Reduced Basis method and Variational Inequalities

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Variational inequalities
Optimization and saddle points

- Variational equalities:

\[
\min_{u \in V} \frac{1}{2} a(u, u) - f(u) \Rightarrow a(u, v) = f(v) \quad \forall v \in V.
\]

- Variational inequalities: Denote

\[
X = \{u \in V, \quad b(u, \eta) \leq g(\eta), \quad \eta \in M\}, \quad M \text{ closed convex set},
\]

\[
\min_{u \in X} \frac{1}{2} a(u, u) - f(u) \Rightarrow a(u, v - u) \geq f(v - u), \quad \forall v \in X,
\]

or equivalently:

\[
\begin{align*}
    a(u, v) + b(v, \lambda) & = f(v), \quad \forall v \in V, \\
    b(u, \eta - \lambda) & \leq g(\eta - \lambda), \quad \forall \eta \in M.
\end{align*}
\]
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Consider the saddle point problem:

**Standard Variational inequality**

Given $\mu \in \mathcal{P}$, $V$, $W$ two Hilbert spaces and $M$ a convex cone in $W$, find $(u(\mu), \lambda(\mu)) \in V \times M$ such that

$$a(u(\mu), v; \mu) + b(v, \lambda(\mu)) = f(v; \mu), \quad v \in V$$

$$b(u(\mu), \eta - \lambda(\mu)) \leq g(\eta - \lambda(\mu); \mu), \quad \eta \in M.$$ 

Equivalently, if $a$ is symmetric:

$$\inf_{u \in X(\mu)} \frac{1}{2} a(u, u; \mu) - f(u; \mu)$$
Moreover, we assume that:

• $a$ is uniformly coercive and continuous w.r. to $\mu$,

$$a(u, v; \mu) \leq \gamma_a \|u\|_V \|v\|_V \quad \alpha \|u\|_V^2 \leq a(u, u; \mu),$$

• $b$ is continuous and inf-sup stable,

$${\inf}_{\eta \in W} {\sup}_{v \in V} b(v, \eta)/(\|v\|_V \|\eta\|_W) \geq \beta > 0,$$

• $f$ and $g$ are continuous,

$$f(v) \leq \gamma_f \|v\|_V, \quad g(\eta) \leq \gamma_g \|\eta\|_W,$$

• $a, f, g$ are Lipschitz with respect to $\mu$. 
Problem setting
Examples of applications

- Mechanics: obstacle problems
- Finance: pricing of American Options
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Now, consider the standard Galerkin approximation: let $V_N$ and $W_N$ some finite dimensional linear sub-space of $V$ and $W$.

**Galerkin Approximation**

Find $(u_N(\mu), \lambda_N(\mu)) \in V_N \times M_N$ such that

\[
\begin{align*}
    a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) &= f(v_N; \mu), \quad v_N \in V_N \\
    b(u_N(\mu), \eta_N - \lambda_N(\mu)) &\leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N
\end{align*}
\]
In the R-B setting, $V_N$ and $W_N$ are built thanks to "snapshots", i.e. fine solutions of the initial problem corresponding to a set of parameters $(\mu_1, ..., \mu_{N_S})$.

In our case, the construction is done as follows:

$$V_N = \text{span}\{u(\mu_i), B\lambda(\mu_i), \ i = 1, ..., N_S\},$$
$$W_N = \text{span}\{\lambda(\mu_i), \ i = 1, ..., N_S\},$$
$$M_N = \text{span}_+\{\lambda(\mu_i), \ i = 1, ..., N_S\},$$

where $B$ is the operator defined through:

$$\langle B\lambda(\mu_i), v \rangle_V = b(v, \lambda(\mu_i)), \ v \in V.$$ 

This approach consists in enriching the primal basis with supremizers.

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Inf-sup stability:

\[ \beta_N := \inf_{\eta_N \in W_N} \sup_{v_N \in V_N} \frac{b(v_N, \eta_N)}{\|v_N\|_V \|\eta_N\|_W} = \inf_{\eta_N \in W_N} \sup_{v_N \in V_N} \frac{\langle v_N, B\eta_N \rangle_V}{\|v_N\|_V \|\eta_N\|_W} \]

\[ = \inf_{\eta_N \in W_N} \frac{\langle B\eta_N, B\eta_N \rangle_V}{\|B\eta_N\|_V \|\eta_N\|_W} \]

\[ \geq \inf_{\eta \in W} \frac{\langle B\eta, B\eta \rangle_V}{\|B\eta\|_V \|\eta\|_W} = \inf_{\eta \in W} \sup_{v \in V} \frac{\langle v, B\eta \rangle_V}{\|v\|_V \|\eta\|_W} = \beta > 0. \]

Hence, existence and uniqueness of the reduced solution \((u_N, \lambda_N)\).
R-B method
Analytical results

Stability of the scheme:

\[ \| u_N(\mu) \|_V \leq \frac{1}{2\alpha} \left( \gamma_f + \frac{\gamma_a}{\beta_N} \gamma_g \right) + \sqrt{\frac{1}{4\alpha^2} \left( \gamma_f + \frac{\gamma_a}{\beta_N} \gamma_g \right)^2 + \frac{\gamma_g \gamma_f}{\alpha \beta_N}} \]

:= \gamma_u,

\[ \| \lambda_N(\mu) \|_W \leq \frac{1}{\beta_N} \left( \gamma_f + \gamma_a \gamma_u \right). \]
Lipschitz continuity:
For all $\mu, \mu'$ there exist $L_u, L_\lambda$ such that

\[
\|u_N(\mu) - u_N(\mu')\|_V \leq L_u \|\mu - \mu'\|_P,
\]
\[
\|\lambda_N(\mu) - \lambda_N(\mu')\|_W \leq L_\lambda \|\mu - \mu'\|_P.
\]
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First, we define the equality residual $r(\cdot; \mu) \in V'$ and $s(\cdot; \mu) \in W'$ by

$$r(v; \mu) := f(v; \mu) - a(u_N(\mu), v; \mu) - b(v, \lambda_N(\mu)),$$

$$s(\eta; \mu) := b(u_N(\mu), \eta) - g(\eta; \mu) =: \langle \eta, \eta_s(\mu) \rangle_W.$$

The residual $r$ represents the right hand side of the error-equation

$$a(u(\mu) - u_N(\mu), v; \mu) + b(v, \lambda(\mu) - \lambda_N(\mu)) = r(v; \mu).$$
Then define:

\[
\begin{align*}
\delta_r(\mu) & := \| r(\cdot; \mu) \|_{V'} \\
\delta_{s1}(\mu) & := \| \pi(\eta_s(\mu)) \|_W \\
\delta_{s2}(\mu) & := \langle \lambda_N(\mu), \pi(\eta_s(\mu)) - \eta_s(\mu) \rangle_W,
\end{align*}
\]

with \( \pi : W \rightarrow M \), the orthogonal projection on \( M \), and \( \eta_s \):

\[
\langle \eta, \eta_s(\mu) \rangle_W = s(\eta; \mu), \quad \eta \in W.
\]
Upper a posteriori Error Bound

For any $\mu$, the reduced basis errors can be bounded by

$$\|u(\mu) - u_N(\mu)\|_V \leq \Delta u(\mu) := c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)},$$

$$\|\lambda(\mu) - \lambda_N(\mu)\|_W \leq \Delta \lambda(\mu) := \frac{1}{\beta_N} (\delta_r(\mu) + \gamma_a(\mu)\Delta u(\mu)), $$

with constants

$$c_1(\mu) := \frac{1}{2\alpha(\mu)} \left( \delta_r(\mu) + \frac{\delta_{s1}(\mu)\gamma_a(\mu)}{\beta_N} \right),$$

$$c_2(\mu) := \frac{1}{\alpha(\mu)} \left( \frac{\delta_{s1}(\mu)\delta_r(\mu)}{\beta_N} + \delta_{s2}(\mu) \right).$$
Sketch of the proof:

\[ \alpha(\mu) \|e_u\|_V^2 \leq a(e_u, e_u) = r(e_u) + b(e_\lambda, e_u). \]

\[
\begin{align*}
b(e_\lambda, e_u) &= b(\lambda_N, u_N) - b(\lambda, u_N) - b(\lambda_N, u) + b(\lambda, u) \\
&\leq g(\lambda_N) - s(\lambda) - g(\lambda) - g(\lambda_N) + g(\lambda) \\
&= -s(\lambda) = s(e_\lambda) = \langle e_\lambda, \eta_s \rangle_W \\
&= \langle e_\lambda, \pi(\eta_s) \rangle_W + \langle e_\lambda, \eta_s - \pi(\eta_s) \rangle_W \\
&\leq \|e_\lambda\|_W \|\eta_s - \pi(\eta_s)\|_W + \langle e_\lambda, \pi(\eta_s) \rangle_W \\
&\leq \delta_{s1} \|e_\lambda\|_W + \delta_{s2}.
\end{align*}
\]
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Obstacle example: $\mu = (\mu_1, \mu_2)$

$$a(u, v; \mu) := \int_{\Omega} \nu(\mu)(x) \nabla u(x) \cdot \nabla v(x) \, dx, \quad v, u \in V$$

$$b(u, \eta) := -\eta(u), \quad u \in V, \eta \in W$$

with $\nu(\mu)(x) = \mu_1 \text{Ind}_{[0,1/2]}(x) + \nu_0 \text{Ind}_{[1/2,1]}(x)$. The obstacle is given by:

$$g(\eta; \mu) = \int \eta(x) h(x; \mu)$$

$$h(x; \mu) = -0.2(\sin(\pi x) - \sin(3\pi x)) - 0.5 + \mu_2 x.$$
Numerical methods:

- Snapshot computation (large problems): Primal-Dual Active Set Strategy.


- Reduced problems (small problems): Standard QP-solver.
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Numerical experiments

Obstacle problem

Figure: Left-middle: Primal solutions and obstacle. Right column: Exact and reduced solutions for a particular parameter. Solid line: exact solutions, dashed line: reduced solutions.
Numerical experiments
Obstacle problem

Figure: Left-middle: Dual solutions. Right column: Exact and reduced solutions for a particular parameter. Solid line: exact solutions, dashed line: reduced solutions.
Numerical experiments
Obstacle problem

Figure: Eight first vectors of the reduced basis \( \{ \varphi_i \}_{i=1}^{N_V} \) forming \( V_N \) (left), of the dual reduced family \( \{ \lambda(\mu_i) \}_{i=1}^{N_S} \) (middle), and the corresponding supremizers \( \{ B \lambda(\mu_i) \}_{i=1}^{N_S} \) (right).
**Numerical experiments**

**Results**

\[
N_S \quad \beta_N \text{ for } V_N^{(2)} \quad \log_{10}(\beta_N) \text{ for } V_N^{(1)}
\]

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<tr>
<th>(N_S)</th>
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<th>(\log_{10}(\beta_N)) for (V_N^{(1)})</th>
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</tr>
<tr>
<td>25</td>
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</tr>
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</table>

**Figure:** Effect of the inclusion of supremizers. Inf-sup stability constants (left) and number of iterations (right) required to solve the reduced problem. Dots: \(V_N = V_N^{(2)}\) with supremizers; crosses: \(V_N = V_N^{(1)}\) without supremizers.
Basis generation via Greedy Algorithm.

Figure: Numerical values of the error $\varepsilon_N(\mu) = e_u(\mu) + e_\lambda(\mu)$ when selecting the parameters on an uniform grid (left) or thanks to the a posteriori estimators (middle).
"A Reduced Basis Method for Parametrized Variational Inequalities",
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We now consider:

\[ \langle \partial_t u, v \rangle_V + a(u, v; \mu) - b(\lambda, v) = f(v; \mu), \]
\[ b(\eta - \lambda, u) \geq g(\eta - \lambda; \mu). \]

Required adaptations:

- **Time solver:** Crank-Nicholson
- **Primal Basis construction:** POD-greedy algorithm.
  
- **Dual Basis construction:** Angle-greedy algorithm.
Angle-greedy algorithm:
Given $N_W$, $\mathcal{P}_{train} \subset \mathcal{P}$, choose arbitrarily $0 \leq n_1 \leq L$ and $\mu_1 \in \mathcal{P}_{train}$ and do

1. set $\Xi^1_N = \left\{ \frac{\lambda^{n_1}(\mu_1)}{\|\lambda^{n_1}(\mu_1)\|_W} \right\}$, $W^1_N := \text{span}(\Xi^1_N)$,
2. for $k = 1, \ldots, N_W - 1$, do
   1. find $(n_{k+1}, \mu_{k+1}) := \arg\max_{n=0,\ldots,L, \mu \in \mathcal{P}_{train}} \left( \angle (\lambda^n(\mu), W^k_N) \right)$,
   2. set $\xi_{k+1} := \frac{\lambda^{n_{k+1}}(\mu_{k+1})}{\|\lambda^{n_{k+1}}(\mu_{k+1})\|_W}$,
   3. define $\Xi^{k+1}_N := \Xi^k_N \cup \{\xi_{k+1}\}$, $W^{k+1}_N := \text{span}(\Xi^{k+1}_N)$,
3. define $\Xi_N := \Xi^W_N$, $W_N := \text{span}(\Xi_N)$. 
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\[ \partial_t P - \frac{1}{2} \sigma^2 s^2 \partial^2_{ss} P - (r - q) s \partial_s P + r P \geq 0, \quad P - \psi \geq 0, \]

\[ \left( \partial_t P - \frac{1}{2} \sigma^2 s^2 \partial^2_{ss} P - (r - q) s \partial_s P + r P \right) \cdot (P - \psi) = 0, \]

where

- \( P = P(s, t) \) is the price of an American put,
- \( s \in \mathbb{R}_+ \) the asset’s value,
- \( \sigma, r, q \) are the volatility, the interest rate and the dividend payment,
- \( \psi = \psi(s, t) \) is the payoff function.
The boundary and initial conditions are as follows: \( P(s, 0) = \psi(s) \), \( P(0, t) = K \), \( \lim_{s \to +\infty} P(s, t) = 0 \), where \( K > 0 \) is a fixed strike price that satisfies \( K = \psi(0, 0) \). In what follows, we use \( \psi(s, t) = (K - s)_+ \) with \((\cdot)_+ = \max(0, \cdot)\).
Extension to time-dependent systems
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Figure: Eight first vectors of the reduced basis $\Psi_N$, $\Xi_N$ and the corresponding supremizers.
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\[\varepsilon^u_N := \max_{\mu \in \mathcal{P}_{\text{train}}} \sqrt{\sum_{n=0}^{L} \|u^n(\mu) - \Pi_{V^k_N}(u^n(\mu))\|^2_V},\]

\[\varepsilon^\lambda_N := \max_{n = 0, \ldots, L, \mu \in \mathcal{P}_{\text{train}}} \left(\angle (\lambda^n(\mu), W^k_N)\right)\]

\[\text{err}_N(\mu) = \sqrt{\Delta t \sum_{n=0}^{L} \|u^n(\mu) - u^n_N(\mu)\|^2_V}, \quad \text{Err}^L_\infty = \max_{\mu \in \mathcal{P}_{\text{test}}} (\text{err}_N(\mu)).\]
Figure: Values of $\varepsilon^u_N$ and $\varepsilon^\lambda_N$ during the iterations of POD-greedy Algorithm (left) and Angle-greedy (middle). Right: Values of $\text{Err}_N^\infty$ with respect to $N_V$ and $N_W$. 

Extension to time-dependent systems
Application to American Option Pricing
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Conclusions and perspectives

Conclusions:

- Theoretical and numerical improvement when using supremizers
- Better accuracy for the primal variable as for the dual
- Adaptation to time dependent systems

Perspectives:

- Better dual cone generation
- Full decomposition of a posteriori estimators
- A posteriori estimators for the time-dependent case
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Also: Another approach this morning, see the work of K. Veroy et al
→ primal-dual approach.
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Also: Another approach tomorrow, see the talk of K. Urban → time-space setting.