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WELL-POSEDNESS THEORY FOR A NONCONSERVATIVE BURGERS-TYPE SYSTEM ARISING IN DISLOCATION DYNAMICS[*]

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Abstract. In this work we study a system of nonconservative Burgers type in one space dimension, arising in modeling the dynamics of dislocation densities in crystals. Starting from physically relevant initial data that are of a special form, namely nondecreasing, periodic plus linear functions, we prove the global existence and uniqueness of a solution in $H^1_{loc}(\mathbb{R}\times[0,+\infty))$ that preserves the nature of the initial data. The approach is made by adding some viscosity to the system, obtaining energy estimates, and passing to the limit for vanishing viscosity. A comparison principle is shown for this system as well as an application in the case of the classical Burgers equation.

Key words. system of Burgers equations, system of nonlinear transport equations, nonlinear hyperbolic system, dynamics of dislocation densities

AMS subject classifications. 35L45, 35Q53, 35Q72, 74H20, 74H25

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1. Introduction.

1.1. Physical motivations and presentation of the model.

Real crystals comprise certain defects in the organization of their crystalline structure called dislocations. In a particular case where these defects are parallel straight lines in the three-dimensional space, they can be viewed as points in a plan. Under the effect of exterior constraints, dislocations can move in a certain crystallographic direction called the slip direction. This slip direction is given by a vector called the "Burgers vector." The norm of this vector represents the amplitude of the generated deformation. (We refer the reader to [12] for further physical explanation.)

In this work, we are interested in the study of a one-dimensional submodel of a problem introduced by Groma and Balogh [11], initially proposed in the two-dimensional case. In fact, this one-dimensional submodel was defined by El Hajj and Forcadel [8, Lemme 3.1].

This two-dimensional model is characterized by the fact that dislocations propagate in the plane (x_1, x_2) following the two Burgers vectors $\pm\vec{b}$ with $\vec{b} = (1, 0)$. In this one-dimensional submodel we suppose also that dislocation densities depend only on the variable $x = x_1 + x_2$, which transforms the two-dimensional into a one-dimensional model (see El Hajj and Forcadel [8] for more modeling details).

More precisely this one-dimensional model is given by the following coupled

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equations of nonconservative Burgers type:

$$
(1)\quad \begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) &= -\left(a(t) + (\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^+}{\partial x}(x,t) \\ & \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho^-}{\partial t}(x,t) &= \left(a(t) + (\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^-}{\partial x}(x,t) \\ & \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)). \end{cases}
$$

The unknowns ρ^+ and ρ^- are scalar-valued functions, which we denote for simplicity by $\rho^\pm$. Their spatial derivatives $\frac{\partial \rho^\pm}{\partial x}$ are the dislocation densities of the Burgers vector $\pm\vec{b} = \pm(1,0)$. The function $a = a(t)$, representing the field of the imposed exterior constraint, is supposed to be independent of x, and the constant α depends on the elastic coefficients and the material size.

We consider the following initial conditions for (1):

$$
(2)\qquad \rho^\pm(x, t=0) = \rho_0^\pm(x) = \rho_0^{\pm,per}(x) + L_0 x, \quad x \in \mathbb{R},
$$

where $\rho_0^{\pm,per}$ are 1-periodic functions. We thus modelize a periodic distribution for the ± dislocations, with a spatial period of length 1. Note that each type of ± dislocation has a mean density equal to L_0. In fact, the use of the periodic boundary conditions is a way of regarding what is going on in the interior of the material away from its boundary.

1.2. A brief review of some related literature. From a mathematical point of view, system (1) is related to other similar models such as transport equations based on vector fields with low regularity. Such equations were, for instance, studied by DiPerna and Lions in [7]. They proved the existence and uniqueness of a solution (in the renormalized sense) for vector fields in $L^1((0,+\infty); W^{1,1}_{loc}(\mathbb{R}^N))$ whose divergence is in $L^1((0,+\infty); L^\infty(\mathbb{R}^N))$. This study was generalized by Ambrosio [3], who considered vector fields in $L^1((0,+\infty); BV_{loc}(\mathbb{R}^N))$ with bounded divergence. In the present paper, we work in dimension $N = 1$ and prove the existence and uniqueness of solutions of the system (1)–(2) with a vector field (i.e., the velocity) only in $L^\infty((0,+\infty), H^1_{loc}(\mathbb{R}))$.

We also refer the reader to the works of LeFloch [13] and LeFloch and Liu [14], in which they considered the study in the framework of functions of bounded variation for a system of the form

$$
(3)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) + A(u)\dfrac{\partial u}{\partial x}(x,t) = 0, & u(x,t) \in U,\ x \in \mathbb{R},\ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}
$$

where the space of states U is an open subset of $\mathbb{R}^p$, and A is a $(p \times p)$ matrix which is of class C^1 on U. Moreover, $A(u)$ have p scalar distinct eigenvalues that we denote by $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_p(u)$. We remark that this condition on the eigenvalues does not enter into our framework even in the case where $\alpha = a = 0$, because we have not sign property on $\rho^+ - \rho^-$. LeFloch and Liu proved that if the initial condition u_0 is sufficiently close to a constant state, and if the total variation $TV(u_0)$ is assumed to be small enough, then system (3) admits a unique solution in

$L^\infty(\mathbb{R}\times(0,+\infty))\cap BV(\mathbb{R}\times(0,+\infty))$, in the sense of weak entropy solutions with respect to admissible function (see LeFloch [13, Definition 3.2]).

When the system is hyperbolic and symmetric, this corresponds to the case $\alpha = a = 0$ in our system (1); it is proved in Serre [17, Thm. 3.6.1] and [18] to be a result of local existence and uniqueness in $C([0,T);H^s(\mathbb{R}^N))\cap C^1([0,T);H^{s-1}(\mathbb{R}^N))$, with $s>\frac{N}{2}+1$, this result being only local in time, even in dimension $N=1$.

The assumptions of increasing initial conditions were also considered in the study of the Euler equation for compressible fluids in dimension one. With regard to these studies, we refer the reader to Chen and Wang [6, Thm. 3.1] for an existence and uniqueness result in $C^1(\mathbb{R}\times[0,+\infty))$ based on the method of characteristic. The result of Chen and Wang shows that the Euler equation of compressible fluids does not create shocks for suitable increasing and $C^1(\mathbb{R})$ initial conditions. In our case, we already knew that solutions of (1) are Lipschitz continuous; see El Hajj and Forcadel [8]. Even if this regularity question is not addressed in the present paper, we may expect some $C^1(\mathbb{R}\times[0,+\infty))$ regularity of the solution for $C^1(\mathbb{R})$ initial data.

1.3. Main result.

The main result of this paper is the existence and uniqueness of global in time solutions for the system (1)–(2), modeling the dynamics of dislocation densities. This result ensures the mathematical well-posedness of the Groma–Balogh model [11] in the particular case of our interest.

THEOREM 1.1 (existence and uniqueness). *For all* $T, L_0\geq 0$, $\alpha\in\mathbb{R}$, *and* $\rho_0^\pm\in H^1_{loc}(\mathbb{R})$, *and under the assumptions*

(H1) $\rho_0^\pm(x+1)=\rho_0^\pm(x)+L_0$ *(1-periodic function + linear function),*

(H2) $\dfrac{\partial\rho_0^\pm}{\partial x}\geq 0$ *a.e. in* $\mathbb{R}$ *(*$\rho_0^\pm$ *nondecreasing),*

(H3) $a\in L^\infty(0,T)$,

the system (1)–(2) *admits a unique solution* $\rho^\pm\in H^1_{loc}(\mathbb{R}\times[0,T))$ *such that, for a.e.* $t\in(0,T)$, *the function* $\rho^\pm(.,t): x\longmapsto\rho^\pm(x,t)$ *verifies* (H1) *and* (H2).

The preceding theorem gives a global existence and uniqueness result of the system (1). Its proof is based on the following steps. First, we regularize the system (1); then we show a uniform a priori estimate in $L^\infty((0,T);H^1_{loc}(\mathbb{R}))$ for this regularized system. These estimates lead to a result of existence for long time solution and ensure the passage to the limit by compactness. Finally, the demonstration of uniqueness is done in a direct way.

THEOREM 1.2 (comparison principle for (1) with $\alpha=0$). *Let* $a(\cdot)$ *satisfy* (H3) *and* $\rho_1^{\pm}$, $\rho_2^\pm\in H^1_{loc}(\mathbb{R}\times[0,T))$ *be two solutions of the system* (1) *with* $\alpha=0$. *Moreover, let* $\rho_1^\pm(.,t)$, $\rho_2^\pm(.,t)$ *verify* (H1) *and* (H2) *for a.e.* $t\in(0,T)$. *Then, if* $\rho_1^\pm(\cdot,0)\leq\rho_2^\pm(\cdot,0)$ *in* $\mathbb{R}$, *we have* $\rho_1^\pm\leq\rho_2^\pm$ *a.e. in* $\mathbb{R}\times(0,T)$.

This comparison result was crucial in a previous work [8], for the demonstration of existence and uniqueness of a Lipschitz solution to problem (1), in the sense of viscosity solution, for Lipschitz initial conditions. Here the interest of this result is a little bit secondary. Indeed, thanks to this comparison principle, we have been able to obtain indirectly $H^1_{loc}(\mathbb{R}\times[0,T))$ estimates. These estimates in turn lead to a result of existence in $H^1_{loc}(\mathbb{R}\times[0,T))$.

Our work focuses on the study of the dynamics of dislocation densities. In a different direction, let us quote some recent results on the dynamics of dislocation lines, taken individually, that are represented by nonlocal Hamilton–Jacobi equations (see [2, 9] and [1, 4] for local and global in time results, respectively).

Remark 1.3 (existence and uniqueness for the Burgers equation). We remark that these techniques can be applied to the case of the classical Burgers equations in

$W^{1,p}_{loc}(\mathbb{R}\times[0,T))$ for all $1 \le p < +\infty$.

Indeed, if we consider for a given function f and initial data u_0 the equation

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(f(u)) = 0 & \text{in} \quad \mathcal{D}'(\mathbb{R}\times(0,T)), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{array}\right. \tag{4}$$

then we have the following theorem.

THEOREM 1.4. *Let* $p \in [1,+\infty)$ *and* f *be locally Lipschitz and convex; then, for all* $T, L_0 \ge 0$, *and* $u_0 \in W^{1,p}_{loc}(\mathbb{R})$, *they satisfy* (H1) *and* (H2). *Equation* (4) *admits a solution* $u \in W^{1,p}_{loc}(\mathbb{R}\times[0,T))$, *unique in the class of solutions satisfying* (H1) *and* (H2), *a.e.* $t \in (0,T)$.

1.4. Organization of the paper.

In section 2, we regularize the function $a(\cdot)$ and the initial conditions and prove that the system (1)–(2) modified by the term $(\varepsilon\{\frac{\partial^2\rho^\pm}{\partial x^2}\})$ admits local in time solutions (in the "mild" sense). This will be achieved by using an application of a fixed point theorem in the space of functions in $C([0,T);H^1_{loc}(\mathbb{R}))$ and verifying (H1) for all $t\in(0,T)$. In section 3, we prove that the obtained solutions are regular and verify (H2) for all $t \in (0,T)$, with initial conditions verifying (H2). In section 4, we prove some uniform a priori estimates on the regularized solution obtained in section 3. Thanks to these estimates, we also prove the existence of global in time solutions. In section 5, we give the demonstration of Theorem 1.1, and in section 6 we prove a comparison principle result of the system (1) in the case $\alpha = 0$. Finally, in section 7 we give an application of the previous results in the case of the classical Burgers equation.

2. Existence of solutions for an approximated system.

In this section, we prove a theorem of existence of solutions, local in time, for the system (1) modified by the term $\varepsilon\{\frac{\partial^2\rho^\pm}{\partial x^2}\}$ after the regularization of the function $a(\cdot)$ and the initial conditions. This approximation brings us back to the study, for every $0<\varepsilon<1$, of the following system:

$$\left\{\begin{array}{rcl} \dfrac{\partial\rho^{+,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2\rho^{+,\varepsilon}}{\partial x^2} &=& -\left(a^\varepsilon(t) + (\rho^{+,\varepsilon}-\rho^{-,\varepsilon}) + \alpha\displaystyle\int_0^1(\rho^{+,\varepsilon}-\rho^{-,\varepsilon})(y,t)dy\right)\dfrac{\partial\rho^{+,\varepsilon}}{\partial x} \\ && \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \\ \dfrac{\partial\rho^{-,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2\rho^{-,\varepsilon}}{\partial x^2} &=& \left(a^\varepsilon(t) + (\rho^{+,\varepsilon}-\rho^{-,\varepsilon}) + \alpha\displaystyle\int_0^1(\rho^{+,\varepsilon}-\rho^{-,\varepsilon})(y,t)dy\right)\dfrac{\partial\rho^{-,\varepsilon}}{\partial x} \\ && \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \end{array}\right. \tag{5}$$

where $a^\varepsilon = \tilde{a} * \eta_\varepsilon$, with $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C^\infty_c(\mathbb{R})$, η is positive, and $\int_{\mathbb{R}}\eta = 1$. The function $\tilde{a}(\cdot)$ is an extension in $\mathbb{R}$ of the function $a(\cdot)$ by 0.

We also consider the regularized initial conditions of the system (5):

$$\rho^{\pm,\varepsilon}(x,0) = \rho_0^{\pm,\varepsilon}(x) = \rho_0^{\pm,\varepsilon,per}(x) + L_0x = \rho_0^{\pm,per} *_{\mathbb{T}} \eta_\varepsilon(x) + L_0x. \tag{6}$$

We have the following local in time existence result for the approximated system.

THEOREM 2.1 (short time existence). *Assume* (H1) *and* (H3). *For all* $\alpha \in \mathbb{R}$ *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$ *there exists*

$$T^\star(\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0$$

such that the system (5)–(6) *admits a solution* $\rho^{\pm,\varepsilon} \in C([0,T^\star); H^1_{loc}(\mathbb{R}))$ *with* $\rho^{\pm,\varepsilon}(.,t)$ *verifying* (H1).

For the proof of this theorem, see subsection 2.3. Before going on, we need to give some notation and preliminary results that will be used throughout the paper.

2.1. Notation.

In what follows, we are going to use the following notation:

1. $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$.
2. $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_0 x$.
3. $\mathbb{T} = (\mathbb{R}/\mathbb{Z})$ is the $[0,1)$-periodic interval.
4. Let $f = (f_1, f_2)$ be a vector such that $f_i \in H^1(\mathbb{T})$ for $i \in \{1,2\}$. The norm of f in $\left(H^1(\mathbb{T})\right)^2$ will be defined by $\|f\|_{H^1(\mathbb{T})} = \max(\|f_1\|_{H^1(\mathbb{T})}, \|f_2\|_{H^1(\mathbb{T})})$.
5. Let f be a function from $\mathbb{R} \times (0,T)$ to $\mathbb{R}$. We note by $f(t) = f(.,t) : x \longmapsto f(x,t)$.

Remark 2.2 (periodicity). According to (H1)–(H2), it is clear that ρ^ε, $\rho^{\pm,\varepsilon,per}$, and $\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}$ are 1-periodic in space functions.

Under the notation of section 2.1, we know that the system (5) is equivalent to

$$\begin{aligned} (7)\quad \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial t} - \varepsilon \frac{\partial^2 \rho^{\pm,\varepsilon,per}}{\partial x^2} = \overbrace{\mp C_\alpha[\rho^\varepsilon(t)] \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}}^{\text{bilinear term}} \overbrace{\mp a^\varepsilon(t) \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x} \mp L_0 C_\alpha[\rho^\varepsilon(t)]}^{\text{linear term}} \\ \mp L_0 a^\varepsilon(t) \quad \text{in } \mathbb{T} \times (0,T), \end{aligned}$$

where $C_\alpha[\rho^\varepsilon(t)](x) = (\rho^\varepsilon(x,t) + \alpha \int_0^1 \rho^\varepsilon(y,t)dy)$, with the periodic initial conditions

$$(8)\qquad \rho^{\pm,\varepsilon,per}(x,0) = \rho_0^{\pm,\varepsilon,per}(x) \quad \text{in} \quad \mathbb{T}.$$

2.2. Preliminary results.

Lemma 2.3 (properties of the regularized sequence). *Under hypotheses* (H1) *and* (H3) *and for every* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$, *we have the following:*

1. *The functions* $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T})$ *and verify the following estimate:*
$$\|\rho_0^{\pm,\varepsilon,per}\|_{H^1(\mathbb{T})} \le C \|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}.$$
2. *The function* $a^\varepsilon(\cdot) \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and verifies the following estimate:*
$$\|a^\varepsilon\|_{L^\infty(\mathbb{R})} \le \|a\|_{L^\infty(0,T)}.$$
3. *The sequence* $a^\varepsilon(\cdot)$ *strongly converges to* $a(\cdot)$ *in* $L^2(0,T)$. *The sequences* $\rho_0^{\pm,\varepsilon,per}$ *strongly converge to* $\rho_0^{\pm,per}$ *in* $H^1(\mathbb{T})$.

The proof of this lemma is a classical property of the regularizing sequence $(\eta_\varepsilon)_\varepsilon$.

Lemma 2.4 (mild solution). *Assume* (H3). *For every* $T \ge 0$, *if* $\rho^{\pm,\varepsilon,per} \in C([0,T); H^1(\mathbb{T}))$ *are solutions of the equation*

$$(9)\quad \begin{aligned} \rho^{\pm,\varepsilon,per}(x,t) &= S_\varepsilon(t)\rho_0^{\pm,\varepsilon,per} \\ &\mp L_0 \int_0^t a^\varepsilon(s)ds \mp \int_0^t S_\varepsilon(t-s) \left(C_\alpha[\rho^\varepsilon(s)] \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}(s) \right) ds \\ &\mp \int_0^t S_\varepsilon(t-s) \left(L_0 C_\alpha[\rho^\varepsilon(s)] + a(t) \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}(s) \right) ds, \end{aligned}$$

where $S_\varepsilon(t) = e^{\varepsilon t \Delta}$ *is the heat semigroup, then* $\rho^{\pm,\varepsilon,per}$ *is a solution of the system* (7)–(8) *in the sense of distributions.*

For the proof of this lemma, we refer the reader to Pazy [16, Thm. 5.2, p. 146].

Lemma 2.5 (fixed point). *Let* E *be a Banach space,* B *be a continuous bilinear application from* $E \times E$ *to* E*, and* L *be a continuous linear application from* E *to* E *such that*

$$\|B(x,y)\|_E \le \lambda \|x\|_E \|y\|_E \quad \textit{for all} \quad x, y \in E,$$

$$\|L(x)\|_E \le \mu \|x\|_E \quad \textit{for all} \quad x \in E,$$

where $\lambda > 0$ *and* $\mu \in (0,1)$ *are given constants. Then, for all* $x_0 \in E$ *such that*

$$\|x_0\|_E < \frac{1}{4\lambda}(\mu - 1)^2,$$

the equation $x = x_0 + B(x,x) + L(x)$ *admits a solution in* E*.*

For the proof of this lemma we refer the reader to Cannone [5, Lem. 4.2.14].

In order to show the existence of a solution within the framework of Lemma 2.4, we apply Lemma 2.5 in the space $E = \left(L^\infty((0,T); H^1(\mathbb{T}))\right)^2$, where x_0, B, and L are defined, for $u = (u_1, u_2),\ v = (v_1, v_2) \in E$, by

(10)
$$x_0 = S_\varepsilon(t)\rho^\varepsilon_{0,vec} + L_0 \vec{i} \int_0^t a^\varepsilon(s) ds, \quad \text{where} \quad \rho^\varepsilon_{0,vec} = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per}),\ \vec{i} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

(11)
$$B(u,v)(t) = \bar{I}_1 \int_0^t S_\varepsilon(t-s)\left(C_\alpha[u_1(s) - u_2(s)]\frac{\partial v}{\partial x}(s)\right) ds, \text{ where } \bar{I}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

(12)
$$L(u)(t) = L_0 \vec{i} \int_0^t S_\varepsilon(t-s) C_\alpha[u_1(s) - u_2(s)] ds + \bar{I}_1 \int_0^t S_\varepsilon(t-s)\left(a^\varepsilon(s)\frac{\partial u}{\partial x}(s)\right) ds.$$

Lemma 2.6 (decreasing estimates). *If* $f \in L^q(\mathbb{T})$ *with* $2 \le q \le +\infty$ *and* $g \in L^2(\mathbb{T})$*, then for all* $t > 0$ *we have the following estimates:*

(i)
$$\|S_\varepsilon(t)(fg)\|_{L^\infty(\mathbb{T})} \le C t^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{T})} \|g\|_{L^2(\mathbb{T})}.$$

(ii)
$$\left\|\frac{\partial}{\partial x}(S_\varepsilon(t) f)\right\|_{L^2(\mathbb{T})} \le C t^{-\frac{1}{2}} \|S_\varepsilon\left(\frac{t}{2}\right) f\|_{L^2(\mathbb{T})}.$$

(iii)
$$\left\|\frac{\partial}{\partial x}\left(S_\varepsilon(t)(fg)\right)\right\|_{L^2(\mathbb{T})} \le C t^{-\frac{1}{2}(1+\frac{1}{q})} \|f\|_{L^q(\mathbb{T})} \|g\|_{L^2(\mathbb{T})},$$

where $C = C(\varepsilon)$ is a positive constant depending on ε.

For the proof of this lemma, see Pazy [16, Lem. 1.1.8 and Thm. 6.4.5].

PROPOSITION 2.7 (bilinear operator). *Let $F_T = \left(L^\infty((0,T);H^1(\mathbb{T}))\right)^2$. Then for every $T \geq 0$, $\alpha \in \mathbb{R}$, $u = (u_1, u_2) \in F_T$, and $v = (v_1, v_2) \in F_T$ the bilinear operator B defined in* (11) *is continuous from $F_T \times F_T$ to F_T. Moreover, there exists a positive constant $C = C(\alpha, \varepsilon)$ such that for all $u, v \in F_T$ we have*

$$\|B(u,v)\|_{F_T} \leq CT^{\frac{1}{2}}\|u\|_{F_T}\|v\|_{F_T}.$$

Proof of Proposition 2.7. First, we know that

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &= \left\|\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)ds\right\|_{H^1(\mathbb{T})}\\ &\leq \int_0^t \left\|S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{H^1(\mathbb{T})} ds.\end{aligned}$$

Then, since $L^\infty(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$, we have

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} \leq &\int_0^t \left\|S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{L^\infty(\mathbb{T})} ds\\ &+\int_0^t \left\|\frac{\partial}{\partial x}S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{L^2(\mathbb{T})} ds.\end{aligned}$$

Using Lemma 2.6(i) for the first term and Lemma 2.6(iii) with $q = \infty$ for the second term, we can conclude that

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &\leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|C_\alpha[u_1(s)-u_2(s)]\|_{L^\infty(\mathbb{T})}\left\|\frac{\partial v}{\partial x}(s)\right\|_{L^2(\mathbb{T})} ds\\ &\leq C\sup_{0\leq t<T}\left(\|u(t)\|_{H^1(\mathbb{T})}\right)\sup_{0\leq t<T}\left(\|v(t)\|_{H^1(\mathbb{T})}\right)\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}ds.\end{aligned}$$

Then, for all $t \in (0,T)$, we have

$$\begin{aligned}\|B(u,v)(t)\|_{H^1(\mathbb{T})} &\leq Ct^{\frac{1}{2}}\|u\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\|v\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\\ &\leq CT^{\frac{1}{2}}\|u\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\|v\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}.\end{aligned} \tag{13}$$

□

PROPOSITION 2.8 (linear operator). *Let $F_T = \left(L^\infty((0,T);H^1(\mathbb{T}))\right)^2$ and $a(\cdot)$ satisfy* (H3)*. Then for all L_0 $T \geq 0$, and $u = (u_1, u_2) \in F_T$, the linear operator L defined in* (12) *is continuous from F_T to F_T. Moreover, there exists a positive constant $C = C(\alpha, \varepsilon, \|a\|_{L^\infty(0,T)}, L_0)$ such that*

$$\|L(u)\|_{F_T} \leq CT^{\frac{1}{2}}\|u\|_{F_T}.$$

The proof of Proposition 2.8 is similar to the one used in Proposition 2.7.

LEMMA 2.9. *For all L_0, $T \geq 0$, and $a(\cdot)$ satisfying* (H3)*, if*

$$X_{a^\varepsilon}(t) = L_0\vec{i}\int_0^t a^\varepsilon(s)ds, \quad t \in (0,T),$$

then

$$\|X_{a^\varepsilon}\|_{(L^\infty(0,T))^2} \leq L_0 T\|a\|_{L^\infty(0,T)}.$$

The proof of Lemma 2.9 is trivial (from Lemma 2.3(2)).

LEMMA 2.10 (continuity of the semigroup). *For all* $f \in W^{2,2}(\mathbb{T})$ *and* $0 \leq \theta < t$, *we have the following estimates:*

(i)

$$\|(S_\varepsilon(t-\theta) - Id)f\|_{L^2(\mathbb{T})} \leq C(t-\theta) \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{L^2(\mathbb{T})}.$$

(ii)

$$\|(S_\varepsilon(t-\theta) - Id)f\|_{L^2(\mathbb{T})} \leq 2\|f\|_{L^2(\mathbb{T})},$$

where $C = C(\varepsilon)$ *is a positive constant depending on* ε.

We refer the reader to Pazy [16, Lem. 6.2, p. 151] for the proof of this lemma.

LEMMA 2.11 (time continuity). *Assume* (H3). *If* $\rho_{0,vec} = (\rho_0^{+,per}, \rho_0^{-,per}) \in (H^1(\mathbb{T}))^2$, *then for all* $T \geq 0$ *and* $u = (u_1, u_2) \in \big(L^\infty((0,T); H^1(\mathbb{T}))\big)^2$, *we have the following applications:*

(A1) $t \to X_{a^\varepsilon}(t)$;
(A2) $t \to S_\varepsilon(t)\rho^\varepsilon_{0,vec}$, *where* $\rho^\varepsilon_{0,vec} = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$;
(A3) $t \to B(u,u)(t)$;
(A4) $t \to L(u)(t)$ *are* $\big(C([0,T); H^1(\mathbb{T}))\big)^2$, *where* X_{a^ε}, B, *and* L *are defined in Lemma* 2.9, (11), *and* (12), *respectively.*

Proof of Lemma 2.11. The continuity of (A1) is trivial since $a \in L^\infty(0,T)$. From the fact that the semigroup $S_\varepsilon(\cdot)$ is continuous from $[0,T)$ to $(H^1(\mathbb{T}))^2$, we deduce the continuity of (A2).

It remains to prove the continuity of (A3) and (A4). Indeed, the continuity of (A3) at 0 is a consequence of inequality (13). Now we are going to prove the continuity of (A3) for all $\theta \in (0,T)$. For all t, such that $\theta < t \leq \min(T, \frac{3\theta}{2})$, we write $t = (1+\gamma)\theta$ and denote $\tau = (1-\gamma)\theta$ (where $0 < \gamma \leq \frac{1}{2}$), and we write

$$\begin{aligned} B(u,u)(t) - B(u,u)(\theta) &= \int_0^\tau (S(t-s) - S(\theta-s)) \left(C_\alpha[u_1(s) - u_2(s)] \frac{\partial u}{\partial x}(s) \right) ds \\ &\quad + \int_\tau^\theta (S(t-s) - S(\theta-s)) \left(C_\alpha[u_1(s) - u_2(s)] \frac{\partial u}{\partial x}(s) \right) ds \\ &\quad + \int_\theta^t S(t-s) \left(C_\alpha[u_1(s) - u_2(s)] \frac{\partial u}{\partial x}(s) \right) ds \end{aligned}$$

$$\begin{aligned} &= \overbrace{\int_0^\tau ((S(t-\theta) - Id)S(\theta-s)) \left(C_\alpha[u_1(s) - u_2(s)] \frac{\partial u}{\partial x}(s) \right) ds}^{I_1} \\ &\quad + \overbrace{\int_\tau^\theta ((S(t-\theta) - Id)S(\theta-s)) \left(C_\alpha[u_1(s) - u_2(s)] \frac{\partial u}{\partial x}(s) \right) ds}^{I_2} \\ &\quad + \int_\theta^t S(t-s) \left(C_\alpha[u_1(s) - u_2(s)] \frac{\partial u}{\partial x}(s) \right) ds. \end{aligned}$$

We apply Lemma 2.10(i) and Lemma 2.6(ii) to find an upper bound to I_1. We then apply Lemma 2.10(ii) to find an upper bound to I_2. After that, we follow the same steps of the proof of Proposition 2.7 to conclude that

$$\begin{aligned}\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \le\ & C(t-\theta)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_0^\tau \frac{1}{(\theta-s)^{\frac{3}{2}}}ds \\ & +C\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_\tau^\theta \frac{1}{(\theta-s)^{\frac{1}{2}}}ds \\ & +C\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_\theta^t \frac{1}{(t-s)^{\frac{1}{2}}}ds.\end{aligned}$$

After the computation of each integral we deduce that

$$\begin{aligned}\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \le\ & C(t-\theta)\left(\tfrac{1}{(\theta-\tau)^{\frac{1}{2}}}-\tfrac{1}{\theta^{\frac{1}{2}}}\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2} \\ & +C\left((\theta-\tau)^{\frac{1}{2}}+(t-\theta)^{\frac{1}{2}}\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}.\end{aligned}$$

Observing that $t-\theta=\theta-\tau=\gamma\theta$ we finally obtain the following inequality:

$$\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \le C(\theta,\gamma)\left((t-\theta)^{\frac{1}{2}}+(t-\theta)\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2},$$

and hence we get the continuity of (A3). In the same way we get the continuity in time of (A4). ◻

2.3. Proof of Theorem 2.1. We rewrite the system (9) in the following vectorial form:

$$\begin{aligned}\rho^\varepsilon_{vec}(\cdot,t) =\ & S_\varepsilon(t)\rho^\varepsilon_{0,vec}+L_0\vec{i}\int_0^t a^\varepsilon(s)ds+\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(C_\alpha[\rho^\varepsilon(s)]\frac{\partial\rho^\varepsilon_{vec}}{\partial x}(s)\right)ds \\ & +L_0\vec{i}\int_0^t S_\varepsilon(t-s)C_\alpha[\rho^\varepsilon(s)]ds+\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(a^\varepsilon(s)\frac{\partial\rho^\varepsilon_{vec}}{\partial x}(s)\right)ds\end{aligned}$$

such that ρ^ε_{vec} is the vector $(\rho^{+,\varepsilon,per},\rho^{-,\varepsilon,per})$ and $\rho^\varepsilon_{0,vec}$ is the vector $(\rho_0^{+,\varepsilon,per},\rho_0^{-,\varepsilon,per})$. $\vec{i}$ and $\bar{I}_1$ are defined in (10) and (11), respectively.

This altogether leads to the following equation:

$$\rho^\varepsilon_{vec}(\cdot,t)=S_\varepsilon(t)\rho^\varepsilon_{0,vec}+X_{a^\varepsilon}(t)+B(\rho^\varepsilon_{vec},\rho^\varepsilon_{vec})(t)+L(\rho^\varepsilon_{vec})(t), \tag{14}$$

where B is the bilinear application and L is the linear application defined in (11) and (12), respectively, and X_{a^ε} is defined in Lemma 2.9. Moreover, according to Lemmas 2.9 and 2.3 we know that

$$\begin{aligned}\|S(t)\rho^\varepsilon_{0,vec}+X_{a^\varepsilon}(t)\|_{(L^\infty((0,T);H^1(\mathbb{T})))^2} & \le \|\rho^\varepsilon_{0,vec}\|_{H^1(\mathbb{T})}+L_0T\|a^\varepsilon\|_{L^\infty(\mathbb{R})} \\ & \le C_0\|\rho_{0,vec}\|_{H^1(\mathbb{T})}+L_0T\|a\|_{L^\infty(0,T)}.\end{aligned}$$

In order to apply Lemma 2.5, we want, for a well-chosen time T, that the following inequality holds:

$$C_0\|\rho^0_{vec}\|_{H^1(\mathbb{T})}+L_0T\|a\|_{L^\infty(0,T)}<\frac{1}{4CT^{\frac{1}{2}}}(CT^{\frac{1}{2}}-1)^2,\quad \text{and } CT^{\frac{1}{2}}<1, \tag{15}$$

where C is the largest constant between the two constants computed in Propositions 2.8 and 2.7. For

$$(T^\star)^{\frac{1}{2}}(\|\rho_{0,vec}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \varepsilon) \tag{16}$$
$$= \min\left(1, \frac{1}{2C}, \frac{1}{16C(C_0\|\rho^0_{vec}\|_{H^1(\mathbb{T})} + L_0\|a\|_{L^\infty(0,T)})}\right),$$

we can easily verify that $T^\star$ satisfies the inequality (15). We apply Lemma 2.5 over the space $F_{T^\star} = \left(L^\infty((0,T^\star); H^1(\mathbb{T}))\right)^2$ to prove the existence of a solution for the system (14) in $F_{T^\star}$.

Then, according to Lemma 2.11, we deduce that the obtained solution is $(C([0,T^\star]; H^1(\mathbb{T})))^2$.

This proves, by Lemma 2.4, the existence of a solution in the sense of distributions for the system (5)–(6) in $C([0,T^\star); H^1_{loc}(\mathbb{R}))$ that verifies (H1). ∎

3. Properties of the solution to the approximated system.

In this section we show that the solutions of system (5)–(6) obtained in the previous section are regular and verify (H2), provided that initial conditions verify (H2).

LEMMA 3.1 (regularity of the solution). *Assume* (H1), (H3), *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$*; if* $\rho^{\pm,\varepsilon} \in C([0,T); H^1_{loc}(\mathbb{R}))$ *are solutions of the system* (5)–(6), *then* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$.

Proof of Lemma 3.1. If we denote the second term of the system (7) by

$$f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] = \mp a^\varepsilon(t)\left(L_0 + \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}\right) \mp C_\alpha[\rho^\varepsilon(t)]\left(\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x} + L_0\right),$$

we know that $f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon] \in L^2(\mathbb{T}\times(0,T))$. Moreover, we know that the initial conditions $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T})$, which allows us to apply the L^2 regularity of the heat equation over the system (7)–(8) (see Lions and Magenes [15, Thm. 8.2]). Then we deduce by induction that the solution is $C^\infty(\mathbb{T}\times[0,T))$. ∎

LEMMA 3.2 (monotonicity of the solution in space). *Assume* (H1), (H2), (H3), *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$*; if* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ *are solutions of the system* (5)–(6), *then* $\rho^{\pm,\varepsilon}(.,t)$ *verifies* (H2) *for all* $t \in (0,T)$.

Proof of Lemma 3.2. First, we remark that if $\frac{\partial \rho_0^\pm}{\partial x} \geq 0$, then $\frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x} \geq 0$. Indeed, we have

$$\begin{aligned}\frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x} = \frac{\partial \rho_0^{\pm,per}}{\partial x} * \eta_\varepsilon + L_0 \quad &= \left(\frac{\partial \rho_0^{\pm,per}}{\partial x} + L_0\right) * \eta_\varepsilon \\ &= \left(\frac{\partial \rho_0^{\pm}}{\partial x}\right) * \eta_\varepsilon \geq 0, \quad \text{because } \eta \text{ is positive.}\end{aligned}$$

We apply the maximum principle over the derived system of (5)–(6):

$$\begin{cases} \dfrac{\partial \theta^{\pm,\varepsilon}}{\partial t} - \varepsilon\dfrac{\partial^2 \theta^{\pm,\varepsilon}}{\partial x^2} \pm (C_\alpha[\rho^\varepsilon(t)] + a^\varepsilon(t))\dfrac{\partial \theta^{\pm,\varepsilon}}{\partial x} \pm (\theta^{+,\varepsilon} - \theta^{-,\varepsilon})\theta^{\pm,\varepsilon} = 0 \\ \qquad\qquad \text{in} \quad \mathbb{T}\times(0,T), \\ \theta^{\pm,\varepsilon}(x,0) = \dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x}, \end{cases}$$

where $\theta^{\pm,\varepsilon} = \frac{\partial \rho^{\pm,\varepsilon}}{\partial x}$ (see Gilbarg and Trudinger [10, Thm. 8.1]). Since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$, we deduce that $\theta^{\pm,\varepsilon} \geq 0$ belongs to $\mathbb{T} \times (0,T)$. ☐

COROLLARY 3.3 (short time existence of nondecreasing regular solutions). *For all* $\alpha \in \mathbb{R}$ *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$, *under the assumptions* (H1), (H2), *and* (H3), *there exists*

$$T^\star(\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0$$

such that the system (5)–(6) *admits a solution* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T^\star))$ *with* $\rho^{\pm,\varepsilon}(.,t)$ *verifying* (H1) *and* (H2).

The proof of Corollary 3.3 is a consequence of Theorem 2.1 and Lemmas 3.1 and 3.2 (with $T = T^\star$).

Remark 3.4. Here we remark that the case of nondecreasing solutions corresponds to a nonshock case in the Burgers equation. On the other hand, the decreasing solutions represent the shock case.

4. A priori estimates and long time existence for the approximated system. In this section, we are going to show some ε-uniform estimates on the solutions of the system (5)–(6). These estimates will be used in section 4 for the passage to the limit as ε tends to zero.

LEMMA 4.1 (L^2 estimates over the space derivatives of the solutions). *Assume* (H1), (H2), (H3), *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$; *if* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$ *is a solution of the system* (5)–(6) *for all* $T \geq 0$, *then*

$$\left\| \frac{\partial \rho^{+,\varepsilon}}{\partial x} \right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\| \frac{\partial \rho^{-,\varepsilon}}{\partial x} \right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} \leq CB_0,$$

with $B_0 = (\|\frac{\partial \rho_0^+}{\partial x}\|^2_{L^2(\mathbb{T})} + \|\frac{\partial \rho_0^-}{\partial x}\|^2_{L^2(\mathbb{T})})$.

Proof of Lemma 4.1. If we denote $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$ and $k^\varepsilon = \rho^{+,\varepsilon} + \rho^{-,\varepsilon}$, then, according to (H1), it is clear that ρ^ε, $\frac{\partial \rho^\varepsilon}{\partial x}$, and $\frac{\partial k^\varepsilon}{\partial x}$ are 1-periodic functions. Moreover, by Lemma 3.2, we know that $\frac{\partial k^\varepsilon}{\partial x} \geq 0$.

If we take into consideration the equations of the system (5), we can conclude that ρ^ε and k^ε verify the following system:

$$(17) \quad \begin{cases} \dfrac{\partial \rho^\varepsilon}{\partial t} - \varepsilon \dfrac{\partial^2 \rho^\varepsilon}{\partial x^2} & = -\left(\rho^\varepsilon + \alpha \displaystyle\int_0^1 \rho^\varepsilon dx + a^\varepsilon(t)\right) \dfrac{\partial k^\varepsilon}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial k^\varepsilon}{\partial t} - \varepsilon \dfrac{\partial^2 k^\varepsilon}{\partial x^2} & = -\left(\rho^\varepsilon + \alpha \displaystyle\int_0^1 \rho^\varepsilon dx + a^\varepsilon(t)\right) \dfrac{\partial \rho^\varepsilon}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)). \end{cases}$$

We derive the first equation of the system (17) with respect to x, then we multiply the result by $\frac{\partial \rho^\varepsilon}{\partial x}$, and, finally, we integrate in space. For all $t \in (0,T)$, we then obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\| \frac{\partial \rho^\varepsilon}{\partial x}(t) \right\|^2_{L^2(\mathbb{T})} + \varepsilon \left\| \frac{\partial^2 \rho^\varepsilon}{\partial x^2}(t) \right\|^2_{L^2(\mathbb{T})} = & -\int_0^1 \left(\frac{\partial \rho^\varepsilon}{\partial x}\right)^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \rho^\varepsilon \frac{\partial \rho^\varepsilon}{\partial x} \frac{\partial^2 k^\varepsilon}{\partial x^2} \\ & -\left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right) \int_0^1 \frac{\partial^2 k^\varepsilon}{\partial x^2} \frac{\partial \rho^\varepsilon}{\partial x}. \end{aligned}$$

Now we proceed in the same way as for the previous equation, but we multiply the second equation of the system (17) by $\frac{\partial k^\varepsilon}{\partial x}$. For every $t \in (0,T)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \varepsilon\left\|\frac{\partial^2 k^\varepsilon}{\partial x^2}(t)\right\|^2_{L^2(\mathbb{T})} = {} & -\int_0^1 \left(\frac{\partial \rho^\varepsilon}{\partial x}\right)^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \rho^\varepsilon \frac{\partial k^\varepsilon}{\partial x}\frac{\partial^2 \rho^\varepsilon}{\partial x^2} \\ & - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \frac{\partial^2 \rho^\varepsilon}{\partial x^2}\frac{\partial k^\varepsilon}{\partial x}.\end{aligned}$$

Adding the two previous equations, thanks to the periodicity of ρ^ε and $\frac{\partial k^\varepsilon}{\partial x}$, we infer that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} \leq {} & -\int_0^1 \left(\frac{\partial \rho^\varepsilon}{\partial x}\right)^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \frac{\partial}{\partial x}\left(\rho^\varepsilon \frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial k^\varepsilon}{\partial x}\right) \\ & - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \frac{\partial}{\partial x}\left(\frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial k^\varepsilon}{\partial x}\right) \\ \leq {} & -\int_0^1 \left(\frac{\partial \rho^\varepsilon}{\partial x}\right)^2 \frac{\partial k^\varepsilon}{\partial x} \leq 0.\end{aligned}$$

We integrate in time and use the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ and Lemma 2.3. We obtain, in particular,

$$\begin{aligned}& \sup_{t\in(0,T)}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \sup_{t\in(0,T)}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} \\ & \qquad \leq C\left(\left\|\frac{\partial(\rho_0^+ - \rho_0^-)}{\partial x}\right\|^2_{L^2(\mathbb{T})} + \left\|\frac{\partial(\rho_0^+ + \rho_0^-)}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right).\end{aligned}$$

That leads to the desired result. ∎

Lemma 4.2 (L^2 estimates of the solutions). *Assume* (H1), (H2), (H3), *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$; *if* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ *are solutions of the system* (5)–(6) *for every* $T \geq 0$, *then*

$$\begin{aligned}& \|\rho^{+,\varepsilon}\|^2_{L^\infty((0,T);L^2(0,1))} + \|\rho^{-,\varepsilon}\|^2_{L^\infty((0,T);L^2(0,1))} \\ & \qquad \leq C\left(M_0 + (B_0 + \|a\|^2_{L^\infty(0,T)})\right)e^{4L_0(1+\alpha^2)T},\end{aligned}$$

where B_0 *is defined in Lemma* 4.1, *and* $M_0 = (\|\rho_0^+\|^2_{L^2(0,1)} + \|\rho_0^-\|^2_{L^2(0,1)})$.

Proof of Lemma 4.2. We will use the same procedure of the proof of Lemma 4.1. We multiply the first equation of the system (17) by ρ^ε; then we integrate in space. For every $t \in (0,T)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \varepsilon\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} = -\int_0^1 (\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x} - \left(\alpha\int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \rho^\varepsilon \frac{\partial k^\varepsilon}{\partial x}.$$

Similarly, we multiply the second equation of the system (17) by k^ε and integrate in space. For every $t \in (0,T)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)} + \varepsilon\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} = -\int_0^1 \rho^\varepsilon\frac{\partial \rho^\varepsilon}{\partial x}k^\varepsilon - \left(\alpha\int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 k^\varepsilon\frac{\partial \rho^\varepsilon}{\partial x}.$$

Now we add the two previous equations and get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}+\frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)}\right)\\
&\qquad\le -\int_0^1\left((\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x}+\frac{1}{2}k^\varepsilon\frac{\partial(\rho^\varepsilon)^2}{\partial x}\right)\\
&\qquad\quad -\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\left(\int_0^1 k^\varepsilon\frac{\partial\rho^\varepsilon}{\partial x}+\int_0^1\rho^\varepsilon\frac{\partial k^\varepsilon}{\partial x}\right)\\
&\qquad\le -\frac{1}{2}\int_0^1(\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x}-\frac{1}{2}\int_0^1\frac{\partial((\rho^\varepsilon)^2k^\varepsilon)}{\partial x}\\
&\qquad\quad -\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}.
\end{aligned}$$

Recalling that ρ^ε is periodic and k^ε is nondecreasing, we see that

$$\frac{1}{2}\left(\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}+\frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)}\right)\le-\left(\alpha\int_0^1\rho^\varepsilon+a^\varepsilon(t)\right)\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}.$$

But we know from (H1) that ρ^ε and $(k^\varepsilon-2L_0x)$ are 1-periodic functions, which implies that

$$\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}=\int_0^1\frac{\partial((k^\varepsilon-2L_0x)\rho^\varepsilon)}{\partial x}+2L_0\int_0^1\frac{\partial(x\rho^\varepsilon)}{\partial x}=2L_0\int_0^1x\frac{\partial\rho^\varepsilon}{\partial x}+2L_0\int_0^1\rho^\varepsilon.$$

We use Lemmas 4.1 and 2.3 and the fact that ($ab\le\frac{1}{2}(a^2+b^2)$ and $(a+b)^2\le 2(a^2+b^2)$) to deduce that

$$\begin{aligned}
&\frac{d}{dt}\left(\|k^\varepsilon(t)\|^2_{L^2(0,1)}+\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}\right)\\
&\qquad\le 4L_0\left(|\alpha|\,\|\rho^\varepsilon(t)\|_{L^2(\mathbb{T})}+\|a\|_{L^\infty(0,T)}\right)\left(\left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|_{L^2(\mathbb{T})}+\|\rho^\varepsilon(t)\|_{L^2(\mathbb{T})}\right)\\
&\qquad\le 4L_0\left(\|a\|^2_{L^\infty(0,T)}+(1+\alpha^2)\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}+\left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}\right)\\
&\qquad\le 4L_0\left(CB_0+\|a\|^2_{L^\infty(0,T)}\right)+4L_0(1+\alpha^2)\left(\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}+\|k^\varepsilon(t)\|^2_{L^2(0,1)}\right).
\end{aligned}$$

Using the previous estimate and the fact that $\rho^{\pm,\varepsilon}\in C^\infty(\mathbb{R}\times[0,T))$, we finally obtain

$$\|\rho^\varepsilon\|^2_{L^\infty((0,T);L^2(\mathbb{T}))}+\|k^\varepsilon\|^2_{L^\infty((0,T)L^2(0,1))}\le C\left(M_0+B_0+\|a\|^2_{L^\infty(0,T)}\right)e^{4L_0(1+\alpha^2)T}.$$

This leads to the desired result. □

LEMMA 4.3 (L^2 estimate on the time derivatives of the solutions). *Assume* (H1), (H2), (H3), *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$; *if* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R} \times [0,T))$ *is a solution of the system* (5)–(6) *for every* $T \geq 0$, *then there exists a constant* $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ *independent of* ε *such that*

$$\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))} \leq C.$$

Proof of Lemma 4.3. For the proof of Lemma 4.3, it is sufficient to show that the second term of the system (5),

$$f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] = \mp\left(a^\varepsilon(t) + \rho^\varepsilon + \alpha\int_0^1 \rho^\varepsilon dx\right)\frac{\partial \rho^{\pm,\varepsilon}}{\partial x},$$

is bounded in $L^\infty((0,T); L^2(\mathbb{T}))$ uniformly in ε. Indeed,

$$\begin{aligned}
&\left\|f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|_{L^\infty((0,T^\star);L^2(\mathbb{T}))} \\
&\quad\leq \left\|\left(a^\varepsilon(\cdot) + \rho^\varepsilon + \alpha\int_0^1 \rho^\varepsilon dx\right)\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))} \\
&\quad\leq C\left(\|\rho^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,T))} + \|a\|_{L^\infty(0,T)}\right)\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))}.
\end{aligned}$$

We use Lemmas 4.1 and 4.2 and the Sobolev injections to deduce that there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ such that

$$\left\|f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|_{L^\infty((0,T);L^2(\mathbb{T}))} \leq C.$$

To conclude, we multiply the first and the second equations of the system (5) by $\frac{\partial \rho^{+,\varepsilon}}{\partial t}$ and $\frac{\partial \rho^{-,\varepsilon}}{\partial t}$, respectively, and we integrate in space. We deduce that for every $t \in (0,T)$ we have

$$\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}(t)\right\|^2_{L^2(\mathbb{T})} + \frac{\varepsilon}{2}\frac{d}{dt}\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} = \int_0^1 f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)]\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}.$$

Integrating in time and using the fact that $\rho^{\pm,\varepsilon} \in C(\mathbb{R} \times [0,T))$ for all $T \geq 0$, we get

$$\begin{aligned}
&\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))} + \frac{\varepsilon}{2}\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}(T)\right\|^2_{L^2(\mathbb{T})} \\
&\qquad= \int_0^T\int_0^1 f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)]\frac{\partial \rho^{\pm,\varepsilon}}{\partial t} + \frac{\varepsilon}{2}\left\|\frac{\partial {\rho_0}^{\pm,\varepsilon}}{\partial x}\right\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

We apply Hölder's inequality and the fact that $\varepsilon < 1$ and $ab \leq \frac{1}{2}(a^2 + b^2)$ to obtain that

$$\begin{aligned}
&\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))}\\
&\quad\le \left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|_{L^2(\mathbb{T}\times(0,T))}\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))}+\frac{1}{2}\left\|\frac{\partial {\rho_0}^{\pm,\varepsilon}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\\
&\quad\le \frac{C}{2}\left(\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^2(\mathbb{T}\times(0,T))}+\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))}+\left\|\frac{\partial {\rho_0}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right),
\end{aligned}$$

which leads to

$$\begin{aligned}
\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))}&\le C\left(\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^2(\mathbb{T}\times(0,T))}+\left\|\frac{\partial {\rho_0}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right)\\
&\le C\left(T\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))}+\left\|\frac{\partial {\rho_0}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right)\le C,
\end{aligned}$$

where $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$. □

Remark 4.4 (the sense of the initial conditions). According to Lemma 4.3, we have $\rho^{\pm,\varepsilon,per} \in C([0,T), L^2(\mathbb{T}))$ uniformly in ε. This will give a sense to the limit of the initial conditions.

THEOREM 4.5 (long time existence). *Assume* (H1), (H2), *and* (H3); *for all* L_0, $T \ge 0$, $\alpha \in \mathbb{R}$, *and* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, *the system* (5)–(6) *admits the solutions* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$, *with* $\rho^{\pm,\varepsilon}(.,t)$ *verifying* (H1) *and* (H2). *Moreover, there exists a constant* $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ *independent of* ε, *with* B_0 *and* M_0 *defined in Lemmas* 4.1 *and* 4.2, *respectively, such that*

$$\|\rho^{\pm,\varepsilon,per}\|_{L^\infty((0,T);L^2(\mathbb{T}))}+\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))}+\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))}\le C, \tag{18}$$

where $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_0 x$.

Proof of Theorem 4.5. We are going to prove that local time solutions obtained by Corollary 3.3 can be extended to global time solutions for the same system.

We argue by contradiction: Assume that there exists a maximum time T_{max} such that we have the existence of solutions of the system (5)–(6) in the function space $C^\infty(\mathbb{R}\times[0,T_{max}))$.

For every $\delta > 0$, we consider the system (5) with the initial conditions

$$\rho^{\pm,\varepsilon}_{\delta,max} = \rho^{\pm,\varepsilon}(x, T_{max}-\delta).$$

We apply for the second time the same technique of Corollary 3.3 to deduce that there exists a time

$$T^\star_{\delta,max}(\|\rho^{\pm,\varepsilon,per}_{\delta,max}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0, \qquad \text{where } \rho^{\pm,\varepsilon,per}_{\delta,max} = \rho^{\pm,\varepsilon}_{\delta,max} - L_0 x,$$

such that the system (5)–(6) admits a solution defined until the time

$$T_0 = (T_{max}-\delta) + T^\star_{\delta,max}.$$

Moreover, according to Lemmas 4.1 and 4.2, we know that $\rho^{\pm,\varepsilon,per}_{\delta,max}$ are δ-uniformly bounded in $H^1(\mathbb{T})$. We use (16) to deduce that there exists a constant $C(\varepsilon, T_{max}, \alpha, \|a\|_{L^\infty(0,T)}, L_0) > 0$ independent of δ such that $T^\star_{\delta,max} \ge C > 0$; then $\lim_{\delta\to 0} T^\star_{\delta,max} \ge C > 0$, which implies that $T_0 > T_{max}$, and so we have a contradiction.

The estimation (18) is a consequence of Lemmas 4.1, 4.2, and 4.3. □

5. Existence and uniqueness of the solution of (1)–(2). In this section, we are going to prove that the system (1)–(2) admits a unique solution $\rho^{\pm}$ (in the distribution sense) which is the limit as $\varepsilon \to 0$ of $\rho^{\pm,\varepsilon}$ given by Theorem 4.5. In order to do that, we pass to the limit when ε tends to 0 in the system (7)–(8), and we use (18) in order to ensure the compactness. The proof of the uniqueness uses direct arguments.

Proof of Theorem 1.1. We first prove the existence and then establish the uniqueness.

Step 1 *(existence).* Let $\rho^{\pm,\varepsilon}$ be the solution of the system (5) given by Theorem 4.5. According to (18) we know that $\rho^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $H^1(\mathbb{T}\times(0,T))$; then we can extract a subsequence that converges weakly in $H^1(\mathbb{T}\times(0,T))$. Knowing that $H^1(\mathbb{T}\times(0,T))$ is compact in $L^2(\mathbb{T}\times(0,T))$, this subsequence strongly converges in $L^2(\mathbb{T}\times(0,T))$. If we denote by $\rho^{\pm,per}$ the limit of this subsequence, we have to prove that $\rho^{\pm,per}+L_0x$ is a solution of the system (1)–(2) in the sense of distribution. Indeed, by Lemma 2.3, the term $\mp(L_0a^{\varepsilon})$ of (7) converges strongly to $(\mp L_0 a)$ in $L^2(0,T)$.

The linear term

$$\mp\left(L_0C_\alpha[\rho^\varepsilon]+a^\varepsilon(t)\frac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}\right)$$

of (7) weakly converges in $L^1(\mathbb{T}\times(0,T))$, and the reason is, on the one hand, that $\frac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $L^2(\mathbb{T}\times(0,T))$, which gives us the weak convergence in $L^2(\mathbb{T}\times(0,T))$, and, on the other hand, that a^ε strongly converges in $L^2(0,T)$. Then the linear term converges in the sense of distributions (i.e., in $\mathcal{D}'(\mathbb{T}\times(0,T))$). It remains to prove that the bilinear term

$$C_\alpha[\rho^\varepsilon]\frac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}$$

of (7) also converges in the sense of distributions. We have the following:

1. The sequence $C_\alpha[\rho^\varepsilon]$ is compact in $L^2(\mathbb{T}\times(0,T))$.
2. The functions $\frac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $L^2(\mathbb{T}\times(0,T))$.

This gives us a strong convergence in $L^2(\mathbb{T}\times(0,T))$ times a weak convergence in $L^2(\mathbb{T}\times(0,T))$ and hence a weak convergence of the product in $L^1(\mathbb{T}\times(0,T))$. This leads, as a consequence, to the convergence in the distribution sense. This altogether shows that $\rho^{\pm,per}+L_0x$ is a solution in the sense of distribution of the system (1)–(2) and $\rho^{\pm,per}$ verifies estimate (18).

It remains to prove that the initial condition is satisfied by the limit function $\rho^{\pm,per}$. In fact, according to the estimate (18) on $\rho^{\pm,\varepsilon,per}$, $\frac{\partial\rho}{\partial t}^{\pm,\varepsilon}$, and $\frac{\partial\rho}{\partial x}^{\pm,\varepsilon}$, we see that $\rho^{\pm,\varepsilon,per}$ is ε-uniformly bounded in $H^1(\mathbb{T}\times(0,T))$.

From the fact that the injection of $H^1(\mathbb{T}\times(0,T))$ in $C([0,T];L^2(\mathbb{T}))$ is continuous and compact by classical arguments, we see that, for all $v\in L^2(\mathbb{T})$, the application $\gamma : U \longmapsto \int_0^1 U(0)v$ is a continuous linear form for $U\in C([0,T];L^2(\mathbb{T}))$ and hence $\gamma(\rho^{\pm,\varepsilon,per})\to\gamma(\rho^{\pm,per})$ as $\varepsilon\to 0$, because up to a subsequence $\rho^{\pm,\varepsilon,per}$ converges strongly in $C([0,T];L^2(\mathbb{T}))$. This altogether proves that the solution verifies the initial conditions (2).

Step 2 *(uniqueness).* Let $\rho_1^{\pm}$ and $\rho_2^{\pm}$ be two solutions of the system (1) such that $\rho_1^{\pm}(\cdot,0)=\rho_2^{\pm}(\cdot,0)=\rho_0^{\pm}$ and $\rho_i^{\pm}(\cdot,t)$ verify (H1), (H2), and estimate (18) for $i=1,2$, $t\in(0,T)$.

If we denote $\rho_i = \rho_i^+ - \rho_i^-$, $k_i = \rho_i^+ + \rho_i^-$ for $i = 1, 2$, then it is clear that $(\rho_1 - \rho_2)$ and $(k_1 - k_2)$ are 1-periodic functions in space and ρ_i, k_i verify the following system for $i = 1, 2$:

$$\left\{ \begin{aligned} \frac{\partial \rho_i}{\partial t} &= -\left(\rho_i + \alpha \int_0^1 \rho_i dx + a(t)\right) \frac{\partial k_i}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)),\\ \frac{\partial k_i}{\partial t} &= -\left(\rho_i + \alpha \int_0^1 \rho_i dx + a(t)\right) \frac{\partial \rho_i}{\partial x} \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)). \end{aligned} \right. \tag{19}$$

We substract the two systems to obtain that

$$\left\{ \begin{aligned} \frac{\partial(\rho_1 - \rho_2)}{\partial t} &= -\left(\rho_1 + \alpha \int_0^1 \rho_1 dx\right) \frac{\partial k_1}{\partial x} + \left(\rho_2 + \alpha \int_0^1 \rho_2 dx\right) \frac{\partial k_2}{\partial x}\\ &\quad - a(t) \frac{\partial(k_1 - k_2)}{\partial x},\\ \frac{\partial(k_1 - k_2)}{\partial t} &= -\left(\rho_1 + \alpha \int_0^1 \rho_1 dx\right) \frac{\partial \rho_1}{\partial x} + \left(\rho_2 + \alpha \int_0^1 \rho_2 dx\right) \frac{\partial \rho_2}{\partial x}\\ &\quad - a(t) \frac{\partial(\rho_1 - \rho_2)}{\partial x}. \end{aligned} \right.$$

The previous system is equivalent to

$$\left\{ \begin{aligned} \frac{\partial(\rho_1 - \rho_2)}{\partial t} &= -\left((\rho_1 - \rho_2) + \alpha \int_0^1 (\rho_1 - \rho_2) dx\right) \frac{\partial k_1}{\partial x}\\ &\quad - \left(\rho_2 + \alpha \int_0^1 \rho_2 dx\right) \frac{\partial(k_1 - k_2)}{\partial x} - a(t) \frac{\partial(k_1 - k_2)}{\partial x},\\ \frac{\partial(k_1 - k_2)}{\partial t} &= -\left((\rho_1 - \rho_2) + \alpha \int_0^1 (\rho_1 - \rho_2) dx\right) \frac{\partial \rho_1}{\partial x}\\ &\quad - \left(\rho_2 + \alpha \int_0^1 \rho_2 dx\right) \frac{\partial(\rho_1 - \rho_2)}{\partial x} - a(t) \frac{\partial(\rho_1 - \rho_2)}{\partial x}. \end{aligned} \right.$$

We multiply the first equation of this system by $(\rho_1 - \rho_2)$ and integrate in space to obtain, for almost every t, that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(\rho_1 - \rho_2)(t)\|^2_{L^2(\mathbb{T})} = \; & -\int_0^1 \left((\rho_1 - \rho_2)^2 \frac{\partial k_1}{\partial x}\right)\\ & -\alpha \left(\int_0^1 (\rho_1 - \rho_2)\right) \int_0^1 \left((\rho_1 - \rho_2) \frac{\partial k_1}{\partial x}\right)\\ & -\int_0^1 \left((\rho_1 - \rho_2) \left(\rho_2 + \alpha \int_0^1 \rho_2\right) \frac{\partial(k_1 - k_2)}{\partial x}\right)\\ & -a(t) \int_0^1 \left((\rho_1 - \rho_2) \frac{\partial(k_1 - k_2)}{\partial x}\right). \end{aligned}$$

Similarly, we multiply the second equation by $(k_1 - k_2)$ and integrate in space to get, for almost every time t,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} = \quad & -\int_0^1 \left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial\rho_1}{\partial x}\right)\\ & -\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1 (k_1-k_2)\frac{\partial\rho_1}{\partial x}\\ & -\int_0^1\left((k_1-k_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)\\ & -a(t)\int_0^1\left((k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right).\end{aligned}$$

We add the two previous equations to obtain, for almost every time t,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})}+\|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})}\right)\\ &\quad = -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)\\ &\quad\quad -\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((k_1-k_2)\frac{\partial\rho_1}{\partial x}\right)\\ &\quad\quad -\int_0^1\left(\frac{\partial}{\partial x}\left((\rho_1-\rho_2)(k_1-k_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\right)\right)\\ &\quad\quad -\int_0^1\left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)-a(t)\int_0^1\left(\frac{\partial}{\partial x}\left((\rho_1-\rho_2)(k_1-k_2)\right)\right).\end{aligned}$$

From the fact that ρ_i, $i=1,2$, and (k_1-k_2) are 1-periodic functions in space, the previous equation becomes

$$\begin{aligned}&= \overbrace{-\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)}^{I_1}\ \overbrace{-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)}^{I_2}\\ &\overbrace{-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((k_1-k_2)\frac{\partial\rho_1}{\partial x}\right)}^{I_3}\ \overbrace{-\int_0^1\left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)}^{I_4}.\end{aligned}$$

And since $\frac{\partial k_i}{\partial x}\geq 0$ for $i=1,2$, we know that

$$\begin{aligned}I_1+I_4 &= -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)-\frac{1}{2}\int_0^1\left((k_1-k_2)\frac{\partial}{\partial x}\left((\rho_1-\rho_2)^2\right)\right)\\ &= -\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)+\frac{1}{2}\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial(k_1-k_2)}{\partial x}\right)\\ &= -\frac{1}{2}\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial(k_1+k_2)}{\partial x}\right)\leq 0.\end{aligned}$$

Moreover, from (18), we have, for almost every t,

$$\begin{aligned} I_2 &\le |\alpha| \|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})} \|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})} \left\| \frac{\partial k_1}{\partial x}(t) \right\|_{L^2(\mathbb{T})} \\ &\le C \|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})}. \end{aligned}$$

Similarly, from (18), we have, for almost every t,

$$\begin{aligned} I_3 &\le |\alpha| \|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})} \|(k_1-k_2)(t)\|_{L^2(\mathbb{T})} \left\| \frac{\partial \rho_1}{\partial x}(t) \right\|_{L^2(\mathbb{T})} \\ &\le C \left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \right). \end{aligned}$$

Then

$$\begin{aligned} &\frac{d}{dt} \left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \right) \\ &\qquad \le C \left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \right). \end{aligned}$$

Now we integrate in time and use the fact that ρ_i, $k_i \in C([0,T), L^2_{loc}(\mathbb{R}))$, $\rho_1(\cdot,0) = \rho_2(\cdot,0)$, and $k_1(\cdot,0) = k_2(\cdot,0)$ to obtain that

$$\sup_{t\in(0,T)} \|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \sup_{t\in(0,T)} \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \le 0.$$

This achieves the proof of uniqueness. ☐

Remark 5.1. In Theorem 1.1, we have proved a result of existence and uniqueness in $H^1_{loc}(\mathbb{R}\times[0,T))$ depending on some uniform estimates in this space. These estimates give a sufficient compactness in order to ensure the passage to the limit as ε tends to 0 in the bilinear term. However, the space $W^{1,1}_{loc}(\mathbb{R}\times[0,T))$ does not give enough compactness. On the other hand, the space of functions $L^2_{loc}(\mathbb{R}\times[0,T))$ having their derivatives in $L^\infty((0,T); (L^1 \log L^1)_{loc}(\mathbb{R}))$ requires the minimal properties to ensure the passage to the limit in the bilinear term. The result of existence in this space will be the core of a paper in preparation.

6. Further properties: Comparison principle with case $\alpha = 0$.

In this section, we are going to prove a comparison principle result of the system (1) in the case $\alpha = 0$ (i.e., Theorem 1.2). In order to do this, first we prove in the following subsection the same result for the approximate system (5). Then we give the proof of Theorem 1.2.

6.1. Comparison principle for the regularized system with case $\alpha = 0$.

LEMMA 6.1 (comparison principle). *Let* $a(\cdot)$ *satisfy* (H3) *and* $\rho_1^{\pm,\varepsilon}$, $\rho_2^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ *be two solutions of the system* (5) *with* $\alpha = 0$. *Moreover, let* $\rho_1^{\pm,\varepsilon}(.,t)$, $\rho_2^{\pm,\varepsilon}(.,t)$ *verify* (H1) *and* (H2) *for all* $t \in [0,T)$. *Then, if* $\rho_1^{\pm,\varepsilon}(\cdot,0) \le \rho_2^{\pm,\varepsilon}(\cdot,0)$ *in* $\mathbb{R}$, *we have* $\rho_1^{\pm,\varepsilon} \le \rho_2^{\pm,\varepsilon}$ *on* $\mathbb{R}\times[0,T)$.

Proof of Lemma 6.1. We know that $\rho_1^{\pm,\varepsilon}$ and $\rho_2^{\pm,\varepsilon}$ verify the following systems:

$$\begin{cases} \dfrac{\partial \rho_1^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_1^{+,\varepsilon}}{\partial x^2} = & -\left(\rho_1^{+,\varepsilon} - \rho_1^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_1^{+,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho_1^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_1^{-,\varepsilon}}{\partial x^2} = & \left(\rho_1^{+,\varepsilon} - \rho_1^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_1^{-,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \end{cases}$$

$$\begin{cases} \dfrac{\partial \rho_2^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_2^{+,\varepsilon}}{\partial x^2} = & -\left(\rho_2^{+,\varepsilon} - \rho_2^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_2^{+,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho_2^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_2^{-,\varepsilon}}{\partial x^2} = & \left(\rho_2^{+,\varepsilon} - \rho_2^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_2^{-,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \end{cases}$$

respectively.

If we denote $w^{\pm,\varepsilon}$ by $\tilde{\rho}_2^{\pm,\varepsilon} - \tilde{\rho}_1^{\pm,\varepsilon}$, where

$$\tilde{\rho}_2^{\pm,\varepsilon} = \rho_2^{\pm,\varepsilon} e^{-\gamma t} \quad \text{and} \quad \tilde{\rho}_1^{\pm,\varepsilon} = \rho_1^{\pm,\varepsilon} e^{-\gamma t} \quad \text{with} \quad \gamma > 0,$$

we can easily check that $w^{\pm,\varepsilon}$ are solutions of the following system:

$$\text{(20)} \quad \begin{cases} \dfrac{\partial w^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 w^{+,\varepsilon}}{\partial x^2} + \gamma w^{+,\varepsilon} & = - e^{\gamma t} \left(w^{+,\varepsilon} - w^{-,\varepsilon}\right) \dfrac{\partial \tilde{\rho}_2^{+,\varepsilon}}{\partial x} \\ & \quad - e^{\gamma t} \left(\tilde{\rho}_1^{+,\varepsilon} - \tilde{\rho}_1^{-,\varepsilon} + e^{-\gamma t} a^\varepsilon(t)\right) \dfrac{\partial w^{+,\varepsilon}}{\partial x}, \\ \dfrac{\partial w^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 w^{-,\varepsilon}}{\partial x^2} + \gamma w^{-,\varepsilon} & = \ e^{\gamma t} \left(w^{+,\varepsilon} - w^{-,\varepsilon}\right) \dfrac{\partial \tilde{\rho}_2^{-,\varepsilon}}{\partial x} \\ & \quad + e^{\gamma t} \left(\tilde{\rho}_1^{+,\varepsilon} - \tilde{\rho}_1^{-,\varepsilon} + e^{-\gamma t} a^\varepsilon(t)\right) \dfrac{\partial w^{-,\varepsilon}}{\partial x}. \end{cases}$$

We are interested in the $\min_{(k,x,t) \in \{+,-\} \times \mathbb{T} \times (0,T)} (w^{k,\varepsilon}(x,t))$. Our result follows if we can prove that this minimum is positive. However, this minimum is attained at a point $(k_0, x_0, t_0) \in \{+,-\} \times \mathbb{T} \times [0,T]$ (because $w^{+,\varepsilon}$ and $w^{-,\varepsilon}$ are $C^\infty(\mathbb{T} \times (0,T))$). Two cases may occur:

1. Case $t_0 = 0$. We have

$$\min_{(k,x,t) \in \{+,-\} \times \mathbb{T} \times (0,T)} (w^{k,\varepsilon}(x,t))$$
$$= w^{k_0,\varepsilon}(x_0,t_0) = \left(\rho_2^{k_0,\varepsilon}(x_0,0) - \rho_1^{k_0,\varepsilon}(x_0,0)\right) e^{-\gamma t_0} \geq 0;$$

   and we are done.

2. Case $t_0 \in (0,T]$. We have that (k_0, x_0, t_0) is a minimum point; then

$$\text{(21)} \qquad \frac{\partial^2 w^{k_0,\varepsilon}}{\partial x^2}(x_0,t_0) \geq 0,$$

$$\text{(22)} \qquad \frac{\partial w^{k_0,\varepsilon}}{\partial t}(x_0,t_0) \leq 0,$$

$$\text{(23)} \qquad \frac{\partial w^{k_0,\varepsilon}}{\partial x}(x_0,t_0) = 0.$$

Combining (21), (22), (23) and taking into consideration that $w^{\pm,\varepsilon}$ verifies the system (20), we obtain that

$$\begin{aligned}\gamma w^{k_0,\varepsilon}(x_0,t_0) &\geq e^{\gamma t_0} sign(w^{+,\varepsilon}(x_0,t_0) - w^{-,\varepsilon}(x_0,t_0))(w^{+,\varepsilon}(x_0,t_0)\\ &\quad - w^{-,\varepsilon}(x_0,t_0))\frac{\partial \tilde{\rho}_2^{k_0,\varepsilon}}{\partial x}\\ &\geq e^{\gamma t_0}|w^{+,\varepsilon}(x_0,t_0) - w^{-,\varepsilon}(x_0,t_0)|\frac{\partial \tilde{\rho}_2^{k_0,\varepsilon}}{\partial x} \geq 0.\end{aligned}$$

Then $\tilde{\rho}_1^{\pm,\varepsilon} \leq \tilde{\rho}_2^{\pm,\varepsilon}$ in $\mathbb{R}\times(0,T)$, which gives $\rho_1^{\pm,\varepsilon} \leq \rho_2^{\pm,\varepsilon}$. ∎

We now give the proof of Theorem 1.2.

6.2. Proof of Theorem 1.2.

Let

$$\rho_1^{\pm}(x,0) = \rho_{1,0}^{\pm}(x) = \rho_{1,0}^{\pm,per}(x) + L_0 x \quad \text{and} \quad \rho_2^{\pm}(x,0) = \rho_{2,0}^{\pm}(x) = \rho_{2,0}^{\pm,per}(x) + L_0 x.$$

If we denote

$$\rho_{1,0}^{\pm,\varepsilon}(x) = \rho_{1,0}^{\pm,per} * \eta_\varepsilon(x) + L_0 x \quad \text{and} \quad \rho_{2,0}^{\pm,\varepsilon}(x) = \rho_{2,0}^{\pm,per} * \eta_\varepsilon(x) + L_0 x,$$

where η_ε is a regularization sequence, we can easily check that $\rho_{1,0}^{\pm,\varepsilon} \leq \rho_{2,0}^{\pm,\varepsilon}$.

Moreover, according to the uniqueness of the solution, we know that there exist $\rho_1^{\pm,\varepsilon}$, $\rho_2^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$, verifying (H2) for all $t \in (0,T)$, which are solutions of the system (5), such that

$$\rho_1^{\pm} = \lim_{\varepsilon\to 0}\rho_1^{\pm,\varepsilon}, \quad \rho_2^{\pm} = \lim_{\varepsilon\to 0}\rho_2^{\pm,\varepsilon},$$

$$\rho_1^{\pm,\varepsilon}(x,0) = \rho_{1,0}^{\pm,\varepsilon}(x) \quad \text{and} \quad \rho_2^{\pm,\varepsilon}(x,0) = \rho_{2,0}^{\pm,\varepsilon}(x).$$

We apply Lemma 6.1 to obtain that $\rho_1^{\pm,\varepsilon} \leq \rho_2^{\pm,\varepsilon}$. We pass to the limit as $\varepsilon \to 0$ to deduce that $\rho_1^{\pm} \leq \rho_2^{\pm}$ a.e. in $\mathbb{R}\times(0,T)$. ∎

Remark 6.2. Thanks to this comparison result, we proved in a previous paper [8] the existence and the uniqueness of a solution (in the viscosity sense). Here this comparison result is an indirect explanation of our estimates obtained in Lemmas 4.1, 4.2, and 4.3 that have ensured our principal theorem, Theorem 1.1.

7. Application in the case of the classical Burgers equation.

In this paragraph we are going to prove that this technique can be also applied to the classical Burgers equation, even in the frame of functions in $W^{1,p}_{loc}(\mathbb{R}\times(0,T))$ for all $1 \leq p < +\infty$, constituting the proof of Theorem 1.4.

Proof of Theorem 1.4. First, we remark that the existence of solution to the regularized problem can be done thanks to the continuous injection $W^{1,p}(\mathbb{T})$ in $L^\infty(\mathbb{T})$. Now, for the proof of this theorem, it suffices to show an estimation over the space derivatives of the solution (i.e., a result similar to that of Lemma 4.1).

First, we put ourselves in the hypothesis of Lemma 4.1. We derive the equation (4) with respect to x, then we multiply it by $(\frac{\partial u}{\partial x})^{p-1}$, and, finally, we integrate over

$(0, 1)$; since u verifies (H2), we obtain that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\left\|\frac{\partial u}{\partial x}(t)\right\|_{L^p(\mathbb{T})}^p &= -\int_0^1 f''(u)\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \int_0^1 f'(u)\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^{p-1}\\ &= -\int_0^1 \frac{\partial(f'(u))}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \frac{1}{p}\int_0^1 f'(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p\\ &= -\frac{1}{p}\int_0^1 \frac{\partial}{\partial x}\left(f'(u)\left(\frac{\partial u}{\partial x}\right)^p\right) - \left(1-\frac{1}{p}\right)\int_0^1 f''(u)\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p \le 0,\end{aligned}$$

because f is convex, u verifies (H2), and $p \geq 1$. To terminate the demonstration, we follow the same steps of the proof of Theorem 1.1. We remark that here we do not need the L^2 bound over the solution and also the compactness in the passage to the limit, because (4) is in the conservative form, which was not the case of our study.

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REFERENCES

[1] O. Alvarez, P. Cardaliaguet, and R. Monneau, *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces Free Bound., 7 (2005), pp. 415–434.

[2] O. Alvarez, P. Hoch, Y. Le Bouar, and R. Monneau, *Dislocation dynamics: Short-time existence and uniqueness of the solution*, Arch. Ration. Mech. Anal., 181 (2006), pp. 449–504.

[3] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math., 158 (2004), pp. 227–260.

[4] G. Barles and O. Ley, *Nonlocal first-order Hamilton-Jacobi equations modelling dislocations dynamics*, Comm. Partial Differential Equations, 31 (2006), pp. 1191–1208.

[5] M. Cannone, *Ondelettes, paraproduits et Navier-Stokes*, Diderot Editeur, Paris, 1995.

[6] G.-Q. Chen and D. Wang, *The Cauchy problem for the Euler equations for compressible fluids*, in Handbook of Mathematical Fluid Dynamics, Vol. I, S. Friedlander and D. Serre, eds., North–Holland, Amsterdam, 2002, pp. 421–543.

[7] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98 (1989), pp. 511–547.

[8] A. El Hajj and N. Forcadel, *A convergent scheme for a non-local coupled system modelling dislocations densities dynamics*, Math. Comp., to appear.

[9] N. Forcadel, *Dislocations Dynamics with a Mean Curvature Term: Short Time Existence and Uniqueness*, preprint, 2005.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics Math., Springer-Verlag, Berlin, 2001.

[11] I. Groma and P. Balogh, *Investigation of dislocation pattern formation in a two-dimensional self-consistent field approximation*, Acta Mater., 47 (1999), pp. 3647–3654.

[12] J. P. Hirth and J. Lothe, *Theory of Dislocations*, 2nd ed., Krieger, Malabar, FL, 1992.

[13] P. LeFloch, *Entropy weak solutions to nonlinear hyperbolic systems under nonconservative form*, Comm. Partial Differential Equations, 13 (1988), pp. 669–727.

[14] P. LeFloch and T.-P. Liu, *Existence theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., 5 (1993), pp. 261–280.

[15] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Travaux et Recherches Mathématiques 17, Dunod, Paris, 1968.

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[17] D. Serre, *Systems of Conservation Laws.* I, Cambridge University Press, Cambridge, UK, 1999.

[18] D. Serre, *Systems of Conservation Laws.* II, Cambridge University Press, Cambridge, UK, 2000.