lemma 2.6.1 (if $\lambda_0 = 0$) for the fourth. Since $o_h(1)$ is bounded and converges uniformly to 0, we get that the convergence of $M_t^{(h),\phi_0}$ towards 1 holds $\mathrm{P}_x^{\phi_0}$-a.s. and in $L^1(\mathrm{P}_x^{\phi_0})$.

Similar arguments relying on (2.78) instead of (2.77) imply that $M_{h/2}^{(h),\phi_0} = 1 + o_h(1)$ for $\lambda_0 > 0$. Since $o_h(1)$ is bounded and converges uniformly to 0, we get that the convergence of $M_{h/2}^{(h),\phi_0}$ towards 1 holds $\mathrm{P}_x^{\phi_0}$-a.s. and in $L^1(\mathrm{P}_x^{\phi_0})$. □

The previous Lemma enables us to conclude about $(H4)$.

Proposition 2.6.8. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Then $(H4)$ holds with $\mathrm{P}_x^{(\infty)} = \mathrm{P}_x^{\phi_0}$.*

Proof. Notice that:

$$M_t^{(h)} = \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}} = M_t^{(h),\phi_0} \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}}.$$

The convergence $\lim_{h\to+\infty} M_t^{(h),\phi_0} = 1$ $\mathrm{P}_x^{\phi_0}$-a.s. and in $\mathrm{L}^1(\mathrm{P}_x^{\phi_0})$ readily implies $(H4)$. Then, use (2.58) to get $\mathrm{P}^{(\infty)} = \mathrm{P}^{\phi_0}$. □

Notice that $(H5)_\nu$ is a direct consequence of Lemma 2.6.6.

Corollary 2.6.9. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Then $(H5)_\nu$ holds with $\rho = \phi_0/\nu(\phi_0)$.*

Proof. We deduce from (2.72) and (2.77) that:

$$\partial_t v_h(x) = f(h)\phi_0(x)\,(1 + o_h(1))\,\mathrm{e}^{-\lambda_0 h},$$

for some positive function f of h. Then we get:

$$\frac{\partial_h v_h(x)}{\nu(\partial_h v_h)} = \frac{\phi_0(x)}{\nu(\phi_0)}(1 + o_h(1)).$$

This gives $(H5)_\nu$, as $o_h(1)$ is bounded, with $\rho = \phi_0/\nu(\phi_0)$. □

Our next goal is to prove $(H6)$ from $(H8)$-$(H9)$, see Proposition 2.6.12.

Observe from (2.53), (2.63) and (2.69) that $\mathrm{P}_\pi^{(-h)}$ is absolutely continuous with respect to $\mathrm{P}_\pi^{\phi_0}$ on $\mathcal{D}_{[-h,0]}$. We define $L^{(-h)}$ the corresponding Radon-Nikodym derivative:

$$L^{(-h)} = \frac{d\mathrm{P}^{(-h)}_{\pi\ |\mathcal{D}_{[-h,0]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{[-h,0]}}} = \frac{1}{\alpha(Y_0)\phi_0(Y_0)} \frac{\partial_h v_h^0\,\mathrm{e}^{\beta_0 h}}{\partial_h v_h^{\phi_0}(Y_{-h})\,\mathrm{e}^{\lambda_0 h}}\,\mathrm{e}^{-2\int_{-h}^0 (\alpha(Y_s)v_{-s}(Y_s) - v_{-s}^0)\,ds}. \tag{2.80}$$

The next Lemma insures the convergence of $L^{(-h)}$ to a limit, say $L^{(-\infty)}$.

Lemma 2.6.10. *Assume $\lambda_0 > 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. We have:*

$$L^{(-h)} \xrightarrow[h\to+\infty]{} L^{(-\infty)} \quad \mathrm{P}_\pi^{\phi_0}\text{-}a.s.\ and\ in\ \mathrm{L}^1(\mathrm{P}_\pi^{\phi_0}).$$

Proof. Notice that $\lim_{h\to+\infty} \partial_h v_h^0\,\mathrm{e}^{\beta_0 h} = -\beta_0^2$. We also deduce from (2.48), (2.49) and (2.75) that $\int_{-h}^0 (\alpha(Y_s)v_{-s}(Y_s) - v_{-s}^0)\,ds$ increases, as h goes to infinity, to $\int_{-\infty}^0 (\alpha(Y_s)v_{-s}(Y_s) - v_{-s}^0)\,ds$ which is finite. For fixed $t > 0$, we also deduce from (2.78) (with h replaced by $h - t$ and ε by t) that $\mathrm{P}_\pi^{\phi_0}$ a.s.:

$$\lim_{h\to+\infty} \partial_t v_h^{\phi_0}(Y_{-h})\,\mathrm{e}^{\lambda_0 h} = \mathrm{e}^{\lambda_0 t}\,\mathrm{E}_\pi^{\phi_0}\big[\,\mathrm{e}^{-2\int_{-\infty}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\,v_{-s}^{\phi_0}(Y_s)}\,\partial_t v_t^{\phi_0}(Y_{-t})\big].$$

We deduce from (2.80) the $\mathrm{P}_\pi^{\phi_0}$ a.s. convergence of $(L^{(-h)}, h > 0)$ to $L^{(-\infty)}$. Notice from (2.76) that, for fixed t, the sequence $(L^{(-h)}, h > t)$ is bounded. Hence the previous convergence holds also in $\mathrm{L}^1(\mathrm{P}_\pi^{\phi_0})$. □

As $\mathrm{E}_\pi^{\phi_0}\big[L^{(-h)}\big]=1$, we deduce that $\mathrm{E}_\pi^{\phi_0}\big[L^{(-\infty)}\big]=1$. We define the probability measure $\mathrm{P}_\pi^{(-\infty),\phi_0}$ on $(D^-,\mathcal{D}_{(-\infty,0]})$ by its Radon Nikodym derivative:

$$\frac{d\mathrm{P}^{(-\infty),\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,0]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,0]}}}=L^{(-\infty)}. \tag{2.81}$$

Remark 2.6.11. Assume $\lambda_0>0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Define for $h>t>0$:

$$L_{-t}^{(-h)}=\mathrm{E}_\pi^{\phi_0}[L^{(-h)}|\mathcal{D}_{(-\infty,-t]}]=\frac{d\mathrm{P}^{(-h)}_{\pi\ |\mathcal{D}_{[-h,-t]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{[-h,-t]}}}$$

$$L_{-t}^{(-\infty)}=\mathrm{E}_\pi^{\phi_0}[L^{(-\infty)}|\mathcal{D}_{(-\infty,-t]}]=\frac{d\mathrm{P}^{(-\infty),\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,-t]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,-t]}}}\cdot$$

Using (2.55) and Lemma 2.6.3, we get:

$$\begin{aligned}L_{-t}^{(-h)}&=\frac{\partial_t v_t(Y_{-t})}{\partial_t v_h(Y_{-h})}\frac{\phi_0(Y_{-h})}{\phi_0(Y_{-t})}\,\mathrm{e}^{-\lambda_0(h-t)}\,\mathrm{e}^{-2\int_{-h}^{-t}ds\ \alpha(Y_s)\,v_{-s}(Y_s)}\\&=\frac{\mathrm{e}^{-2\int_{-h}^{-t}ds\ \alpha(Y_s)\ \phi_0(Y_s)\,v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})}{\mathrm{E}_{Y_{-h}}^{\phi_0}\big[\,\mathrm{e}^{-2\int_{-h}^{-t}ds\ \alpha(Y_s)\ \phi_0(Y_s)\,v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})\big]}\cdot\end{aligned}$$

Using Lemma 2.6.6 and convergence of $(L_{-t}^{(-h)},h>t)$ to $L_{-t}^{(-\infty)}$, which is a consequence of Lemma 2.6.10, we also get that for $t>0$:

$$L_{-t}^{(-\infty)}=\frac{\mathrm{e}^{-2\int_{-\infty}^{-t}ds\ \alpha(Y_s)\ \phi_0(Y_s)\,v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})}{\mathrm{E}_{\pi}^{\phi_0}\big[\,\mathrm{e}^{-2\int_{-\infty}^{-t}ds\ \alpha(Y_s)\ \phi_0(Y_s)\,v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})\big]}\cdot$$

Those formulas are more self-contained than (2.80) and the definition of $L^{(-\infty)}$ as a limit, but they only hold for $t>0$.

The following Proposition gives that $(H6)$ holds.

Proposition 2.6.12. *Assume $\lambda_0>0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Then $(H6)$ holds with* $\mathrm{P}^{(-\infty)}=\mathrm{P}^{(-\infty),\phi_0}$.

Proof. Let $0<t$ and F be a bounded and $\mathcal{D}_{[-t,0]}$ measurable function. For h large enough, we have:

$$\begin{aligned}\mathrm{E}_x^{(-h)}\big[F(Y_{[-t,0]})\big]&=\mathrm{E}_x^{(h)}\big[\mathrm{E}_{Y_{h/2}}^{(h/2)}\big[F(\theta_{h/2}(Y)_{[-t,-s]})\big]\big]\\&=\mathrm{E}_x^{\phi_0}\big[M_{h/2}^{(h),\phi_0}\ \mathrm{E}_{Y_{h/2}}^{(h/2)}\big[F(\theta_{h/2}(Y)_{[-t,0]})\big]\big]\\&=\mathrm{E}_x^{\phi_0}\big[\mathrm{E}_{Y_{h/2}}^{(h/2)}\big[F(\theta_{h/2}(Y)_{[-t,0]})\big]\big]+o_h(1)\\&=\mathrm{E}_\pi^{(h/2)}\big[F(\theta_{h/2}(Y)_{[-t,0]})\big]+o_h(1)\\&=\mathrm{E}_\pi^{(-h/2)}\big[F(Y_{[-t,0]})\big]+o_h(1),\end{aligned}$$

where we used the definition of $\mathrm{P}^{(-h)}$ and the Markov property for the first equality, Lemma 2.6.7 together with F bounded by $\|F\|_\infty$ for the third, and assumption $(H9)$ for the fourth. We continue the computations as follows:

$$\begin{aligned}\mathrm{E}_x^{(-h)}\left[F(Y_{[-t,0]})\right] &= \mathrm{E}_\pi^{\phi_0}\left[L^{(-h/2)}F(Y_{[-t,0]})\right] + o_h(1)\\ &= \mathrm{E}_\pi^{\phi_0}\left[L^{(-\infty)}F(Y_{[-t,0]})\right] + o_h(1)\\ &= \mathrm{E}_\pi^{(-\infty),\phi_0}\left[F(Y_{[-t,0]})\right] + o_h(1),\end{aligned}$$

where we used Lemma 2.6.10 for the second equality. This gives $(H6)$ with $\mathrm{P}^{(-\infty)} = \mathrm{P}^{(-\infty),\phi_0}$. □

2.6.2 Proof of Lemmas 2.6.1, 2.6.2, 2.6.3, 2.6.4 and 2.6.6

Proof of Lemma 2.6.1. From $(H2)$ and $(H8)$, there exist $m, M \in \mathbb{R}$ such that

$$\forall x \in E, 0 < m \le \alpha\phi_0(x) \le M < \infty.$$

Let W be a $(\frac{M}{\alpha\phi_0}\mathcal{L}, 0, M)$ Brownian snake and define the time change Φ for every $w \in \mathcal{W}$ by $\Phi_t(w) = \int_0^t ds\ \frac{M}{\alpha\phi_0}(w(s))$. As $\partial_t\Phi_t(w) \ge 1$, we have that $t \to \Phi_t(w)$ is strictly increasing. Let $t \to \Phi_t^{(-1)}(w)$ denote its inverse. Then, using Proposition 12 of [33], first step of the proof, we have that the time changed snake $W \circ \Phi^{-1}$, with value

$$(W \circ \Phi^{-1})_s = (W_s(\Phi_t^{-1}(W_s)), t \in [0, \Phi^{-1}(W_s, H_s)])$$

at time s, is a $(\mathcal{L}, 0, \alpha\phi_0)$ Brownian snake. Noting the obvious bound on the time change $\Phi_t^{-1}(w) \le t$, we have, according to Theorem 14 of [33]:

$$\mathbf{P}_{\frac{\alpha\phi_0(x)}{M}\delta_x}^{\left(\frac{M}{\alpha\phi_0}\ \mathcal{L}^{\phi_0},0,M\right)}(H_{\max} \le t) \ge \mathbf{P}_{\delta_x}^{(\mathcal{L}^{\phi_0},0,\alpha\phi_0)}(H_{\max} \le t)$$

which implies:

$$\frac{\alpha\phi_0(x)}{M}\ \mathbf{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) \le \mathbf{N}_x^{(\mathcal{L}^{\phi_0},0,\alpha\phi_0)}(H_{\max} > t)$$

from the exponential formula for Poisson point measures. Now, the left hand side of this inequality can be computed explicitly:

$$\mathbf{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) = \mathbb{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) = \frac{1}{Mt}$$

and the right hand side of this inequality is $v_t^{\phi_0}(x)$ from (2.71). We thus have proved that:

$$\frac{\alpha\phi_0(x)}{M^2 t} \le v_t^{\phi_0}(x),$$

and this yields the first part of the inequality of Lemma 2.6.1. The second part is obtained in the same way using the coupling with the $(\frac{m}{\alpha\phi_0}\mathcal{L}^{\phi_0}, 0, m)$ Brownian snake. □

Proof of lemma 2.6.2. Assumption $(H2)$ and $(H8)$ allow us to apply Lemma 2.6.1 for the case $\lambda_0 = 0$, which yields that $v_\infty^{\phi_0} = 0$, and then $v_\infty = 0$ thanks to (2.72). This in turn implies that $(H1)$ holds in the case $\lambda_0 = 0$ according to Lemma 2.2.5. For $\lambda_0 > 0$, we may use item 5 of Proposition 13 of [33] (which itself relies on a Girsanov theorem) with $\mathbb{P}^{(\mathcal{L},0,\alpha\phi_0)}$ in the rôle of $\mathbb{P}^c$ and $\mathbb{P}^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}$ in the rôle of $\mathbb{P}^{b,c}$ to conclude that the extinction property $(H1)$ holds under $\mathbb{P}^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}$. □

Proof of Lemma 2.6.3. Let $\varepsilon > 0$. The function v^{ϕ_0} is known to solve the following mild form of the Laplace equation, see equation (2.3):

$$v_{t+s}^{\phi_0}(x) + \mathrm{E}_x^{\phi_0}\left[\int_0^t dr\ \left(\lambda_0\, v_{t+s-r}^{\phi_0}(Y_r) + \alpha(Y_r)\phi_0(Y_r)(v_{t+s-r}^{\phi_0}(Y_r))^2\right)\right] = \mathrm{E}_x^{\phi_0}\left[v_s^{\phi_0}(Y_t)\right].$$

By differentiating with respect to s and taking $t = t - s$, we deduce from dominated convergence and the bounds (2.46), (2.47) and (2.49) on $v^{\phi_0} = v/\phi_0$ and its time derivative (valid under the assumptions $(H1)$-$(H3)$) the following mild form on the time derivative $\partial_t v^{\phi_0}$:

$$\partial_t v_t^{\phi_0}(x) + \mathrm{E}_x^{\phi_0}\left[\int_0^{t-s} dr\ \left(\lambda_0\ + 2\alpha(Y_r)\phi_0(Y_r)v_{t-r}^{\phi_0}(Y_r)\right)\partial_t v_{t-r}^{\phi_0}(Y_r)\right] = \mathrm{E}_x^{\phi_0}\left[\partial_t v_s^{\phi_0}(Y_{t-s})\right].$$

From the Markov property, for fixed $t > 0$, the two following processes:

$$\left(v_{t-s}^{\phi_0}(Y_s) - \int_0^s dr\ \left(\lambda_0 + \alpha(Y_r)\phi_0(Y_r)v_{t-r}^{\phi_0}(Y_r)\right)\, v_{t-r}^{\phi_0}(Y_r), 0 \le s < t\right)$$

and

$$\left(\partial_t v_{t-s}^{\phi_0}(Y_s) - \int_0^s dr\ \left(\lambda_0\ + 2\alpha(Y_r)\phi_0(Y_r)v_{t-r}^{\phi_0}(Y_r)\right)\partial_t v_{t-r}^{\phi_0}(Y_r), 0 \le s < t\right)$$

are $\mathcal{D}_s$-martingale under $\mathrm{P}_\pi^{\phi_0}$. A Feynman-Kac manipulation, as done in the proof of Lemma 2.3.1, enables us to conclude that for fixed $t > 0$:

$$\left(v_{t-s}^{\phi_0}(Y_s)\, \mathrm{e}^{-\int_0^s dr\ \left(\lambda_0+\alpha(Y_r)\phi_0(Y_r)v_{t-r}^{\phi_0}(Y_r)\right)}, 0 \le s < t\right)$$

and

$$\left(\partial_t v_{t-s}^{\phi_0}(Y_s)\, \mathrm{e}^{-\int_0^s dr\ \left(\lambda_0\ +2\alpha(Y_r)\phi_0(Y_r)v_{t-r}^{\phi_0}(Y_r)\right)}, 0 \le s < t\right)$$

are $\mathcal{D}_s$-martingale under $\mathrm{P}_\pi^{\phi_0}$. Taking expectations at time $s = 0$ and $s = h$ with $t = h + \varepsilon$, we get the representations formulae stated in the Lemma:

$$v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\, v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\, v_\varepsilon^{\phi_0}(Y_h)\right],$$

$$\partial_h v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\, v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\, \partial_h v_\varepsilon^{\phi_0}(Y_h)\right].$$

□

Proof of Lemma 2.6.4. Since $v_\varepsilon^{\phi_0} = v_\varepsilon/\phi_0 = \tilde{v}_\varepsilon/(\alpha\phi_0)$, we can conclude from (2.46), $(H2)$ and $(H8)$ that $v_\varepsilon^{\phi_0}$ is bounded from above and from below by positive constants. Similarly, we also get from (2.47), (2.48) and (2.49) that $|\partial_h \tilde v_\varepsilon|$ is bounded from above and from below by two positive constants. Thus, we have the existence of four positive constants, D_1, D_2, D_3 and D_4, such that, for all $x \in E$:

$$D_1 \le v_\varepsilon^{\phi_0}(x) \le D_2, \tag{2.82}$$

$$D_3 \le |\partial_t v_\varepsilon^{\phi_0}(x)| \le D_4. \tag{2.83}$$

From equations (2.73), (2.82) and the positivity of v^{ϕ_0}, we deduce that:

$$v_{h+\varepsilon}^{\phi_0}(x) \le D_2 \,\mathrm{e}^{-\lambda_0 h}. \tag{2.84}$$

Putting back (2.84) into (2.73), we have the converse inequality $D_5 \,\mathrm{e}^{-\lambda_0 h} \le v_{h+\varepsilon}^{\phi_0}(x)$ with $D_5 = D_1 \exp\{-D_2 \,\|\alpha\|_\infty \,\|\phi_0\|_\infty / \lambda_0\} > 0$. This gives (2.75).

Similar arguments using (2.74) and (2.83) instead of (2.73) and (2.82), gives (2.76). □

Proof of Lemma 2.6.6. Using the Feynman-Kac representation of $\partial_h v_{h+\varepsilon}^{\phi_0}$ from (2.73) and the Markov property, we have:

$$\begin{aligned}\partial_h v_{h+\varepsilon}^{\phi_0}(x)\,\mathrm{e}^{\lambda_0 h} &= \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_h)\right]\\ &= \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^{\sqrt h} ds\ \alpha\,\phi_0\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\,\mathrm{E}_{Y_{\sqrt h}}^{\phi_0}\big[\mathrm{e}^{-2\int_0^{h-\sqrt h} ds\,\alpha(Y_s)\,\phi_0(Y_s)\,v_{h-\sqrt h+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_{h-\sqrt h})\big]\right].\end{aligned}$$

Notice that

$$\left|\int_0^{\sqrt h} ds\ \alpha\,\phi_0\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)\right| \le \|\alpha\|_\infty \,\|\phi_0\|_\infty \sqrt h \,\|v_{h+\varepsilon-\sqrt h}^{\phi_0}\|_\infty = o_h(1), \tag{2.85}$$

according to Lemma 2.6.4 if $\lambda_0 > 0$ and Lemma 2.6.1 if $\lambda_0 = 0$. We get:

$$\begin{aligned}\partial_h v_{h+\varepsilon}^{\phi_0}(x)\,\mathrm{e}^{\lambda_0 h} &= \mathrm{E}_x^{\phi_0}\left[\mathrm{E}_{Y_{\sqrt h}}^{\phi_0}\big[\mathrm{e}^{-2\int_0^{h-\sqrt h} ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h-\sqrt h+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_{h-\sqrt h})\big]\right](1+o_h(1))\\ &= \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_0^{h-\sqrt h} ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h-\sqrt h+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_{h-\sqrt h})\right](1+o_h(1))\\ &= \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-(h-\sqrt h)}^0 ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right](1+o_h(1))\\ &= \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-h}^0 ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right](1+o_h(1)),\end{aligned}$$

where we used (2.85) for the first equality, $(H9)$ for the second, stationarity of Y under $\mathrm{P}_\pi^{\phi_0}$ for the third and (2.85) again for the last. This gives (2.77).

Moreover, if $\lambda_0 > 0$, we get that:

$$\mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^0 ds\ \alpha\,\phi_0\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right]$$

is finite and that:

$$\lim_{h'\to+\infty} \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-h'}^0 ds\ \alpha\,\phi_0\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right] = \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^0 ds\ \alpha\,\phi_0\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right].$$

Therefore, we deduce (2.78) from (2.77). □

2.6.3 About the Bismut spine.

Choosing uniformly an individual at random at height t under $\mathbf{N}_x$ and letting $t \to \infty$, we will see that the law of the ancestral lineage should converge in some sense to the law of the oldest ancestral lineage which itself converges to $\mathrm{P}_x^{(\infty)}$ defined in (2.58), according to Lemma 2.6.8.

We have defined in (2.45) the following family of probability measure indexed by $t \geq 0$:

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\right]}.$$

Lemma 2.6.13. *Assume* $(H8)$*-*$(H9)$*. We have, for every* $0 \leq t_0 \leq t$*:*

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} \xrightarrow[t\to+\infty]{} \frac{d\mathrm{P}^{(\infty)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} \quad \mathrm{P}_x\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x).$$

Note that there is no restriction on the sign of λ_0 for this Lemma to hold.

Remark 2.6.14. This result correspond to the so called globular phase in the random polymers litterature (see [27], Theorem 8.3).

Proof. We have:

$$\begin{aligned}
\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} &= \mathrm{e}^{-\int_0^{t_0} ds\ \beta(Y_s)}\ \frac{\mathrm{E}_{Y_{t_0}}\left[\mathrm{e}^{-\int_0^{t-t_0} ds\ \beta(Y_s)}\right]}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^{t} ds\ \beta(Y_s)}\right]}\\
&= \mathrm{e}^{-\int_0^{t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\mathrm{E}_{Y_{t_0}}\left[\mathrm{e}^{-\int_0^{t-t_0} ds\ (\beta(Y_s)-\lambda_0)}\right]}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^{t} ds\ (\beta(Y_s)-\lambda_0)}\right]}\\
&= \mathrm{e}^{-\int_0^{t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_{t_0})}{\phi_0(Y_0)}\ \frac{\mathrm{E}_{Y_{t_0}}\left[\mathrm{e}^{-\int_0^{t-t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_{t-t_0})}{\phi_0(Y_0)}\ \frac{1}{\phi_0(Y_{t-t_0})}\right]}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^{t} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_t)}{\phi_0(x)}\ \frac{1}{\phi_0(Y_t)}\right]}\\
&= \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}}\ \frac{\mathrm{E}^{\phi_0}_{Y_{t_0}}\left[1/\phi_0(Y_{t-t_0})\right]}{\mathrm{E}^{\phi_0}_x\left[1/\phi_0(Y_t)\right]}\\
&\xrightarrow[t\to\infty]{} \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}}\ \frac{\pi(\frac{1}{\phi_0})}{\pi(\frac{1}{\phi_0})} = \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}},
\end{aligned}$$

where we use the Markov property at the first equality, we force the apparition of λ_0 at the second equality and we force the apparition of ϕ_0 at the third equality in order to obtain the Radon Nikodym derivative of $\mathrm{P}_x^{\phi_0}$ with respect to P_x: this observation gives the fourth equality. The ergodic assumption $(H9)$ ensures the P_x-a.s. convergence to 1 of the fraction in the fourth equality as t goes to ∞. Since

$$\Big((t,y) \to \mathrm{E}_y^{\phi_0}\left[1/\phi_0(Y_{t-t_0})\right]/\mathrm{E}_x^{\phi_0}\left[1/\phi_0(Y_t)\right]\Big)$$

is bounded according to $(H8)$, we conclude that the convergence also holds in $\mathrm{L}^1(\mathrm{P}_x)$. Then use Lemma 2.6.8 to get that $\mathrm{P}_x^{\phi_0} = \mathrm{P}_x^{(\infty)}$. □

2.7 Two examples

In this section, we specialize the results of the previous sections to the case of the multitype Feller process and of the superdiffusion.

2.7.1 The multitype Feller diffusion

The multitype Feller diffusion is the superprocess with finite state space: $E = \{1, \ldots, K\}$ for K integer. In this case, the spatial motion is a pure jump Markov process, which will be assumed irreducible. Its generator $\mathcal{L}$ is a square matrix $(q_{ij})_{1\leq i,j\leq K}$ of size K with lines summing up to 0, where q_{ij} gives the transition rate from i to j for $i \neq j$. The functions β and α defining the branching mechanism (2.2) are vectors of size K: this implies that $(H2)$ and $(H3)$ automatically hold. For more details about the construction of finite state space superprocess, we refer to [41], example 2, p. 10, and to [25] for investigation of the Q-process.

The generalized eigenvalue λ_0 is defined by:

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u > 0 \text{ such that } (\mathrm{Diag}(\beta) - \mathcal{L})u = \ell u\}, \tag{2.86}$$

where $\mathrm{Diag}(\beta)$ is the diagonal $K \times K$ matrix with diagonal coefficients derived from the vector β. We stress that the generalized eigenvalue is also the Perron Frobenius eigenvalue, *i.e.* the eigenvalue with the maximum real part, which is real by Perron Frobenius theorem, see [121], Exercice 2.11. Moreover, the associated eigenspace is one-dimensional. We will denote by ϕ_0 and $\tilde{\phi}_0$ its generating left, resp. right, eigenvectors, normalized so that $\sum_{i=1}^K \phi_0(i)\tilde{\phi}_0(i) = 1$, and the coordinates of ϕ_0 and $\tilde{\phi}_0$ are positive.

We first check that the two assumptions we made in Section 6 are satisfied.

Lemma 2.7.1. *Assumptions $(H8)$ and $(H9)$ hold with $\pi = \phi_0\,\tilde{\phi}_0$.*

Proof. Assumption $(H8)$ is obvious in the finite state space setting. Assumption $(H9)$ is a classical statement about irreducible finite state space Markov Chains. □

Lemma 2.7.2. *Assume $\lambda_0 \geq 0$. Then $(H1)$, $(H4)$ and $(H5)_\nu$ hold.*

Proof. Assumption $(H2)$ and $(H8)$ hold according to Lemma 2.7.1. Together with $\lambda_0 \geq 0$, this allows us to apply Lemma 2.6.2 to obtain $(H1)$. Then use Proposition 2.6.8 to get $(H4)$ and Corollary 2.6.9 to get $(H5)_\nu$. □

Lemma 2.7.3. *Assume $\lambda_0 > 0$. Then $(H6)$ and $(H7)$ holds.*

Proof. We apply Proposition 2.6.12 to prove $(H6)$ and Lemma 2.6.5 to prove $(H7)$. □

Recall that $\mathrm{P}_x^{(h)}$ and $\mathrm{P}_x^{(\infty)}$ were defined in (2.54) and (2.58) respectively.

Lemma 2.7.4. *We have:*

(i) $\mathrm{P}_x^{(h)}$ *is a continuous time non-homogeneous Markov chain on* $[0, h)$ *issued from* x *with transition rates from* i *to* j, $i \neq j$, *equal to* $\frac{\partial_h v_{h-t}(j)}{\partial_h v_{h-t}(i)}\, q_{ij}$ *at time* t, $0 \leq t < h$.

(ii) $\mathrm{P}_x^{(\infty)}$ *is a continuous time homogeneous Markov chain on* $[0, \infty)$ *issued from* x *with transition rates from* i *to* j, $i \neq j$, *equal to* $\frac{\phi_0(j)}{\phi_0(i)}\, q_{ij}$.

Let us comment on this Lemma. The logarithmic derivative of the function $x \to \partial_h v_{h-t}(x)$ is used to bias the rate at time t of the Markov process formed by the ancestral lineage of the oldest individual alive, given this lineage extincts precisely at h. When looking at the unique infinite ancestral lineage in the genealogy of the Q-process instead, the same statement holds with the map $x \to \phi_0(x)$ instead. Notice that the bias does no more depend on t in this case.

Proof. The first item is a consequence of a small adaptation of Lemma 2.3.2 for time dependent function. Namely, let $g_t(x)$ be a time dependent function. Consider the law of process (t, Y_t) and consider the probability measure P^g defined by (2.9) with $g(t, Y_t) = g_t(Y_t)$. Denoting by $\mathcal{L}_t^g$ the generator of (the non-homogeneous Markov process) Y_t under P^g, we have that:

$$\forall u \in \mathcal{D}_g(\mathcal{L}), \quad \mathcal{L}_t^g(u) = \frac{\mathcal{L}(g_t u) - \mathcal{L}(g_t) u}{g_t}. \tag{2.87}$$

Recall that for all vector u, $\mathcal{L}(u)(i) = \sum_{j \neq i} q_{ij}(u(j) - u(i))$. Then apply (2.87) to the time dependent function $g_t(x) = \partial_t v_{h-t}(x)$, and note that $\mathrm{P}^g = \mathrm{P}^{(h)}$ thanks to (2.54). For the second item, observe that Proposition 2.6.8 identifies $\mathrm{P}_x^{(\infty)}$ with P^{ϕ_0}. Use then Lemma 2.3.2 to conclude. □

Williams' decomposition under $\mathbb{N}_x^{(h)}$ (Propositions 2.4.14) together with the convergence of this decomposition (Theorem 2.5.5) then hold under the assumption $\lambda_0 \geq 0$. Convergence of the distribution of the superprocess near its extinction time under $\mathbb{N}_x^{(h)}$ (Proposition 2.5.9) holds under the stronger assumption $\lambda_0 > 0$. We were unable to derive an easier formula for $\mathrm{P}^{(-\infty)}$ in this context.

Remark that Lemma 2.5.1, Definition 2.5.7 and Corollary 2.5.8 give a precise meaning to the "interactive immigration" suggested by Champagnat and Roelly in Remark 2.8. of [25].

2.7.2 The superdiffusion

The superprocess associated to a diffusion is called superdiffusion. We first define the diffusion and the relevant quantities associated to it, and take for that the general setup from [107]. Here E is an arbitrary domain of $\mathbb{R}^K$ for K integer. Let a_{ij} and b_i be in $\mathcal{C}^{1,\mu}(E)$, the usual Hölder space of order $\mu \in [0, 1)$, which consists of functions whose first order derivatives are locally Hölder continuous with exponent μ, for each i, j in $\{1, \ldots, K\}$. Moreover, assume that the functions $a_{i,j}$ are such that the matrix $(a_{ij})_{(i,j) \in \{1 \ldots K\}^2}$ is positive definite. Define now the generator $\mathcal{L}$ of the diffusion to be the elliptic operator:

$$\mathcal{L}(u) = \sum_{i=1}^{K} b_i\, \partial_{x_i} u + \frac{1}{2} \sum_{i,j=1}^{K} a_{ij}\, \partial_{x_i, x_j} u.$$

The generalized eigenvalue λ_0 of the operator $\beta - \mathcal{L}$ is defined by:

$$\lambda_0 = \sup \{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L})u = \ell\, u\}\cdot \tag{2.88}$$

Denoting E the expectation operator associated to the process with generator $\mathcal{L}$, we recall an equivalent probabilistic definition of the generalized eigenvalue λ_0:

$$\lambda_0 = - \sup_{A \subset \mathbb{R}^K} \lim_{t\to\infty} \frac{1}{t} \log \mathrm{E}_x \big[\, \mathrm{e}^{-\int_0^t ds\, \beta(Y_s)}\, \mathbf{1}_{\{\tau_{A^c} > t\}} \big],$$

for any $x \in \mathbb{R}^K$, where $\tau_{A^c} = \inf \{t > 0 : Y(t) \notin A\}$ and the supremum runs over the compactly embedded subsets A of $\mathbb{R}^K$. We assume that the operator $(\beta - \lambda_0) - \mathcal{L}$ is critical in the sense that the space of positive harmonic functions for $(\beta - \lambda_0) - \mathcal{L}$ is one dimensional, generated by ϕ_0. In that case, the space of positive harmonic functions of the adjoint of $(\beta - \lambda_0) - \mathcal{L}$ is also one dimensional, and we denote by $\tilde{\phi}_0$ a generator of this space. We further assume that the operator $(\beta - \lambda_0) - \mathcal{L}$ is **product-critical**, *i.e.* $\int_E dx\, \phi_0(x)\, \tilde{\phi}_0(x) < \infty$, in which case we can normalize the eigenvectors in such a way that $\int_E dx\, \phi_0(x)\, \tilde{\phi}_0(x) = 1$. This assumption (already appearing in [42]) is a rather strong one and implies in particular that P^{ϕ_0} is the law of a recurrent Markov process, see Lemma 2.7.5 below.

Concerning the branching mechanism, we will assume, in addition to the conditions stated in section 2.2, that $\alpha \in \mathcal{C}^4(E)$.

Lemma 2.7.5. *Assume* $(H8)$*. Assumption* $(H9)$ *holds with* $\pi(dx) = \phi_0(x)\, \tilde{\phi}_0(x)\, dx$*.*

Proof. We repeat the argument developed in Remark 5 of [42]. We first note that $-\mathcal{L}^{\phi_0}$ is the (usual) h-transform of the operator $(\beta - \lambda_0) - \mathcal{L}$ with $h = \phi_0$, where the h-transform of $\mathcal{L}(\cdot)$ is $\frac{\mathcal{L}(h\cdot)}{h}$. Now, we assumed above that $(\beta - \lambda_0) - \mathcal{L}$ is a critical operator and criticality is invariant under h-transforms for operators. Moreover, a calculus shows that $\tilde{\phi}_0$ and ϕ_0 transforms into $\phi_0\, \tilde{\phi}_0$ and 1 respectively when turning from $(\beta - \lambda_0) - \mathcal{L}$ to $-\mathcal{L}^{\phi_0}$, which is thus again product critical. We may apply Theorem 9.9 p.192 of [107] which states that $(H9)$ holds, with ϕ_0 replaced by 1 and $\tilde{\phi}_0$ by $\phi_0\, \tilde{\phi}_0$. □

Note that the non negativity of the generalized eigenvalue of the operator $(\beta - \mathcal{L})$ now characterizes in general the local extinction property (the superprocess Z satisfies local extinction if its restrictions to compact domains of E satisfies global extinction); see [43] for more details on this topic. However, under the boundedness assumption we just made on α and ϕ_0, the extinction property $(H1)$ holds, as will be proved (among other things) in the following Lemma.

Lemma 2.7.6. *Assume* $\lambda_0 \geq 0$ *and* $(H8)$*. Then* $(H1)$*-*$(H4)$ *and* $(H5)_\nu$ *hold. If moreover* $\lambda_0 > 0$*, then* $(H6)$ *and* $(H7)$ *holds.*

Proof. The assumption $\alpha \in \mathcal{C}^4(E)$ ensures that $(H2)$ and $(H3)$ hold. Then the end of the proof is similar to the end of the proof of Lemma 2.7.2 and the proof of Lemma 2.7.3. □

Recall that $\mathrm{P}_x^{(h)}$ and $\mathrm{P}_x^{(\infty)}$ were defined in (2.54) and (2.58).

Lemma 2.7.7. *We have:*

– $\mathrm{P}_x^{(h)}$ *is a non-homogeneous diffusion on* $[0, h)$ *issued from* x *with generator* $(\mathcal{L} + a \frac{\nabla \partial_h v_{h-t}}{\partial_h v_{h-t}} \nabla .)$ *at time* t*,* $0 \leq t < h$*.*

– $\mathrm{P}_x^{(\infty)}$ *is an homogeneous diffusion on* $[0,\infty)$ *issued from* x *with generator* $(\mathcal{L}+a\frac{\nabla\phi_0}{\phi_0}\nabla.)$.

Proof. The proof is similar to the proof of Lemma 2.7.4. □

Williams' decomposition under $\mathbb{N}_x^{(h)}$ (Propositions 2.4.14) together with the convergence of this decomposition (Theorem 2.5.5) then hold under the assumption $\lambda_0 \geq 0$ and $(H8)$. Convergence of the distribution of the superprocess near its extinction time under $\mathbb{N}_x^{(h)}$ (Proposition 2.5.9) holds under the stronger assumption $\lambda_0 > 0$.

Remark 2.7.8. Engländer and Pinsky offer in [44] a decomposition of supercritical non-homogeneous superdiffusion using immigration on the backbone formed by the prolific individuals (as denominated further in Bertoin, Fontbona and Martinez [13]). It is interesting to note that the generator of the backbone is $\mathcal{L}^w$ where w formally satifies the equation $\mathcal{L}w = \psi(w)$, whereas the generator of the spine of the Q-process investigated in Theorem 2.5.5 is $\mathcal{L}^{\phi_0}$ where ϕ_0 formally satisfies $\mathcal{L}\phi_0 = (\beta-\lambda_0)\phi_0$. In particular, we notice that the generator of the backbone $\mathcal{L}^w$ depends on both β and α and that the generator $\mathcal{L}^{\phi_0}$ of our spine does not depend on α.

3

Change of measure in the lookdown particle system

3.1 Introduction

Measure valued processes are usually defined as rescaled limit of particle systems. At the limit, the particle picture is lost. It is nevertheless often useful to keep track of the particles in the limiting process. First attempts to do that were concerned with a single particle, the most persistent one: this generated the so called spinal decompositions of superprocesses, see Roelly and Rouault [112], Evans [48] and Overbeck [102]. Second attempts deal with many particles, still the most persistent, as the infinite lineages of a supercritical superprocess, see Evans and O'Connell [50]. Most interesting is to keep track of all the particles; this can be achieved by the following trick: ordering the particles by persistence (and giving them a label called the "level" accordingly) allows one to keep a particle representation of the full system *after taking the limit*; the measure-valued process is then represented by (a multiple of) the de Finetti measure of the exchangeable sequence formed by the types of the particles. This program was realized by Donnelly and Kurtz [35] with the construction of the look-down particle system.

Our aim in this article is to explain that some transformations of the law of measure-valued processes, which belong to the class of h-transforms, admit a simple interpretation when considered from the look-down particle system point of view.

We recall Doob h-transform refers to the following operation: given a transition kernel $p_t(x, dy)$ of a Markov process and a positive space time harmonic function $H(t, y)$ for this kernel, meaning that:

$$\int H(t, y)\; p_t(x, dy) = H(0, x)$$

for every x and t, a new transition kernel is defined by $p_t(x, dy)H(t, y)/H(0, x)$, and the associated Markov process is called an h-transform. Working with measure-valued processes, we may choose $H(t, y)$ to be a linear functional of the measure y, in which case the h-transform is called additive.

Our contribution is the following: we observe that the Radon Nikodym derivative $H(t, y)/H(0, x)$ may be simply interpreted in term of the look-down particle system in the following two cases:

- For a probability measure-valued process on $\{1 \dots K'\}$ called the Λ-Fleming-Viot process without mutation, the choice $H(t, y) = \mathrm{e}^{r_K t} \prod_{i=1}^{K} y(\{i\})$ for $1 \leq K \leq K'$, y a probability measure on $\{1, \dots, K'\}$ and r_K a non negative constant chosen so that H is harmonic, amounts to allocate the first K types to the first K particles. The corresponding h-transform is the process conditioned on coexistence of the first K types in remote time, the associated look-down particle system is obtained by just "forgetting" some reproduction events in the original particle system, which may be understood as additional immigration. Thus, much as in the case of a branching population conditioned on non extinction in remote time, see the conceptual proofs of Lyons, Pemantle and Peres [97], the conditioning on non extinction of the types in a constant size population amounts to add immigration. We also take the opportunity to present an intertwining relationship for the Wright Fisher diffusion and explicit the associated pathwise decomposition. This adds another decomposition to the striking one of Swart, see [126].

– For a more general measure-valued process on a Polish space E (incorporating mutation and nonconstant population size), the choice $H(t, y) = \int y(du)h(t, u)$ for a suitable function $h(t, u) : \mathbb{R}^+ \times E \to \mathbb{R}^+$ of the underlying mutation process and y a finite measure on E, amounts to force the first level particle to move like an h-transform of the underlying spatial motion (or mutation process), and to bias the total mass process. This confirms a suggestion of Overbeck about the additive h-transform of Fleming-Viot processes, see [103] p. 183. This also relates in the branching setting to decompositions of the additive h-transforms of superprocesses found by the same author [102] using Palm measures.

Our two examples, although similar, are independent: the first one may not be reduced to the second one, and vice versa. We stress on the change of filtration technique, learnt in Hardy and Harris [63], which allows us to give simple proofs of the main results.

We first recall in Section 3.2.1 the look-down construction of [35] in the case of the Λ-Fleming-Viot process without mutation in finite state space. We look in Section 3.2.2 at the aforementioned product-type h-transform, and prove in Section 3.2.3 that it may be interpreted as the process conditioned on coexistence of some genetic types. In Section 3.2.4, we compute the generator of the conditioned process in case the finite state space is composed of only two types, and recognize it as the generator of a Λ-Fleming-Viot process with immigration. Section 3.2.4 also contains the statement and the interpretation of the intertwining relationship. Section 3.3 starts with the introduction of a look-down construction allowing for mutation and nonconstant population size (also extracted from [35]). We then present in Section 3.3.2 the additive h-transform of the associated measure-valued process. Section 3.3.3 is concerned with applications in two classical cases: Dawson Watanabe processes and Λ-Fleming-Viot processes.

3.2 A product type h-transform

3.2.1 The construction of the Λ-Fleming-Viot Process without mutation.

Donnelly and Kurtz introduced in [35] a population model evolving in continuous time. We present in this Section the particular case of the Λ-Fleming-Viot process without mutation in which we are interested. A more general framework will be introduced in Section 3.3.

We denote by $\mathcal{P}_\infty$ the space of partitions of the set of integers $\mathbb{N} = \{1, 2, 3, \ldots\}$. We assume that $c \geq 0$ and we define μ^k as the measure on $\mathcal{P}_\infty$ assigning mass one to partitions with a unique non trivial block consisting of two different integers, and call μ^k the Kingman measure. We assume that ν is a measure on $(0, 1]$ satisfying $\int_{(0,1]} x^2 \nu(dx) < \infty$. We denote by dt the Lebesgue measure on $\mathbb{R}_+$, and by ρ_x the law of the exchangeable partition of $\mathbb{N}$ with a unique non trivial block with asymptotic frequency x: If $(U_i)_{i\in\mathbb{N}}$ is a sequence of independent Bernoulli random variables with parameter x, then the partition π whose unique non trivial block contains the integers i such that $U_i = 1$ has law ρ_x. Finally, we define $N(dt, d\pi)$ the Poisson point measure on $\mathbb{R}_+ \times \mathcal{P}_\infty$ with intensity

$$dt \times \mu(d\pi) := dt \times \left(c\mu^k(d\pi) + \int_{(0,1]} \nu(dx)\rho_x(d\pi) \right).$$

Let R_0 be a random probability measure on the finite state space of the *types* $E = \{1, 2, \dots, K'\}$ for $K' \geq 2$. Assume R_0 is independent of N. Conditionally on (R_0, N), we define the look-down particle system $\xi = (\xi_t(n), t \geq 0, n \in \mathbb{N})$:

- The initial types $(\xi_0(n), n \in \mathbb{N})$ form an exchangeable sequence valued in E with de Finetti's measure R_0: Conditionally on R_0, $(\xi_0(n), n \in \mathbb{N})$ is a sequence of independent random variables with law R_0.
- At each atom (t, π) of N, we associate a *reproduction event* as follows: let $j_1 < j_2 < \dots$ be the elements of the unique block of the partition π which is not a singleton (either it is a doubleton or an infinite set). The individuals $j_1 < j_2 < \dots$ at time t are declared to be the children of the individual j_1 at time $t-$, and receive the type of the parent j_1, whereas the types of all the other individuals are shifted upwards accordingly, keeping the order they had before the birth event: for each integer ℓ, $\xi_t(j_\ell) = \xi_{t-}(j_1)$ and for each $k \notin \{j_\ell, \ell \in \mathbb{N}\}$, $\xi_t(k) = \xi_{t-}(k - \#J_k)$ with $\#J_k$ the cardinality of the set $J_k := \{\ell > 1, j_\ell \leq k\}$.
- For each $n \in \mathbb{N}$, the type $\xi_t(n)$ of the particle at level n do not evolve between reproduction events which affect level n.

Remark 3.2.1. The integrability condition $\int_{(0,1]} x^2 \nu(dx) < \infty$ ensures that finitely many reproduction events change the type of a particle at a given level in a finite time interval.

For a fixed $t \geq 0$, the sequence $(\xi_t(n), n \in \mathbb{N})$ is exchangeable according to Proposition 3.1 of [35]. This allows to define the random probability measure R_t on E as the de Finetti measure of the sequence $(\xi_t(n), n \in \mathbb{N})$:

$$R_t(dx) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi_t(n)}(dx) \tag{3.1}$$

The process $R = (R_t, t \geq 0)$ takes values in the space $\mathcal{M}_f(E)$ of finite measures on E (in fact, in the space of probability measures), and we endow $\mathcal{M}_f(E)$ with the topology of weak convergence. We shall work with the càdlàg version of the process R (such a version exists according to Theorem 3.2 of [35]). The process R is called the Λ-Fleming-Viot process without mutation. We stress that, conditionally given R_t, the random variables $(\xi_t(n), n \in \mathbb{N})$ on E are independent and identically distributed according to the probability measure R_t thanks to de Finetti's Theorem. This key fact will be used several times in the following.
We will denote by $\mathbb{P}$ the law of ξ. We now introduce the relevant filtrations we will work with:
- $(\mathcal{F}_t = \sigma\{(\xi_s(n), n \in \mathbb{N}), 0 \leq s \leq t\})$ corresponds to the filtration of the particle system.
- $(\mathcal{G}_t = \sigma\{R_s, 0 \leq s \leq t\})$ corresponds to the filtration of the measure-valued process R.

Notice that ξ is a Markov process with respect to the filtration $\mathcal{F}$, and that R is a Markov process with respect to the filtration $\mathcal{G}$.

3.2.2 A pathwise construction of an h-transform

Results

The proofs of the results enounced here may be found in the next Subsection. Fix $1 \leq K \leq K'$. We assume from now on and until the end of Section 3.2 that:

$$\mathbb{E}(\prod_{i=1}^{K} R_0\{i\}) > 0, \tag{3.2}$$

to avoid empty definitions in the following. Recall the definition of the particle system ξ associated with R. We define from ξ a new particle system ξ^∞ as follows:

(i) The finite sequence $(\xi_0^\infty(j), 1 \le j \le K)$ is a uniform permutation of $\{1, \dots, K\}$, and, independently, the sequence $(\xi_0^\infty(j), j \ge K+1)$ is exchangeable with asymptotic frequencies R_0^H, where R_0^H is the random probability measure with law:

$$\mathbb{P}(R_0^H \in A) = \mathbb{E}\left(\mathbf{1}_A(R_0) \frac{\prod_{i=1}^K R_0\{i\}}{\mathbb{E}(\prod_{i=1}^K R_0\{i\})}\right).$$

(ii) The reproduction events are given by the *restriction* of the Poisson point measure N (defined as in Subsection 3.2.1) to $V := \Big\{(s,\pi), \pi_{|[K]} = \{\{1\},\{2\},\dots,\{K\}\}\Big\}$, where $\pi_{|[K]}$ is the restriction of the partition π of $\mathbb{N}$ to $\{1,\dots,K\}$, that is the atoms of N for which the reproductions events do not involve more than one of the first K levels.

Remark 3.2.1 ensures that this definition of the particle system ξ^∞ makes sense.

Remark 3.2.2. Note that the particle system $(\xi_0^\infty(j), j \ge 1)$ is no more exchangeable due to the constraint on the K initial levels. Nevertheless, the particle system $(\xi_0^\infty(j), j > K)$ is still exchangeable, and we shall view the first K levels as K independent sources of immigration. This approach will be used in Section 3.2.4.

We also need the definition of the first level $L(t)$ at which the first K types appear:

$$L(t) = \inf\{i \ge K, \{1,\dots,K\} \subset \{\xi_t(1),\dots,\xi_t(i)\}\}, \tag{3.3}$$

with the convention that $\inf\{\varnothing\} = \infty$. The random variable $L(0)$ is finite if and only if $\prod_{i=1}^K R_0\{i\} > 0$, $\mathbb{P}$-a.s., thanks to de Finetti's Theorem. The process $L(t)$ is $\mathcal{F}_t$ measurable, but not $\mathcal{G}_t$ measurable. Notice the random variable $L(t)$ is an instance of the coupon collector problem, based here on a random probability measure R: how many levels do we need to check for seeing the first K types? We define, for $i \ge 1$, the *pushing rates* r_i at level i:

$$r_i = \frac{i(i-1)}{2}\, c + \int_{(0,1]} \nu(dx) \left(1 - (1-x)^i - ix(1-x)^{i-1}\right).$$

Notice that $r_1 = 0$ and that r_i is finite for every $i \ge 1$ since $\int_{(0,1]} x^2\ \nu(dx) < \infty$. From the construction of the look-down particle system, these pushing rates r_i may be understood as the rate at which a type at level i is pushed up to higher levels (not necessarily $i+1$) by reproduction events at lower levels. Let us define a process $Q = (Q_t, t \ge 0)$ as follows:

$$Q_t = \frac{\mathbf{1}_{\{L(t)=K\}}}{\mathbb{P}(L(0)=K)}\, \mathrm{e}^{r_K t}.$$

Lemma 3.2.3. *The process $Q = (Q_t, t \ge 0)$ is a non negative $\mathcal{F}$-martingale, and*

$$\forall A \in \mathcal{F}_t,\ \mathbb{P}(\xi^\infty \in A) = \mathbb{E}\left(\mathbf{1}_A(\xi)\ Q_t\right). \tag{3.4}$$

We need the following definition of the process:

$$M_t = \frac{\prod_{i=1}^K R_t\{i\}}{\mathbb{E}(\prod_{i=1}^K R_0\{i\})}\, \mathrm{e}^{r_K t}.$$

By projection on the smaller filtration $\mathcal{G}_t$, we deduce Lemma 3.2.4.

Lemma 3.2.4. *The process $M = (M_t, t \geq 0)$ is a non negative $\mathcal{G}$-martingale.*

This fact allows to define the process $R^H = (R_t^H, t \geq 0)$ absolutely continuous with respect to $R = (R_t, t \geq 0)$ on each $\mathcal{G}_t$, $t \geq 0$, with Radon Nykodim derivative:

$$\forall A \in \mathcal{G}_t,\ \mathbb{P}(R^H \in A) = \mathbb{E}\left(\mathbf{1}_A(R)\ M_t\right). \tag{3.5}$$

The process R^H is the product type h-transform of interest. Intuitively, the ponderation by M favours the paths in which the first K types are present in equal proportion. Also notice that equation (3.5) agrees with the definition of R_0^H. We shall deduce from Lemma 3.2.3 and Lemma 3.2.4 the following Theorem, which gives the pathwise construction of the h-transform R^H of R.

Theorem 3.2.5. *Let $1 \leq K \leq K'$. We have that:*
(a) The limit of the empirical measure:

$$R_t^\infty(dx) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi_t^\infty(n)}(dx)$$

exists a.s.
(b) The process $(R_t^\infty, t \geq 0)$ is distributed as $(R_t^H, t \geq 0)$.

Let us comment on these results. The process ξ^∞ is constructed by changing the initial condition and forgetting (as soon as $K \geq 2$) specific reproduction events in the look-down particle system of ξ. Lemma 3.2.3 tells us that this procedure selects the configurations of ξ in which the first K levels are filled with the first K types at initial time without any "interaction" between these first K levels at a further time. Theorem 3.2.5 tells us that the process R^∞ constructed in this way is an h-transform of R and Lemma 3.2.4 yields the following simple probabilistic interpretation of the Radon Nikodym derivative in equation (3.5): the numerator is proportional to the probability that the first K levels are occupied by the first K types at time t, whereas the denominator is proportional to the probability that the first K levels are occupied by the first K types at time 0. We shall see in Section 3.2.3 that the processes ξ^∞ and R^∞ also arise by conditioning the processes ξ and R on coexistence of the first K types.

Proofs

Proof of Lemma 3.2.3. From the de Finetti theorem, conditionally on R_t, the random variables $(\xi_t(i), i \in \mathbb{N})$ are independent and identically distributed according to R_t. This implies that:

$$\mathbb{P}(L(t) = K|\mathcal{G}_t) = K!\ \prod_{i=1}^{K} R_t\{i\}. \tag{3.6}$$

In particular, we have:

$$\mathbb{P}(L(0) = K) = K!\ \mathbb{E}(\prod_{i=1}^{K} R_0\{i\}),$$

which, together with (3.2), ensures that Q_t is well defined.

Then, let us define $W = \{\pi, \pi_{|[K]} = \{\{1\}, \{2\}, \ldots, \{K\}\}\}$, and $V_t = \{(s,\pi), 0 \leq s \leq t, \pi \in W\}$, and also the set difference $W^c = \mathcal{P}_\infty \setminus W$ and $V_t^c = \{(s,\pi), 0 \leq s \leq t, \pi \in W^c\}$. We observe that:

– From the de Finetti Theorem, the law of ξ_0^∞, as defined in (i), is that of ξ_0 conditioned on $\{L(0) = K\}$.
– The law of the restriction of a Poisson point measure on a given subset is that of a Poisson point measure conditioned on having no atoms outside this subset: thus N conditioned on having no atoms in V_t^c (this event has positive probability) is the restriction of N to V_t.

Since the two conditionings are independent, we have, for $A \in \mathcal{F}_t$:

$$\begin{aligned}\mathbb{P}(\xi^\infty \in A) &= \mathbb{P}(\xi \in A | \{L(0) = K\} \cap \{N(V_t^c) = 0\}) \\ &= \mathbb{E}\left(\mathbf{1}_A(\xi) \frac{\mathbf{1}_{\{L(0)=K\}\cap\{N(V_t^c)=0\}}}{\mathbb{P}(L(0) = K)\mathbb{P}(N(V_t^c) = 0)}\right)\end{aligned} \tag{3.7}$$

We compute:

$$\begin{aligned}\mu(W^c) &= c\mu^k(W^c) + \int_{(0,1]} \nu(dx)\rho_x(W^c) \\ &= \frac{K(K-1)}{2}\, c \;+\; \int_{(0,1]} \nu(dx) \left(1 - (1-x)^K - Kx(1-x)^{K-1}\right) \\ &= r_K.\end{aligned}$$

This implies from the construction of N that:

$$\mathbb{P}(N(V_t^c) = 0) = \mathrm{e}^{-\mu(W^c)t} = e^{-r_K t}. \tag{3.8}$$

Notice that

$$\{L(t) = K\} = \{L(0) = K\} \cap \{N(V_t^c) = 0\}. \tag{3.9}$$

From (3.7), (3.8) and (3.9), we deduce that:

$$\mathbb{P}(\xi^\infty \in A) = \mathbb{E}\left(\mathbf{1}_A(\xi) \frac{\mathbf{1}_{\{L(t)=K\}}}{\mathbb{P}(L(0) = K)}\, \mathrm{e}^{r_K t}\right) = \mathbb{E}\left(\mathbf{1}_A(\xi) Q_t\right).$$

Observe now that A also belongs to $\mathcal{F}_s$ as soon as $s \geq t$, which yields:

$$\mathbb{P}(\xi^\infty \in A) = \mathbb{E}(\mathbf{1}_A(\xi) Q_s).$$

Comparing the two last equalities ensures that $(Q_t, t \geq 0)$ is a $\mathcal{F}$-martingale. □

Proof of Lemma 3.2.4. We know from Lemma 3.2.3 that $(Q_t, t \geq 0)$ is a $\mathcal{F}$-martingale. Since $\mathcal{G}_t \subset \mathcal{F}_t$ for every $t \geq 0$, we deduce that $(\mathbb{E}(Q_t|\mathcal{G}_t), t \geq 0)$ is a $\mathcal{G}$-martingale. But

$$\mathbb{E}(Q_t|\mathcal{G}_t) = \mathbb{E}\left(\frac{\mathbf{1}_{\{L(t)=K\}}}{\mathbb{P}(L(0) = K)}\, \mathrm{e}^{r_K t} |\mathcal{G}_t\right) = \frac{\prod_{i=1}^K R_t\{i\}}{\mathbb{E}(\prod_{i=1}^K R_0\{i\})}\, \mathrm{e}^{r_K t} = M_t,$$

using (3.6) for the second equality, so that $(M_t, t \geq 0)$ is a $\mathcal{G}$-martingale.

□

Proof of Theorem 3.2.5. From Lemma 3.2.3, ξ^∞ is absolutely continuous with respect to ξ on $\mathcal{F}_t$. The existence of the almost sure limit of the empirical measure claimed in point (a) follows from (3.1). We now project on $\mathcal{G}_t$ the absolute continuity relationship on $\mathcal{F}_t$ given in Lemma 3.2.4. Let $A \in \mathcal{G}_t$:

$$\mathbb{P}(R^\infty \in A) = \mathbb{E}\left(\mathbf{1}_A(R) Q_t\right) = \mathbb{E}\left(\mathbf{1}_A(R)\mathbb{E}(Q_t|\mathcal{G}_t)\right) = \mathbb{E}\left(\mathbf{1}_A(R) M_t\right) = \mathbb{P}(R^H \in A),$$

where we use Lemma 3.2.3 for the first equality and the definition of R^H for the last equality. This proves point (b). □

3.2.3 The h-transform as a conditioned process

We gave a pathwise construction of the h-transform R^H in the previous Subsection. We now study the conditioning associated with this h-transform.

Let $1 \leq K \leq K'$. Assumption (3.2) allows us to define a family of processes $R^{(\geq t)}$ on $\mathcal{G}$ by:

$$\forall A \in \mathcal{G}_t,\ \mathbb{P}(R^{(\geq t)} \in A) = \mathbb{P}(R \in A | \prod_{i=1}^{K} R_t\{i\} \neq 0),$$

and the associated particle system $\xi^{(\geq t)}$ on $\mathcal{F}$ by:

$$\forall A \in \mathcal{F}_t,\ \mathbb{P}(\xi^{(\geq t)} \in A) = \mathbb{P}(\xi \in A | \prod_{i=1}^{K} R_t\{i\} \neq 0).$$

The process $R^{(\geq t)}$ thus corresponds to the process R conditioned on coexistence of the *first* K types at time t. It is not easy to derive the probabilistic structure of the particle system $\xi^{(\geq t)}$ on all $\mathcal{F}_t$. Nevertheless, for fixed $s \geq 0$, the probabilistic structure of $\xi^{(\geq t)}$ on the sigma algebra $\mathcal{F}_s$ simplifies as t goes to infinity, as shown by the following Theorem, which may be seen as a generalization of Theorem 3.7.1.1 of Lambert [86]. The latter Theorem builds on the work of Kimura [73] and corresponds to the case $\nu = 0$. We need some notations: We write $\mathbb{P}_i$ for the law of L (defined in (3.3)) conditionally on $\{L(0) = i\}$. For I an interval of $\mathbb{R}^+$ and F a process indexed by $\mathbb{R}^+$, we denote by F_I the restriction of F on the interval I.

Theorem 3.2.6. *Let $s \geq 0$ be fixed. Assume that*

$$\lim_{t\to\infty} \frac{\mathbb{P}_{K+1}(L(t) < \infty)}{\mathbb{P}_K(L(t) < \infty)} = 0. \tag{3.10}$$

Then:
(i) The family of processes $\xi^{(\geq t)}_{[0,s]}$ weakly converges as $t \to \infty$ towards the process $\xi^{\infty}_{[0,s]}$.
(ii) The family of processes $R^{(\geq t)}_{[0,s]}$ weakly converges as $t \to \infty$ towards the process $R^{\infty}_{[0,s]}$.

We refer to Lemma 3.2.10 for a sufficient condition for (3.10) to be satisfied, and notice that the case $K = 1$ corresponds to a non degenerate conditioning since the event $\{R_t\{1\} \neq 0 \text{ for every } t\}$ has positive probability under (3.2).

Remark 3.2.7. Assume $K \geq 2$. The following property

$$\text{(CDI)} \qquad \mathbb{P}(\inf\{t > 0, L(t) = \infty\} < \infty) = 1,$$

is independent of K used to define L in (3.3). (CDI) property corresponds to the Coming Down from Infinity property for the Λ-coalescent associated with the Λ-Fleming-Viot process R, whence the acronym (CDI).

The key points to see this connection are:
- the fact that $L(0)$ is an upper bound on the number of blocks in the standard coalescent started at any time greater than $\inf\{t > 0, L(t) = \infty\}$.
- the $0 - 1$ law of Pitman, according to which the number of blocks in a standard Λ-coalescent either stays infinite at each nonnegative time $t \geq 0$ with probability 1, either is finite at each positive time $t > 0$ with probability 1.

Remark 3.2.8. It should still be possible to interpret the processes ξ^∞ and R^∞ as conditioned processes, without assuming (3.10). In that more general case, ξ^∞ corresponds intuitively to ξ conditioned by the event $\{\limsup_{t\to\infty}\prod_{i=1}^K R_t\{i\} > 0\}$ (which has null probability as soon as $K \geq 2$).

Proof. First observation is that, from the Kingman's paintbox construction for exchangeable random partition, we have: $\prod_{i=1}^K R_t\{i\} \neq 0$ if and only if $L(t) < \infty$, $\mathbb{P}$ a.s. This gives, for any $A \in \mathcal{F}_s$:

$$\mathbb{P}\Big(A| \prod_{i=1}^K R_t\{i\} \neq 0\Big) = \frac{\mathbb{P}\left(A \cap \{\prod_{i=1}^K R_t\{i\} \neq 0\}\right)}{\mathbb{P}(\prod_{i=1}^K R_t\{i\} \neq 0)}$$
$$= \frac{\mathbb{P}\left(A \cap \{L(t) < \infty\}\right)}{\mathbb{P}(L(t) < \infty)}.$$

Now, using the Markov property, we have:

$$\begin{aligned}
&\mathbb{P}(A \cap \{L(t) < \infty\}) \\
&= \mathbb{P}(A \cap \{L(s) = K\} \cap \{L(t) < \infty\}) + \mathbb{P}(A \cap \{L(s) \geq K+1\} \cap \{L(t) < \infty\}) \\
&= \mathbb{P}(A \cap \{L(s) = K\})\mathbb{P}_K(L(t-s) < \infty) + \mathbb{E}(\mathbf{1}_{A\cap\{L(s)\geq K+1\}}\, \mathbb{P}_{L(s)}(\tilde{L}(t-s) < \infty)),
\end{aligned}$$

where $\tilde{L}$ is an independent copy of L.

Let $\ell \in \mathbb{N}$. We can couple the processes L under $\mathbb{P}_\ell$ and L under $\mathbb{P}_{\ell+1}$ on the same look-down graph by using the same reproduction events. More precisely, imagine that we distinguish the particles at level ℓ and $\ell+1$ at initial time, giving each of them a special type shared by no other particles. Then the first two levels $L_\ell(t)$ and $L_{\ell+1}(t)$ at which these two types may be found at time t yield the required coupling, in that: L_ℓ is distributed as L under $\mathbb{P}_\ell$ and $L_{\ell+1}$ is distributed as L under $\mathbb{P}_{\ell+1}$. By the ordering by persistence property of the look-down graph, we have that, for every $t \geq 0$:

$$L_\ell(t) \leq L_{\ell+1}(t),$$

whence:

$$\mathbb{P}_{\ell+1}(L(t) < \infty) \leq \mathbb{P}_\ell(L(t) < \infty) \tag{3.11}$$

for every integer ℓ. Therefore, we have:

$$\mathbb{E}(\mathbf{1}_{A\cap\{L(s)\geq K+1\}}\mathbb{P}_{L(s)}(\tilde{L}(t-s) < \infty)) \leq \mathbb{P}(A \cap \{L(s) \geq K+1\})\, \mathbb{P}_{K+1}(L(t-s) < \infty).$$

Our assumption (3.10) now implies:

$$\frac{\mathbb{P}(A \cap \{L(t) < \infty\})}{\mathbb{P}_K(L(t-s) < \infty)} \underset{t\to\infty}{\to} \mathbb{P}(A \cap \{L(s) = K\}).$$

Setting $A = \Omega$, this also yields:

$$\frac{\mathbb{P}(L(t) < \infty)}{\mathbb{P}_K(L(t-s) < \infty)} \underset{t\to\infty}{\to} \mathbb{P}(L(s) = K).$$

Taking the ratio, we find that:

$$\frac{\mathbb{P}\left(A \cap \{L(t) < \infty\}\right)}{\mathbb{P}(L(t) < \infty)} \underset{t\to\infty}{\to} \frac{\mathbb{P}(A \cap \{L(s) = K\})}{\mathbb{P}(L(s) = K)}.$$

We also have that $\mathbb{P}(L(s) = K) = \mathbb{P}(L(0) = K)\,\mathrm{e}^{-r_K s}$ since Q is a $\mathcal{G}$-martingale from Lemma 3.2.3. Altogether, we find that:

$$\lim_{t\to\infty} \mathbb{P}(A|\prod_{i=1}^{K} R_t\{i\} \neq 0\}) = \mathbb{E}\left(\mathbf{1}_A(\xi)\frac{\mathbf{1}_{\{L(s)=K\}}}{\mathbb{P}(L(0) = K)}\,\mathrm{e}^{r_K s}\right) = \mathbb{P}(\xi^\infty \in A)$$

where the last equality corresponds to Lemma 3.2.3. This implies the convergence in law of $\xi^{(\geq t)}$ towards ξ^∞ as $t \to \infty$, and proves (i). The proof of (ii) is similar to the one for (i). □

Remark 3.2.9. Having introduced in the previous proof the coupling (L_K, L_{K+1}), we may complete the Remark 3.2.7: It is possible to prove that, if (CDI) holds and for each $t \geq 0$,

$$(j \to \mathbb{P}(L_{K+1}(t) < \infty | L_K(t) \leq j)) \text{ is non increasing}$$

then (3.10) holds.

We now give a sufficient condition for (3.10) to be satisfied.

Lemma 3.2.10. *If $\sum_{j\geq K} \frac{1}{r_j} < \infty$, then (3.10) holds.*

Proof. A lower bound for $\mathbb{P}_K(L(t) < \infty)$ is easily found:

$$\mathrm{e}^{-r_K t} = \mathbb{P}_K(L(t) = K) \leq \mathbb{P}_K(L(t) < \infty). \tag{3.12}$$

We now look for an upper bound for $\mathbb{P}_{K+1}(L(t) < \infty)$. Recall the non decreasing pure jump process L jumps with intensity r_j when $L = j$.

We may write, under $\mathbb{P}_{K+1}$:

$$\sup\{t, L(t) < \infty\} = \sum_{j\geq K+1} \tilde{T}_j$$

where, conditionally given the range $\{L(t), t \geq 0\} = \{L^{K+1}, L^{K+2}, \ldots\}$ of the random function L, the sequence $(\tilde{T}_j, j \geq K+1)$ is a sequence of independent exponential random variable with parameter r_{L^j}. Since $(r_j)_{j\geq K+1}$ forms an increasing sequence and the function L has jumps greater than or equal to one, we have for each $j \geq K+1$,

$$r_{L^j} \geq r_j. \tag{3.13}$$

Let $(T_j, j \geq K+1)$ be a sequence of independent exponential random variables with parameter $(r_j, j \geq K)$. We compute, for $0 < \lambda < r_{K+1}$:

$$\begin{aligned}
\mathbb{P}_{K+1}(L(t)<\infty) &= \mathbb{P}(\sum_{j\geq K+1} \tilde{T}_j > t) \\
&\leq \mathbb{P}(\sum_{j\geq K+1} T_j > t) \\
&= \mathbb{P}(\exp(\lambda \sum_{j\geq K+1} T_j) > \exp(\lambda t)) \\
&\leq \exp(-\lambda t)\, \mathbb{E}(\exp(\lambda \sum_{j\geq K+1} T_j)) \\
&= \exp(-\lambda t) \prod_{j\geq K+1} \frac{r_j}{r_j-\lambda} \\
&= \exp\big(-\lambda t + \sum_{j\geq K+1} \log(1+\frac{\lambda}{r_j-\lambda})\big) \\
&\leq \exp\big(-\lambda t + \lambda \sum_{j\geq K+1} \frac{1}{r_j-\lambda}\big),
\end{aligned}$$

where we use (3.13) for the first inequality and the Markov inequality for the second inequality. From the assumption, $\sum_{j\geq K+1} 1/r_j$ is finite, which implies also that $\sum_{j\geq K+1} 1/(r_j-\lambda)$ is finite. Taking $\lambda = (r_K + r_{K+1})/2$, we obtain that:

$$\mathbb{P}_{K+1}(L(t)<\infty) < C \exp\left(-\frac{r_K + r_{K+1}}{2} t\right) \tag{3.14}$$

for the finite constant $C = \exp \lambda \sum_{j\geq K} 1/(r_j-\lambda)$ associated with this choice of λ. Using (3.12) and (3.14), we have that:

$$0 \leq \frac{\mathbb{P}_{K+1}(L(t)<\infty)}{\mathbb{P}_K(L(t)<\infty)} \leq C \exp\left(-\frac{r_{K+1} - r_K}{2} t\right).$$

Letting t tend to ∞, we get the required limit. □

The following Corollary ensures that (3.10) is satisfied in the most interesting cases.

Corollary 3.2.11. *If $c>0$, or $c=0$ and there exists $\alpha \in (1,2)$ such that $\nu(dx) = f(x)dx$ with $\liminf_{x\to 0} f(x)x^{\alpha+1} > 0$, then (3.10) holds.*

Remark 3.2.12. Notice that, for $1<\alpha<2$, the Beta$(2-\alpha,\alpha)$-Fleming-Viot, associated with $\nu(dx) = x^{-1-\alpha}(1-x)^{\alpha-1}\mathbf{1}_{(0,1)}(x)dx$, satifies this assumption.

Proof. If $c>0$, $r_j \geq cj(j-1)/2$, and thus $\sum_{j\geq K} 1/r_j < \infty$. Assume now $c=0$ and $\liminf_{x\to 0} f(x)x^{\alpha+1} > 0$ for some $1<\alpha<2$. From Lemma 2 of Limic and Sturm [96], we have the equality:

$$r_{j+1} - r_j = \int_{(0,1]} j(1-x)^{j-1} x^2 \nu(dx)$$

We deduce that there exists an integer n, and a positive constant C such that:

$$r_{j+1}-r_j > C\int_{(0,1/n]} j(1-x)^{j-1}x^{1-\alpha}dx \geq \frac{C}{n}\int_{(0,1]} j(1-x)^{j-1}x^{1-\alpha}dx = \frac{C}{n}\, j \ \mathrm{Beta}(2-\alpha, j+\alpha-1),$$

using the definition of lim inf at the first inequality, and the fact that the map $x \mapsto (1-x)^{j-1}x^{1-\alpha}$ is non-increasing at the second inequality. Let us define a sequence $(s_j, j \geq K)$ by:

$$s_K = 0 \text{ and } s_{j+1} - s_j = \frac{C}{n}\, j \text{ Beta}(2-\alpha, j+\alpha-1) \text{ for } j \geq K.$$

Since:

$$\text{Beta}(2-\alpha, j+\alpha-1) \underset{j\to\infty}{\sim} \Gamma(2-\alpha) j^{\alpha-2},$$

we deduce that:

$$s_j \underset{j\to\infty}{\sim} \frac{C}{n}\Gamma(2-\alpha) j^{\alpha}/\alpha.$$

By definition of the sequence $(s_j)_{j\geq K}$, we have the inequality $r_j \geq s_j$ for $j \geq K$, and we deduce that

$$\sum_{j\geq K} 1/r_j \leq \sum_{j \geq K} 1/s_j < \infty.$$

Lemma 3.2.10 allows to conclude that (1.33) holds in both cases. □

3.2.4 The immigration interpretation

We develop further the two following examples:

(i) $K = K' = 2$: this amounts (provided condition (3.10) is satisfied) on conditioning a two-type Λ-Fleming-Viot process on coexistence of each type.

(ii) $1 = K < K' = 2$: this amounts (provided (3.10) is satisfied) on conditioning a two-type Λ-Fleming-Viot process on absorbtion by the first type.

We regard the $K(= 1 \text{ or } 2)$ first level particles in ξ^∞ as K external sources of immigration in the population now assimilated to the particle system $(\xi^\infty(n), n \geq K+1)$ and decompose the generator of the process R^∞ accordingly. We refer to Foucart [54] for a study of Λ-Fleming-Viot processes with one source of immigration ($K = 1$ here).

Since $K' = 2$, the resulting probability measure-valued process $R = (R_t, t \geq 0)$ and $R^\infty = (R_t^\infty, t \geq 0)$ on $\{1,2\}$ may be simply described by the $[0,1]$-valued processes $R\{1\} = (R_t\{1\}, t \geq 0)$ and $R^h\{1\} = (R_t^\infty\{1\}, t \geq 0)$ respectively. For the sake of simplicity, we will just write R for $R\{1\}$ and R^∞ for $R^\infty\{1\}$ respectively. We recall that the infinitesimal generator of R is given by:

$$Gf(x) = \frac{1}{2}cx(1-x)f''(x) + x\int_{(0,1]} \nu(dy)[f(x(1-y)+y) - f(x)] + (1-x)\int_{(0,1]} \nu(dy)[f(x(1-y)) - f(x)]$$

for all $f \in \mathcal{C}^2([0,1])$, the space of twice differentiable functions with continuous derivatives, and $x \in [0,1]$, see Bertoin and Le Gall [15].

We assume $K = K' = 2$

We define, for $f \in \mathcal{C}^2([0,1])$, and $x \in [0,1]$:

$$G^0 f(x) = c(1-2x)f'(x) + \int_{(0,1]} y(1-y)\nu(dy)[f(x(1-y)+y) - f(x)] + \int_{(0,1]} y(1-y)\nu(dy)[f(x(1-y)) - f(x)],$$

and

$$G^1 f(x) = \frac{1}{2}cx(1-x)f''(x) + x\int_{(0,1]}(1-y)^2\nu(dy)[f(x(1-y)+y)) - f(x)] \\ +(1-x)\int_{(0,1]}(1-y)^2\nu(dy)[f(x(1-y)) - f(x)].$$

Proposition 3.1. *Assume $K = K' = 2$. The operator $G^0 + G^1$ is a generator for R^∞.*

Remark 3.2.13. When the measure ν is null, the process R is called a Wright Fisher diffusion (WF diffusion in the following). In that case, the process R^∞ may be seen as a WF diffusion with immigration, where the two first level particles induce continuous immigration (according to G^0) of both types 1 and 2 in the original population (which evolves according to $G^1 = G$ in that case).

When the measure ν is not null, the process R^∞ is a Λ-Fleming-Viot process with immigration, but the generator G^1 is no more that of the initial Λ-Fleming-Viot process G: the two first level particles induce both continuous and discontinuous immigration (according to G^0) of types 1 and 2 in a population with a reduced reproduction (the measure $\nu(dy)$ is ponderated by a factor $(1-y)^2 \leq 1$ in G^1).

Proof. Let us denote by G^∞ the generator of R^∞. The process R^h is the Doob h-transform of R for the following function H:

$$H(t,x) = x(1-x)\,\mathrm{e}^{r_2 t}\,.$$

Since $(H(t,R_t), t \geq 0)$ is a $\mathcal{G}$-martingale according to Lemma 3.2.4, the nonnegative function H is (by definition) space time harmonic. From the definition of the generator, for $f \in \mathcal{C}^2([0,1])$, and $x \in [0,1]$:

$$f(R_t)H(t,R_t) - f(R_0)H(0,R_0) - \int_0^t ds\, G(H(s,.)f)(R_s) - \int_0^t ds\, \partial_t H(.,R_s)(s) f(R_s)$$

is $\mathcal{G}$ martingale, where in the first integrand G acts on $x \to f(x)H(s,x)$. Therefore, on $\{H(0,R_0) \neq 0\}$, the process

$$\frac{H(t,R_t)}{H(0,R_0)}f(R_t) - f(R_0) - \int_0^t ds\, \frac{H(s,R_s)}{H(0,R_0)}\frac{G(H(s,.)f)(R_s)}{H(s,R_s)} - \int_0^t ds\, \frac{H(s,R_s)}{H(0,R_0)}\frac{\partial_t H(s,R_s)}{H(s,R_s)}f(R_s)$$

is again $\mathcal{G}$ martingale under $\mathbb{P}$. This implies that:

$$f(R_t^\infty) - f(R_0^\infty) - \int_0^t ds\, \frac{G(H(s,.)f)(R_s^\infty)}{H(s,R_s^\infty)} - \int_0^t ds\, \frac{\partial_t H(s,R_s^\infty)}{H(s,R_s^\infty)}f(R_s^\infty)$$

is a $\mathcal{G}$ martingale under $\mathbb{P}$. We thus have:

$$G^\infty f(x) = \left(\frac{G(H(t,.)f) + \partial_t H(t,.)f}{H(t,.)}\right)(x) \\ = \frac{G(H(t,.)f)}{H(t,.)}(x) + r_2 f(x), \tag{3.15}$$

A simple computation ensures that this last expression does not depend on t and completes the proof of the proposition. □

Using the particle system ξ^∞, we also have the following intuitive interpretation of the generator G^∞ in the case of a pure jump Λ-Fleming-Viot process ($c = 0$). Let us decompose the measure ν as follows:

$$\nu(dy) = 2y(1-y)\nu(dy) + (1-y)^2\nu(dy) + y^2\nu(dy).$$

1. The first term is the sum of the two measures $y(1-y)\nu(dy)$ appearing in each integrand of the generator G^0 and each of these measures corresponds to the intensity of the reproduction events involving level 1 and not level 2, or level 2 and not level 1 (these events have probability $y(1-y)$ when the reproduction involves a fraction y of the population). We interpret them as immigration events.
2. The second term is the measure $(1-y)^2\nu(dy)$ appearing in the generator G^1 and corresponds to the intensity of the reproduction events involving neither level 1 nor level 2 (this event has probability $(1-y)^2$ when the reproduction involves a fraction y of the population). We interpret them as reproduction events.
3. The third term does not appear in the generators G^0 and G^1: it corresponds to the intensity of the reproduction events involving both level 1 and 2, and these events have been discarded in the construction of ξ^∞.

We assume $K = 1, K' = 2$

Note that the case $K = 1$ differs from the case $K = 2$, since the event $\{R_t \neq 0 \text{ for every } t\}$ has positive probability under (3.2). Let us define, for $f \in \mathcal{C}^2([0,1])$, and $x \in [0,1]$:

$$I^0 f(x) = c(1-x)f'(x) + \int_{(0,1]} y\nu(dy)[f(x(1-y)+y) - f(x)]$$

and

$$I^1 f(x) = \frac{1}{2}cx(1-x)f''(x) + x\int_{(0,1]}(1-y)\nu(dy)[f(x(1-y)+y)) - f(x)] \\ +(1-x)\int_{(0,1]}(1-y)\nu(dy)[f(x(1-y)) - f(x)].$$

We can then prove the analog of Proposition 3.1 in that setting.

Proposition 3.2. *Assume $K = 1, K' = 2$. The operator $I^0 + I^1$ is a generator for the Markov process R^∞.*

In particular, we recover the well known fact that a WF diffusion conditioned on fixation at 1 (that is, $R_t = 1$ for t large enough) may be viewed as a WF process with immigration, see [39] for instance.

Proof. The proof is similar to that of Proposition 3.1. Here we use an h-transform with the function

$$H(t,x) = x.$$

This function is space time harmonic according to Lemma 3.2.4 (recall $r_1 = 0$). □

Here again, we have the following intuitive interpretation of the generator $I^0 + I^1$ in the case $c = 0$. We decompose the measure ν as follows:

$$\nu(dy) = y\nu(dy) + (1-y)\nu(dy).$$

1. The first term is the measure $y\nu(dy)$ appearing in the generator I^0. This is the intensity of the reproduction events involving level 1 particle. We interpret them as immigration events.
2. The second term is the measure $(1-y)\nu(dy)$ appearing in the generator I^1. This is the intensity of the reproduction events not involving level 1 particle. We interpret them as reproduction events.
3. Summing the two measures $y\nu(dy)$ and $(1-y)\nu(dy)$, we recover this time the full measure $\nu(dy)$ since no reproduction events are discarded in the case $K = 1$.

Intertwining

Let us recall the following piece of intertwining theory. Given a Markov process (A, B), or more precisely its generator, we ask wether A is a Markov process on its own and, in that case, what is his generator. The following Theorem, due to Rogers and Pitman [113], answers by the affirmative under an algebraic relationship (3.16), that we shall call the intertwining relationship.

Theorem 3.2.14. *Let $((A_t, B_t), t \geq 0)$ be a Markov process with state space $S \times T$ and with generator $\hat{G}$, let K be a probability kernel from S to T. Define the operator $\hat{K}$ by*

$$\hat{K}f(x) = \sum_{y \in T} K(x, y) f(x, y).$$

Let G be the generator of a Markov process in S and assume that, for each $f : S \times T \to S$,

$$\hat{K}\hat{G}(f)(x) = G\hat{K}(f)(x),\ x \in S. \tag{3.16}$$

Then:

$$\mathbb{P}(B_0 = y | A_0) = K(A_0, y) \quad a.s.$$

implies that for each $t \geq 0$

$$\mathbb{P}(B_t = y | (A_s, 0 \leq s \leq t)) = K(A_t, y) \quad a.s.$$

and $(A_t, t \geq 0)$ is, on its own, a Markov process on S with generator G.

In this Subsection, we shall prove the intertwining relationship is satisfied with (A, B) where B is a process related to L, and A is a Wright-Fisher diffusion with immigration driven by B. This will add another path decomposition for the Wright-Fisher diffusion to the striking one of Swart, see [126], which does not seem to admit a clear interpretation from the look-down particle system.

We assume $K' = 2$ and $\nu = 0$ (for the sake of simplicity). In fact, we find it more convenient to work with L^1 rather than L, where:

$$L^1(t) = \inf \{i \geq 1, 1 \in \{\xi_t(1), \dots, \xi_t(i)\}\}$$

is the first level occupied by a type 1 particle. The process L^1 is a Markov process valued in $\mathbb{N} \cup \{\infty\}$, and jumps by 1 at rate $c\ell(\ell-1)/2$ when at $\ell \in \mathbb{N}$, and has ∞ as an absorbing point. Notice that also 1 is an absorbing point for L^1. In fact, the process R^h studied in Subsection 3.2.4 is the process R conditioned on $\{L^1 = 1\}$.

Let us define the following kernel:

$$K(x,\ell) = (1-x)^{\ell-1}x, \quad x \in (0,1], \ell \in \mathbb{N}.$$

acting on function $f(x,\ell)$ as follows:

$$\hat{K}f(x) = \sum_{\ell \geq 1} K(x,\ell) f(x,\ell).$$

We slightly abuse of notation, still denoting by G the generator of the Wright-Fisher diffusion:

$$Gf(x) = \frac{1}{2}cx(1-x)f''(x)$$

acting on $f \in \mathcal{C}^2([0,1])$. We denote by $\hat{G}$ the generator defined for $\ell \in \mathbb{N}$ and $x \in (0,1]$ by:

$$\hat{G}f(x,\ell) = \frac{1}{2}cx(1-x)\partial_{xx}f(x,\ell) + c\left[(1-x)-(\ell-1)x\right]\partial_x f(x,\ell) + c\frac{\ell(\ell-1)}{2}\left[f(x,\ell+1)-f(x,\ell)\right].$$

This generator acts on functions f such that f, as a function of x, belongs to $\mathcal{C}^2([0,1])$. The intertwining relationship reads as follows.

Proposition 3.3. *Let f be in the domain of $\hat{G}$ and $x \in (0,1]$. The kernel $\hat{K}$ intertwins the generators G and $\hat{G}$ in the sense that:*

$$\hat{K}\hat{G}(f)(x) = G\hat{K}(f)(x).$$

The proof consists in a long but simple calculation and is eluded. A similar intertwining relation also holds for $\nu \neq 0$, but the generator $\hat{G}$ is then more complicated (because L^1 and R may jump together in that case). The intertwining relation implies that the first coordinate of the process with generator $\hat{G}$ is an autonomous Markov process with generator G, i.e. a Wright-Fisher diffusion.

The generator $\hat{G}$ is in fact the generator of (R, L^1) up to the hitting time of 0 by R. Let us explain why. The process L^1 is plainly Markov in its own filtration and jumps from ℓ to $\ell+1$ at rate $c\ell(\ell-1)/2$. Then conditionally on the value of $L^1 = \ell$, we view the ℓ first particles as ℓ sources of immigration, $\ell-1$ sources of type 2 and one source of type 1, whence the drift term $c\left[(1-x)-(\ell-1)x\right]$ thanks to similar calculations as in 3.2.4. The process R may be seen as a WF diffusion with multitype immigration, We thus obtain the following pathwise decomposition of a Wright-Fisher diffusion, which is another way to express the intertwining relationship:

- Conditionally on $\{R_0 = x\}$, the initial value $L^1(0)$ has law:

$$\mathbb{P}(L^1(0) = \ell) = (1-x)^{\ell-1}x + \mathbf{1}_{\{\infty\}}(\ell)\mathbf{1}_{\{0\}}(x), \ \ell \geq 1.$$

- Conditionally on $(R_0, L^1(0))$, the process L^1 is a pure jump Markov process, which jumps from ℓ to $\ell+1$ at rate $c\ell(\ell-1)/2$ if $\ell < \infty$, and has $+\infty$ as an absorbing point.
- Conditionally on (R_0, L^1), the process R is a Wright-Fisher diffusion with immigration, with generator given by:

$$\frac{1}{2}cx(1-x)f''(x) + \mathbf{1}_{\{L^1<\infty\}}\ c\left[(1-x)-(L^1-1)x\right]f'(x).$$

3.3 The additive h-transform

In this Section, we derive another example of an h-transform (of measure-valued processes) admitting a simple construction from the look-down particle system.

3.3.1 The general construction of the look-down particle system

We first present a more general construction of an exchangeable particle system, which allows to deal with type mutation and nonconstant population size. We recall this model was defined (in greater generality) in [35].

Let E be a Polish space. We consider a triple (R_0, Y, U) constructed as follows. R_0 stands for a probability measure on E, $Y = (Y_t, t \geq 0)$ and $U = (U_t, t \geq 0)$ for two non negative real valued processes. We assume that $U_0 = 0$ and U is non decreasing, so that U admits a unique decomposition $U_t = U_t^k + \sum_{s \leq t} \Delta U_s$ where U^k is continuous (with Stieltjes measure denoted by dU^k) and $\Delta U_s = U_s - U_{s-}$. We assume that 0 is an absorbing point for Y, and set $\tau(Y) = \inf\{t > 0, Y_t = 0\}$ the extinction time of Y. We also assume that for each $t \geq 0$, $\Delta U_t \leq Y_t^2$. Conditionally on U and Y, we define two point measures N^ρ and N^k on $\mathbb{R}_+ \times \mathcal{P}_\infty$, where $\mathcal{P}_\infty$ denotes the set of partition of $\mathbb{N}$:

– $N^\rho = \sum_{0 \leq t < \tau(Y), \Delta U_t \neq 0} \delta_{(t,\pi)}(dt, d\pi)$ where the exchangeable partitions π of $\mathbb{N}$ are independent and have a unique non trivial block with asymptotic frequency $\sqrt{\Delta U_t}/Y_t$.
– $N^k = \sum_{0 \leq t < \tau(Y)} \delta_{(t,\pi)}(dt, d\pi)$ is an independent Poisson point measure with intensity

$$(dU_t^k/(Y_t)^2) \times \mu^k,$$

and the Kingman measure μ^k assigns mass one to partitions with a unique non trivial block consisting of two different integers, and mass 0 to the others.

Conditionally on (R_0, Y, U), we then define a particle system $\xi = (\xi_t(n), 0 \leq t < \tau(Y), n \in \mathbb{N})$ as follows:

– The initial state $(\xi_0(n), n \in \mathbb{N})$ is an exchangeable sequence valued in E with de Finetti's measure R_0.
– At each atom (t, π) of $N := N^k + N^\rho$, we associate a *reproduction event* as follows: let $j_1 < j_2 < \ldots$ be the elements of the unique block of the partition π which is not a singleton (either it is a doubleton if (t, π) is an atom of N^k or an infinite set if (t, π) is an atom of N^ρ). The individuals $j_1 < j_2 < \ldots$ at time t are declared to be the children of the individual j_1 at time $t-$, and receive the type of the parent j_1, whereas the types of all the other individuals are shifted upwards accordingly, keeping the order they had before the birth event: for each integer ℓ, $\xi_t(j_\ell) = \xi_{t-}(j_1)$ and for each $k \notin \{j_\ell, \ell \in \mathbb{N}\}$, $\xi_t(k) = \xi_{t-}(k - \#J_k)$ with $\#J_k$ the cardinality of the set $J_k := \{\ell > 1, j_\ell \leq k\}$.
– Between the reproduction events, the type $\xi_t(n)$ of the particle at level n mutates according to a Markov process with càdlàg paths in E and without fixed discontinuities, with law $(\mathrm{P}_x, x \in E)$ when started at $x \in E$, independently for each n.

This defines the particle system ξ on $[0, \tau(Y))$. The process $\xi_s(j)$ admits a limit as s goes to $\tau(Y)$ for each j, and we set $\xi_t(j) = \lim_{s \to \tau(Y)} \xi_s(j)$ for $t > \tau(Y)$. The sequence $(\xi_t(j), j \in \mathbb{N})$ is still exchangeable for $t \geq \tau_Y$ according to Proposition 3.1 of [35]. Conditionally on (R_0, Y, U), the sequence $(\xi_t(n), n \in \mathbb{N})$ is well defined for each $t \in \mathbb{R}^+$, exchangeable, and we denote by R_t its de Finetti measure:

$$R_t(dx) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi_t(n)}(dx),$$

The probability measure-valued process $R = (R_t, t \geq 0)$ has a càdlàg version according to Theorem 3.2 of [35]. We shall work with this version from now on. We finally define the càdlàg $\mathcal{M}_f(E)$ valued process of interest Z by:

$$(Z_t, t \geq 0) = (Y_t\, R_t, t \geq 0). \tag{3.17}$$

The finite measure Z represents the distribution of a population distributed in a space E, the process Y corresponds to the total population size, and U tracks the resampling inside the population. We stress that, conditionally given R_t, the random variables $(\xi_t(n), n \in \mathbb{N})$ on E are independent and identically distributed according to the probability measure R_t thanks to the de Finetti Theorem.

We will denote by $\mathbb{P}$ the law of the triple (Y, U, ξ). We introduce the relevant filtrations:
- $(\mathcal{F}_t = \sigma((Y_s, s \leq t), (\xi_s, s \leq t)))$ corresponds to the filtration of the particle system and the total population size.
- $(\mathcal{G}_t = \sigma(Z_s, s \leq t))$ corresponds to the filtration of the resulting measure-valued process.
- $\mathcal{D}_t$ is the filtration induced by the canonical process under P.

We shall use the classical notation $\mu(f) = \int \mu(dx) f(x)$ for a non negative map $f : E \to \mathbb{R}$ and $\mu \in \mathcal{M}_f$. Note that $Y_t = Z_t(\mathbf{1})$, and thus Y is $\mathcal{G}$-measurable.

3.3.2 A pathwise construction of the additive *h*-transform

We call a nonnegative function H on $[0, \infty) \times \mathcal{M}_f$ a space-time harmonic function for $\mathbb{P}$ when the process $(H(t, Z_t), t \geq 0)$ is a martingale under $\mathbb{P}$. The h-transform Z^H of Z associated with H is then defined by:

$$\forall A \in \mathcal{G}_t,\ \mathbb{P}(Z^H \in A) = \frac{H(t, Z_t)}{\mathbb{E}(H(0, Z_0))} \mathbb{P}\left(Z \in A\right). \tag{3.18}$$

for every $t \geq 0$. Furthermore, an h-transform is called additive if there exists a nonnegative function $(h_t(x), t \geq 0, x \in E)$ such that $H(t, Z_t) = Z_t(h_t)$. An additive h-transform intuitively favours the paths for which the population (represented by the measure-valued process) is large where h is large.

Statement of the results

Let ξ be the canonical process under P_x. We assume there exists a deterministic positive function m such that $(Y_t/m(t), t \geq 0)$ and $(m(t)h_t(\xi_t), t \geq 0)$ are martingales in their own filtrations. We also assume from now on that

$$\mathbb{E}(Y_0 R_0(h_0)) > 0.$$

Under this assumption, we define (the law of) a new process

$$(Y^h, U^h, \xi^h)$$

by the following requirements:

(i) The initial condition satisfies:

$$\forall A \in \mathcal{G}_t, \mathbb{P}((Y_0^h, R_0^h) \in A) = \mathbb{E}\left(\frac{Y_0 R_0(h_0)}{\mathbb{E}(Y_0 R_0(h_0))} \mathbf{1}_A(Y_0, R_0)\right).$$

(ii) Conditionally on (Y_0^h, R_0^h), and provided $R_0^h(h_0) > 0$, $\xi_0^h(1)$ is distributed according to:

$$\forall A \in \mathcal{D}_0, \mathbb{P}(\xi_0^h(1) \in A | R_0^h = \mu) = \mathbb{E}\left(\frac{h_0(\xi_0(1))}{\mu(h_0)} \mathbf{1}_A(\xi_0(1)) | R_0 = \mu\right),$$

and $(\xi_0^h(n), n \geq 2)$ is a random sequence with de Finetti's measure R_0^h.

(iii) Conditionally on $(Y_0^h, R_0^h, \xi_0^h(1))$, the process (Y^h, U^h) is distributed according to:

$$\forall A \in \mathcal{G}_t,\ \mathbb{P}((Y^h, U^h) \in A | Y_0^h = x) = \mathbb{E}\left(\frac{Y_t}{x} \frac{m(0)}{m(t)} \mathbf{1}_A(Y, U) | Y_0 = x\right). \tag{3.19}$$

(iv) Conditionally on $(Y^h, U^h, R_0^h, \xi_0^h(1))$, $\xi^h(1)$ is distributed according to:

$$\forall A \in \mathcal{D}_t,\ \mathbb{P}(\xi^h(1) \in A | \xi_0^h(1) = x) = \mathbb{E}\left(\frac{h_t(\xi_t(1))}{h_0(x)} \frac{m(t)}{m(0)} \mathbf{1}_A(\xi(1)) | \xi_0(1) = x\right). \tag{3.20}$$

(v) The rest of the definition of ξ^h is the same as the one given for ξ, namely:
- for $n \geq 2$, between the reproduction events, the type $\xi_t^h(n)$ of the particle at level n mutates according to a Markov process in E with law $(\mathrm{P}_x, x \in E)$ when started at $x \in E$, independently for each n.
- at each atom (t, π) of $N = N^k + N^\rho$, *with* N^k *and* N^ρ *derived from* U^h *and* Y^h, a reproduction event is associated as previously.

Note that the law of the initial condition Z_0^h specified by (i) possibly differs from that of Z_0 only for random Z_0. Also, notice that items (iii) and (iv) are meaningful since both $(Y_t/m(t), t \geq 0)$ and $(m(t)h_t(\xi_t(1)), t \geq 0)$ are assumed to be martingales. Last, observe from (3.19) that $\mathbb{P}(Y_t^h = 0) = 0$ for each $t \geq 0$, which implies $\mathbb{P}(\tau(Y^h) = \infty) = 1$ since 0 is assumed to be absorbing. We will assume that (Y, U, ξ) and (Y^h, U^h, ξ^h) are defined on a common probability space with probability measure $\mathbb{P}$, and denote the expectation by $\mathbb{E}$.

Let us define a process $S = (S_t, t \geq 0)$ by:

$$S_t = \frac{h_t(\xi_t(1))\, Y_t}{\mathbb{E}(Z_0(h_0))}.$$

Lemma 3.3.1. *The process $(S = S_t, t \geq 0)$ is a non negative $\mathcal{F}$-martingale, and*

$$\forall A \in \mathcal{F}_t,\ \mathbb{P}(\xi^h \in A) = \mathbb{E}\left(\mathbf{1}_A(\xi)\ S_t\right). \tag{3.21}$$

We then define the process T:

$$T_t = \frac{Z_t(h_t)}{\mathbb{E}(Z_0(h_0))}.$$

Using Lemma 3.3.1, and projecting on the filtration $\mathcal{G}_t$, we deduce Lemma 3.3.2.

Lemma 3.3.2. *The process $T = (T_t, t \geq 0)$ is a non negative $\mathcal{G}$-martingale.*

This fact allows to define the process $Z^H := (Z_t^H, t \geq 0)$ absolutely continuous with respect to $Z := (Z_t, t \geq 0)$ on each $\mathcal{G}_t$, $t \geq 0$, with Radon Nykodim derivative:

$$\forall A \in \mathcal{G}_t,\ \mathbb{P}(Z^H \in A) = \mathbb{E}\left(\mathbf{1}_A(Z)\ T_t\right).$$

We deduce from Lemma 3.3.1 and Lemma 3.3.2 the following Theorem, which gives a pathwise construction of the additive h-transform.

Theorem 3.3.3. *We have that:*
(a) The limit of the empirical measure:

$$R_t^h(dx) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi_t^h(n)}(dx)$$

exists a.s.
(b) The process $(Z_t^h := Y_t^h R_t^h, t \geq 0)$ is distributed as $(Z_t^H, t \geq 0)$.

We may interpret Theorem 3.3.3 as follows. The effect of the additive h-transform factorizes in two parts, according to the decomposition of the Radon Nikodym derivative:

$$Z_t(h_t) = Y_t\ R_t(h_t).$$

The first term Y_t induces a size bias of the total population size $Z^h(\mathbf{1}) = Y^h$, see formula (3.19), whereas the second term $R_t(h_t)$ forces the first level particle to follow an h-transform of P, see formula (3.20).

The sequence $(\xi_t^h(n), n \in \mathbb{N})$ is not exchangeable in general, which contrasts with the initial sequence $(\xi_t(n), n \in \mathbb{N})$. The following Proposition shows that, loosely speaking, the first level particle is precursory.

Proposition 3.3.4. *Conditionally on $\{R_t^h = \mu\}$, $\xi_1^h(t)$ is distributed according to:*

$$\mathbb{P}(\xi_t^h(1) \in dx) = \frac{h_t(x)}{\mu(h_t)}\mu(dx),$$

and $(\xi_t^h(n))_{n\geq 2}$ is an independent exchangeable random sequence with de Finetti's measure μ.

Proofs

Proof of Lemma 3.3.1. It is enough to observe that, by construction, the law of (Y^h, U^h, ξ^h) is absolutely continuous with respect to the law of (Y, U, ξ) on $\mathcal{F}_t$, with Radon Nykodim derivative given by:

$$\begin{aligned}\forall A \in \mathcal{F}_t,\ \mathbb{P}((Y^h, U^h, \xi^h) \in A) &= \mathbb{E}\left(\frac{Y_0 R_0(h_0)}{\mathbb{E}(Y_0 R_0(h_0))} \frac{h_0(\xi_0^h(1))}{R_0(h_0)} \frac{Y_t}{Y_0} \frac{m(0)}{m(t)} \frac{h_t(\xi_t(1))}{h_0(\xi_0^h(1))} \frac{m(t)}{m(0)} \mathbf{1}_A(Y, U, \xi)\right) \\ &= \mathbb{E}\left(\frac{Y_t\ h_t(\xi_t(1))}{\mathbb{E}(Z_0(h_0))} \mathbf{1}_A(Y, U, \xi)\right)\end{aligned}$$

This also yields (the obvious fact) that $(S_t, t \geq 0)$ is a $\mathcal{F}$-martingale, arguing as in the proof of Lemma 3.2.3. □

Proof of Lemma 3.3.2. Since $\mathcal{G}_t \subset \mathcal{F}_t$ and S is a $\mathcal{F}$-martingale, the projection $\mathbb{E}(S_t|\mathcal{G}_t)$ is a $\mathcal{G}$-martingale. We also have that:

$$\mathbb{E}(S_t|\mathcal{G}_t) = \mathbb{E}\left(\frac{Y_t\; h_t(\xi_t(1))}{\mathbb{E}(Z_0(h_0))}|\mathcal{G}_t\right) = \frac{Z_t(h_t)}{\mathbb{E}(Z_0(h_0))} = T_t,$$

where we used that $\xi_t(1)$ has law R_t conditionally on $\mathcal{G}_t$ for the third equality. Thus $(T_t, t \geq 0)$ is a $\mathcal{G}$-martingale. □

Proof of Theorem 3.3.3. From Lemma 3.3.1, the law of ξ^h is absolutely continuous with respect to the law of ξ. The existence of the a.s. limit of the empirical measure of ξ^h follows from that of ξ (but not the exchangeability of the sequence) and yields point (a). We prove point (b) now. Take $A \in \mathcal{G}_t$.

$$\begin{aligned}
\mathbb{P}(Z^h \in A) &= \mathbb{E}\left(S_t \mathbf{1}_A(Z)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(S_t|\mathcal{G}_t\right) \mathbf{1}_A(Z)\right) \\
&= \mathbb{P}(T_t\; \mathbf{1}_A(Z)) \\
&= \mathbb{P}(Z^H \in A),
\end{aligned}$$

where we use Lemma 3.3.2 at the third equality and the definition of Z^H for the last equality.

□

Proof of Proposition 3.3.4. Let $n \in \mathbb{N}$ be fixed, and let $(\phi_i)_{(1\leq i\leq n)}$ be a collection of bounded and measurable functions on E.

$$\begin{aligned}
\mathbb{E}\left(\prod_{1\leq i\leq n} \phi_i(\xi_i^h(t))\right) &= \mathbb{E}\left(\frac{Y_t\; h_t(\xi_t(1))}{\mathbb{E}(Z_0(h_0))} \prod_{1\leq i\leq n} \phi_i(\xi_t(i))\right) \\
&= \frac{1}{\mathbb{E}(Z_0(h_0))}\mathbb{E}\left(Y_t\; \mathbb{E}\left(h_t(\xi_t(1))\phi_1(\xi_t(1)) \prod_{2\leq i\leq n} \phi_i(\xi_t(i))|\mathcal{G}_t\right)\right) \\
&= \frac{1}{\mathbb{E}(Z_0(h_0))}\mathbb{E}\left(Y_t\; R_t(h_t\; \phi_1) \prod_{2\leq i\leq n} R_t(\phi_i)\right) \\
&= \mathbb{E}\left(\frac{Z_t(h_t)}{\mathbb{E}(Z_0(h_0))}\; R_t\left(\frac{h_t\; \phi_1}{R_t(h_t)}\right) \prod_{2\leq i\leq n} R_t(\phi_i)\right) \\
&= \mathbb{E}\left(R_t^h\left(\frac{h_t\; \phi_1}{R_t^h(h_t)}\right) \prod_{2\leq i\leq n} R_t^h(\phi_i)\right),
\end{aligned}$$

where we use Lemma 3.3.1 at the first equality, the de Finetti Theorem at the third equality, and Theorem 3.3.3 at the last equality. Since functions of the type $\prod_{1\leq i\leq n} \phi_i$ characterize the law of n-uple, this proves the Proposition. □

3.3.3 Applications

Overbeck investigated in [103] h-transform of measure-valued diffusions, among which the Dawson Watanabe process (with quadratic branching mechanism) and the Fleming-Viot process (which is the Λ-Fleming-Viot process for $\nu = 0$) using a martingale problem approach. He also provided a pathwise constructions in the first case, see [102]. We shall see in this last Section how Theorem 3.3.3 applies in both cases and sheds new light on Overbeck's results.

Λ-Fleming-Viot processes

The Λ-Fleming-Viot process (with mutation) is the process Z constructed in Section 3.3.1 when setting:
– $Y = \mathbf{1}$,
– U is a subordinator with jumps no greater than 1,
It corresponds to the process R introduced in Section 3.2.1 when allowing for mutations. We denote by ϕ the Laplace exponent of the subordinator U:

$$\phi(\lambda) = c\lambda + \int_{(0,1]} (1 - \mathrm{e}^{-\lambda x})\nu^U(dx)$$

where $c \geq 0$ and the Lévy measure ν^U satisfies $\int_{(0,1]} x \ \nu^U(dx) < \infty$. The genealogy of the look-down particle system is by construction described by the Λ-coalescent of Pitman [108]. The finite measure Λ is related to ν and c by $x^{-2}\Lambda_{|(0,1]}(dx) = \nu(dx)$ and $\Lambda\{0\} = c$, and may be recovered from ϕ through the identity:

$$\int_{[0,1]} g(x)\Lambda(dx) = cg(0) + \int_{(0,1]} g(\sqrt{x})x\nu^U(dx),$$

see Sections 3.1.4 and 5.1 of [35].

Since $Y_t = 1$, Y is a martingale and we may apply results of Section 3.3.2 for any nonnegative space time harmonic function $(h_t(x), t \geq 0, x \in E)$ for the spatial motion P, that is any function such that $(h_t(\xi_t), t \geq 0)$ is a nonnegative martingale where ξ stands for the canonical process under P. Notice the construction of the particle system ξ^h simplifies here since $(U^h, Y^h) \overset{(law)}{=} (U, Y) = (U, \mathbf{1})$.

Overbeck suggested in [103] that in the particular case of the Fleming-Viot process (which is the Λ-Fleming-Viot process for $\nu^U = 0$), an additive h-transform looks like a FV process where "*the gene type of at least one family mutates as an h-transform of the one particle motion*". This suggestion was made "*plausible*" by similar results known for superprocesses, see [102] or the next Subsection, and a well known connection between superprocesses and Fleming-Viot processes which goes back to Shiga [124]. We did not attempt to derive the pathwise construction of the additive h-transform of Λ-Fleming-Viot processes in this way, since the connection between superprocesses and Λ-Fleming-Viot processes is restricted to stable superprocesses and Beta-Fleming-Viot processes, see Birkner *et. al* [18]. The construction is provided by Theorem 3.3.3. This Theorem allows us to see at first glance that the family which "*mutates as an h-transform*" is the family generated by the first level particle in the look-down process. Notice also that our Theorem applies for Λ-Fleming-Viot processes.

Remark 3.3.5. If $\nu^U = 0$ (or, equivalently, $\Lambda(dx) = \Lambda\{0\}\ \delta_0(dx)$), the truncated processes obtained by considering the first N particles:

$$Z_t^N(dx) := \frac{1}{N}\sum_{1\le n\le N} \delta_{\xi_t(n)}(dx) \text{ and } Z_t^{N,h}(dx) := \frac{1}{N}\sum_{1\le n\le N} \delta_{\xi_t^h(n)}(dx)$$

correspond respectively to the Moran model with N particles (see [35]) and its additive h-transform. This proves our approach is robust, in the sense that we can also consider discrete population.

Finally, we may interpret the h-transform as a conditioned process. For fixed $s \ge 0$, the additive h-transform of the Λ-Fleming-Viot process on $[0, s]$ may be obtained by conditioning a random particle chosen at time t, t large, to move as an h-transform. For other conditionings on boundary statistics in the context of measure-valued branching processes this time, we refer to Salisbury and Sezer [117].

The Dawson Watanabe superprocess

Recall a continuous state branching process is a strong Markov process characterized by a branching mechanism ψ taking the form

$$\psi(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \beta\lambda + \int_{(0,\infty)} (\mathrm{e}^{-\lambda u} - 1 + \lambda u \mathbf{1}_{u\le 1})\nu^Y(du), \tag{3.22}$$

for ν^Y a Lévy measure such that $\int_{(0,\infty)}(1\wedge u^2)\nu^Y(du) < \infty$, $\beta \in \mathbb{R}$, and $\sigma^2 \in \mathbb{R}^+$. We will denote it CB($\psi$) for short. More precisely, the CB(ψ) process is the strong Markov process Y with Laplace transform given by:

$$\mathbb{E}(\mathrm{e}^{-\lambda Y_t}|Y_0 = x) = \mathrm{e}^{-xu(\lambda,t)},$$

where u is the unique nonnegative solution of the integral equation, holding for all $t \ge 0$, $\lambda \ge 0$:

$$u(\lambda, t) + \int_0^t ds\ \psi\left(u(\lambda, s)\right) = \lambda. \tag{3.23}$$

We assume from now on that $\psi'(0+) > -\infty$, so that the CB(ψ) has integrable marginals, and

$$(Y_t\,\mathrm{e}^{\psi'(0+)t}, t \ge 0)$$

is a martingale. The Dawson Watanabe process with general branching mechanism given by ψ is the measure-valued process $(Z_t, t \ge 0)$ constructed in Section 3.3.1 when:
– $(Y_t, t \ge 0)$ is a CB.
– $(U_t, t \ge 0)$ is the quadratic variation process of Y, $U_t = [Y](t)$. Therefore, $\Delta U_t = (\Delta Y_t)^2 \le Y_t^2$.
Since the process $(Y_t\,\mathrm{e}^{\psi'(0+)t}, t \ge 0)$ is a martingale, we may apply our results for any nonnegative function $(h_t(x), t \ge 0, x \in E)$ such that $(h_t(\xi_t)\,\mathrm{e}^{-\psi'(0+)t}, t \ge 0)$ is a martingale.

We now link our results with the literature:

1. When Y is a subcritical CB process, meaning that $\psi'(0+) \ge 0$, setting $m(t) = \mathrm{e}^{-\psi'(0+)t}$ and $h_t(x) = \mathrm{e}^{\psi'(0+)t}$, we recover from Theorem 3.3.3 part of the Roelly & Rouault [112] and Evans [48] decomposition. This h-transform may be interpreted as the process conditioned on non extinction in remote time, see Lambert [85].

2. When Y is a critical Feller diffusion and P the law of a Brownian motion, if we assume we are given $h_t(x)$ a space time harmonic function for P and set $m(t) = 1$, we get from Theorem 3.3.3 the decomposition of the h-transform of the Dawson Watanabe process provided by Overbeck in [102].

Recall the two effects of the additive h-transform: the total population is size biased and the first level particle follows an h-transform of P. We shall now concentrate on the first effect, and explain how a "spinal" decomposition may be partly recovered from Theorem 3.3.3: Lemma 3.3.6 identifies the size biased total mass process $Y^h = Z^h(\mathbf{1})$ with a branching process with immigration, and Lemma 3.3.7 recognizes the first level particle as the source of this immigration.

Let ϕ be the Laplace exponent of a subordinator. Recall a continuous state branching process with immigration with branching mechanism ψ and immigration mechanism ϕ, CBI(ψ,ϕ) for short, is a strong Markov process $(Y_t^i, t \geq 0)$ characterized by the Laplace transform:

$$\mathbb{E}(\mathrm{e}^{-\lambda Y_t^i} | Y_0^i = x) = \mathrm{e}^{-xu(\lambda,t) - \int_0^t ds \ \phi(u(\lambda,s))} .$$

We recall for the ease of reference the following well known lemma, and we stress that this Lemma also holds in the supercritical case $\psi'(0+) < 0$.

Lemma 3.3.6. *The process Y^h defined by (3.19) with $m(t) = \mathrm{e}^{-\psi'(0+)t}$ is a CBI(ψ,ϕ) with immigration mechanism given by $\psi'(\lambda) - \psi'(0+)$.*

The proof is classical and relies on computation of the Laplace transforms. Notice that in the case where the CB process Y extincts almost surely, the CBI process Y^h may also be interpreted as the CB process Y conditioned on non extinction in remote time, see Lambert [85].

The total mass process $Y^h = Z^h(\mathbf{1})$ is thus a CBI process. We may wonder "who" are the immigrants in the population represented by the particle system ξ^h. The following Lemma shows that the offsprings of the first level particle are the immigrants when $c = 0$ (see the following Remark for the general case). Recall j_1 refers to the first level sampled in the look-down construction. Let us denote $j_1(s)$ instead of j_1 for indicating the dependence in s.

Lemma 3.3.7. *The process $\left(\sum_{0\leq s\leq t} \Delta Y_s^h \, \mathbf{1}_{\{j_1(s)=1\}}, t \geq 0\right)$ is a pure jump subordinator with Lévy measure $u\nu^Y(du)$.*

Proof. By assumption, the process Y is a CB(ψ) and from Lemma 3.3.6, Y^h is a CBI($\psi, \psi'(\cdot) - \psi'(0+)$). From the Poissonian construction of CBI, we have that the point measure

$$\sum_{0\leq s\leq t} \delta_{(s,\Delta Y_s^h)}(ds, du)$$

has for predictable compensator

$$ds \ (Y_{s-}^h \nu^Y(du) + u\nu^Y(du)).$$

The expression of the compensator may be explained as follows. The term $ds \ Y_{s-}^h \nu^Y(du)$ comes from the time change of the underlying spectrally positive Lévy process, called the Lamperti time change (for CBs). The term $ds \ u\nu^Y(du)$ is independent of the current state of the population and corresponds to the immigration term. Then, conditionally on the value of the jump $\Delta Y_s^h = u$, the event $\{j_1(s) = 1\}$ has probability

$$\frac{u}{Y_s^h} = \frac{u}{Y_{s-}^h + u}$$

independently for each jump. Therefore, the predictable compensator of the point measure

$$\sum_{0 \le s \le t} \delta_{(s, \Delta Y_s^h)}(ds, du) \mathbf{1}_{\{j_1(s)=1\}}$$

is

$$ds \left(\frac{u}{Y_{s-}^h + u} \right) (Y_{s-}^h \nu^Y(du) + u\nu^Y(du)) = ds \ u\nu^Y(du),$$

This ends up the proof. □

Remark 3.3.8. Understanding the action of the continuous part of the subordinator requires to work with the discrete particle system generated by the first N particles. Namely, it is possible to prove that the family of processes

$$\left(\sum_{0 \le s \le t} Y_s^h \frac{\#\{1 \le i \le N, j_i(s) \le N\}}{N} \mathbf{1}_{\{j_1(s)=1, j_2(s) \le N\}}, t \ge 0 \right)$$

converges almost surely as $N \to \infty$ in the Skorohod topology towards a subordinator with Laplace exponent $\psi'(\lambda) - \psi'(0+)$.

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4

Stable CBI and Beta-Fleming-Viot

4.1 Introduction

The connections between the Fleming-Viot processes and the continuous-state branching processes have been intensively studied. Shiga established in 1990 that a Fleming-Viot process may be recovered from the *ratio process* associated with a Feller diffusion up to a random time change, see [124]. This result has been generalized in 2005 by Birkner *et al* in [18] in the setting of Λ-Fleming-Viot processes and continuous-state branching processes (CBs for short). In that paper they proved that the ratio process associated with an α-stable branching process is a time-changed Beta$(2-\alpha,\alpha)$-Fleming-Viot process for $\alpha \in (0,2)$. The main goal of this article is to study such connections when immigration is incorporated in the underlying population. The continuous-state branching processes with immigration (CBIs for short) are a class of time-homogeneous Markov processes with values in $\mathbb{R}_+$. They have been introduced by Kawazu and Watanabe in 1971, see [72], as limits of rescaled Galton-Watson processes with immigration. These processes are characterized by two functions Φ and Ψ respectively called the immigration mechanism and the branching mechanism. A new class of measure-valued processes with immigration has been recently set up in Foucart [54]. These processes, called M-Fleming-Viot processes are valued in the space of probability measures on $[0,1]$. The notation M stands for a couple of finite measures (Λ_0,Λ_1) encoding respectively the rates of immigration and of reproduction. The genealogies of the M-Fleming-Viot processes are given by the so-called M-coalescents. These processes are valued in the space of the partitions of $\mathbb{Z}_+$, denoted by $\mathcal{P}^0_\infty$.

In the same manner as Birkner *et al.* in [18], Perkins in [105] and Shiga in [124], we shall establish some relations between continuous-state branching processes with immigration and M-Fleming-Viot processes. A notion of continuous population with immigration may be defined using a flow of CBIs in the same spirit as Bertoin and Le Gall in [14]. This allows us to compare the two notions of continuous populations provided respectively by the CBIs and by the M-Fleming-Viot processes. Using calculations of generators, we show in Theorem 4.3.3 that the following self-similar CBIs admit time-changed M-Fleming-Viot processes for ratio processes:

- the Feller branching diffusion with branching rate σ^2 and immigration rate β (namely the CBI with $\Phi(q)=\beta q$ and $\Psi(q)=\frac{1}{2}\sigma^2q^2$) which has for ratio process a time-changed M-Fleming-Viot process where $M=(\beta\delta_0,\sigma^2\delta_0)$,
- the CBI process with $\Phi(q)=d'\alpha q^{\alpha-1}$ and $\Psi(q)=dq^\alpha$ for some $d,d'\geq 0$, $\alpha\in(1,2)$ which has for ratio process a time-changed M-Fleming-Viot process where the couple of measures M satifies $M=(c'\mathrm{Beta}(2-\alpha,\alpha-1), c\mathrm{Beta}(2-\alpha,\alpha))$, $c'=\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d'$ and $c=\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d$.

We stress that the CBIs may reach 0, see Proposition 4.3.1, in which case the M-Fleming-Viot processes involved describe the ratio process up to this hitting time only. When $d = d'$ or $\beta = \sigma^2$, the corresponding CBIs are respectively the α-stable branching process and the Feller branching diffusion *conditioned to be never extinct.* In that case, the M-coalescents are genuine Λ-coalescent viewed on $\mathcal{P}_\infty^0$. We get respectively a Beta$(2-\alpha, \alpha-1)$-coalescent when $\alpha \in (1,2)$ and a Kingman's coalescent for $\alpha = 2$, see Theorem 4.4.4. This differs from the α-stable branching process *without immigration* (already studied in [18]) for which the coalescent involved is a Beta$(2-\alpha, \alpha)$-coalescent.
Last, ideas provided to establish our main theorem have been used by Handa [62] to study stationary distributions for another class of Λ-Fleming-Viot processes.

Outline. The paper is organized as follows. In Section 4.2, we recall the definition of a continuous-state branching process with immigration and of an M-Fleming-Viot process. We describe briefly how to define from a flow of CBIs a continuous population represented by a measure-valued process. We state in Section 4.3 the connections between the CBIs and M-Fleming-Viot processes, mentioned in the Introduction, and study the random time change. Recalling the definition of an M-coalescent, we focus in Section 4.4 on the genealogy of the M-Fleming-Viot processes involved. We establish that, when the CBIs correspond with CB-processes conditioned to be never extinct, the M-coalescents involved are actually classical Λ-coalescents. We identify them and, as mentioned, the Beta$(2-\alpha, \alpha-1)$-coalescent arises. In Section 4.5, we compare the generators of the M-FV and CBI processes and prove the main result.

4.2 A continuous population embedded in a flow of CBIs and the M-Fleming-Viot

4.2.1 Background on continuous state branching processes with immigration

We will focus on critical continuous-state branching processes with immigration characterized by two functions of the variable $q \geq 0$:

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\hat{\nu_1}(du)$$
$$\Phi(q) = \beta q + \int_0^\infty (1 - e^{-qu})\hat{\nu_0}(du)$$

where $\sigma^2, \beta \geq 0$ and $\hat{\nu_0}$, $\hat{\nu_1}$ are two Lévy measures such that $\int_0^\infty (1 \wedge u)\hat{\nu_0}(du) < \infty$ and $\int_0^\infty (u \wedge u^2)\hat{\nu_1}(du) < \infty$. The measure $\hat{\nu_1}$ is the Lévy measure of a spectrally positive Lévy process which characterizes the reproduction. The measure $\hat{\nu_0}$ characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The non-negative constants σ^2 and β correspond respectively to the continuous reproduction and the continuous immigration. Let $\mathbb{P}_x$ be the law of a CBI $(Y_t, t \geq 0)$ started at x, and denote by $\mathbb{E}_x$ the associated expectation. The law of the Markov process $(Y_t, t \geq 0)$ can then be characterized by the Laplace transform of its marginal as follows: for every $q > 0$ and $x \in \mathbb{R}_+$,

$$\mathbb{E}_x[e^{-qY_t}] = \exp\left(-x v_t(q) - \int_0^t \Phi(v_s(q))ds\right)$$

where v is the unique non-negative solution of $\frac{\partial}{\partial t} v_t(q) = -\Psi(v_t(q))$, $v_0(q) = q$.

The pair (Ψ, Φ) is known as the branching-immigration mechanism. A CBI process $(Y_t, t \geq 0)$ is said to be conservative if for every $t > 0$ and $x \in [0, \infty[, \mathbb{P}_x[Y_t < \infty] = 1$. A result of Kawazu and Watanabe [72] states that $(Y_t, t \geq 0)$ is conservative if and only if for every $\epsilon > 0$

$$\int_0^\epsilon \frac{1}{|\Psi(q)|} dq = \infty.$$

Moreover, we shall say that the CBI process is *critical* when $\Psi'(0) = 0$: in that case, the CBI process is necessarily conservative. We follow the seminal idea of Bertoin and Le Gall in [15] to define a genuine continuous population model with immigration on $[0, 1]$ associated with a CBI. Emphasizing the rôle of the initial value, we denote by $(Y_t(x), t \geq 0)$ a CBI started at $x \in \mathbb{R}_+$. The branching property ensures that $(Y_t(x + y), t \geq 0) \overset{law}{=} (Y_t(x) + X_t(y), t \geq 0)$ where $(X_t(y), t \geq 0)$ is a CBI$(\Psi, 0)$ starting from y (that is a CB-process without immigration and with branching mechanism Ψ) independent of $(Y_t(x), t \geq 0)$. The Kolmogorov's extension theorem allows one to construct a flow $(Y_t(x), t \geq 0, x \geq 0)$ such that for every $y \geq 0$, $(Y_t(x + y) - Y_t(x), t \geq 0)$ has the same law as $(X_t(y), t \geq 0)$ a CB-process started from y. We denote by $(Z_t, t \geq 0)$ the Stieltjes-measure associated with the increasing process $x \in [0, 1] \mapsto Y_t(x)$. Namely, define

$$\begin{aligned} &Z_t(]x, y]) := Y_t(y) - Y_t(x), \quad 0 \leq x \leq y \leq 1. \\ &Z_t(\{0\}) := Y_t(0). \end{aligned}$$

The process $(Y_t(1), t \geq 0)$ is assumed to be conservative, therefore the process $(Z_t, t \geq 0)$ is valued in the space $\mathcal{M}_f$ of finite measures on $[0, 1]$. By a slight abuse of notation, we denote by $(Y_t, t \geq 0)$ the process $(Y_t(1), t \geq 0)$. The framework of measure-valued processes allows us to consider an infinitely many types model. Namely each individual has initially its own type (which lies in $[0, 1]$) and transmits it to its progeny. People issued from the immigration have a *distinguished* type fixed at 0. Since the types do not evolve in time, they allow us to track the ancestors at time 0. This model can be viewed as a superprocess without spatial motion (or without mutation in population genetics vocable).
Let $\mathcal{C}$ be the class of functions on $\mathcal{M}_f$ of the form

$$F(\eta) := G\left(\langle f_1, \eta\rangle, ..., \langle f_n, \eta\rangle\right),$$

where $\langle f, \eta\rangle := \int_{[0,1]} f(x)\eta(dx)$, $G \in C^2(\mathbb{R}^n)$ and $f_1, ..., f_n$ are bounded measurable functions on $[0, 1]$. Section 9.3 of Li's book [95] (see Theorem 9.18 p. 218) ensures that the following operator acting on the space $\mathcal{M}_f$ is an extended generator of $(Z_t, t \geq 0)$. For any $\eta \in \mathcal{M}_f$,

$$\begin{aligned} \mathcal{L}F(\eta) := &\ \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) && (4.1)\\ &+ \beta F'(\eta; 0) && (4.2)\\ &+ \int_0^1 \eta(da) \int_0^\infty \hat{\nu_1}(dh)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)] && (4.3)\\ &+ \int_0^\infty \hat{\nu_0}(dh)[F(\eta + h\delta_0) - F(\eta)] && (4.4) \end{aligned}$$

where $F'(\eta; a) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(\eta + \epsilon\delta_a) - F(\eta)]$ is the Gateaux derivative of F at η in direction δ_a, and $F''(\eta; a, b) := G'(\eta; b)$ with $G(\eta) = F'(\eta; a)$. The terms (1) and (3) correspond to the reproduction, see for instance Section 6.1 p. 106 of Dawson [30]. The terms (2) and (4) correspond to the immigration. We stress that in our model the immigration is concentrated on 0, contrary to other works which consider infinitely many types for the immigrants. For the interested reader, the operator $\mathcal{L}$ corresponds with that given in equation (9.25) of Section 9 of Li [95] by setting $H(d\mu) = \int_0^\infty \hat{\nu}_0(dh)\delta_{h\delta_0}(d\mu)$ and $\eta = \beta\delta_0$.

For all $\eta \in \mathcal{M}_f$, we denote by $|\eta|$ the total mass $|\eta| := \eta([0,1])$. If $(Z_t, t \geq 0)$ is a Markov process with the above operator for generator, the process $(|Z_t|, t \geq 0)$ is by construction a CBI. This is also plain from the form of the generator $\mathcal{L}$: let ψ be a twice differentiable function on $\mathbb{R}_+$ and define $F : \eta \mapsto \psi(|\eta|)$, we find $\mathcal{L}F(\eta) = zG_B\psi(z) + G_I\psi(z)$ for $z = |\eta|$, where

$$G_B\psi(z) = \frac{\sigma^2}{2}\psi''(z) + \int_0^\infty [\psi(z+h) - \psi(z) - h\psi'(z)]\hat{\nu_1}(dh) \tag{4.5}$$

$$G_I\psi(z) = \beta\psi'(z) + \int_0^\infty [\psi(z+h) - \psi(z)]\hat{\nu_0}(dh). \tag{4.6}$$

4.2.2 Background on M-Fleming-Viot processes

We denote by $\mathcal{M}_1$ the space of probability measures on $[0,1]$. Let c_0, c_1 be two non-negative real numbers and ν_0, ν_1 be two measures on $[0,1]$ such that $\int_0^1 x\nu_0(dx) < \infty$ and $\int_0^1 x^2\nu_1(dx) < \infty$. Following the notation of [54], we define the couple of finite measures $M = (\Lambda_0, \Lambda_1)$ such that

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx),\ \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx).$$

The M-Fleming-Viot process describes a population with *constant size* which evolves by resampling. Let $(\rho_t, t \geq 0)$ be an M-Fleming-Viot process. The evolution of this process is a superposition of a continuous evolution, and a discontinuous one. The continuous evolution can be described as follows: every couple of individuals is sampled at constant rate c_1, in which case one of the two individuals gives its type to the other: this is a reproduction event. Furthermore, any individual is picked at constant rate c_0, and its type replaced by the distinguished type 0 (the immigrant type): this is an immigration event. The discontinuous evolution is prescribed by two independent Poisson point measures N_0 and N_1 on $\mathbb{R}_+ \times [0,1]$ with respective intensity $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$. More precisely, if (t,x) is an atom of $N_0 + N_1$ then t is a jump time of the process $(\rho_t, t \geq 0)$ and the conditional law of ρ_t given ρ_{t-} is:
– $(1-x)\rho_{t-} + x\delta_U$, if (t,x) is an atom of N_1, where U is distributed according to ρ_{t-}
– $(1-x)\rho_{t-} + x\delta_0$, if (t,x) is an atom of N_0.
If (t,x) is an atom of N_1, an individual is picked at random in the population at generation $t-$ and generates a proportion x of the population at time t: this is a reproduction event, as for the genuine Λ-Fleming-Viot process (see [15] p. 278). If (t,x) is an atom of N_0, the individual 0 at time $t-$ generates a proportion x of the population at time t: this is an immigration event. In both cases, the population at time $t-$ is reduced by a factor $1-x$ so that, at time t, the total size is still 1. The genealogy of this population (which is identified as a probability measure on $[0,1]$) is given by an M-coalescent (see Section 4.4 below). This description is purely heuristic (we stress for instance that the atoms of $N_0 + N_1$ may form an infinite dense set), to make a rigorous construction of such processes, we refer to the Section 5.2 of [54] (or alternatively Section 3.2 of [53]).
For any $p \in \mathbb{N}$ and any continuous function f on $[0,1]^p$, we denote by G_f the map

$$\rho \in \mathcal{M}_1 \mapsto \langle f, \rho^{\otimes p}\rangle := \int_{[0,1]^p} f(x)\rho^{\otimes p}(dx) = \int_{[0,1]^p} f(x_1, ..., x_p)\rho(dx_1)...\rho(dx_p).$$

Let $(\mathcal{F}, \mathcal{D})$ denote the generator of $(\rho_t, t \geq 0)$ and its domain. The vector space generated by the functionals of the type G_f forms a core of $(\mathcal{F}, \mathcal{D})$ and we have (see Lemma 5.2 in [54]):

$$\mathcal{F}G_f(\rho) = c_1 \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes p}(dx) \tag{1'}$$

$$+ c_0 \sum_{1\leq j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{0,j}) - f(\mathrm{x})]\rho^{\otimes p}(dx) \tag{2'}$$

$$+ \int_0^1 \nu_1(dr) \int \rho(da)[G_f((1-r)\rho + r\delta_a) - G_f(\rho)] \tag{3'}$$

$$+ \int_0^1 \nu_0(dr)[G_f((1-r)\rho + r\delta_0) - G_f(\rho)]. \tag{4'}$$

where x denotes the vector $(x_1, ..., x_p)$ and
– the vector $\mathrm{x}^{0,j}$ is defined by $\mathrm{x}^{0,j}_k = x_k$, for all $k \neq j$ and $\mathrm{x}^{0,j}_j = 0$,
– the vector $\mathrm{x}^{i,j}$ is defined by $\mathrm{x}^{i,j}_k = x_k$, for all $k \neq j$ and $\mathrm{x}^{i,j}_j = x_i$.

4.3 Relations between CBIs and M-Fleming-Viot processes

4.3.1 Forward results

The expressions of the generators of $(Z_t, t \geq 0)$ and $(\rho_t, t \geq 0)$ lead us to specify the connections between CBIs and GFVIs. We add a cemetery point Δ to the space $\mathcal{M}_1$ and define $(R_t, t \geq 0) := (\frac{Z_t}{|Z_t|}, t \geq 0)$, the ratio process with lifetime $\tau := \inf\{t \geq 0; |Z_t| = 0\}$. By convention, for all $t \geq \tau$, we set $R_t = \Delta$. As mentioned in the Introduction, we shall focus our study on the two following critical CBIs:
(i) $(Y_t, t \geq 0)$ is CBI with parameters $\sigma^2, \beta \geq 0$ and $\hat{\nu}_0 = \hat{\nu}_1 = 0$, so that $\Psi(q) = \frac{\sigma^2}{2}q^2$ and $\Phi(q) = \beta q$.
(ii) $(Y_t, t \geq 0)$ is a CBI with $\sigma^2 = \beta = 0$, $\hat{\nu_0}(dh) = c' h^{-\alpha} 1_{h>0} dh$ and $\hat{\nu_1}(dh) = c h^{-1-\alpha} 1_{h>0} dh$ for $1 < \alpha < 2$, so that $\Psi(q) = dq^\alpha$ and $\Phi(q) = d'\alpha q^{\alpha-1}$ with $d' = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}c'$ and $d = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}c$
Notice that the CBI in (i) may be seen as a limit case of the CBIs in (ii) for $\alpha = 2$. We first establish in the following proposition a dichotomy for the finiteness of the lifetime, depending on the ratio immigration over reproduction.

Proposition 4.3.1. *Recall the notation* $\tau = \inf\{t \geq 0, Y_t = 0\}$.
– *If* $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ *in case (i) or* $\frac{c'}{c} \geq \frac{\alpha-1}{\alpha}$ *in case (ii), then* $\mathbb{P}[\tau = \infty] = 1$.
– *If* $\frac{\beta}{\sigma^2} < \frac{1}{2}$ *in case (i) or* $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ *in case (ii), then* $\mathbb{P}[\tau < \infty] = 1$.

We then deal with the random change of time. In the case of a CB-process (that is a CBI process without immigration), Birkner *et al.* used the Lamperti representation and worked on the embedded stable spectrally positive Lévy process. We shall work directly on the CBI process instead. For $0 \leq t \leq \tau$, we define:

$$C(t) = \int_0^t Y_s^{1-\alpha} ds,$$

in case (ii) and set $\alpha = 2$ in case (i).

Proposition 4.3.2. *In both cases (i) and (ii), we have:*

$$\mathbb{P}\left(C(\tau) = \infty\right) = 1.$$

In other words, the additive functional C maps $[0, \tau[$ to $[0, \infty[$.

By convention, if τ is almost surely finite we set $C(t) = C(\tau) = \infty$ for all $t \geq \tau$. Denote by C^{-1} the right continuous inverse of the functional C. This maps $[0, \infty[$ to $[0, \tau[$, a.s. We stress that in most cases, $(R_t, t \geq 0)$ is not a Markov process. Nevertheless, in some cases, through a change of time, the process $(R_t, t \geq 0)$ may be changed into a Markov process. This shall be stated in the following Theorem where the functional C is central.
For every $x, y > 0$, denote by $\mathrm{Beta}(x, y)(dr)$ the finite measure with density

$$r^{x-1}(1-r)^{y-1}1_{(0,1)}(r)dr,$$

and recall that its total mass is given by the Beta function $B(x, y)$.

Theorem 4.3.3. *Let $(Z_t, t \geq 0)$ be the measure-valued process associated to a process $(Y_t(x), x \in [0, 1], t \geq 0)$.*
- In case (i), the process $(R_{C^{-1}(t)})_{t\geq 0}$ is a M-Fleming-Viot process with
$$\Lambda_0(dr) = \beta\delta_0(dr) \text{ and } \Lambda_1(dr) = \sigma^2\delta_0(dr).$$
- In case (ii), the process $(R_{C^{-1}(t)})_{t\geq 0}$ is a M-Fleming-Viot process with
$$\Lambda_0(dr) = c' \,\mathrm{Beta}(2-\alpha, \alpha-1)(dr) \text{ and } \Lambda_1(dr) = c\,\mathrm{Beta}(2-\alpha, \alpha)(dr).$$

The proof requires rather technical arguments on the generators and is given in Section 4.5.

Remark 4.1. – *The CBIs in the statement of Theorem 4.3.3 with $\sigma^2 = \beta$ in case (i) or $c = c'$ in case (ii), are also CBs conditioned on non extinction and are studied further in Section 4.4.*
– *Contrary to the case without immigration, see Theorem 1.1 in [18], we have to restrict ourselves to $\alpha \in (1, 2]$.*

So far, we state that the ratio process $(R_t, t \geq 0)$ associated to $(Z_t, t \geq 0)$, once time changed by C^{-1}, is a M-Fleming-Viot process. Conversely, starting from a M-Fleming-Viot process, we could wonder how to recover the measure-valued CBI process $(Z_t, t \geq 0)$. This lead us to investigate the relation between the time changed ratio process $(R_{C^{-1}(t)}, t \geq 0)$ and the process $(Y_t, t \geq 0)$.

Proposition 4.3.4. *In case (i) of Theorem 4.3.3, the additive functional $(C(t), t \geq 0)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ are independent.*

This proves that in case (i) we need additional randomness to reconstruct M from the M-Fleming-Viot process. On the contrary, in case (ii), the process $(Y_t, t \geq 0)$ is clearly not independent of the ratio process $(R_t, t \geq 0)$, since both processes jump at the same time.
The proof of Propositions 4.3.1, 4.3.2 are given in the next Subsection. Some rather technical arguments are needed to prove Proposition 4.3.4. We postpone its proof to the end of Section 4.5.

4.3.2 Proofs of Propositions 4.3.1, 4.3.2

Proof of Proposition 4.3.1. Let $(X_t(x), t \geq 0)$ denote an α-stable branching process started at x (with $\alpha \in (1, 2]$). Denote ζ its absorption time, $\zeta := \inf\{t \geq 0; X_t(x) = 0\}$. The following construction of the process $(Y_t(0), t \geq 0)$ may be deduced from the expression of the Laplace transform of the CBI process. We shall need the canonical measure $\mathbb{N}$ which is a sigma-finite measure on càdlàg paths and represents informally the "law" of the population generated by one single individual in a CB(Ψ), see Li [95]. We write:

$$(Y_t(0), t \geq 0) = \left(\sum_{i \in \mathcal{I}} X^i_{(t-t_i)_+}, t \geq 0 \right) \tag{4.7}$$

with $\sum_i \delta_{(t_i, X^i)}$ a Poisson random measure on $\mathbb{R}_+ \times \mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity $dt \otimes \mu$, where $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ denotes the space of càdlàg functions, and μ is defined as follows:

- in case (ii), $\mu(dX) = \int \hat{\nu_0}(dx)\mathbb{P}_x(dX)$, where $\mathbb{P}_x$ is the law of a CB(Ψ) with $\Psi(q) = dq^\alpha$. Formula (4.7) may be understood as follows: at the jump times t_i of a pure jump stable subordinator with Lévy measure $\hat{\nu_0}$, a new arrival of immigrants, of size X^i_0, occurs in the population. Each of these "packs", labelled by $i \in \mathcal{I}$, generates its own descendance $(X^i_t, t \geq 0)$, which is a CB(Ψ) process.
- in case (i), $\mu(dX) = \beta\ \mathbb{N}(dX)$, where $\mathbb{N}$ is the canonical measure associated to the CB(Ψ) with $\Psi(q) = \frac{\sigma^2}{2} q^2$. The canonical measure may be thought of as the "law" of the population generated by one single individual. The link with case (ii) is the following: the pure jump subordinator degenerates into a continuous subordinator equal to $(t \mapsto \beta t)$. The immigrants no more arrive by packs, but appear continuously.

Actually, the canonical measure $\mathbb{N}$ is defined in both cases (i) and (ii), and we may always write $\mu(dX) = \Phi(\mathbb{N}(dX))$. The process $(Y_t(0), t \geq 0)$ is a CBI(Ψ, Φ) started at 0. We call $\mathcal{Z}$ the set of zeros of $(Y_t(0), t > 0)$:

$$\mathcal{Z} := \{t > 0; Y_t(0) = 0\}.$$

Denote $\zeta_i = \inf\{t > 0, X^i_t = 0\}$ the lifetime of the branching process X^i. The intervals $]t_i, t_i + \zeta_i[$ and $[t_i, t_i + \zeta_i[$ represent respectively the time where X^i is alive in case (i) and in case (ii) (in this case, we have $X^i_{t_i} > 0$.) Therefore, if we define $\tilde{\mathcal{Z}}$ as the set of the positive real numbers left uncovered by the random intervals $]t_i, t_i + \zeta_i[$, that is:

$$\tilde{\mathcal{Z}} := \mathbb{R}^\star_+ \setminus \bigcup_{i \in \mathcal{I}}]t_i, t_i + \zeta_i[.$$

we have $\mathcal{Z} \subset \tilde{\mathcal{Z}}$ with equality in case (i) only.

The lengths ζ_i have law $\mu(\zeta \in dt)$ thanks to the Poisson construction of $Y(0)$. We now distinguish the two cases:

- Feller case: this corresponds to $\alpha = 2$. We have $\Psi(q) := \frac{\sigma^2}{2} q$ and $\Phi(q) := \beta q$, and thus

$$\mu[\zeta > t] = \beta\ \mathbb{N}[\zeta > t] = \frac{2\beta}{\sigma^2} \frac{1}{t}$$

  see Li [95] p. 62. Using Example 1 p. 180 of Fitzsimmons et al. [52], we deduce that

$$\tilde{\mathcal{Z}} = \emptyset \ \text{ a.s. if and only if } \ \frac{2\beta}{\sigma^2} \geq 1. \tag{4.8}$$

- Stable case: this corresponds to $\alpha \in (1, 2)$. Recall $\Psi(q) := dq^\alpha, \Phi(q) := d'\alpha q^{\alpha-1}$. In that case, we have,

$$\mathbb{N}(\zeta > t) = d^{-\frac{1}{\alpha-1}}[(\alpha - 1)t]^{-\frac{1}{\alpha-1}}.$$

Thus, $\mu[\zeta > t] = \Phi(\mathbb{N}(\zeta > t)) = \frac{\alpha}{\alpha-1}\frac{d'}{d}\frac{1}{t}$. Recall that $\frac{d'}{d} = \frac{c'}{c}$. Therefore, using reference [52], we deduce that

$$\tilde{\mathcal{Z}} = \emptyset \text{ a.s. if and only if } \frac{c'}{c} \geq \frac{\alpha - 1}{\alpha}. \tag{4.9}$$

This allows us to establish the first point of Proposition 4.3.1: we get $\mathcal{Z} \subset \tilde{\mathcal{Z}} = \emptyset$, and the inequality $Y_t(1) \geq Y_t(0)$ for all t ensures that $\tau = \infty$.

We deal now with the second point of Proposition 4.3.1. Assume that $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ or $\frac{\beta}{\sigma^2} < \frac{1}{2}$. By assertions (4.8) and (4.9), we already know that $\tilde{\mathcal{Z}} \neq \emptyset$. However, what we really need is that $\tilde{\mathcal{Z}}$ is a.s. not bounded. To that aim, observe that, in both cases (i) and (ii),

$$\mu[\zeta > s] = \Phi(\mathbb{N}(\zeta > s)) = \frac{\kappa}{s}$$

with $\kappa = \frac{\alpha}{\alpha-1}\frac{d'}{d} = \frac{\alpha}{\alpha-1}\frac{c'}{c} < 1$ if $1 < \alpha < 2$ and $\kappa = \frac{2\beta}{\sigma^2} < 1$ if $\alpha = 2$. Thus $\int_1^u \mu[\zeta > s]ds = \kappa \ln(u)$ and we obtain

$$\exp\left(-\int_1^u \mu[\zeta > s]ds\right) = \left(\frac{1}{u}\right)^{\kappa}.$$

Therefore, since $\kappa < 1$,

$$\int_1^{\infty} \exp\left(-\int_1^u \mu[\zeta > s]ds\right) du = \infty,$$

which implies thanks to Corollary 4 (Equation 17 p 183) of [52] that $\tilde{\mathcal{Z}}$ is a.s. not bounded.

Since $\mathcal{Z} = \tilde{\mathcal{Z}}$ in case (i), the set $\mathcal{Z}$ is a.s. not bounded in that case. Now, we prove that $\mathcal{Z}$ is a.s. not bounded in case (ii). The set $\tilde{\mathcal{Z}}$ is almost surely not empty and not bounded. Moreover this is a perfect set (Corollary 1 of [52]). Since there are only countable points $(t_i, i \in \mathcal{I})$, the set $\tilde{\mathcal{Z}} = \mathcal{Z} \setminus \bigcup_{i \in I}\{t_i\}$ is also uncountable and not bounded.

Last, recall from Subsection 4.2.1 that we may write $Y_t(1) = Y_t(0) + X_t(1)$ for all $t \geq 0$ with $(X_t(1), t \geq 0)$ a CB-process independent of $(Y_t(0), t \geq 0)$. Let $\xi := \inf\{t \geq 0, X_t(1) = 0\}$ be the extinction time of $(X_t(1), t \geq 0)$. Since $\mathcal{Z}$ is a.s. not bounded in both cases (i) and (ii), $\mathcal{Z} \cap (\xi, \infty) \neq \emptyset$, and $\tau < \infty$ almost surely.

Proof of Proposition 4.3.2. Recall that $Y_t(x)$ is the value of the CBI started at x at time t. We will denote by $\tau^x(0) := \inf\{t > 0, Y_t(x) = 0\}$. With this notation, $\tau^1(0) = \tau$ introduced in Section 4.3.1. In both cases (i) and (ii), the processes are self-similar, see Kyprianou and Pardo [80]. Namely, we have

$$(xY_{x^{1-\alpha}t}(1), t \geq 0) \overset{law}{=} (Y_t(x), t \geq 0),$$

where we take $\alpha = 2$ in case (i). Performing the change of variable $s = x^{1-\alpha}t$, we obtain

$$\int_0^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\tau^1(0)} ds\ Y_s(1)^{1-\alpha}. \tag{4.10}$$

According to Proposition 4.3.1, depending on the values of the parameters:

– Either $\mathbb{P}(\tau^x(0) < \infty) = 1$ for every x. Let $x > 1$. Denote $\tau^x(1) = \inf\{t > 0, Y_t(x) \le 1\}$. We have $\mathbb{P}(\tau^x(1) < \infty) = 1$. We have:

$$\int_0^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha} = \int_0^{\tau^x(1)} dt\ Y_t(x)^{1-\alpha} + \int_{\tau^x(1)}^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha}$$

By the strong Markov property applied at the stopping time $\tau^x(1)$, since Y has no negative jumps:

$$\int_{\tau^x(1)}^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\tau^1(0)} dt\ \tilde{Y}_t(1)^{1-\alpha},$$

with $(\tilde{Y}_t(1), t \ge 0)$ an independent copy started from 1. Since

$$\int_0^{\tau^x(1)} dt\ Y_t(x)^{1-\alpha} > 0,\ a.s.,$$

the equality (4.10) is impossible unless both sides of the equality are infinite almost surely. We thus get that $C(\tau) = \infty$ almost surely in that case.

– Either $\mathbb{P}(\tau^x(0) = \infty) = 1$ for every x, on which case we may rewrite (4.10) as follows:

$$\int_0^{\infty} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\infty} ds\ Y_s(1)^{1-\alpha}.$$

Since, for $x > 1$, the difference $(Y_t(x) - Y_t(1), t \ge 0)$ is an α-stable CB-process started at $x - 1 > 0$, we deduce that $C(\tau) = \infty$ almost surely again.

This proves the statement.

Remark 4.2. *The situation is quite different when the CBI process starts at* 0*, in which case the time change also diverges in the neighbourhood of* 0*. The same change of variables as in (4.10) yields, for all* $0 < x < k$*,*

$$\int_0^{\iota^x(k)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\iota^1(k/x)} dt\ Y_t(1)^{1-\alpha},$$

with $\iota^x(k) = \inf\{t > 0, Y_t(x) \ge k\} \in [0, \infty]$*. Letting* x *tend to* 0*, we get* $\iota^1(k/x) \longrightarrow \infty$ *and the right hand side diverges to infinity. Thus, the left hand side also diverges, which implies that:*

$$\mathbb{P}\left(\int_0^{\iota^0(k)} dt\ Y_t(0)^{1-\alpha} = \infty\right) = 1.$$

4.4 Genealogy of the Beta-Fleming-Viot processes

To describe the genealogy associated with stable CBs, Bertoin and Le Gall [16] and Birkner et al. [18] used partition-valued processes called Beta-coalescents. These processes form a subclass of Λ-coalescents, introduced independently by Pitman and Sagitov in 1999. A Λ-coalescent is an exchangeable process in the sense that its law is invariant under the action of any permutation. In words, there is no distinction between the individuals. Although these processes arise as models of genealogy for a wide range of stochastic populations, they are not in general adapted to describe the genealogy of a population with immigration. Recently, a larger class of processes called M-coalescents has been defined in [54] (see Section 5). These processes are precisely those describing the genealogy of M-Fleming-Viot processes.

Remark 4.3. *We mention that the use of the lookdown construction in Birkner et al. [18] may be easily adapted to our framework and yields a genealogy for any conservative CBI. Moreover, other genealogies, based on continuous trees, have been investigated by Lambert [83] and Duquesne [36].*

4.4.1 Background on M-coalescents

Before focusing on the M-coalescents involved in the context of Theorem 4.3.3, we recall their general definition and the duality with the M-Fleming-Viot processes. Contrary to the Λ-coalescents, the M-coalescents are only invariant by permutations letting 0 fixed. The individual 0 represents the immigrant lineage and is distinguished from the others. We denote by $\mathcal{P}^0_\infty$ the space of partitions of $\mathbb{Z}_+ := \{0\} \bigcup \mathbb{N}$. Let $\pi \in \mathcal{P}^0_\infty$. By convention, we identify π with the sequence $(\pi_0, \pi_1, ...)$ of the blocks of π enumerated in increasing order of their smallest element: for every $i \leq j$, $\min \pi_i \leq \min \pi_j$. Let $[n]$ denote the set $\{0, ..., n\}$ and $\mathcal{P}^0_n$ the space of partitions of $[n]$. The partition of $[n]$ into singletons is denoted by $0_{[n]}$. As in Section 2.2, the notation M stands for a pair of finite measures (Λ_0, Λ_1) such that:

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx), \ \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx),$$

where c_0, c_1 are two non-negative real numbers and ν_0, ν_1 are two measures on $[0, 1]$ subject to the same conditions as in Section 4.2.2. Let N_0 and N_1 be two Poisson point measures with intensity respectively $dt \otimes \nu_0$ and $dt \otimes \nu_1$. An M-coalescent is a Feller process $(\Pi(t), t \geq 0)$ valued in $\mathcal{P}^0_\infty$ with the following dynamics.
- At an atom (t, x) of N_1, flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" are merged immediately in one block. At time t, a proportion x share a common parent in the population.
- At an atom (t, x) of N_0, flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" coagulate immediately with the distinguished block. At time t, a proportion x of the population is children of immigrant.

In order to take into account the parameters c_0 and c_1, imagine that at constant rate c_1, two blocks (not containing 0) merge *continuously* in time, and at constant rate c_0, one block (not containing 0) merged with the distinguished one. We refer to Section 4.2 of [54] for a rigorous definition. Let $\pi \in \mathcal{P}^0_n$. The jump rate of an M-coalescent from $0_{[n]}$ to π, denoted by q_π, is given as follows:
- If π has one block not containing 0 with k elements and $2 \leq k \leq n$, then

$$q_\pi = \lambda_{n,k} := \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx).$$

- If the distinguished block of π has $k+1$ elements (counting 0) and $1 \leq k \leq n$ then

$$q_\pi = r_{n,k} := \int_0^1 x^{k-1}(1-x)^{n-k}\Lambda_0(dx).$$

The next duality property is a key result and links the M-Fleming-Viot processes to the M-coalescents. For any π in $\mathcal{P}^0_\infty$, define

$$\alpha_\pi : k \mapsto \text{the index of the block of } \pi \text{ containing } k.$$

We have the duality relation (see Lemma 4 in [53]): for any $p \geq 1$ and $f \in C([0,1]^p)$,

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, ..., x_{\alpha_{\Pi(t)}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)\rho_t(dx_1)...\rho_t(dx_p)\right],$$

where $(\rho_t, t \geq 0)$ is a M-FV started from the Lebesgue measure on $[0,1]$. We establish a useful lemma relating genuine Λ-coalescents and M-coalescents. Consider a Λ-coalescent taking values in the set $\mathcal{P}_\infty^0$; this differs from the usual convention, according to which they are valued in the set $\mathcal{P}_\infty$ of the partitions of $\mathbb{N}$ (see Chapters 1 and 3 of [9] for a complete introduction to these processes). In that framework, Λ-coalescents appear as a subclass of M-coalescents and the integer 0 may be viewed as a typical individual. The proof is postponed in Section 4.4.3.

Lemma 4.4.1. *A M-coalescent, with $M = (\Lambda_0, \Lambda_1)$ is also a Λ-coalescent on $\mathcal{P}_\infty^0$ if and only if*

$$(1-x)\Lambda_0(dx) = \Lambda_1(dx).$$

In that case $\Lambda = \Lambda_0$.

4.4.2 The Beta$(2-\alpha, \alpha-1)$-coalescent

The aim of this Section is to show how a Beta$(2-\alpha, \alpha-1)$-coalescent is embedded in the genealogy of an α-stable CB-process conditioned to be never extinct. Along the way, we also derive the fixed time genealogy of the Feller CBI.

We first state the following straightforward Corollary of Theorem 4.3.3, which gives the genealogy of the ratio process at the random time $C^{-1}(t)$:

Corollary 4.4.2. *Let $(R_t, t \geq 0)$ be the ratio process of a CBI in case (i) or (ii). We have for all $t \geq 0$:*

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, ..., x_{\alpha_{\Pi(t)}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_{C^{-1}(t)}(dx_1)...R_{C^{-1}(t)}(dx_p)\right],$$

where:
– In case (i), $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$,
– In case (ii), $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (c'\mathrm{Beta}(2-\alpha, \alpha-1), c\mathrm{Beta}(2-\alpha, \alpha))$.

In general, we cannot set the random quantity $C(t)$ instead of t in the equation of Corollary 4.4.2. Nevertheless, using the independence property proved in Proposition 4.3.4, we get the following Corollary, whose proof may be found in Section 4.4.3..

Corollary 4.4.3. *In case (i), assume $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$, then for all $t \geq 0$,*

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, ..., x_{\alpha_{\Pi(C(t))}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_t(dx_1)...R_t(dx_p)\right],$$

where $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$, $(Y_t, t \geq 0)$ is a CBI in case (i) independent of $(\Pi(t), t \geq 0)$ and $(C(t), t \geq 0) = \left(\int_0^t \frac{1}{Y_s}ds, t \geq 0\right)$.

We stress on a fundamental difference between Corollaries 4.4.2 and 4.4.3. Whereas the first gives the genealogy of the ratio process R at the random time $C^{-1}(t)$, the second gives the genealogy of the ratio process R at a fixed time t. Notice that we impose the additional assumption that $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ in Corollary 4.4.3 for ensuring that the lifetime is infinite. Therefore, $R_t \neq \Delta$ for all $t \geq 0$, and we may consider its genealogy.

We easily check that the M-coalescents for which $M = (\sigma^2\delta_0, \sigma^2\delta_0)$ and $M = (c\text{Beta}(2-\alpha, \alpha - 1), c\text{Beta}(2-\alpha, \alpha))$ fulfill the conditions of Lemma 4.4.1. Recall from Section 4.3.1 the definitions of the CBIs in case (i) and (ii) .

Theorem 4.4.4. *(i) If the process* $(Y_t, t \geq 0)$ *is a CBI such that* $\sigma^2 = \beta > 0$, $\hat{\nu}_1 = \hat{\nu}_0 = 0$, *then the process* $(\Pi(t/\sigma^2), t \geq 0)$ *defined in Corollary 4.4.2 is a Kingman's coalescent valued in* $\mathcal{P}_\infty^0$.
(ii) If the process $(Y_t, t \geq 0)$ *is a CBI such that* $\sigma^2 = \beta = 0$ *and* $\hat{\nu}_0(dh) = ch^{-\alpha}dh$, $\hat{\nu}_1(dh) = ch^{-\alpha-1}dh$ *for some constant* $c > 0$ *then the process* $(\Pi(t/c), t \geq 0)$ *defined in Corollary 4.4.2 is a Beta(*$2-\alpha, \alpha-1$*)-coalescent valued in* $\mathcal{P}_\infty^0$.

In both cases, the process $(Y_t, t \geq 0)$ involved in that Theorem may be interpreted as a CB-process $(X_t, t \geq 0)$ without immigration ($\beta = 0$ or $c' = 0$) conditioned on non-extinction, see Lambert [85]. We then notice that both the genealogies of the time changed Feller diffusion and of the time changed Feller diffusion conditioned on non extinction are given by the same Kingman's coalescent. On the contrary, the genealogy of the time changed α-stable CB-process is a Beta$(2-\alpha, \alpha)$-coalescent, whereas the genealogy of the time changed α-stable CB-process conditioned on non-extinction is a Beta$(2-\alpha, \alpha-1)$-coalescent. We stress that for any $\alpha \in (1, 2)$ and any borelian B of $[0, 1]$, we have Beta$(2-\alpha, \alpha-1)(B) \geq$ Beta$(2-\alpha, \alpha)(B)$. This may be interpreted as the additional reproduction events needed for the process to be never extinct.

4.4.3 Proofs.

Proof of Lemma 4.4.1. Let $(\Pi'(t), t \geq 0)$ be a Λ-coalescent on $\mathcal{P}_\infty^0$. Let $n \geq 1$, we may express the jump rate of $(\Pi'_{|[n]}(t), t \geq 0)$ from $0_{[n]}$ to π by

$$q'_\pi = \begin{cases} 0 \text{ if } \pi \text{ has more than one non-trivial block} \\ \\ \int_{[0,1]} x^k (1-x)^{n+1-k} x^{-2} \Lambda(dx) \text{ if the non trivial block has } k \text{ elements.} \end{cases}$$

Consider now a M-coalescent, denoting by q_π the jump rate from $0_{[n]}$ to π, we have

$$q_\pi = \begin{cases} 0 \text{ if } \pi \text{ has more than one non-trivial block} \\ \\ \int_{[0,1]} x^k (1-x)^{n-k} x^{-2} \Lambda_1(dx) \text{ if } \pi_0 = \{0\} \text{ and the non trivial block has } k \text{ elements} \\ \\ \int_{[0,1]} x^{k-1} (1-x)^{n+1-k} x^{-1} \Lambda_0(dx) \text{ if } \#\pi_0 = k. \end{cases}$$

Since the law of a Λ-coalescent is entirely described by the family of the jump rates of its restriction on $[n]$ from $0_{[n]}$ to π for π belonging to $\mathcal{P}_n^0$ (see Section 4.2 of [11]), the processes Π and Π' have the same law if and only if for all $n \geq 0$ and $\pi \in \mathcal{P}_n^0$, we have $q_\pi = q'_\pi$, that is if and only if $(1-x)\Lambda_0(dx) = \Lambda_1(dx)$.

Proof of Corollary 4.4.3. Since $C^{-1}(C(t)) = t$,

$$\mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_t(dx_1)...R_t(dx_p)\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(C(t))}(dx_1)...R_{C^{-1}(C(t))}(dx_p)\right].$$

Then, using the independence between $R_{C^{-1}}$ and C, the right hand side above is also equal to:

$$\int \mathbb{P}(C(t) \in ds)\ \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(s)}(dx_1)...R_{C^{-1}(s)}(dx_p)\right].$$

Using Corollary 4.4.2 and choosing $(\Pi(t), t \geq 0)$ independent of $(C(t), t \geq 0)$, we find:

$$\begin{aligned}
&\int \mathbb{P}(C(t) \in ds)\ \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(s)}(dx_1)...R_{C^{-1}(s)}(dx_p)\right] \\
&= \int \mathbb{P}(C(t) \in ds)\ \mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(s)}(1)}, ..., x_{\alpha_{\Pi(s)}(p)}) \delta_0(dx_0) dx_1...dx_p\right] \\
&= \mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, ..., x_{\alpha_{\Pi(C(t))}(p)}) \delta_0(dx_0) dx_1...dx_p\right].
\end{aligned}$$

Remark 4.4. *Notice the crucial rôle of the independence in order to establish Corollary 4.4.3. When this property fails, as in the case (ii), the question of describing the fixed time genealogy of the α-stable CB or CBI remains open. We refer to the discussion in Section 2.2 of Berestycki et. al [7].*

4.5 Proof of Theorem 4.3.3 and Proposition 4.3.4

We first deal with Theorem 4.3.3. The proof of Proposition 4.3.4 is rather technical ans is postponed at the end of this Section. In order to get the connection between the two measure-valued processes $(R_t, t \geq 0)$ and $(Z_t, t \geq 0)$, we may follow the ideas of Birkner *et al.* [18] and rewrite the generator of the process $(Z_t, t \geq 0)$ using the "polar coordinates": for any $\eta \in \mathcal{M}_f$, we define

$$z := |\eta| \text{ and } \rho := \frac{\eta}{|\eta|}.$$

The proof relies on five lemmas. Lemma 4.5.1 establishes that the law of a M-Fleming-Viot process is entirely determined by the generator $\mathcal{F}$ on the test functions of the form $\rho \mapsto \langle \phi, \rho \rangle^m$ with ϕ a measurable non-negative bounded map and $m \in \mathbb{N}$. Lemmas 4.5.2, 4.5.3 and 4.5.5 allow us to study the generator $\mathcal{L}$ on the class of functions of the type $F : \eta \mapsto \frac{1}{|\eta|^m} \langle \phi, \eta \rangle^m$. Lemma 4.5.4 (lifted from Lemma 3.5 of [18]) relates stable Lévy-measures and Beta-measures. We end the proof using results on time change by the inverse of an additive functional. We conclude thanks to a result due to Volkonskiĭ in [127] about the generator of a time-changed process.

Lemma 4.5.1. *The following martingale problem is well-posed: for any function f of the form:*

$$(x_1, ..., x_p) \mapsto \prod_{i=1}^{p} \phi(x_i)$$

with ϕ a non-negative measurable bounded map and $p \geq 1$, the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale.

Proof. Only the uniqueness has to be checked. We shall establish that the martingale problem of the statement is equivalent to the following martingale problem: for any continuous function f on $[0,1]^p$, the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale. This martingale problem is well posed, see Proposition 5.2 of [54]. Notice that we can focus on continuous and symmetric functions since for any continuous f, $G_f = G_{\tilde{f}}$ with $\tilde{f}$ the symmetrized version of f. Moreover, by the Stone-Weierstrass theorem, any symmetric continuous function f from $[0,1]^p$ to $\mathbb{R}$ can be uniformly approximated by linear combination of functions of the form $(x_1,...,x_p) \mapsto \prod_{i=1}^p \phi(x_i)$ for some function ϕ continuous on $[0,1]$. We now take f symmetric and continuous, and let f_k be an approximating sequence. Plainly, we have

$$|G_{f_k}(\rho) - G_f(\rho)| \leq ||f_k - f||_\infty$$

Assume that $(\rho_t, t \geq 0)$ is a solution of the martingale problem stated in the lemma. Since the map $h \mapsto G_h$ is linear, the process

$$G_{f_k}(\rho_t) - \int_0^t \mathcal{F}G_{f_k}(\rho_s)ds$$

is a martingale for each $k \geq 1$. We want to prove that the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale, knowing it holds for each f_k. We will show the following convergence

$$\mathcal{F}G_{f_k}(\rho) \underset{k\to\infty}{\longrightarrow} \mathcal{F}G_f(\rho) \text{ uniformly in } \rho.$$

Recall expressions (1') and (2') in Subsection 4.2.2, one can check that the following limits are uniform in the variable ρ

$$\sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f_k(\mathrm{x}^{i,j}) - f_k(\mathrm{x})]\rho^{\otimes p}(dx) \underset{k\to\infty}{\longrightarrow} \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes p}(dx)$$

and

$$\sum_{1\leq i\leq m} \int_{[0,1]^p} [f_k(\mathrm{x}^{0,i}) - f_k(\mathrm{x})]\rho^{\otimes p}(dx) \underset{k\to\infty}{\longrightarrow} \sum_{1\leq i\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{0,i}) - f(\mathrm{x})]\rho^{\otimes p}(dx).$$

We have now to deal with the terms (3') and (4'). In order to get that the quantity

$$\int_0^1 \nu(dr) \int_0^1 [G_{f_k}((1-r)\rho + r\delta_a) - G_{f_k}(\rho)]\rho(da)$$

converges toward

$$\int_0^1 \nu(dr) \int_0^1 [G_f((1-r)\rho + r\delta_a) - G_f(\rho)]\rho(da),$$

we compute

$$\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle.$$

Since the function $f_k - f$ is symmetric, we may expand the p-fold product $\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle$, this yields

$$\begin{aligned}
&\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle \\
&\qquad = \sum_{i=0}^{p} \binom{p}{i} r^i (1-r)^{p-i} \left(\langle f_k - f, \rho^{\otimes p-i} \otimes \delta_a^{\otimes i}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle \right) \\
&\qquad = pr(1-r)^{p-1} \left(\langle f_k - f, \rho^{\otimes p-1} \otimes \delta_a\rangle - \langle f_k - f, \rho^{\otimes p}\rangle \right) \\
&\qquad\quad + \sum_{i=2}^{p} \binom{p}{i} r^i (1-r)^{p-i} \left(\langle f_k - f, \rho^{\otimes p-i} \otimes \delta_a^{\otimes i}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle \right).
\end{aligned}$$

We use here the notation

$$\langle g, \mu^{\otimes m-i} \otimes \delta_a^{\otimes i}\rangle := \int g(x_1, ..., x_{m-i}, \underbrace{a, ..., a}_{i \text{ terms}})\mu(dx_1)...\mu(dx_{m-i}).$$

Therefore, integrating with respect to ρ, the first term in the last equality vanishes and we get

$$\left| \int_0^1 \rho(da) \left(G_{f-f_k}((1-r)\rho + r\delta_a) - G_{f-f_k}(\rho) \right) \right| \le 2^{p+1} ||f - f_k||_\infty r^2$$

where $||f_k - f||_\infty$ denotes the supremum of the function $|f_k - f|$. Recall that the measure ν_1 verifies $\int_0^1 r^2 \nu_1(dr) < \infty$, moreover the quantity $||f_k - f||_\infty$ is bounded. Thus appealing to the Lebesgue Theorem, we get the sought-after convergence. Same arguments hold for the immigration part (4') of the operator $\mathcal{F}$. Namely we have

$$|G_{f-f_k}((1-r)\rho + r\delta_0) - G_{f-f_k}(\rho)| \le 2^{p+1} r ||f_k - f||_\infty$$

and the measure ν_0 satisfies $\int_0^1 r\nu_0(dr) < \infty$. Combining our results, we obtain

$$|\mathcal{F}G_{f_k}(\rho) - \mathcal{F}G_f(\rho)| \le C||f - f_k||_\infty$$

for a positive constant C independent of ρ. Therefore the sequence of martingales $G_{f_k}(\rho_t) - \int_0^t \mathcal{F}G_{f_k}(\rho_s)ds$ converges toward

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds,$$

which is then a martingale.

Lemma 4.5.2. *Assume that $\hat{\nu_0} = \hat{\nu_1} = 0$ the generator $\mathcal{L}$ of $(Z_t, t \ge 0)$ is reduced to the expressions (1) and (2):*

$$\mathcal{L}F(\eta) = \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db) F''(\eta; a, b) + \beta F'(\eta; 0)$$

Let ϕ be a measurable bounded function on $[0,1]$ and F be the map $\eta \mapsto G_f(\rho) := \langle f, \rho^{\otimes m}\rangle$ with $f(x_1, ..., x_p) = \prod_{i=1}^p \phi(x_i)$. We have the following identity

$$|\eta|\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

for $\eta \ne 0$, where $\mathcal{F}$ is the generator of a M-Fleming-Viot process with reproduction rate $c_1 = \sigma^2$ and immigration rate $c_0 = \beta$, see expressions (1') and (2').

Proof. By the calculations in Section 4.3 of Etheridge [46] (but in a non-spatial setting, see also the proof of Theorem 2.1 p. 249 of Shiga [124]), we get:

$$\frac{\sigma^2}{2}\int_0^1\int_0^1 \eta(da)\delta_a(db)F''(\eta;a,b) = |\eta|^{-1}\frac{\sigma^2}{2}\int_0^1\int_0^1 \frac{\partial^2 G_f}{\partial\rho(a)\partial\rho(b)}(\rho)[\delta_a(db)-\rho(db)]\rho(da)$$
$$= |\eta|^{-1}\sigma^2 \sum_{1\le i<j\le m}\int_{[0,1]^p}[f(\mathrm{x}^{i,j})-f(\mathrm{x})]\rho^{\otimes m}(dx).$$

We focus now on the immigration part. We take f a function of the form $f:(x_1,...,x_m)\mapsto \prod_{i=1}^m \phi(x_i)$ for some function ϕ, and consider $F(\eta):=G_f(\rho)=\langle f,\rho^{\otimes m}\rangle$. We may compute:

$$F(\eta+h\delta_a)-F(\eta) = \left\langle \phi, \frac{\eta+h\delta_a}{z+h}\right\rangle^m - \langle\phi,\rho\rangle^m$$
$$= \sum_{j=2}^m \binom{m}{j}\left(\frac{z}{z+h}\right)^{m-j}\left(\frac{h}{z+h}\right)^j [\langle\phi,\rho\rangle^{m-j}\phi(a)^j - \langle\phi,\rho\rangle^m] \tag{4.11}$$
$$+ m\left(\frac{z}{z+h}\right)^{m-1}\left(\frac{h}{z+h}\right)[\langle\phi,\rho\rangle^{m-1}\phi(a)-\langle\phi,\rho\rangle^m]\,. \tag{4.12}$$

We get that:

$$F'(\eta;a) = \frac{m}{z}\left[\phi(a)\langle\phi,\rho\rangle^{m-1}-\langle\phi,\rho\rangle^m\right].$$

Thus,

$$F'(\eta;0) = |\eta|^{-1}\sum_{1\le i\le m}\int_{[0,1]^p}[f(\mathrm{x}^{0,i})-f(\mathrm{x})]\rho^{\otimes m}(dx)$$

and

$$\int F'(\eta;a)\eta(da) = 0 \tag{4.13}$$

for such function f. This proves the Lemma.

This first lemma will allow us to prove the case (i) of Theorem 4.3.3. We now focus on the case (ii). Assuming that $\sigma^2=\beta=0$, the generator of $(Z_t, t\ge 0)$ reduces to

$$\mathcal{L}F(\eta) = \mathcal{L}_0F(\eta) + \mathcal{L}_1F(\eta) \tag{4.14}$$

where, as in equations (3) and (4) of Subsection 4.2.1,

$$\mathcal{L}_0F(\eta) = \int_0^\infty \hat{\nu_0}(dh)[F(\eta+h\delta_0)-F(\eta)]$$
$$\mathcal{L}_1F(\eta) = \int_0^1 \eta(da)\int_0^\infty \hat{\nu_1}(dh)[F(\eta+h\delta_a)-F(\eta)-hF'(\eta,a)].$$

The following lemma is a first step to understand the infinitesimal evolution of the non-markovian process $(R_t, t\ge 0)$ in the purely discontinuous case.

Lemma 4.5.3. *Let f be a continuous function on $[0,1]^p$ of the form $f(x_1,...,x_p)=\prod_{i=1}^p\phi(x_i)$ and F be the map $\eta\mapsto G_f(\rho)=\langle\phi,\rho\rangle^p$. Recall the notation $\rho:=\eta/|\eta|$ and $z=|\eta|$. We have the identities:*

$$\mathcal{L}_0F(\eta) = \int_0^\infty \hat{\nu_0}(dh)\left[G_f\left([1-\frac{h}{z+h}]\rho+\frac{h}{z+h}\delta_0\right)-G_f(\rho)\right]$$
$$\mathcal{L}_1F(\eta) = z\int_0^\infty \hat{\nu_1}(dh)\int_0^1\rho(da)\left[G_f\left([1-\frac{h}{z+h}]\rho+\frac{h}{z+h}\delta_a\right)-G_f(\rho)\right].$$

Proof. The identity for $\mathcal{L}_0$ is plain, we thus focus on $\mathcal{L}_1$. Combining Equation (13) and the term (4.12) we get

$$\int_0^1 \rho(da)\left[m\left(\frac{z}{z+h}\right)^{m-1}\left(\frac{h}{z+h}\right)[\langle\phi,\rho\rangle^{m-1}\phi(a)-\langle\phi,\rho\rangle^m]-hF'(\eta;a)\right]=0.$$

We easily check from the terms of (4.11) that the map $h\mapsto\int_0^1\rho(da)[F(\eta+h\delta_a)-F(\eta)-hF'(\eta,a)]$ is integrable with respect to the measure $\hat{\nu}_1$. This allows us to interchange the integrals and yields:

$$\mathcal{L}_1F(\eta)=z\int_0^\infty\hat{\nu}_1(dh)\int_0^1\rho(da)\left[G_f\left(\frac{\eta+h\delta_a}{z+h}\right)-G_f(\rho)\right]. \tag{4.15}$$

The previous lemma leads us to study the images of the measures $\hat{\nu}_0$ and $\hat{\nu}_1$ by the map $\phi_z : h\mapsto r:=\frac{h}{h+z}$, for every $z>0$. Denote $\lambda_z^0(dr)=\hat{\nu}_0\circ\phi_z^{-1}$ and $\lambda_z^1(dr)=\hat{\nu}_1\circ\phi_z^{-1}$. The following lemma is lifted from Lemma 3.5 of [18].

Lemma 4.5.4. *There exist two measures ν_0, ν_1 such that $\lambda_z^0(dr)=s_0(z)\nu_0(dr)$ and $\lambda_z^1(dr)=s_1(z)\nu_1(dr)$ for some maps s_0,s_1 from $\mathbb{R}_+$ to $\mathbb{R}$ if and only if for some $\alpha\in(0,2),\alpha'\in(0,1)$ and $c,c'>0$:*

$$\hat{\nu}_1(dx)=cx^{-1-\alpha}dx,\ \hat{\nu}_0(dx)=c'x^{-1-\alpha'}dx.$$

In this case:

$$s_1(z)=z^{-\alpha},\ \nu_1(dr)=r^{-2}c\mathrm{Beta}(2-\alpha,\alpha)(dr)$$

and

$$s_0(z)=z^{-\alpha'},\ \nu_0(dr)=r^{-1}c'\mathrm{Beta}(1-\alpha',\alpha')(dr).$$

Proof. The necessary part is given by the same arguments as in Lemma 3.5 of [18]. We focus on the sufficient part. Assuming that $\hat{\nu}_0,\hat{\nu}_1$ are as above, we have

– $\lambda_z^1(dr)=cz^{-\alpha}r^{-1-\alpha}(1-r)^{-1+\alpha}dr=z^{-\alpha}r^{-2}c\mathrm{Beta}(2-\alpha,\alpha)(dr)$, and thus $s_1(z)=z^{-\alpha}$.
– $\lambda_z^0(dr)=c'z^{-\alpha'}r^{-1-\alpha'}(1-r)^{-1+\alpha'}dr=z^{-\alpha'}r^{-1}c'\mathrm{Beta}(1-\alpha',\alpha')(dr)$ and thus $s_0(z)=z^{-\alpha'}$.

The next lemma allows us to deal with the second statement of Theorem 4.3.3.

Lemma 4.5.5. *Assume that $\sigma^2=\beta=0$, $\hat{\nu}_0(dh)=ch^{-\alpha}1_{h>0}dh$ and $\hat{\nu}_1(dh)=ch^{-1-\alpha}1_{h>0}dh$. Let f be a function on $[0,1]^p$ of the form $f(x_1,...,x_p)=\prod_{i=1}^p\phi(x_i)$, and F be the map $\eta\mapsto G_f(\rho)$. We have*

$$|\eta|^{\alpha-1}\mathcal{L}F(\eta)=\mathcal{F}G_f(\rho),$$

for $\eta\neq0$, where $\mathcal{F}$ is the generator of a M-Fleming-Viot process, with $M=(c'\mathrm{Beta}(2-\alpha,\alpha-1),c\mathrm{Beta}(2-\alpha,\alpha))$, see expressions (3′), (4′).

Proof. Recall Equation (4.14):

$$\mathcal{L}F(\eta)=\mathcal{L}_0F(\eta)+\mathcal{L}_1F(\eta)$$

Recall from Equation (13) that we have $\int_0^1F'(\eta;a)\eta(da)=0$ for $F(\eta)=G_f(\rho)$. Applying Lemma 4.5.3 and Lemma 4.5.4, we get that in the case $\sigma^2=\beta=0$ and $\hat{\nu}_1(dx)=cx^{-1-\alpha}dx,\hat{\nu}_0(dx)=c'x^{-1-\alpha'}dx$:

$$\mathcal{L}F(\eta)=\mathcal{L}G_f(\rho)= \quad s_0(z)\int_0^1 r^{-1}c'Beta(1-\alpha',\alpha')(dr)[G_f((1-r)\rho+r\delta_0)-G_f(\rho)]$$
$$+zs_1(z)\int_0^1 r^{-2}cBeta(2-\alpha,\alpha)(dr)\int_0^1\rho(da)[G_f((1-r)\rho+r\delta_a)-G_f(\rho)].$$

Recalling the expressions (3'), (4'), the factorization $h(z)\mathcal{L}F(\eta)=\mathcal{F}G(\rho)$ holds for some function h if

$$s_0(z)=zs_1(z),$$

if $\alpha'=\alpha-1$. In that case, $h(z)=z^{\alpha-1}$.

We are now ready to prove Theorem 4.3.3. To treat the case (i), replace α by 2 in the sequel. The process $(Y_t,R_t)_{t\geq 0}$ with lifetime τ has the Markov property. The additive functional $C(t)=\int_0^t\frac{1}{Y_s^{\alpha-1}}ds$ maps $[0,\tau)$ to $[0,\infty)$. From Theorem 65.9 of [123] and Proposition 4.3.2, the process $(Y_{C^{-1}(t)},R_{C^{-1}(t)})_{t\geq 0}$ is a strong Markov process with infinite lifetime. Denote by $\mathcal{U}$ the generator of $(Y_t,R_t)_{t\geq 0}$. As explained in Birkner et al. [18] (Equation (2.6) p314), the law of $(Y_t,R_t)_{t\geq 0}$ is characterized by $\mathcal{U}$ acting on the following class of test functions:

$$(z,\rho)\in\mathbb{R}_+\times\mathcal{M}_1\mapsto F(z,\rho):=\psi(z)\langle\phi,\rho\rangle^m$$

for ϕ a non-negative measurable bounded function on $[0,1]$, $m\geq 1$ and ψ a twice differentiable non-negative map. Theorem 3 of Volkonskiĭ, see [127] (or Theorem 1.4 Chapter 6 of [47]) states that the Markov process with generator

$$\tilde{\mathcal{U}}F(z,\rho):=z^{\alpha-1}\mathcal{U}F(z,\rho)$$

coincides with $(Y_{C^{-1}(t)},R_{C^{-1}(t)})_{t\geq 0}$. We establish now that $(R_{C^{-1}(t)},t\geq 0)$ is a Markov process with the same generator as the M-Fleming-Viot processes involved in Theorem 4.3.3. Let $G(z,\rho)=G_f(\rho)=\langle\phi,\rho\rangle^m$ (taking $f:(x_1,...,x_m)\mapsto\prod_{i=1}^m\phi(x_i)$). In both cases (i) and (ii) of Theorem 4.3.3, we have:

$$\begin{aligned} z^{\alpha-1}\mathcal{U}G(z,\rho)&=z^{\alpha-1}\mathcal{L}F(\eta)\text{ with }F:\eta\mapsto G_f(\rho)\\ &=\mathcal{F}G_f(\rho).\end{aligned}$$

First equality holds since we took $\psi\equiv 1$ and the second uses Lemma 4.5.2 and Lemma 4.5.5. Since it does not depend on z, the process $(R_{C^{-1}(t)},t\geq 0)$ is a Markov process, moreover it is a M-Fleming-Viot process with parameters as stated.

Proof of Proposition 4.3.4. Let $(Y_t)_{t\geq 0}$ be a Feller branching diffusion with continuous immigration with parameters (σ^2,β). Consider an independent M-Fleming-Viot $(\rho_t,t\geq 0)$ with $M=(\beta\delta_0,\sigma^2\delta_0)$. We first establish that $(Y_t\rho_{C(t)},0\leq t<\tau)$ has the same law as the measure-valued branching process $(Z_t,0\leq t<\tau)$. Recall that $\mathcal{L}$ denote the generator of $(Z_t,t\geq 0)$ (here only the terms (1) and (2) are considered). Consider $F(\eta):=\psi(z)\langle\phi,\rho\rangle^m$ with $z=|\eta|$, ψ a twice differentiable map valued in $\mathbb{R}_+$ and ϕ a non-negative bounded measurable function. Note that the generator acting on such functions F characterizes the law of $(Z_{t\wedge\tau},t\geq 0)$. First we easily obtain that

$$F'(\eta;0)=\psi'(z)\langle\phi,\rho\rangle^m+m\frac{\psi(z)}{z}[\phi(0)\langle\phi,\rho\rangle^{m-1}-\langle\phi,\rho\rangle^m],$$
$$F''(\eta;a,b)=\psi''(z)\langle\phi,\rho\rangle^m+m\frac{\psi'(z)}{z}\left[(\phi(b)+\phi(a))\langle\phi,\rho\rangle^{m-1}-2\langle\phi,\rho\rangle^m\right]$$
$$+m\frac{\psi(z)}{z^2}\left[(m-1)\phi(a)\phi(b)\langle\phi,\rho\rangle^{m-2}-m(\phi(a)+\phi(b))\langle\phi,\rho\rangle^{m-1}+(m+1)\langle\phi,\rho\rangle^m\right].$$

Simple calculations yield,

$$\mathcal{L}F(\eta) = \left[z\left(\frac{\sigma^2}{2}\psi''(z)\right) + \beta\psi'(z)\right]\langle\phi,\rho\rangle^m$$
$$+\frac{\psi(z)}{z}\left[\sigma^2\frac{m(m-1)}{2}\left(\langle\phi^2,\rho\rangle\langle\phi,\rho\rangle^{m-2} - \langle\phi,\rho\rangle^m\right) + \beta m\left(\phi(0)\langle\phi,\rho\rangle^{m-1} - \langle\phi,\rho\rangle^m\right)\right].$$

We recognize in the first line the generator of $(Y_t, t \geq 0)$ and in the second, $\frac{1}{z}\mathcal{F}G_f(\rho)$ with $f(x_1, ..., x_m) = \prod_{i=1}^m \phi(x_i)$ and $c_0 = \beta, c_1 = \sigma^2$. We easily get that this is the generator of the Markov process $(Y_t\rho_{C(t)}, t \geq 0)$ with lifetime τ. We conclude that it has the same law as $(Z_{t\wedge\tau}, t \geq 0)$. We rewrite this equality in law as follows:

$$(Y_t\ \rho_{C(t)}, 0 \leq t < \tau) \overset{law}{=} (|Z_t|\ R_{C^{-1}(C(t))}, 0 \leq t < \tau), \tag{4.16}$$

with C defined by $C(t) = \int_0^t |Z_s|^{-1}ds$ for $0 \leq t < \tau$ on the right hand side. Since $(C(t), t \geq 0)$ and $(\rho_t, t \geq 0)$ are independent on the left hand side and the decomposition in (4.16) is unique, we have also $(C(t), 0 \leq t < \tau)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ independent on the right hand side.

Concerning the case *(ii)* of Theorem 4.3.3, we easily observe that the presence of jumps implies that such a decomposition of the generator cannot hold. See for instance Equation (2.7) of [18] p344. The processes $(R_{C^{-1}(t)}, t \geq 0)$ and $(Y_t, \geq 0)$ are not independent.

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5

The excursions of the Q-process

5.1 Introduction

The regenerative processes form a broad class of processes which are conveniently described in term of the Poisson point measure of their excursions. We focus in this work on confining the excursions of a regenerative process X with law $\mathbb{P}$ in a certain subset Ω^1, whose complement Ω^0 has a positive and finite excursion measure, denoted by n. Assuming that Ω^0 contains the excursions with infinite length, we have that the probability that X has no excursion in Ω^0 is null, and therefore, conditioning on this event is degenerated. It is well known, after the works of Knight [76], Bertoin and Doney [12], Hirano [67], Najnudel, Roynette and Yor [99], that different approximations of null events may lead to different confined processes. For example, letting T^0 be the *left endpoint* of the first excursion in Ω^0, and the subordinator σ be the inverse local time of X, we may either condition the process X on

$$\{T^0 > \sigma_\ell\} \text{ or on } \{T^0 > t\}$$

and let ℓ and t tend to ∞. We denote the limiting probability measures by $\mathbb{P}^{(loc)}$ and $\mathbb{P}^{(\infty)}$ respectively.

A naive way to confine the process X would be to erase the excursions in Ω^0, or equivalently, to glue together the successive excursions in Ω^1: this results in a sample path with law $\mathbb{P}^{(loc)}$. Consider a sample path under $\mathbb{P}^{(loc)}$. Under a Cramer type assumption, we shall prove that the insertion at rate $n(\Omega^0)$ in the local time scale of long excursions in Ω^1 yields a sample path under $\mathbb{P}^{(\infty)}$. Conversely, from a sample path under $\mathbb{P}^{(\infty)}$, we recover a sample path under $\mathbb{P}^{(loc)}$ by marking the time axis at a certain rate ρ and erasing the marked excursions. Our study will be focused on the inverse local time σ, since the problem of conditioning on $\{T^0 > t\}$ may be reduced to that of conditioning a subordinator to reach a high level t before an independent exponential clock with parameter $n(\Omega^0)$.

Applications are mainly concerned with the case where Ω^0 consists of those excursions hitting some measurable subset E_0 of the state space E. In that case, conditioning by $\{\bar{T}^0 > t\}$ with $\bar{T}^0$ the hitting time of E_0 by X is more natural. Under a Cramer type assumption, we shall give a condition on $\Delta = \bar{T}^0 - T^0$ for ensuring that the conditioned process has still law $\mathbb{P}^{(\infty)}$.

Our results may be seen as a complement of those of Glynn and Thorisson [56, 57], which already conditioned regenerative processes under Cramer type assumptions. The originality of this work relies on the change of filtration technique, and on the subordinator picture, which suggests a new behaviour when the Cramer assumption does not hold.

Let us first consider a subordinator σ. Its state space will be endowed with the natural filtration $\mathcal{G} = (\mathcal{G}_\ell, \ell \geq 0)$ generated by the process σ. We denote by ϕ its Laplace exponent. Our results read as follows. We shall prove that the exponential rate of decay of $\mathbb{P}(\sigma_{\mathbf{e}} > t)$ is:

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(\sigma_{\mathbf{e}} > t)}{t} = \sup\{\lambda \geq 0, \mathbb{E}(\mathrm{e}^{\lambda\sigma_1}) \leq \mathrm{e}^{\kappa}\} := \rho$$

where κ is the parameter of the independent exponential random variable $\mathbf{e}$. Under the following Cramer type assumption: $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{\kappa}$ and $\mathbb{E}(\sigma_1\, \mathrm{e}^{\rho\sigma_1}) < \infty$, and using the renewal Theorem, we also obtain the following reinforcement:

$$\lim_{t\to\infty} \frac{\mathbb{P}(\sigma_{\mathbf{e}} > t-s)}{\mathbb{P}(\sigma_{\mathbf{e}} > t)} = \mathrm{e}^{\rho s}\,.$$

This implies a functional limit theorem: The law of $(\sigma_u, 0 \leq u \leq \ell)$ conditioned on $\{\sigma_{\mathbf{e}} > s\}$ weakly converges as $s \to \infty$ to the restriction on $\mathcal{G}_\ell$ of the h-transformed probability measure of $\mathbb{P}$ for the harmonic function $h_\ell(x) = \mathrm{e}^{\rho x + \phi(-\rho)\ell}$.

Turning now to the case of regenerative process X, we shall prove that the probability measure $\mathbb{P}^{(loc)}$ satisfies

$$\mathbb{P}^{(loc)}(A) = \mathbb{E}(\mathrm{e}^{\ell n(\Omega^0)}, T^0 > \sigma_\ell, A),\ \ A \in \mathcal{G}_\ell.$$

We then introduce a Cramer type assumption, ensuring that there exists a unique $\rho > 0$ such that $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) = 1$. Under this assumption, the limiting probability measure $\mathbb{P}^{(\infty)}$ exists and satisfies:

$$\mathbb{P}^{(\infty)}(A) = \mathbb{E}(\mathrm{e}^{\rho\sigma_\ell}, T^0 > \sigma_\ell, A),\ \ A \in \mathcal{G}_\ell.$$

We shall deduce from these formulas that the regenerative property is preserved under $\mathbb{P}^{(loc)}$ and $\mathbb{P}^{(\infty)}$, and also that the Markov property is preserved if X is a Markov process.

Assuming that X is Markov, and letting $\bar{T}^0$ be the hitting time of E_0, with Ω^0 consisting of those excursions hitting E_0, we have:

$$\mathbb{P}^{(loc)}(A) = \mathbb{E}(\mathrm{e}^{L_t\, n(\Omega^0)}\, h_0(X_t), \bar{T}^0 > t, A),\ \ A \in \mathcal{F}_t$$

and

$$\mathbb{P}^{(\infty)}(A) = \mathbb{E}(\mathrm{e}^{\rho t}\, h_\rho(X_t), \bar{T}^0 > t, A),\ \ A \in \mathcal{F}_t,$$

where

$$h_0(x) = \mathbb{P}_x(\bar{T}^0 > T_a) \text{ and } h_\rho(x) = \mathbb{E}_x(\mathrm{e}^{\rho T_a}, \bar{T}^0 > T_a).$$

As a consequence, the Markov property is preserved under $\mathbb{P}^{(loc)}$ and $\mathbb{P}^{(\infty)}$.

In fact, as the notations h_0 and h_ρ suggest, $\mathbb{P}^{(loc)}$ and $\mathbb{P}^{(\infty)}$ are two particular instances of a parametrized family of probability measures, whose associated sample paths have their excursions confined in Ω^1. In the Markov setting, this family consists of h-transforms of the killed process $X_t \mathbf{1}_{\{\bar{T}^0 \geq t\}}$ *together with* its local time.

The intuition why conditioning on $\{T^0 > t\}$ favours long excursions is the following. The local time is a measure of how much the process has regenerated. Each time the process regenerates, it has a new chance to start an excursion in Ω^0. Therefore, for minimizing these chances we have to reduce the local time and this may be done by favorizing the long excursions of Ω^1.

5.2 Organization of the paper

The paper is organized as follows: Section 5.3 is concerned with the conditioned subordinator. We first derive expressions of the tail $\mathbb{P}(\sigma_{\mathbf{e}} > t)$ of the subordinator σ evaluated at an independent exponential random time $\mathbf{e}$, and use these expressions to study its asymptotic behaviour as t tends to ∞, first under a Cramer type assumption, then under a regular variation assumption. Section 5.4 is an application of Section 5.3 to the problem of confining the excursions of a regenerative process in remote time. The two conditionings, in the local and in the real time, are introduced and studied in detail. Examples are provided in Section 5.5: we apply there our results to the simple random walk and to the Brownian motion.

5.3 A conditioned subordinator

A subordinator is a non-decreasing $\mathbb{R}^+$-valued process with independent and homogeneous increments. It is characterized by its Laplace exponent:

$$\phi(\lambda) = \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\eta(dx),$$

for $\mathtt{d} \geq 0$ and η a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)}(1 \wedge x)\eta(dx) < \infty$. The non-decreasing function ϕ maps $\mathbb{R}$ to $[-\infty, +\infty)$ and we set

$$-\rho_\phi = \inf\{\lambda, \phi(\lambda) > -\infty\}.$$

Notice that ϕ is analytic on $(-\rho_\phi, \infty)$. We set

$$\Theta^\phi = \{\lambda \in \mathbb{R}, \phi(\lambda) > -\infty\}. \tag{5.1}$$

Either $\Theta^\phi = [-\rho_\phi, \infty)$, in which case, $\phi(-\rho_\phi) > -\infty$ by definition, or $\Theta^\phi = (-\rho_\phi, \infty)$, in which case, $\lim_{\lambda \downarrow -\rho_\phi} \phi(\lambda) = -\infty$ by monotone convergence.

Let σ be a subordinator with Laplace exponent ϕ. We denote by $\mathbb{P}_x$ the law of σ started at $x \geq 0$, and by $\mathbb{E}_x$ the associated expectation. We will use $\mathbb{P}$ and $\mathbb{E}$ as a shorthand for $\mathbb{P}_0$ and $\mathbb{E}_0$, respectively. The process σ is the unique Markov process with marginals given by:

$$\mathbb{E}_x(\mathrm{e}^{-\lambda\sigma_\ell}) = \mathbb{E}(\mathrm{e}^{-\lambda(x+\sigma_\ell)}) = \mathrm{e}^{-\lambda x - \ell\phi(\lambda)},\ \ell \geq 0.$$

The filtration $(\mathcal{G}_\ell, \ell \geq 0)$ is the filtration generated by the process σ. We set $\mathcal{G} = \bigcup_{\ell \geq 0} \mathcal{G}_\ell$. We define $\mathbb{P}^s$ as the law of the process σ conditioned on reaching level s before an independent exponential random time $\mathbf{e}$ with parameter $\kappa > 0$:

$$\mathbb{P}^s(\sigma \in A) := \mathbb{P}(\sigma \in A | \sigma_{\mathbf{e}} > s),\ A \in \mathcal{G}.$$

Our aim is to characterize the limit of the family of probability measures $\mathbb{P}^s$ as $s \to \infty$ on $\mathcal{G}_\ell$ for a fixed $\ell \geq 0$.

Setting $f(s) := \mathbb{P}(\sigma_{\mathbf{e}} > s)$, we have, according to the Markov property of σ, for $A \in \mathcal{G}_\ell$:

$$\mathbb{P}^s(A) = \frac{\mathbb{P}(\sigma_{\mathbf{e}} > s, A)}{\mathbb{P}(\sigma_{\mathbf{e}} > s)} = \frac{\mathbb{P}(\sigma_{\mathbf{e}} > s, \sigma_\ell > s, A)}{\mathbb{P}(\sigma_{\mathbf{e}} > s)} + \mathbb{E}\Big(\mathrm{e}^{-\kappa\ell}\,\frac{f(s-\sigma_\ell)}{f(s)}, \sigma_\ell \leq s, A\Big). \tag{5.2}$$

This computation explains why the asymptotic behaviour of the ratio $f(s-t)/f(s)$ as $s \to \infty$ plays a crucial rôle in the analysis. The following quantities will appear to be relevant:

$$\rho := \sup\{\lambda \geq 0, \mathbb{E}(\mathrm{e}^{\lambda\sigma_1}) \leq \mathrm{e}^{\kappa}\} = -\inf\{\lambda \leq 0, \phi(\lambda) \geq -\kappa\}. \tag{5.3}$$

Lemma 5.3.1. *We have $\phi(-\rho) = \max\{\phi(-\rho_\phi), -\kappa\}$, and the process $\left(\mathrm{e}^{\rho\sigma_\ell - \phi(-\rho)\ell}, \ell \geq 0\right)$ is a non-negative $\mathcal{G}_\ell$-martingale.*

Proof. By the monotone convergence Theorem, $\phi(-\lambda_n)$ decreases to $\phi(-\rho)$ for an increasing sequence λ_n converging to ρ. Since we have $\phi(-\lambda_n) \geq -\kappa$, also $\phi(-\rho) \geq -\kappa$ holds. We deduce that $\mathbb{E}(\mathrm{e}^{\rho\sigma_\ell}) = \mathrm{e}^{-\ell\phi(-\rho)}$ and the martingale property of the process $\left(\mathrm{e}^{\rho\sigma_\ell - \phi(-\rho)\ell}, \ell \geq 0\right)$ follows from an application of the Markov property. If $\phi(-\rho_\phi) < -\kappa$, then $\phi(-\rho) = -\kappa$ by continuity of the map $\lambda \to \phi(\lambda)$ on $(-\rho_\phi, \infty)$, to which $-\rho$ belongs. If $\phi(-\rho_\phi) \geq -\kappa$, then $\rho = \rho_\phi < \infty$ and $\phi(-\rho) = \phi(-\rho_\phi)$. □

We may therefore define the h-transformed probability measure $\mathbb{P}^\infty$ of $\mathbb{P}$ for the space-time harmonic function $h(t,x) = \mathrm{e}^{\rho x + \phi(-\rho)t}$.

$$\mathbb{P}^\infty(A) := \mathbb{E}(\mathrm{e}^{\rho\sigma_\ell + \phi(-\rho)\ell}, A),\ A \in \mathcal{G}_\ell. \tag{5.4}$$

Notice that in the special case where the subordinator σ under $\mathbb{P}$ has no exponential moments, then $\rho = 0$, and $\mathbb{P}^\infty$ merely reduces to $\mathbb{P}$. The process with law $\mathbb{P}^\infty$ is still a subordinator with Laplace exponent:

$$\phi^\infty(\lambda) = \phi(\lambda - \rho) - \phi(-\rho) = \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\, \mathrm{e}^{\rho x}\, \eta(dx).$$

The probability measure $\mathbb{P}^\infty$ is an instance of an Esscher transform of σ under $\mathbb{P}$. When $-\kappa < \phi(-\rho)$, the subordinator with law $\mathbb{P}^\infty$ has no exponential moments.

Remark 5.3.2. Since

$$\phi^\infty(\lambda) = \phi(\lambda) + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})(\mathrm{e}^{\rho x} - 1)\eta(dx),$$

$\mathbb{P}^\infty$ is the law of the sum $\sigma + \sigma^i$, where σ has law $\mathbb{P}$ and σ^i an independent pure jump subordinator with Lévy measure $(\mathrm{e}^{\rho x} - 1)\eta(dx)$, in fact a compound Poisson process since $\int_{(0,\infty)}(\mathrm{e}^{\rho x} - 1)\eta(dx) < \infty$.

5.3.1 Identities involving the potential measure

We first introduce the potential measure U^∞ on $(0,\infty)$ under $\mathbb{P}^\infty$, defined by:

$$U^\infty(dx) = \int_{(0,\infty)} d\ell\, \mathbb{P}^\infty(\sigma_\ell \in dx). \tag{5.5}$$

For a measure μ on $\mathbb{R}^+$, $\theta_s(\mu)$ will denote the push forward measure of μ by the shift operator $\theta_s : t \to t + s$, satisfying

$$\theta_s(\mu)([0,t]) = \mu([s, s+t]) \text{ for all } t \geq 0.$$

Proposition 5.3.3. *Let* $0 < t < s$.

– *Assume* $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{\kappa}$, *or, equivalently,* $\phi(-\rho) = -\kappa$. *Then the following identity holds:*

$$f(s) = \kappa\,\mathrm{e}^{-\rho s} \int_0^\infty \mathrm{e}^{-\rho x}\ \theta_s(U^\infty)(dx).$$

– *Assume* $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) < \mathrm{e}^{\kappa}$, *or, equivalently,* $\phi(-\rho) > -\kappa$. *Let* $\mathbf{e}'$ *and* $\mathbf{e}''$ *be two independent exponential random variables, independent of* σ *under* $\mathbb{P}^\infty$, *with parameter* $\kappa + \phi(-\rho)$ *and* ρ *respectively. Then:*

$$f(s) = \frac{\kappa}{\kappa + \phi(-\rho)}\,\mathrm{e}^{-\rho s}\,\mathbb{P}^\infty(s < \sigma_{\mathbf{e}'} < s + \mathbf{e}'').$$

Proof. Assume that $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{\kappa}$. We compute:

$$\mathbb{P}(\sigma_{\mathbf{e}} > s) = \int_{(0,\infty)} d\ell\,\kappa\,\mathrm{e}^{-\kappa\ell} \int_{(s,\infty)} \mathbb{P}(\sigma_\ell \in dx) = \kappa \int_{(s,\infty)} \mathrm{e}^{-\rho x}\,U^\infty(dx) = \kappa\,\mathrm{e}^{-\rho s} \int_{(0,\infty)} \mathrm{e}^{-\rho x}\,\theta_s(U^\infty)(dx), \tag{5.6}$$

using the definition (5.4) and (5.5) of $\mathbb{P}^\infty$ and U^∞ respectively, the equality $\phi(-\rho) = -\kappa$ and the Fubini Theorem at the second equality, and performing a basic change of variable at the third equality.

Assume now that $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) < \mathrm{e}^{\kappa}$. A similar computation yields that:

$$\mathbb{P}(\sigma_{\mathbf{e}} > s) = \int_{(0,\infty)} d\ell\,\kappa\,\mathrm{e}^{-\kappa\ell} \int_{(s,\infty)} \mathbb{P}(\sigma_\ell \in dx) = \mathrm{e}^{-\rho s} \int_{(0,\infty)} d\ell\,\kappa\,\mathrm{e}^{-(\kappa+\phi(-\rho))\ell} \int_{(0,\infty)} \mathrm{e}^{-\rho x}\,\mathbb{P}^\infty(\sigma_\ell - s \in dx),$$

and therefore:

$$\mathbb{P}(\sigma_{\mathbf{e}} > s) = \mathrm{e}^{-\rho s}\,\frac{\kappa}{\kappa + \phi(-\rho)}\,\mathbb{E}^\infty(\mathrm{e}^{-\rho(\sigma'_{\mathbf{e}} - s)}, \sigma_{\mathbf{e}'} > s) = \mathrm{e}^{-\rho s}\,\frac{\kappa}{\kappa + \phi(-\rho)}\,\mathbb{P}^\infty(s < \sigma_{\mathbf{e}'} < s + \mathbf{e}'').$$

□

5.3.2 A Cramer type assumption

In order to apply the renewal theorem to the measure U^∞, we shall need the following Cramer type assumption, which implies that $\rho > 0$.

(C) The number $\rho \geq 0$ satisfies $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{\kappa}$ and $\mathbb{E}(\sigma_1\,\mathrm{e}^{\rho\sigma_1}) < \infty$.

This is equivalent to assuming that $\phi(-\rho) = -\kappa$ and $\phi'(-\rho) < \infty$. Notice that assumption (C) is a property of the couple (σ, κ). If there exists $h > 0$ such that $\mathbb{P}(\sigma_t \in h\mathbb{N}$ for every $t \geq 0) = 1$, the subordinator is said to be lattice and the maximal such h is then called the span of the subordinator. Then we set:

$$\varrho = \begin{cases} \rho & \text{if } \sigma \text{ is non lattice,} \\ (\mathrm{e}^{\rho h} - 1)/h & \text{if } \sigma \text{ is lattice with span } h > 0. \end{cases}$$

Proposition 5.3.4. *Assume that* (C) *holds. Then the quantity* $f(s) = \mathbb{P}(\sigma_{\mathbf{e}} > s)$ *satisfies:*

$$f(s)\,\mathrm{e}^{\rho s} \to -\frac{1}{\varrho}\frac{\phi(-\rho)}{\phi'(-\rho)},$$

as s *goes to infinity, through the set* $\{nh, n \in \mathbb{N}\}$ *in the lattice case.*

The proof is an adaptation of that of Bertoin and Doney [12].

Proof. Let $\sigma^{(i)}$ be distributed as σ_γ, where σ has law $\mathbb{P}$ and γ is an independent Gamma random variable with parameter i. We notice that U^∞ may be expressed as follows:

$$U^\infty(dx) = \int_{(0,\infty)} d\ell\, \mathbb{P}^\infty(\sigma_\ell \in dx) = \int_{(0,\infty)} d\ell \sum_{i\geq 1} \mathrm{e}^{-\ell} \frac{\ell^{i-1}}{(i-1)!} \mathbb{P}^\infty(\sigma_\ell \in dx) = \sum_{i\geq 1} \mathbb{P}^\infty(\sigma^{(i)} \in dx).$$

In words, the potential measure also appears as the renewal measure associated with interarrival times with law $\sigma^{(1)}$. Under assumption (C), $\sigma^{(1)}$ has finite expectation under $\mathbb{P}^\infty$, equal to:

$$\mathbb{E}^\infty(\sigma^{(1)}) = \mathbb{E}^\infty(\sigma_1) = (\phi^\infty)'(0) = \phi'(-\rho). \tag{5.7}$$

Recall from Proposition 5.3.3 the following equality:

$$\mathrm{e}^{\rho s}\, \mathbb{P}(\sigma_{\mathbf{e}} > s) = \kappa \int_{(0,\infty)} \mathrm{e}^{-\rho x}\;\; \theta_s(U^\infty)(dx).$$

Now, the function $x \to \mathrm{e}^{-\rho x}$ is nonnegative, decreasing, and Lebesgue integrable. Therefore, using 4.4.1 of Daley and Vere-Jones [28], it is directly Riemann integrable, see formula (5.8) for a definition. Using (5.7) and the key renewal theorem, we obtain, if σ is non lattice:

$$\lim_{s\to\infty} \mathrm{e}^{\rho s}\, \mathbb{P}(\sigma_{\mathbf{e}} > s) = \frac{\kappa}{\varrho \phi'(-\rho)} = -\frac{1}{\varrho} \frac{\phi(-\rho)}{\phi'(-\rho)},$$

and if σ is lattice, the same key renewal theorem applies, but we have to restrict ourselves to $s \in \{nh, n \in \mathbb{N}\}$:

$$\lim_{n\to\infty} \mathrm{e}^{\rho nh}\, \mathbb{P}(\sigma_{\mathbf{e}} > nh) = \frac{\kappa}{\phi'(-\rho)} \sum_{n\in\mathbb{N}} h\, \mathrm{e}^{-\rho nh} = -\frac{1}{\varrho} \frac{\phi(-\rho)}{\phi'(-\rho)}.$$

□

Proposition 5.3.5. *Assume that* (C) *holds. Fix* $\ell \geq 0$. *We have:*

$$\lim_{s\to\infty} \mathbb{P}^s(A) = \mathbb{P}^\infty(A), \quad A \in \mathcal{G}_\ell.$$

This entails the weak convergence of the probability measures $\mathbb{P}^s_{|\mathcal{G}_\ell}$ towards $\mathbb{P}^\infty_{|\mathcal{G}_\ell}$, as $s \to \infty$.

Proof. Let $0 \leq s_0 \leq s$, and $A \in \mathcal{G}_\ell$. We have from the definition of $\mathbb{P}^s$:

$$\mathbb{P}^s(A) \geq \mathbb{P}^s(A, \sigma_\ell \leq s_0) = \mathbb{E}\Big(\, \mathrm{e}^{-\kappa\ell} \frac{f(s-\sigma_\ell)}{f(s)}, A, \sigma_\ell \leq s_0\Big).$$

From Fatou Lemma, we have:

$$\liminf_{s\to\infty} \mathbb{E}\Big(\, \mathrm{e}^{-\kappa\ell} \frac{f(s-\sigma_\ell)}{f(s)}, A, \sigma_\ell \leq s_0\Big) \geq \mathbb{E}\Big(\, \mathrm{e}^{-\kappa\ell} \liminf_{s\to\infty} \frac{f(s-\sigma_\ell)}{f(s)}, A, \sigma_\ell \leq s_0\Big).$$

From Proposition 5.3.4, we have $\liminf_{s\to\infty} \frac{f(s-\sigma_\ell)}{f(s)} = \mathrm{e}^{\rho\sigma_\ell}$ a.s., in both lattice and non-lattice case: indeed, careful inspection reveals that this limit also holds in the lattice case since σ_ℓ a.s. belongs to the set $h\mathbb{N}$. This implies:

$$\liminf_{s\to\infty} \mathbb{P}^s(A) \geq \mathbb{P}^\infty(A, \sigma_\ell \leq s_0).$$

This inequality holds for every $s_0 \geq 0$, whence:

$$\liminf_{s\to\infty} \mathbb{P}^s(A) \geq \mathbb{P}^\infty(A).$$

Considering now the complementary set of A instead of A, we obtain the converse inequality, since $\mathbb{P}^\infty$ is a probability measure:

$$\limsup_{s\to\infty} \mathbb{P}^s(A) \leq \mathbb{P}^\infty(A).$$

This completes the proof. □

As a straightforward corollary, we have that $\lim_{s\to\infty} \mathbb{P}^s(\mathbf{e} < \ell) = 0$. Indeed, for $s \geq s_0$,

$$0 \leq \mathbb{P}^s(\mathbf{e} < \ell) \leq \mathbb{P}^s(\sigma_\ell > s) \leq \mathbb{P}^s(\sigma_\ell > s_0) \to \mathbb{P}^\infty(\sigma_\ell > s_0) \text{ as } s \to \infty,$$

using Proposition 5.3.5 with $A = \{\sigma_\ell > s_0\}$ for the limit. We conclude using that

$$\mathbb{P}^\infty(\sigma_\ell > s_0) \to 0 \text{ as } s_0 \to \infty.$$

A function $g : \mathbb{R} \to \mathbb{R}$ is said directly Riemann integrable when:

$$\sum_{n\in\mathbb{Z}} hg_-^h(nh) \text{ and } \sum_{n\in\mathbb{Z}} hg_+^h(nh) \tag{5.8}$$

converge to a common finite limit as $h \to 0$, where:

$$g_-^h(x) = \inf_{0\leq h'\leq h} g(x - h') \text{ and } g_+^h(x) = \sup_{0\leq h'\leq h} g(x - h').$$

Let Δ be a real valued random variable defined under $\mathbb{P}$, independent of σ. We introduce the assumption:

(C') $x \mapsto g(x) := \mathbb{P}(\Delta > x)\, \mathrm{e}^{\rho x}$ is directly Riemann integrable.

and the notation:

$$\delta = \begin{cases} \int_{(-\infty,\infty)} \mathbb{P}(\Delta > x)\, \mathrm{e}^{\rho x}\, dx = \mathbb{E}(\mathrm{e}^{\rho\Delta})/\rho & \text{if } \sigma \text{ is non lattice,} \\ h \sum_{n\in\mathbb{Z}} \mathbb{P}(\Delta > nh)\, \mathrm{e}^{\rho nh} & \text{if } \sigma \text{ is lattice with span } h > 0. \end{cases}$$

Last, we define:

$$\bar{f}(s) = \mathbb{P}(\sigma_{\mathbf{e}} + \Delta > s) \text{ and } \bar{\mathbb{P}}^s(A) = \mathbb{P}(A | \sigma_{\mathbf{e}} + \Delta > s),\ A \in \mathcal{G}.$$

Proposition 5.3.6. *Assume that* (C) *and* (C') *hold. Then:*

$$\bar{f}(s)\, \mathrm{e}^{\rho s} \to \frac{\kappa\, \delta}{\phi'(-\rho)}.$$

as s goes to infinity, through the set $\{nh, n \in \mathbb{N}\}$ in the lattice case.

Proof. We adapt the calculus (5.6) to our new context:

$$\bar{f}(s) = \mathbb{P}(\sigma_{\mathbf{e}} + \Delta > s) = \kappa \int_{(0,\infty)} \mathbb{P}(\Delta > s - x)\, \mathrm{e}^{-\rho x}\, U^{\infty}(dx) = \kappa\, \mathrm{e}^{-\rho s} \int_{(0,\infty)} g(s-x) U^{\infty}(dx).$$

The key renewal theorem and the equality (5.7), now yield:

$$\phi'(-\rho) \int_{(0,\infty)} g(s-x) U^{\infty}(dx) \to \delta,$$

as $s \to \infty$, through the set $\{nh, n \in \mathbb{N}\}$ in the lattice case.

□

We deduce the following result, whose proof, similar to that of Proposition 5.3.5, is eluded.

Proposition 5.3.7. *Assume that* (C) *and* (C') *hold. Fix* $\ell \geq 0$. *We have:*

$$\lim_{s \to \infty} \bar{\mathbb{P}}^{s}(A) = \mathbb{P}^{\infty}(A), \;\; A \in \mathcal{G}_{\ell}.$$

This proves the robustness of our conditioning, since conditioning on the event $\{\sigma_{\mathbf{e}} > s\}$ or on the event $\{\sigma_{\mathbf{e}} + \Delta > s\}$ result in the same limiting distribution $\mathbb{P}^{\infty}$ as $s \to \infty$ when (C) and (C') are satisfied.

5.3.3 The general case

We now turn to the general case. We first enounce some remarks of general interest and then discuss the analogue of Proposition 5.3.4.

Lemma 5.3.8. *The quantity* $f(s) = \mathbb{P}(\sigma_{\mathbf{e}} > s)$ *satisfies:*

$$f(s+t) \geq f(t) f(s), \;\; s, t \geq 0.$$

Proof. Let $s, t \geq 0$. We denote by L the right-continuous inverse of the subordinator σ,

$$L_t = \inf \{\ell \geq 0, \sigma_{\ell} > t\},$$

and by $o^{+}(t) = \sigma_{L_t} - t \geq 0$ the overshoot at level t. We then compute:

$$\mathbb{P}(\sigma_{\mathbf{e}} > t+s) = \mathbb{P}(\mathbf{1}_{\{\sigma_{\mathbf{e}} > t\}} \mathbb{P}_{\sigma_{L_t}}(\sigma_{\mathbf{e}} > t+s)) = \mathbb{P}(\mathbf{1}_{\{\sigma_{\mathbf{e}} > t\}} \mathbb{P}_{o^{+}(t)}(\sigma_{\mathbf{e}} > s)) \geq \mathbb{P}(\sigma_{\mathbf{e}} > t) \mathbb{P}(\sigma_{\mathbf{e}} > s).$$

since $o^{+}(t) \geq 0$ by definition. □

We now prove that ρ yields the exponential rate of decay of $f(s)$ in the general case:

Proposition 5.3.9. *We have:*

$$\lim_{s \to \infty} \frac{-\log f(s)}{s} = \rho. \tag{5.9}$$

Proof. The function $s \to -\log f(s)$ is subadditive by Lemma 5.3.8. The existence of the limit in the left-hand side of (5.9) follows. Notice also that, under assumption (C), the identification of the limit with ρ follows from Proposition 5.3.4. In the general case, we introduce σ^ε a subordinator with Laplace exponent:

$$\phi^\varepsilon(\lambda) = \mathtt{d}\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) 1_{\{\varepsilon x < 1\}} \eta(dx).$$

Assumption (C) is satisfied by $(\sigma^\varepsilon, \kappa)$. We then apply Proposition 5.3.4:

$$\lim_{s \to \infty} \frac{-\log \mathbb{P}(\sigma^\varepsilon_{\mathbf{e}} > s)}{s} = \rho^\varepsilon,$$

where ρ^ε is the unique solution of the equation: $\phi^\varepsilon(-\lambda) = -\kappa$. We also introduce an exponential random variable $\mathbf{f}$ with parameter γ, $\gamma < -\phi(-\rho)$, independent of σ. Assumption (C) also holds for (σ, γ) by definition of γ and another application of Proposition 5.3.4 yields that:

$$\lim_{s \to \infty} \frac{-\log \mathbb{P}(\sigma_{\mathbf{f}} > s)}{s} = \rho^\gamma,$$

where ρ^γ is the unique solution to $\phi(-\rho^\gamma) = -\gamma$. Since $\sigma^\varepsilon \leq \sigma$ and $\mathbf{e} \leq \mathbf{f}$ for the stochastic order, we have:

$$\mathbb{P}(\sigma^\varepsilon_{\mathbf{e}} > s) \leq \mathbb{P}(\sigma_{\mathbf{e}} > s) \leq \mathbb{P}(\sigma_{\mathbf{f}} > s).$$

Notice that

$$\rho^\varepsilon = \lim_{s \to \infty} \frac{-\log \mathbb{P}(\sigma^\varepsilon_{\mathbf{e}} > s)}{s} \geq \lim_{s \to \infty} \frac{-\log \mathbb{P}(\sigma_{\mathbf{e}} > s)}{s} \geq \lim_{s \to \infty} \frac{-\log \mathbb{P}(\sigma_{\mathbf{f}} > s)}{s} = \rho^\gamma.$$

We now let ε tend to 0. Let $\lambda \leq 0$. By monotone convergence, we have that $\varepsilon \to \phi^\varepsilon(\lambda)$ is a non increasing function which converges to $\phi(\lambda)$ as $\varepsilon \to 0$. Therefore ρ^ε decreases to ρ. Similarly, as $\gamma \to -\phi(-\rho)$, ρ^γ increases to ρ' satisfying $\phi(-\rho') = \phi(-\rho)$. This implies $\rho = \rho'$. □

We consider the assumption:

(R) $f(s)/f(s-t) \to \mathrm{e}^{-\rho t}$ as $s \to \infty$.

We already now that (R) is satisfied under (C), but we were unable to find general conditions ensuring that (R) is satisfied. The following Remark enounces a particular condition for checking (R) when (C) is not satified.

Remark 5.3.10. A function $\ell : (0, \infty) \to (0, \infty)$ is slowly varying at $+\infty$ if it satisfies for every $\lambda > 0$:

$$\frac{\ell(\lambda s)}{\ell(s)} \to 1 \text{ as } s \to \infty.$$

Assume that there exists a slowly varying function ℓ at $+\infty$ and some constant $0 < \alpha \leq 1$ such that $\phi(\lambda) \sim \ell(1/\lambda)\lambda^\alpha$ as $\lambda \to 0+$. Then we have the following asymptotic equivalence

$$\mathbb{P}(\sigma_{\mathbf{e}} > s) \sim \frac{(1-\alpha)\ell(s)}{\Gamma(2-\alpha)\,\kappa} s^{-\alpha} \text{ as } s \to \infty,$$

and (R) is satisfied with $\rho = 0$.

This may be established as follows. First, we compute, for $\lambda \geq 0$, the Laplace transform of $\sigma_{\mathbf{e}}$:

$$\mathbb{E}(\mathrm{e}^{-\lambda\sigma_{\mathbf{e}}}) = \int_0^\infty d\ell\, \kappa\, \mathrm{e}^{-\kappa\ell}\, \mathbb{E}(\mathrm{e}^{-\lambda\sigma_\ell}) = \frac{\kappa}{\kappa+\phi(\lambda)}.$$

We then deduce that:

$$\lambda \int_0^\infty ds\, \mathbb{P}(\sigma_{\mathbf{e}} > s)\, \mathrm{e}^{-\lambda s} = \mathbb{E}(1-\mathrm{e}^{-\lambda\sigma_{\mathbf{e}}}) = \frac{\phi(\lambda)}{\kappa+\phi(\lambda)} \sim \frac{\ell(1/\lambda)\,\lambda^\alpha}{\kappa} \text{ as } \lambda \to 0+\,.$$

The Tauberian theorem on page 10 of [10] allows to deduce that:

$$\int_0^t ds\, \mathbb{P}(\sigma_{\mathbf{e}} > s) \sim \frac{\ell(t)}{\Gamma(2-\alpha)\,\kappa} t^{1-\alpha} \text{ as } t \to \infty.$$

Then the monotone density Theorem, see [10] on page 10, gives the result:

$$\mathbb{P}(\sigma_{\mathbf{e}} > s) \sim \frac{(1-\alpha)\ell(s)}{\Gamma(2-\alpha)\,\kappa} s^{-\alpha} \text{ as } s \to \infty.$$

Now, according to see [10], page 9, the slowly varying function ℓ may be represented under the form:

$$\ell(s) = \exp\left(c(s) + \int_1^s du\, \frac{\varepsilon(u)}{u}\right),$$

for c and $\varepsilon : (0,\infty) \to \mathbb{R}$ two bounded measurable functions admitting limits at $+\infty$, this limit being null in the case of ε. Therefore, for any $t \geq 0$,

$$\lim_{s\to\infty} \ell(s-t)/\ell(s) = 1,$$

and we may conclude that (R) is satisfied with $\rho = 0$.

The assumption (R) allows to obtain the convergence of the probability measures $\mathbb{P}^s$ in the following sense:

Proposition 5.3.11. *Assume* (R)*. Let* $\ell, s_0 \geq 0$*. We have:*

$$\lim_{s\to\infty} \mathbb{P}^s(A, \sigma_\ell \leq s_0) = \mathrm{e}^{-(\kappa+\phi(-\rho))\ell}\ \mathbb{P}^\infty(A, \sigma_\ell \leq s_0),\ \ A \in \mathcal{G}_\ell.$$

We stress that, comparing with Proposition 5.3.5, the convergence here holds towards a defective probability measure if $\kappa + \phi(-\rho) > 0$. This loss of mass may be interpreted as a jump to infinity of the subordinator.

Remark 5.3.12. We refer to Griffin [59] for a recent work on a functional limit theorem for a Lévy process conditioned on reaching a high level at a *fixed* time. The behaviour of the conditioned Lévy process bears some resemblance with that of our subordinator, but the assumption are of different nature. Griffin indeed works under the assumption that the tail of the Lévy measure is convolution equivalent, which allows him to get precise asymptotic on the tail $\mathbb{P}(\sigma_t > s)$ as $s \to \infty$ and to check that (R) is satisfied.

Proof. Let $0 \leq s_0 \leq s$. We have from the definition of $\mathbb{P}^s$:

$$\mathbb{P}^s(A, \sigma_\ell \leq s_0) = \mathbb{E}\left(\mathrm{e}^{-\kappa\ell}\, \frac{f(s-\sigma_\ell)}{f(s)}, A, \sigma_\ell \leq s_0\right).$$

We have the following bound, deduced from Lemma 5.3.8:

$$\frac{f(s-\sigma_\ell)}{f(s)} \le \frac{1}{f(\sigma_\ell)} \le \frac{1}{f(s_0)} \text{ on } \{\sigma_\ell \le s_0\}.$$

The dominated convergence Theorem now implies:

$$\lim_{s\to\infty} \mathbb{E}\Big(\mathrm{e}^{-\kappa\ell} \frac{f(s-\sigma_\ell)}{f(s)}, A, \sigma_\ell \le s_0 \Big) = \mathbb{E}\Big(\mathrm{e}^{-\kappa\ell} \lim_{s\to\infty} \frac{f(s-\sigma_\ell)}{f(s)}, A, \sigma_\ell \le s_0 \Big).$$

Therefore, using assumption (R),

$$\lim_{s\to\infty} \mathbb{P}^s(A, \sigma_\ell \le s_0) = \mathrm{e}^{-(\kappa+\phi(-\rho))\ell}\, \mathbb{P}^\infty(A, \sigma_\ell \le s_0).$$

□

5.4 Application to confined regenerative processes

Excursion theory gives a labelling of the excursions of a regenerative process, such that the point measure of the excursions together with their labels builds a Poisson point measure. The label we shall associate to an excursion is called its local time. We propose in this Section a rapid review of the topic, based on the exposition in Kallenberg [71], see also Blumenthal [20].

Let $X = (X_t, t \ge 0)$ be a càdlàg process with values in a Polish space E, and $a \in E$. Since we will need to define other random variables, we shall assume that X is a random process defined on some abstract underlying probability space with probability measure $\mathbb{P}$, and will use $\mathbb{P}_x$ to indicate that the process X is started at $x \in E$, and simply $\mathbb{P}$ if $x = a$. The push forward measure $\mathbb{P}_x(X \in \cdot)$ is then a probability measure on the set Ω of càdlàg paths on E and we shall equip Ω with the the sigma-field $\mathcal{F}_t = \sigma(X_s, s \le t)$. We also set $\mathcal{F} = \bigcup_{t\ge0} \mathcal{F}_t$.

We assume that the process X under $\mathbb{P}$ is regenerative at a, in the sense that for every $\mathcal{F}$-stopping time T, and for every $x \in E$, $A \in \mathcal{F}_T$ and $B \in \mathcal{F}$,

$$\mathbb{P}_x(X \in A, \theta_T(X) \in B, T < \infty, X_T = a) = \mathbb{P}_x(X \in A, T < \infty, X_T = a)\mathbb{P}_a(X \in B),$$

where θ_t denotes the shift operator, acting on Ω, according to $\theta_t(\omega) = \omega(t + \cdot)$ for $\omega \in \Omega$. The set $\mathcal{Z} = \{t \ge 0, X_t = a\}$ is called the regenerative set. The interior of $\mathbb{R}^+ \setminus \mathcal{Z}$ may be written as the union of maximal open intervals, and $\mathbb{R}^+ \setminus \mathcal{Z}$ as a union of maximal intervals open to the right since X is right-continuous. Each of these intervals is of the form (u, v) or $[u, v)$, and is associated with a stopped càdlàg path e as follows:

$$e = (X_{(u+s)\wedge v}, s \ge 0).$$

The path e belongs to the set of excursions paths:

$$\Omega^{\mathbf{e}} = \{\omega \in \Omega, (\omega(s) = a \text{ and } s > 0) \Rightarrow \omega(t) = a \text{ for all } t \ge s\}.$$

The set $\Omega^{\mathbf{e}}$, as a subset of Ω, will be equipped with the trace sigma algebra $\mathcal{F}^e$. The excursions $(e^i, i \in \mathcal{I})$ of X may first be labelled using a countable set. According to Proposition 22.7 of [71], the following dichotomy holds: either a.s. all points of $\mathcal{Z}$ are isolated, or a.s. none of them is. Let $T_a = \inf\{t > 0, X_t = a\}$, and $R_a = \inf\{t > 0, X_t \ne a\}$, and define the recurrence time as follows:

$$R_a + T_a \circ \theta_{R_a}.$$

Fix $h > 0$, or $h \geq 0$ if the recurrence time is positive a.s. Then $C_h = \text{Card}\,\{i, T_a(e^i) > h\}$ is a geometric random variable, and the C_h excursions e^i for which $\{T_a > h\}$ are independent and identically distributed according to a distribution we shall denote by n_h. According to Lemma 22.10 of [71], there exists a measure n on Ω^{e} such that $n(T_a > h) < \infty$ for every $h > 0$ and

$$n(\cdot \mid T_a > h) = n_h(\cdot).$$

This measure n is unique up to a normalization constant, with finite mass if the recurrence time is a.s. positive under $\mathbb{P}$.

Assume we are in the case where a.s. none of the points of $\mathcal{Z}$ are isolated. Then Theorem 22.11 of [71] claims that there exists a nondecreasing, continuous, adapted process L on $\mathbb{R}^+$ with support $\bar{\mathcal{Z}}$ a.s. We denote by σ the right-continuous inverse of L, $\sigma_\ell = \inf\{s > 0, L_s > \ell\}$. The process σ, called the inverse local time, is under $\mathbb{P}$ a subordinator started at 0 at initial time, with Laplace exponent:

$$k + \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x}) n(T_a \in dx) \tag{5.10}$$

where $k = n(T_a = \infty)$ is called the killing rate, and $\mathtt{d}$ is called the drift. It is known that the drift of the inverse local time and the Lebesgue measure of $\mathcal{Z}$ are linked as follows:

$$\int_{[0,t]} \mathbf{1}_{\mathcal{Z}}(s) ds = \mathtt{d} L_t, \;\; t \geq 0 \text{ a.s.}$$

Since the non-decreasing process L has $\bar{\mathcal{Z}}$ for support, it is constant on each excursion interval and we may set, for each jump time ℓ of σ:

$$e_\ell(s) := X_{(\sigma_{\ell-} + s) \wedge \sigma_\ell}. \tag{5.11}$$

Denoting by M a Poisson point process on $\mathbb{R}^+ \times \Omega^e$ with intensity measure $ds\, n(de)$, it holds that:

$$\sum_{\ell \geq 0, \sigma_{\ell-} < \sigma_\ell} \delta_{(\ell, e_\ell)}(ds, de) \text{ is the restriction of } M \text{ to } [0, L_\infty] \times \Omega^{\mathrm{e}}. \tag{5.12}$$

Furthermore, the product $n.L$ is a.s. unique.

Assume we are in the case where a.s. all the points of $\mathcal{Z}$ are isolated. Then there is a first excursion, and we define the local time as follows. Let $(L_i, i \geq 1)$ be a sequence of independent and identically distributed exponential random variables with arbitrary positive parameter. Then associate to the i-th excursion a local time equal to $\sum_{1 \leq j \leq i} L_j$. The inverse local time is by construction a subordinator, and we will take again (5.10) to be its Laplace exponent. Notice that the drift $\mathtt{d}$ is null. Defining e_ℓ as in (5.11), we still have (5.12), with the product $n.L$ a.s. unique, and n a finite measure.

5.4.1 Notations

We introduce two notations: To every regenerative process X defined on $\mathbb{R}^+$, we associate the point measure $M(X)$ of its excursions as follows:

$$M(X) := \sum_{\ell \geq 0, \sigma_{\ell-} \neq \sigma_\ell} \delta_{(\ell, e_\ell)}.$$

Conversely, given a Poisson point measure $M = \sum_{\ell \geq 0} \delta_{(\ell, e_\ell)}$ on $\mathbb{R}^+ \times \Omega^{\mathbf{e}}$ and a parameter $\mathtt{d}$, we define a process $Y(M, \mathtt{d})$ as follows:

$$\sigma_\ell = \mathtt{d}\,\ell + \sum_{\ell' \leq \ell} T_a(e_{\ell'}) \in [0, \infty], \tag{5.13}$$

and the local time L as the right-continuous inverse of σ. We finally set, for $t \geq 0$:

$$Y(M, \mathtt{d})_t = \begin{cases} e_{L_t}(t - \sigma_{L_t -}) & \text{if } (L_t, e_{L_t}) \text{ is an atom of } M, \\ a & \text{otherwise.} \end{cases}$$

In this formula, $\sigma_{L_t -}$ locates the left endpoint of the excursion straddling t and $t - \sigma_{L_t -}$ locates the position of t in this excursion.

In general, starting from an arbitrary Poisson point measure M, the sample paths of the process $Y(M, \mathtt{d})$ do not enjoy nice properties. Nevertheless, we shall use this construction only in a very particular setting, and we will always obtain càdlàg processes.

Remark 5.4.1. Notice that for a regenerative process X with point measure of excursion $M(X)$ and inverse local time drift $\mathtt{d}$, we have the following pathwise equality

$$\mathbb{P}(Y(M(X), \mathtt{d}) = X) = 1.$$

Indeed, the process X satisfies:

$$X_t = \begin{cases} e_{L_t}(t - \sigma_{L_t -}) & \text{if } (L_t, e_{L_t}) \text{ is an atom of } M(X), \\ a & \text{otherwise.} \end{cases}$$

Moreover, the local time L of X is the right continuous inverse of σ and this process σ may be constructed pathwise from $\mathtt{d}$ and $M(X)$ as in (5.13). Therefore, the two processes X and $Y(M(X), \mathtt{d})$ are almost surely equal.

5.4.2 A family of confined processes: the regenerative setting

Let Ω^0 be a measurable subset of $\Omega^{\mathbf{e}}$, with complementary set Ω^1 in $\Omega^{\mathbf{e}}$, such that
– Ω^0 has a positive and finite excursion measure: $0 < n(\Omega^0) < \infty$.
– Ω^0 contains the infinite excursions, in the sense that:

$$n(T_a = \infty, \Omega^1) = 0. \tag{5.14}$$

We let T^0 be the left endpoint of the first excursion in Ω^0.

$$T^0 = \inf \{\sigma_{\ell -}, e_\ell \in \Omega^0\}$$

We set:

$$\phi(\lambda) = \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x}) n(T_a \in dx, \Omega^1).$$

Recall the definition (5.1) of Θ^ϕ. We define for $\rho' \in \Theta^\phi$ a Poisson point measure $M^{\rho'}$ on $\mathbb{R}^+ \times \Omega^{\mathbf{e}}$ with intensity $d\ell\; n(\mathrm{e}^{\rho' T_a}\, de, \Omega^1)$ and construct a process $X^{\rho'}$ as follows:

$$X^{\rho'} = Y(M^{\rho'}, \mathtt{d}).$$

We will assume that $X^{\rho'}$ is still defined under $\mathbb{P}$. Intuitively, each member of this family of processes is a confined process in the sense that it has no excursion in Ω^0. We set $\mathcal{G}_\ell = \mathcal{F}_{\sigma_\ell}$ for $\ell \geq 0$.

Proposition 5.4.2. *We have:*

$$\mathbb{P}(X^{\rho'} \in A) = \mathbb{E}(\mathrm{e}^{\rho'\sigma_\ell + \alpha'\ell}, T^0 > \sigma_\ell, X \in A), \ \ A \in \mathcal{G}_\ell, \tag{5.15}$$

where $\alpha' = n(\Omega^0) + \phi(-\rho')$.

Remark 5.4.3. We stress that Propositions 5.4.2, 5.4.4, 5.4.7 and Lemma 5.4.11 generalize results contained in the first author's master thesis [60] (written in German) which deals with the case $\rho' = 0$.

Taking $A = \Omega$ and $\ell = 1$ in (5.15), we obtain, as an immediate Corollary of this Proposition, that:

$$\rho' \in \Theta^\phi \text{ if and only if } \mathbb{E}(\mathrm{e}^{\rho'\sigma_1}, T^0 > \sigma_1) < \infty.$$

Proof. Let M be a Poisson point measure on $\mathbb{R}^+ \times \Omega^{\mathbf{e}}$ with intensity $d\ell\, n(de)$. We set $M_{[0,\ell]}(f) = \sum_{0 \le \ell' \le \ell} f(e_{\ell'})$. Assume f and g are measurable functions on Ω^e, f is non-negative and g is such that $n(1 - \mathrm{e}^{-g}) < \infty$. Then the exponential formula for Poisson point measures yields:

$$\mathbb{E}(\mathrm{e}^{-M_{[0,\ell]}(f)}\, \mathrm{e}^{-M_{[0,\ell]}(g) + \ell n(1-\mathrm{e}^{-g})}) = \mathrm{e}^{-\ell n((1-\mathrm{e}^{-f})\, \mathrm{e}^{-g})}\,. \tag{5.16}$$

On the one hand, setting $\mathrm{e}^{-g} = \mathrm{e}^{\rho' T_a}\, \mathbf{1}_{\Omega^1}$, we may rewrite the left-hand side of (5.16) as follows:

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{-M_{[0,\ell]}(f)}\, \mathrm{e}^{-M_{[0,\ell]}(g) + \ell n(1-\mathrm{e}^{-g})}) &= \mathbb{E}(\mathrm{e}^{-M_{[0,\ell]}(f)}\, \mathrm{e}^{\rho' \sum_{\ell' \le \ell} T_a(e_{\ell'})}\, \mathrm{e}^{+\ell(n(\Omega^0) + n(1-\mathrm{e}^{\rho' T_a}, \Omega^1))}, T^0 > \sigma_\ell)\\ &= \mathbb{E}(\mathrm{e}^{-M_{[0,\ell]}(f)}\, \mathrm{e}^{\rho'\sigma_\ell}\, \mathrm{e}^{\ell(n(\Omega^0) + \phi(-\rho'))}, T^0 > \sigma_\ell),\\ &= \mathbb{E}(\mathrm{e}^{-M_{[0,\ell]}(f)}\, \mathrm{e}^{\rho'\sigma_\ell + \alpha'\ell}, T^0 > \sigma_\ell),\end{aligned}$$

using that $\sum_{\ell' \le \ell} T_a(e_{\ell'}) + \mathtt{d}\ell = \sigma_\ell$ and $-\mathtt{d}\rho' + n(1 - \mathrm{e}^{\rho' T_a}, \Omega^1) = \phi(-\rho')$ at the second equality, and the definition of α' at the third equality. On the other hand, from the definition of $M^{\rho'}$, we may write the right-hand side of (5.16) as follows:

$$\mathrm{e}^{-\ell n((1-\mathrm{e}^{-f})\, \mathrm{e}^{-g})} = \mathbb{E}(\mathrm{e}^{-M^{\rho'}_{[0,\ell]}(f)}).$$

Therefore, for every bounded measurable function F,

$$\mathbb{E}(F(M^{\rho'}_{[0,\ell]})) = \mathbb{E}(F(M_{[0,\ell]})\, \mathrm{e}^{\rho'\sigma_\ell + \alpha'\ell}, T^0 > \sigma_\ell).$$

Let $A \in \mathcal{G}_\ell$. In particular, setting $F = \mathbf{1}_{\{Y(\cdot, \mathtt{d}) \in A\}}$, we obtain:

$$\begin{aligned}\mathbb{P}(X^{\rho'} \in A) &= \mathbb{P}(Y(M^{\rho'}, \mathtt{d}) \in A)\\ &= \mathbb{E}(\mathrm{e}^{\rho'\sigma_\ell + \alpha'\ell}, T^0 > \sigma_\ell, Y(M, \mathtt{d}) \in A)\\ &= \mathbb{E}(\mathrm{e}^{\rho'\sigma_\ell + \alpha'\ell}, T^0 > \sigma_\ell, X \in A),\end{aligned}$$

noting at the third equality that $Y(M, \mathtt{d})$ is distributed as $Y(M(X), \mathtt{d})$, a.s. equal to X, see remark 5.4.1. □

Lemma 5.4.4. *The process* $X^{\rho'}$ *is càdlàg and regenerative.*

Proof. The fact that $X^{\rho'}$ has almost surely càdlàg paths follows from the absolute continuity relationship (5.15). We now prove the regenerative property. Let T be a $\mathcal{F}$-stopping time, $B \in \mathcal{F}_T$ and and $A \in \mathcal{G}_\ell$ for $\ell \geq 0$. We first observe that, from Proposition 1.3, Chapter 8, of Revuz Yor [111], we may replace ℓ by any finite $\mathcal{G}$-stopping time in the relation (5.15), and in particular by $\ell + L_T$:

$$\begin{aligned}
&\mathbb{P}(\theta_T(X^{\rho'}) \in A, X^{\rho'} \in B, X_T^{\rho'} = a, T < \infty)\\
&= \mathbb{P}(\mathrm{e}^{\rho'\sigma_{L_T+\ell}+\alpha'(\ell+L_T)}, T^0 > \sigma_{L_T+\ell}, \theta_T(X) \in A, X \in B, X_T = a, T < \infty)\\
&= \mathbb{P}(\mathrm{e}^{\rho'\sigma_\ell\circ\theta_T+\alpha'\ell}, T^0 \circ \theta_T > \sigma_\ell, \theta_T(X) \in A, \mathrm{e}^{\rho'\sigma_{L_T}+\alpha' L_T}, T^0 > \sigma_{L_T}, X \in B, X_T = a, T < \infty)\\
&= \mathbb{P}(\mathrm{e}^{\rho'\sigma_\ell+\alpha'\ell}, T^0 > \sigma_\ell, X \in A)\, \mathbb{P}(\mathrm{e}^{\rho'\sigma_{L_T}+\alpha' L_T}, T^0 > \sigma_{L_T}, X \in B, T^0 > T, X_T = a, T < \infty)\\
&= \mathbb{P}(X^{\rho'} \in A)\, \mathbb{P}(X^{\rho'} \in B, X_T^{\rho'} = a, T < \infty),
\end{aligned}$$

using the equality $\sigma_{L_T+\ell} = \sigma_{L_T} + \sigma_\ell \circ \theta_T$ at the second equality, the regenerative property of X under $\mathbb{P}$ at the third equality, and once again Lemma 5.4.2 at the fourth equality . We have thus proved the regenerative property for $A \in \mathcal{G}_\ell$. A monotone class argument allows to extend the statement to $A \in \cup_{\ell\geq 0}\mathcal{G}_\ell$. □

We have the following description of the inverse local time under $\mathbb{P}^{(loc)}$:

Lemma 5.4.5. *The inverse local time of $X^{\rho'}$ is a subordinator with Laplace exponent:*

$$\mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x}) n(\mathrm{e}^{\rho' T_a}, T_a \in dx, \Omega^1).$$

Proof. We use the notations of the proof of Lemma (5.4.2). We also denote by $\sigma^{\rho'}$ the inverse local time of $X^{\rho'}$. Since σ_ℓ is $\mathcal{G}_\ell$-measurable, we may use (5.15) and write:

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{-\lambda\sigma_\ell^{\rho'}}) &= \mathbb{E}(\mathrm{e}^{(\rho'-\lambda)\sigma_\ell+\alpha'\ell}, T^0 > \sigma_\ell)\\
&= \mathbb{E}(\mathrm{e}^{(\rho'-\lambda)(M_{[0,\ell]}(T_a)+\mathtt{d}\ell)+\alpha'\ell}, M_{[0,\ell]}(\Omega^0) = 0)\\
&= \mathbb{E}(\mathrm{e}^{(\rho'-\lambda)(M_{[0,\ell]}(T_a,\, \Omega^1)+\mathtt{d}\ell)+\alpha'\ell})\, \mathbb{P}(M_{[0,\ell]}(\Omega^0) = 0)\\
&= \mathrm{e}^{-\phi(\lambda-\rho')\ell+(n(\Omega^0)+\phi(-\rho'))\ell}\, \mathrm{e}^{-\ell n(\Omega^0)}\\
&= \mathrm{e}^{-\ell(\phi(\lambda-\rho')-\phi(-\rho'))},
\end{aligned}$$

using the independence properties of Poisson point measures at the third equality, the exponential formula and the definition of α' at the fourth equality. We conclude by computing:

$$\phi(\lambda - \rho') - \phi(-\rho') = \mathtt{d}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x}) n(\mathrm{e}^{\rho' T_a}, T_a \in dx, \Omega^1).$$

□

Remark 5.4.6. Let $\rho' \in \Theta^\phi$. We choose $(N_t, t \geq 0)$ to be a standard Poisson process with rate $\nu \geq 0$ and use it to mark the excursions of $X^{\rho'}$, that we finally erase. This may be achieved as follows: we define a random time-change by setting

$$C(t) = \int_0^t ds\, \mathbf{1}_{\{N_{\sigma_{L_s-}} = N_{\sigma_{L_s}}\}}$$

where $\sigma_{L_s-} = \lim_{\ell \uparrow L_s} \sigma_\ell$ and then define $(C^{-1}(t), t \geq 0)$ as the right continuous inverse of $(C(t), t \geq 0)$. Then we state that

$$(X^{\rho'}_{C^{-1}(t)}, t \geq 0) \text{ is distributed as } (X^{\rho'-\nu}_t, t \geq 0). \tag{5.17}$$

Intuitively, the process C serves to locate the excursions of $(X^{\rho'}_t, t \geq 0)$ which are not marked by the Poisson process N, and $(X^{\rho'}_{C^{-1}(t)}, t \geq 0)$ then retains only those excursions.

We may justify (5.17) as follows. Let us note $\mathcal{Z} = \{t \geq 0, X^{\rho'}_t = a\}$. The interior of $\mathbb{R}^+ \setminus \mathcal{Z}$ may be written as the union of the maximal open intervals:

$$\bigcup_{s \geq 0, \sigma_{L_s-} \neq \sigma_{L_s}} (\sigma_{L_s-}, \sigma_{L_s}),$$

with σ the inverse local time of $X^{\rho'}$. These intervals are the intervals of excursions. By definition of C^{-1}, only those intervals may be thinned by the random time change C^{-1}. Moreover, if s belongs to an excursion interval with length $\ell = \sigma_{L_s} - \sigma_{L_s-}$, the corresponding excursion is kept with probability:

$$\mathrm{e}^{-\nu(\sigma_{L_s} - \sigma_{L_s-})} = \mathrm{e}^{-\nu\ell},$$

independently of the other excursion intervals. As a consequence, the thinning of the Poisson point measure of the excursions of X^ρ, with law M^ρ, has law $M^{\rho-\nu}$. Therefore, the process $X^{\rho'} = Y(M^{\rho'}, \mathtt{d})$ is time-changed into a process with law $Y(M^{\rho'-\nu}, \mathtt{d}) = X^{\rho'-\nu}$ by the random time change C^{-1}.

5.4.3 A family of confined processes: the Markov setting

Classical examples are concerned with the simple situation when Ω^0 consists of those excursions hitting a measurable subset E_0:

(H) There exists $E_0 \subset E$ such that $\Omega^0 = \{\omega \in \Omega^{\mathbf{e}}, \omega(s) \in E_0 \text{ for some } s > 0\}$.

Notice that necessarily, we have $a \notin E_0$ for the assumptions on Ω^0 enounced at the beginning of Section 5.4.2 to be satisfied. Assuming that Ω^0 is of the form (H), we may wonder if $X^{\rho'}$ is a Markov process when X itself is a Markov process. It would have been possible to use Itô's synthesis theorem to check the Markov property. Nevertheless, absolute continuity properties allow us to obtain this property without much effort in our setting.

Under assumption (H), we may define the following quantity:

$$\bar{T}^0 = \inf\{t > 0, X_t \in E_0\}.$$

Notice that $T^0 \leq \bar{T}^0$, and let us set, for $\rho' \in \Theta^\phi$:

$$h_{\rho'}(x) = \mathbb{E}_x(\mathrm{e}^{\rho' T_a}, \bar{T}^0 > T_a). \tag{5.18}$$

Proposition 5.4.7. *Assume that the process X is Markov with respect to the filtration $(\mathcal{F}_t, t \geq 0)$, and that* (H) *holds. Then we have:*

$$\mathbb{P}(X^{\rho'} \in A) = \mathbb{E}(\mathrm{e}^{\rho' t + \alpha' L_t} h_{\rho'}(X_t), \bar{T}^0 > t, X \in A), \ \ A \in \mathcal{F}_t. \tag{5.19}$$

Remark 5.4.8. Each member of the family of processes $(X^{\rho'}, \rho' \in \Theta^{\phi})$ is therefore an h-transform on $\mathcal{F}$ of the killed process *together with* its local time $(X_t \mathbf{1}_{\{\bar{T}^0 > t\}}, L_t)$.

Proof. First, we can replace ℓ by any finite $\mathcal{G}$-stopping time in the relation (5.15) (see the proof of Proposition 5.4.4). Let $A \in \mathcal{F}_t$. Notice that $\mathcal{F}_t \subset \mathcal{G}_{L_t}$ and write:

$$\mathbb{P}(X^{\rho'} \in A) = \mathbb{E}(\mathrm{e}^{\rho' \sigma_{L_t} + \alpha' L_t}, \bar{T}^0 > \sigma_{L_t}, X \in A).$$

Now, $\sigma_{L_t}(X) = t + T_a(\theta_t X)$. Therefore, using the Markov property of X, we have $\mathbb{P}$ a.s.:

$$\mathbb{E}(\mathrm{e}^{\rho' \sigma_{L_t}}, \bar{T}_0 > \sigma_{L_t} | \mathcal{F}_t) = \mathrm{e}^{\rho' t}\, h_{\rho'}(X_t) \mathbf{1}_{\{\bar{T}^0 > t\}}.$$

This allows us to conclude:

$$\mathbb{P}(X^{\rho'} \in A) = \mathbb{E}(\mathrm{e}^{\rho' t + \alpha' L_t}\, h_{\rho'}(X_t), \bar{T}^0 > t, X \in A).$$

□

The reader interested in the applications of the family of processes may skip the remainder of this subsection at first reading. So far, the law of $X^{\rho'}$ has been defined starting at a only. We propose the following definition of the law of $X^{\rho'}$ starting at x:

$$\mathbb{P}_x(X^{\rho'} \in A) = \mathbb{E}_x \left(\mathrm{e}^{\rho' t + \alpha' L_t} \frac{h_{\rho'}(X_t)}{h_{\rho'}(x)}, \bar{T}^0 > t, X \in A \right), \ A \in \mathcal{F}_t. \tag{5.20}$$

Notice that the definition of $\mathbb{P}_a(X^{\rho'} \in A)$ is consistent with the one given in (5.19) for the process started at a. Indeed, from formula (5.19), the process $(\mathrm{e}^{\rho' t + \alpha' L_t}\, h_{\rho'}(X_t) \mathbf{1}_{\{\bar{T}^0 > t\}}, t \geq 0)$ is a martingale with expectation 1, and, choosing $A = \{X_0\} \in \mathcal{F}_0$, we deduce that $h_{\rho'}(a) = 1$ since $X_0 = a$ a.s.

Proposition 5.4.9. *Assume that the process X is Markov with respect to the filtration $(\mathcal{F}_t, t \geq 0)$, and that* (H) *holds. Then formula* (5.20) *defines a probability measure for each $x \in E$ and the process $X^{\rho'}$ is Markov with respect to the filtration $(\mathcal{F}_t, t \geq 0)$.*

Remark 5.4.10. Knight and Pittenger [77] consider the general problem of the preservation of the Markov property after deletion of excursions. See also Rogers and Williams [114] on Example 58 in Chapter 8 for an example where excursions are censored or reweighted.

Proof. Let us denote by $T_a^t = t + T_a \circ \theta_t$ the hitting time of a after t. First, for $u \geq 0$, we may replace t by the stopping time T_a^u in equation (5.19), which then reads, with $A = \{T_a^u < \infty\}$:

$$\mathbb{E}(\mathrm{e}^{\rho' T_a^u + \alpha' L_{T_a^u}}\, h_{\rho'}(X_{T_a^u}), \bar{T}^0 > T_a^u, T_a^u < \infty) = \mathbb{P}(T_a^u(X^{\rho'}) < \infty) = 1, \tag{5.21}$$

Notice that we may dispense ourselves on taking the intersection with $\{T_a^u < \infty\}$ on the left hand side of (5.21) since $\{T_a^u = \infty\}$ implies that u belongs to an infinite excursion, which, according to (5.14), necessarily belongs to Ω^0, whence $\{\bar{T}^0 < \infty\}$. We thus may write:

$$\mathbb{E}(\mathrm{e}^{\rho' T_a^u + \alpha' L_{T_a^u}}\, h_{\rho'}(X_{T_a^u}), \bar{T}^0 > T_a^u) = \mathbb{P}(T_a^u(X^{\rho'}) < \infty) = 1, \tag{5.22}$$

Let $0 \leq s \leq t$, and $A \in \mathcal{F}_s$. We now compute:

$$
\begin{aligned}
&\mathbb{E}_x\left(\mathrm{e}^{\rho' t+\alpha' L_t}\, h_{\rho'}(X_t), X\in A, \bar{T}^0>t\right)\\
=&\mathbb{E}_x\left(\mathrm{e}^{\rho' T_a^t+\alpha' L_{T_a^t}}, X\in A, \bar{T}^0>T_a^t\right)\\
=&\mathbb{E}_x\left(\mathrm{e}^{\rho' T_a^s+\alpha' L_{T_a^s}}\,\mathbb{E}_a\left(\mathrm{e}^{\rho' T_a^u+\alpha' L_{T_a^u}}, \bar{T}^0>T_a^u\right)_{|u=t-T_a^s}, X\in A, \bar{T}^0>T_a^s, T_a^s\le t\right)\\
+&\mathbb{E}_x\left(\mathrm{e}^{\rho' T_a^s+\alpha' L_{T_a^s}}, X\in A, \bar{T}^0>T_a^s, T_a^s>t\right)\\
=&\mathbb{E}_x\left(\mathrm{e}^{\rho' T_a^s+\alpha' L_{T_a^s}}, X\in A, \bar{T}^0>T_a^s\right)\\
=&\mathbb{E}_x\left(\mathrm{e}^{\rho' s+\alpha' L_s}\, h_{\rho'}(X_s), X\in A, \bar{T}^0>s\right).
\end{aligned}
$$

using the definition (5.18) of $h_{\rho'}$ at the first equality and the Markov property, distinguishing according to $\{T_a^s>t\}$, in which case $T_a^s=T_a^t$ or $\{T_a^s\le t\}$ and using additivity of the local time, the regenerative property at the second equality, the relation (5.22) at the third and the definition (5.18) of $h_{\rho'}$ at the last. The relation (5.20) therefore consistently defines a family of probability measure on $(\mathcal{F}_t, t\ge 0)$.

Recall the Markov property of X reads as follows:

$$\mathbb{P}(\theta_t(X)\in B, X\in A)=\mathbb{E}(\mathbb{P}_{X_t}(X\in B), X\in A),\;\; A\in\mathcal{F}_t, B\in\mathcal{F}.$$

We then compute, for $A\in\mathcal{F}_t$, $B\in\mathcal{F}_s$:

$$
\begin{aligned}
&\mathbb{P}_x(\theta_t(X^{\rho'})\in B, X^{\rho'}\in A)\\
&=\mathbb{E}_x\left(\mathrm{e}^{\rho'(t+s)+\alpha' L_{t+s}}\,\frac{h_{\rho'}(X_{t+s})}{h_{\rho'}(x)}, \bar{T}^0>t+s, \theta_t(X)\in B, X\in A\right)\\
&=\mathbb{E}_x\left(\mathrm{e}^{\rho' t+\alpha' L_t}\,\frac{h_{\rho'}(X_t)}{h_{\rho'}(x)}\mathbb{P}_{X_t}\left(\mathrm{e}^{\rho' s+\alpha'\, L_s}\,\frac{h_{\rho'}(X_s)}{h_{\rho'}(X_0)}, \bar{T}^0>s, X\in B\right), \bar{T}^0>t, X\in A\right)\\
&=\mathbb{E}_x(\mathbb{P}_{X_t^{\rho'}}(X^{\rho'}\in B), X^{\rho'}\in A),
\end{aligned}
\tag{5.23}
$$

using the Markov property of X and the additivity property of local times at the second equality. By a monotone class argument, the equality (5.23) is still valid for $B\in\mathcal{F}$. This proves the Markov property of $X^{\rho'}$. □

5.4.4 Confining in the real time scale and in the local time scale

We shall return in this section to the regenerative framework, and assume that X is under $\mathbb{P}$ the regenerative process introduced in the introduction of Section 5.4.

We propose to approximate the null event $\{T^0=\infty\}$ by

$$\bigcap_{\ell\ge 0}\{T^0>\sigma_\ell\}\text{ and }\bigcap_{\ell\ge 0}\{T^0>t\}.$$

We therefore define, for $t\ge 0$, $\ell\ge 0$, and $A\in\mathcal{F}$,

$$\mathbb{P}^{(t)}(A)=\mathbb{P}(X\in A|T^0>t)\text{ and }\mathbb{P}^{(loc,\ell)}(A)=\mathbb{P}(X\in A|T^0>\sigma_\ell).$$

We shall study the limits of these two family of probability measures, as t and ℓ tend to ∞, that we will call respectively the processes confined in the real time scale and in the local time scale. A simple computation yields:

$$\mathbb{P}^{(loc,\ell)}(A) = \mathbb{E}(e^{n(\Omega^0)\ell}\mathbf{1}_{\{T^0>\sigma_\ell\}}, X \in A),\ A \in \mathcal{G}_\ell. \tag{5.24}$$

Now, equation (5.15) with $\rho' = 0$ shows that the probability measures $\mathbb{P}^{(loc,\ell)}$ are compatible on $\mathcal{G}_{\ell_0}$ as ℓ varies, $\ell \geq \ell_0$. We shall call $\mathbb{P}^{(loc)}$ the unique probability measure such that for any $\ell \geq 0$,

$$\mathbb{P}^{(loc)}(A) = \mathbb{P}^{(loc,\ell)}(A),\ A \in \mathcal{G}_\ell,$$

and we observe that $\mathbb{P}^{(loc)}$ corresponds to the law of $X^{\rho'}$ with $\rho' = 0$.

We now turn to the study of $\mathbb{P}^{(t)}$. To that aim, it is important to understand the structure of the process X before T^0. We shall need the following key identity, which justifies the study of the conditioned subordinator. Let us set $\kappa = n(\Omega^0)$.

Lemma 5.4.11. *We have that*

$$(X_t, t \leq T^0) \text{ under } \mathbb{P} \text{ is distributed as } (X_t, t \leq \sigma_{\mathbf{e}}) \text{ under } \mathbb{P}^{(loc)},$$

where $\mathbf{e}$ *is an exponential random variable with parameter* κ *independent of the inverse local time* σ *under* $\mathbb{P}^{(loc)}$.

Proof. The process X being regenerative under $\mathbb{P}$, we have the almost sure equality $X = Y(M(X), \mathtt{d})$ where $M(X)$ is the restriction on $[0, L_\infty] \times \Omega^{\mathbf{e}}$ of a Poisson point measure M on $\mathbb{R}^+ \times \Omega^{\mathbf{e}}$ with intensity $d\ell\, n(de)$. The quantity $L_\infty \in [0, \infty]$ corresponds to the local time of the first infinite excursion (if any). We may decompose M as the independent sum of M^0 and M^1,

$$M = M^0 + M^1,$$

where M^0 is the restriction of M to $\mathbb{R}^+ \times \Omega^0$ and M^1 the restriction of M to the complementary set $\mathbb{R}^+ \times \Omega^1$. Notice that $Y(M^1, \mathtt{d})$ has law $\mathbb{P}^{(loc)}$. By construction, we have:

$$(Y(M, \mathtt{d})_t, t \leq T^0) = (Y(M^1, \mathtt{d})_t, t \leq T^0),$$

where T^0 refers to $T^0(Y(M, \mathtt{d}))$ in both the right- and the left-hand side. Notice that T^0 may also be written as follows:

$$T^0 = \sigma_{L_{T^0}-},$$

where on the right-hand side, we may choose σ to be the inverse local time of the process $Y(M, \mathtt{d})$, or the inverse local time of $Y(M^1, \mathtt{d})$, since both coincide up to time $L_{T_0}-$. Let us assume we make the second choice and that σ denotes the inverse local time of $Y(M^1, \mathtt{d})$. Since L_{T^0} is independent of $Y(M^1, \mathtt{d})$, and the process σ is stochastically continuous, we have the following equality in distribution:

$$\sigma_{L_{T^0\_}} = \sigma_{L_{T^0}}.$$

We conclude the proof noticing that L_{T^0} is distributed as an independent exponential random variable with parameter κ.

□

Notice in particular that the distribution of T^0 under $\mathbb{P}$ is that of $\sigma_{\mathbf{e}}$ under $\mathbb{P}^{(loc)}$. Therefore, to condition by $\{T^0 > t\}$ for a large t, we need to understand the structure of σ conditionnally on $\sigma_{\mathbf{e}}$ to be large, which has been achieved in Section 5.3.

We are now in position to connect sections 5.3 and 5.4. The quantity $n(T_a \in dx, \Omega^1)$ now plays the rôle of $\eta(dx)$. Therefore, we redefine the quantity ρ as follows:

$$\rho = \sup\{\lambda \geq 0, \mathbb{E}^{(loc)}(\mathrm{e}^{\lambda\sigma_1}) \leq \mathrm{e}^{\kappa}\} = \sup\{\lambda \geq 0, \mathbb{E}(\mathrm{e}^{\lambda\sigma_1}, T^0 > \sigma_1) \leq 1\}. \tag{5.25}$$

We redefine assumption (C) as follows:

(C) $\mathbb{E}^{(loc)}(\mathrm{e}^{\rho\sigma_1}) = \mathrm{e}^{\kappa}$ and $\mathbb{E}^{(loc)}(\sigma_1\,\mathrm{e}^{\rho\sigma_1}) < \infty$.

Notice (C) is equivalent to:

$$\mathbb{E}(\mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) = 1 \text{ and } \mathbb{E}(\sigma_1\,\mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) < \infty.$$

In the case where T_a is positive almost surely, we may also write (C) as follows:

$$\mathbb{E}(\mathrm{e}^{\rho T_a}, T^0 \geq T_a) = 1 \text{ and } \mathbb{E}(T_a\,\mathrm{e}^{\rho T_a}, T^0 > T_a) < \infty.$$

Let $\mathbf{e}$ be an exponential random variable with parameter κ independent of σ under $\mathbb{P}^{(loc)}$. We redefine the assumption (R) as follows:

(R) $\mathbb{P}^{(loc)}(\sigma_{\mathbf{e}} > s)/\mathbb{P}^{(loc)}(\sigma_{\mathbf{e}} > s-t) \to \mathrm{e}^{\rho t}$ as $s \to \infty$.

The process $(\mathrm{e}^{\rho\sigma_\ell + \phi(-\rho)\ell}, \ell \geq 0)$, with ϕ defined at (5.4.2), is a $\mathcal{G}$-martingale under $\mathbb{P}^{(loc)}$. Therefore, the relation

$$\mathbb{P}^{(\infty)}(A) = \mathbb{E}^{(loc)}(\mathrm{e}^{\rho\sigma_\ell + \phi(-\rho)\ell}, A),\ A \in \mathcal{G}_\ell,$$

defines a probability measure $\mathbb{P}^{(\infty)}$, which also satisfies, by definition of $\mathbb{P}^{(loc)}$:

$$\mathbb{P}^{(\infty)}(A) = \mathbb{E}(\mathrm{e}^{\rho\sigma_\ell + (\kappa+\phi(-\rho))\ell}, X \in A, T^0 > \sigma_\ell),\ A \in \mathcal{G}_\ell \tag{5.26}$$

If ρ defined in (5.25) satisfies $\mathbb{E}(\mathrm{e}^{\rho\sigma_1}, T^0 > \sigma_1) = 1$, then $\kappa + \phi(-\rho) = 0$ and the expression (5.26) simplifies. In this case, $\mathbb{P}^{(\infty)}$ is the law of $X^{\rho'}$ for $\rho' = \rho$ in the family of confined processes studied in Subsections 5.4.2 and 5.4.3.

We now obtain our main Theorem.

Theorem 5.4.12. *Let $\ell \geq 0$ be fixed. Assume* (C) *is satisfied. We have:*

$$\lim_{t\to\infty} \mathbb{P}^{(t)}(A) = \mathbb{P}^{(\infty)}(A),\ A \in \mathcal{G}_\ell.$$

Assume (R) *is satisfied, and fix $s_0 \geq 0$. We have:*

$$\lim_{t\to\infty} \mathbb{P}^{(t)}(A, \sigma_\ell \leq s_0) = \mathrm{e}^{-(\kappa+\phi(-\rho))\ell}\,\mathbb{P}^{(\infty)}(A, \sigma_\ell \leq s_0),\ A \in \mathcal{G}_\ell. \tag{5.27}$$

Therefore, the two different approximations of the event $\{T^0 = \infty\}$ yield to two different probability measures $\mathbb{P}^{(loc)}$ and $\mathbb{P}^{(\infty)}$. Notice this would not be the case if T^0 could be infinite with positive probability (which can not arise from our set of assumptions on Ω^0): in that situation, both limits would agree.

We always have $\kappa + \phi(-\rho) \leq 0$ by definition of ρ in (5.25). When furthermore $\kappa + \phi(-\rho) < 0$, the right-hand side of (5.27) defines a defective probability measure.

Proof. Let $s_0 \le t$. Using the regenerative property of $\mathbb{P}$, we have, for $A \in \mathcal{G}_\ell$:

$$\mathbb{P}^{(t)}(A) \ge \mathbb{P}^{(t)}(A, \sigma_\ell \le s_0) = \mathbb{E}\left(\left[\frac{\mathbb{P}(T^0 > t-u)}{\mathbb{P}(T^0 > t)}\right]_{|u=\sigma_\ell}, X \in A, \sigma_\ell \le s_0\right).$$

Then, using Lemma 5.4.11, and in the notations of this Lemma,

$$\frac{\mathbb{P}(T^0 > t-u)}{\mathbb{P}(T^0 > t)} = \frac{\mathbb{P}^{(loc)}(\sigma_{\mathbf{e}} > t-u)}{\mathbb{P}^{(loc)}(\sigma_{\mathbf{e}} > t)}$$

and we conclude mimicking the proof of Proposition 5.3.5. The proof of the second statement follows the lines of the proof of Proposition 5.3.11. □

If (H) holds, the most natural approximation of the confined process is arguably through the decreasing sequence

$$\{\bar{T}^0 > t\} \text{ where } \bar{T}^0 := \inf\{s > 0, X_s \in E_0\}.$$

and we denote by

$$\bar{\mathbb{P}}^{(t)}(A) = \mathbb{P}(X \in A | \bar{T}^0 > t), \ A \in \mathcal{F}. \tag{5.28}$$

the law of the conditioned process. The Q-process is usually defined as the weak limit of the probability measures $\bar{\mathbb{P}}^{(t)}$. The following theorem ensures that, under an appropriate set of assumptions, $\mathbb{P}^{(\infty)}$ is the law of the Q-process. Its proof, similar to that of Theorem 5.4.12, is eluded.

Theorem 5.4.13. *Let $\ell \ge 0$ be fixed. Assume that* (C) *is satisfied, that Ω^0 is of the form* (H) *and that* (C') *holds for $\Delta = \bar{T}^0 - T^0$. Then:*

$$\lim_{t\to\infty} \bar{\mathbb{P}}^{(t)}(A) = \mathbb{P}^{(\infty)}(A), \ A \in \mathcal{G}_\ell.$$

5.5 Examples

5.5.1 A random walk confined in a finite interval

We first exemplify our results on the simplest non trivial cases. We want to confine a simple symmetric random walk X in $\mathbb{Z}$ in a finite symmetric interval: that means, we choose Ω^0 satisfying (H) with $E_0 = (-\infty, -b] \cup [b, \infty)$. This problem may be reduced to that of a random walk in $\mathbb{Z}^+$ reflected at 0 and $E_0 = [b, \infty)$. We therefore assume in the following that X takes its values in $\mathbb{Z}^+$ with transition probabilities:

$$\mathbb{P}_0(X_1 = 1) = 1 \text{ and } \mathbb{P}_n(X_1 = n+1) = \mathbb{P}_n(X_1 = n-1) = 1/2, \ n \in \mathbb{N},$$

and we will choose 0 as the regenerative point. To be coherent with the framework developed in section 5.4, we will define X on $\mathbb{R}^+$ by setting:

$$X_t = (n+1-t)X_n + (t-n)X_{n+1}, \quad n = \lfloor t \rfloor.$$

Case b=3

We first take $b = 3$, so that only states 0, 1 and 2 are allowed, but not the states greater than or equal to 3. The set Ω^0 consists of those excursions from 0 for which $T_3 = \inf\{t > 0, X_t = 3\}$ is finite and thus satisfies (H). Besides, X is a Markov process. We may therefore apply Proposition 5.4.7 with $\rho = 0$, and deduce that X under $\mathbb{P}^{(loc)}$ is a Markov process. Therefore, we just have to specify its transitions. From the construction of X under $\mathbb{P}^{(loc)}$, we have:

$$\mathbb{P}_0^{(loc)}(X_1 = 1) = \mathbb{P}_2^{(loc)}(X_1 = 1) = 1.$$

Computing the two last transitions $\mathbb{P}_1^{(loc)}(X_1 = 0)$ and $\mathbb{P}_1^{(loc)}(X_1 = 2)$ requires a bit more thought. We use the formula (5.20):

$$\mathbb{P}_1^{(loc)}(X_1 = 0) = \frac{h_0(0)}{h_0(1)}\,\mathbb{E}(\mathrm{e}^{n(\Omega^0)\mathbf{e}})\,\mathbb{P}_1(X_1 = 0) \text{ and } \mathbb{P}_1^{(loc)}(X_1 = 2) = \frac{h_0(2)}{h_0(1)}\,\mathbb{P}_1(X_1 = 2), \quad (5.29)$$

where $h_0(x) = \mathbb{P}_x(T_3 > T_0)$ according to (5.18) and $\mathbf{e}$ is an exponential random variable with arbitrary parameter, 1 say. This choice implies $n(\Omega^0) = 1/3$ since $\mathbb{P}_0(T_3 > T_0) = 1/3$. We need the values of $h_0(x) = \mathbb{P}_x(T_3 > T_0)$ for $x = 0, 1, 2$: since $T_x = \inf\{t > 0, X_t = x\}$ where the infimum is on the *positive* time, these numbers are given by $2/3, 2/3, 1/3$ respectively. Altogether, we find:

$$\mathbb{P}_1^{(loc)}(X_1 = 0) = 3/4 \text{ and } \mathbb{P}_1^{(loc)}(X_1 = 2) = 1/4,$$

which are just the transitions of the process conditioned on the event $\{T_3 > T_0\}$.

Remark 5.5.1. Of course, it would have been possible to obtain these transitions more directly. Nevertheless, the goal of this Section is to apply the proven formulas and see how these formulas provide a generic and unified treatment in both the discrete and continuous setting.

Remark 5.5.2. We face the following subtelty here: X is a Markov process with respect to $(\mathcal{F}_n, n \in \mathbb{Z}^+)$ and not $(\mathcal{F}_t, t \geq 0)$. Nevertheless, we may check that the Markov property is just used at integer time in the proof of the formula (5.29), whose derivation is therefore justified.

We now derive the transitions of the Markov process with law $\mathbb{P}^{(\infty)}$. We first have to find

$$\rho = \sup\{\lambda \geq 0, \mathbb{E}_0(\mathrm{e}^{\lambda T_0}, T_0 < T_3) \leq 1\}.$$

In that case, an explicit computation can be achieved:

$$\mathbb{E}_0(\mathrm{e}^{\lambda T_0}, T_0 < T_3) = \sum_{k \geq 1} (\mathrm{e}^{2\lambda k})(1/2)^{2k-1},$$

and yields to:

$$\rho = \log(\frac{2}{\sqrt{3}}) \text{ and } \mathbb{E}_0(\mathrm{e}^{\rho T_0}, T_0 < T_3) = 1. \quad (5.30)$$

Therefore (or, alternatively, because we are working on a finite state space), $\mathbb{P}^{(\infty)}$ is locally absolutely continuous with respect to $\mathbb{P}$, whence the relations:

$$\mathbb{P}_0^{(\infty)}(X_1 = 1) = \mathbb{P}_2^{(\infty)}(X_1 = 1) = 1.$$

and also:

$$\mathbb{P}_1^{(\infty)}(X_1 = 0) + \mathbb{P}_1^{(\infty)}(X_1 = 2) = 1.$$

At this point, it remains to determine the two numbers $\mathbb{P}_1^{(\infty)}(X_2 = 0)$ and $\mathbb{P}_1^{(\infty)}(X_1 = 2)$. But we know from (5.20) that, for any i, j,

$$\mathbb{P}_i^{(\infty)}(X_1 = j) = \mathbb{P}_i(X_1 = j)\ \mathrm{e}^{\rho}\ \frac{\mathbb{E}_j(\mathrm{e}^{\rho T_0}, T_0 < T_3)}{\mathbb{E}_i(\mathrm{e}^{\rho T_0}, T_0 < T_3)},$$

for ρ given in (5.30). Applying this relation with $i = 1$ and $j = 0, 2$, the problem reduces to find a relation between $\mathbb{E}_0(\mathrm{e}^{\rho T_0}, T_0 < T_3)$ and $\mathbb{E}_2(\mathrm{e}^{\rho T_0}, T_0 < T_3)$. Applying the Markov property at time 1, we have:

$$\mathbb{E}_0(\mathrm{e}^{\rho T_0}, T_0 < T_3) = \mathbb{E}_1(\mathrm{e}^{\rho(T_0+1)}, T_0 < T_3),$$

and

$$\mathbb{E}_2(\mathrm{e}^{\rho T_0}, T_0 < T_3), = \frac{1}{2}\mathbb{E}_1(\mathrm{e}^{\rho(T_0+1)}, T_0 < T_3).$$

which gives:

$$\mathbb{P}_1^{(\infty)}(X_1 = 0)/\mathbb{P}_1^{(\infty)}(X_1 = 2) = 2.$$

We conclude that:

$$\mathbb{P}_1^{(\infty)}(X_1 = 0) = 2/3 \text{ and } \mathbb{P}_1^{(\infty)}(X_1 = 2) = 1/3.$$

We now compare $\mathbb{P}^{(\infty)}$ with the law of the Q-process. Rather than checking (C'), we shall use the technique explained in Section 1.8.1 to prove that the Q-process X^∞, defined by

$$\mathbb{P}(X^\infty \in A) = \lim_{n\to\infty} \mathbb{P}(X \in A | T_3 > n) \tag{5.31}$$

for any $A \in \mathcal{F}_m$, m fixed, has law $\mathbb{P}^{(\infty)}$. The technique, detailed in Appendix M of Aldous [3], consists in applying the Perron-Frobenius theorem to the transition matrix of the usual chain *restricted to* $\{0, 1, 2\}$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

The eigenvalue of largest modulus of this substochastic matrix P is real, equal to $\sqrt{3}/2 = \mathrm{e}^{-\rho} < 1$, and associated to the right eigenvector β with entries $(2,\ \sqrt{3},\ 1)$. This implies, see appendix M of Aldous [3], that the Q-process has the following transition from i to j:

$$P_{ij}\ \mathrm{e}^{\rho}\ \frac{\beta_j}{\beta_i}.$$

Simple computations then ensure that X^∞ has law $\mathbb{P}^{(\infty)}$, which is therefore also the law of the Q-process.

Case b=4

We then take $b = 4$, so that only states 0, 1, 2 and 3 are allowed, but not the states greater than or equal to 4.

We again use formulas (5.18) and (5.20) with $\rho' = 0$ to determine the transition probabilities under $\mathbb{P}^{(loc)}$, and we find, by basic calculations, the following transition matrix under $\mathbb{P}^{(loc)}$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The quantity ρ satisfies $\rho = \log(2(2-\sqrt{2}))/2$, and $\mathbb{E}_0(e^{\rho T_0}, T_0 < T_4) = 1$. We then compute, using again (5.18) and (5.20) with $\rho' = \rho$ this time, the transition matrix under $\mathbb{P}^{(\infty)}$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2-\sqrt{2} & 0 & \sqrt{2}-1 & 0 \\ 0 & \sqrt{2}/2 & 0 & (2-\sqrt{2})/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In both cases, it is not difficult to check that assumption (C) is satisfied and therefore Theorem 5.4.12 applies.

Last, let us notice that, using the technique explained in Section 1.8.1, we get that the Q-process has law $\mathbb{P}^{(\infty)}$ again.

5.5.2 Brownian motion confined in a finite interval

We now consider the example of a standard real Brownian motion $(B_t, t \geq 0)$ started at 0 and take Ω^0 associated to $E_0 = (-\infty, -b) \cup (b, \infty)$ for $b > 0$. Once again, we will choose 0 as the regenerative point. We will denote $T_x = \inf\{t > 0, B_t = x\}$ the hitting time of x and $T_{b \wedge -b} = T_b \wedge T_{-b}$ the hitting time of $\{-b\} \cup \{b\}$. We would like to identify the process with law $\mathbb{P}^{(loc)}$ first. We compute $n(\Omega^0) = 1/(2b)$. Then, from equation (5.20) with $\rho' = 0$, we deduce that:

$$\mathbb{P}_x^{(loc)}(A) = \mathbb{E}_x\left(e^{L_t/(2b)} \frac{h_0(X_t)}{h_0(x)}, X \in A, T_{b\wedge -b} > t\right), \ A \in \mathcal{F}_t,$$

with L_t the local time at 0 and:

$$h_0(x) = \mathbb{P}_x(T_{b\wedge -b} > T_0) = \frac{b - |x|}{b}, \ -b \leq x \leq b.$$

It is then classical to deduce from the form of this h-transform that $\mathbb{P}^{(loc)}$ is the law of a diffusion in $(-b, b)$ with generator with generator acting as follows on smooth functions f:

$$f(x) \mapsto f''(x) - \frac{1}{b - |x|} f'(x) \text{ in } (-b, 0) \cup (0, b),$$

see Knight [76]. Formula 4.15.4(1) of Borodin and Salminen [22] yields that:

$$\mathbb{E}(e^{-\lambda \sigma_\ell}, T_{b\wedge -b} > \sigma_\ell) = \exp -\ell\left(\frac{\sqrt{2\lambda}}{2\tanh b\sqrt{2\lambda}}\right). \tag{5.32}$$

Taking $\lambda = 0$ in (5.32), we obtain:

$$\mathbb{E}^{(loc)}(e^{-\lambda \sigma_\ell}) = \exp -\ell\left(\frac{\sqrt{2\lambda}}{2\tanh b\sqrt{2\lambda}} - \frac{1}{2b}\right).$$

so that the Laplace exponent $\phi(\lambda)$ of σ under $\mathbb{P}^{(loc)}$ satisfies:

$$\phi(\lambda) = \frac{\sqrt{2\lambda}}{2\tanh b\sqrt{2\lambda}} - \frac{1}{2b}.$$

Mathematica provides the following identity valid for $\lambda \geq 0$,

$$\frac{\sqrt{2\lambda}}{2\tanh b\sqrt{2\lambda}} - \frac{1}{2b} = \frac{2\lambda}{b}\sum_{k\geq 1}\frac{1}{k^2\pi^2 + 2b^2\lambda}.$$

The right-hand side function is furthermore analytic on $(-\pi^2/(2b^2), +\infty)$, this interval being its maximal domain of analyticity. The function ϕ, also analytic with maximal domain $(-\rho_\phi, \infty)$, therefore coincides with the right-hand side function on $(-\rho_\phi, \infty) = (-\pi^2/(2b^2), +\infty)$. Assumption (C) is satisfied. Mathematica also gives that $\rho = \pi^2/(8b^2)$, meaning that:

$$\mathbb{E}\left(\mathrm{e}^{\frac{\pi^2}{8b^2}\sigma_\ell}, T_{b\wedge -b} > \sigma_\ell\right) = 1.$$

From (5.20), the probability measure $\mathbb{P}_x^{(\infty)}$ satisfies:

$$\mathbb{P}_x^{(\infty)}(A) = \mathbb{E}_x\left(\mathrm{e}^{\pi^2 t/(8b^2)}\,\frac{h_\rho(X_t)}{h_\rho(x)}, X \in A, T_{b\wedge -b} > t\right),\ \ A \in \mathcal{F}_t, \tag{5.33}$$

with $h_\rho(x) = \mathbb{E}_x(\mathrm{e}^{\rho T_0}, T_{b\wedge -b} > T_0)$. Rather than computing h from its definition as we have done so far, we notice that, taking $A = \Omega$ in (5.33), we have

$$\mathbb{E}_x(h_\rho(X_t)\,\mathrm{e}^{\rho t}, T_{b\wedge -b} > t) = h_\rho(x).$$

This relation in turn allows to compute the generator of the Brownian motion evaluated at h_ρ:

$$\begin{cases} \frac{1}{2}h_\rho''(x) = -\rho\, h_\rho(x) \text{ on } (-b, b), \\ h(-b) = h(b) = 0. \end{cases}$$

But this equation has a unique solution:

$$h(x) = \sin\left(\frac{\pi(x+b)}{2b}\right).$$

From that point, it is classic to derive that $\mathbb{P}^{(\infty)}$ is the law of a diffusion in $(-b, b)$ with generator acting as follows on smooth functions f:

$$f(x) \to f''(x) - \frac{\pi}{2b}\tan\left(\frac{\pi x}{2b}\right)f'(x) \text{ in } (-b, b),$$

see Knight [76], where it is also proven that $\mathbb{P}^{(\infty)}$ is the law of the Q-process.

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