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Global Existence for a System of Non-Linear and Non-Local Transport Equations Describing the Dynamics of Dislocation Densities

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Abstract

In this paper, we study the global in time existence problem for the Groma-Balogh model describing the dynamics of dislocation densities. This model is a two-dimensional model where the dislocation densities satisfy a system of transport equations such that the velocity vector field is the shear stress in the material, solving the equations of elasticity. This shear stress can be expressed as some Riesz transform of the dislocation densities. The main tool in the proof of this result is the existence of an entropy for this system.

1. Introduction

1.1. Physical motivation and presentation of the model

Real crystals show certain defects in the organization of their crystalline structure, called dislocations. These defects were introduced in the Thirties by Taylor, Orowan and Polanyi as the principal explanation of plastic deformation at the microscopic scale of materials.

In a particular case where these defects are parallel lines in the three-dimensional space, their cross-section can be viewed as points in a plane. Under the effect of an exterior stress, dislocations can move. In the special case of what is called "edge dislocations", these dislocations move in the direction of their "Burgers vector" which has a fixed direction (cf. Hirth and Lothe [25] for more physical description).

In this work, we are interested in the mathematical study of a model introduced by Groma and Balogh [22,23]. In this model we consider two types of dislocations in the plane (x_1, x_2). Typically for a given velocity field, those dislocations of type $(+)$ propagate in the direction $+\mathbf{b}$ where $\mathbf{b} = (1, 0)$ is the Burgers vector, while those of type $(-)$ propagate in the direction $-\mathbf{b}$ (see Figure 1).

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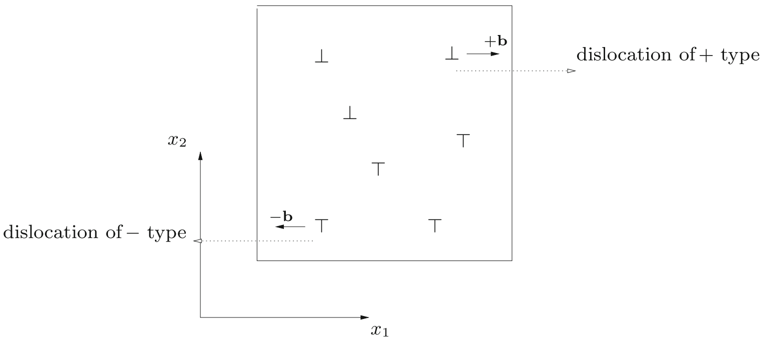

Fig. 1. GROMA-BALOGH 2D model

Here the velocity vector field is the shear stress in the material, solving the equations of elasticity. It turns out that this shear stress can be expressed as some Riesz transform of the solution (see Section 2). More precisely our non-linear and non-local system of transport equations is the following:

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) = -\left(R_1^2 R_2^2 \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x)\right) \dfrac{\partial \rho^+}{\partial x_1}(x,t) \\ \dfrac{\partial \rho^-}{\partial t}(x,t) = \left(R_1^2 R_2^2 \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x)\right) \dfrac{\partial \rho^-}{\partial x_1}(x,t) \end{cases} \tag{P}$$

The unknowns of the system (P) are the scalar functions ρ^+ and ρ^- at the time t and the position $x = (x_1, x_2)$, that we denote for simplification by $\rho^\pm$. These terms correspond to the plastic deformations in a crystal. Their derivative in the x_1 direction (that is the direction of Burgers vector $\mathbf{b}$), $\frac{\partial \rho^\pm}{\partial x_1}$ represents the dislocation densities of $\pm$ type. In our work, we will only consider solutions $\rho^\pm$ such that $\frac{\partial \rho^\pm}{\partial t}$, $\nabla \rho^\pm$ and $\rho^+ - \rho^-$ are $\mathbb{Z}^2$-periodic functions. The operators R_1 (resp. R_2) are the Riesz transformations associated to x_1 (resp. x_2). More precisely, these Riesz transforms are defined as follows:

Definition 1.1. (Riesz transforms in the periodic case) Let the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We define for $i \in \{1,2\}$ the Riesz transforms R_i over $\mathbb{T}^2$ as follows. If $f \in L^2(\mathbb{T}^2)$, the Fourier series coefficients of $R_i f$ are given by:

(i) $c_{(0,0)}(R_i f) = 0$,
(ii) $c_k(R_i f) = \frac{k_i}{|k|} c_k(f)$ for $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

where we recall that $c_k(f) = \int_{\mathbb{T}^2} f(x) e^{-2\pi i k\cdot x} d^2x$.

In fact, this 2D model has been generalized later in 2003 by I. Groma, F. Csikor and M. Zaiser in a model taking into account the back stress describing more carefully boundary layers (see [24] for further details). The Groma-Balogh model neglects in particular the short range dislocation-dislocation correlations in one slip direction. For an extension to multiple slip, see YEFIMOV and

Van der Giessen [38, ch. 5.]. This multiple slip version of the Groma-Balogh model presents some analogies with some traffic flow models (see Biham et al. [8]). See also Deshpande et al. [14] for a similar model with boundary conditions and exterior forces. Recently, EL-Azab [16], Zaiser and Hochrainer [39] and Monneau [29] were interested in modeling the dynamics of dislocation densities in the three-dimensional space, but many more open questions have to be solved for establishing a satisfactory three-dimensional theory of dislocations dynamics and for getting rigorous results.

We stress out the attention of the reader that there was no existence and uniqueness results for (P). In this paper we prove that (P) admits a "global in time" solution.

1.2. Main result

In this work, we consider the following initial conditions:

$$\rho^{\pm}(x_1, x_2, t = 0) = \rho_0^{\pm}(x_1, x_2) = \rho_0^{\pm,\mathrm{per}}(x_1, x_2) + Lx_1, \tag{IC}$$

where $\rho^{\pm,\mathrm{per}}$ is a 1-periodic function in x_1 and x_2. The periodicity is a way of studying the bulk behavior of the material away from its boundary. Here L is a given positive constant that represents the initial total dislocation densities of $\pm$ type on the periodic cell.

Before to give our main result, we want to show that the bilinear term on the right hand side of (P) is well defined. To this end, we need first to recall the following definition:

Definition 1.2. (The space $L \log L$) We define the space $L \log L(\mathbb{T}^2)$

$$L \log L(\mathbb{T}^2) = \left\{ f \in L^1(\mathbb{T}^2) \text{ such that } \int_{\mathbb{T}^2} |f| \ln(e + |f|) < +\infty \right\}.$$

This space is endowed with the (Luxemburg) norm

$$\|f\|_{L \log L(\mathbb{T}^2)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{T}^2} \frac{|f|}{\lambda} \ln\left(e + \frac{|f|}{\lambda}\right) \leqq 1 \right\}.$$

The space $L \log L(\mathbb{T}^2)$ is a special space of Zygmund spaces (see Adams [1, (13), Page 234] and Stein [36, Page 43])

We can now state the following proposition.

Proposition 1.3. (Meaning of the bilinear term) *Let $T > 0$, f and g be two functions defined on $\mathbb{T}^2 \times (0, T)$, such that $f \in L^1((0, T); W^{1,2}(\mathbb{T}^2))$ and $g \in L^\infty((0, T); L \log L(\mathbb{T}^2))$ then,*

$$fg \in L^1(\mathbb{T}^2 \times (0, T)).$$

We will see that the proof of this proposition (given in Section 3.2) is a direct consequence of Trudinger inequality.

We can now state our main result (see also our comments in Section 1.3 on the unknown uniqueness of the solution).

Theorem 1.4. (Global existence) *For all* $T, L > 0$ *, and for every initial data* $\rho_0^\pm \in L^2_{\text{loc}}(\mathbb{R}^2)$ *with*

(H1) $\rho_0^\pm(x_1+1, x_2) = \rho_0^\pm(x_1, x_2) + L$, *almost everywhere on* $\mathbb{R}^2$,
(H2) $\rho_0^\pm(x_1, x_2+1) = \rho_0^\pm(x_1, x_2)$, *almost everywhere on* $\mathbb{R}^2$,
(H3) $\frac{\partial \rho_0{}^\pm}{\partial x_1} \geqq 0$, *almost everywhere on* $\mathbb{R}^2$,
(H4) $\left\| \frac{\partial \rho_0{}^\pm}{\partial x_1} \right\|_{L \log L(\mathbb{T}^2)} \leqq C$, *with* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

the system (P)–(IC) *admits solutions* $\rho^\pm \in C([0,T); L^1_{\text{loc}}(\mathbb{R}^2)) \cap L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ *in the distributional sense, such that,* $\rho^\pm(\cdot, t)$ *satisfy* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$ *for almost everywhere* $t \in (0,T)$. *Moreover, we have:*

(P1) $R_1^2 R_2^2(\rho^+ - \rho^-) \in L^2((0,T); W^{1,2}_{\text{loc}}(\mathbb{R}^2))$.

Remark 1.5. (Bilinear term) It is clear here that the bilinear term on the right hand side of (P) is always defined via $(P1)$ and Proposition 1.3.

In order to prove our main theorem we regularize the system (P) by adding the viscosity term $(\varepsilon \Delta \rho^\pm)$, and regularize also the initial data (IC) by classical convolution. Then, using a fixed point Theorem, we prove that our regularized system admits local in time solutions. Moreover, as we get some ε-independent *a priori* estimates we will be able to extend our local in time solution into a global one. This turns out to be possible thanks to the entropy inequality (1.1). Then, joined with other *a priori* estimates, it will be possible to prove some compactness properties and to pass to the limit as ε goes to 0 in the ε-problem.

Remark 1.6. (Entropy and energy inequalities) It turns out that the constructed solution also satisfies the following fundamental entropy inequality (as a consequence of Lemma 5.4), for almost everywhere $t \in (0,T)$,

$$\begin{aligned} &\int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho^\pm}{\partial x_1} \ln\left(\frac{\partial \rho^\pm}{\partial x_1}\right) + \int_0^t \int_{\mathbb{T}^2} \left(R_1 R_2 \left(\frac{\partial \rho^+}{\partial x_1} - \frac{\partial \rho^-}{\partial x_1}\right)\right)^2 \\ &\qquad \leqq \int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho_0^\pm}{\partial x_1} \ln\left(\frac{\partial \rho_0^\pm}{\partial x_1}\right) \end{aligned} \tag{1.1}$$

Moreover, (at least formally for sufficiently regular solution) the following energy inequality holds:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{T}^2} \left(R_1 R_2(\rho^+ - \rho^-)(\cdot, t)\right)^2 + \int_0^t \int_{\mathbb{T}^2} \left(R_1^2 R_2^2(\rho^+ - \rho^-)\right)^2 \left(\frac{\partial \rho^+}{\partial x_1} + \frac{\partial \rho^-}{\partial x_1}\right) \\ &\quad \leqq \frac{1}{2}\int_{\mathbb{T}^2} \left(R_1 R_2(\rho_0^+ - \rho_0^-)\right)^2. \end{aligned}$$

Remark 1.7. (Bounds on the solution) If we denote $\rho = \rho^+ - \rho^-$, then there exists a constant C independent on T, and a constant C_T depending on T such that,

$$(E1)\|\rho^\pm - Lx_1\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} \leqq C_T, \quad (E4)\|R_1^2 R_2^2 \rho\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \leqq C,$$

$$(E2)\left\|\frac{\partial \rho^\pm}{\partial x_1}\right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))} \leqq C, \qquad (E5)\left\|R_1^2 R_2^2 \frac{\partial \rho}{\partial t}\right\|_{L^2((0,T);W^{-1,2}(\mathbb{T}^2))} \leqq C,$$

$$(E3)\left\|\frac{\partial \rho^\pm}{\partial t}\right\|_{L^2((0,T);L^1(\mathbb{T}^2))} \leqq C,$$

where $W^{-1,2}(\mathbb{T}^2)$ is the dual space of $W^{1,2}(\mathbb{T}^2)$.

In a particular sub-case of model (P) where the dislocation densities depend on a single variable $x = x_1 + x_2$, the existence and uniqueness of a Lipschitz viscosity solution was proved in El Hajj and Forcadel [18]. Also the existence and uniqueness of a strong solution in $W^{1,2}_{\text{loc}}(\mathbb{R} \times [0, T))$ was proved in El Hajj [17]. Concerning the model of Groma et al. [24] which takes into consideration the short range dislocation-dislocation correlations giving a parabolic-hyperbolic system, let us mention the work of Ibrahim [26] where a result of existence and uniqueness of a viscosity solution is given but only for a one-dimensional model.

Our study of the dynamics of dislocation densities in a special geometry is related to the more general dynamics of dislocation lines. We refer the interested reader to the work of Alvarez et al. [3], for a local existence and uniqueness of some non-local Hamilton-Jacobi equation. We also refer to Alvarez et al. [2] and Barles and Ley [6] for some long time existence results.

1.3. Comments on the uniqueness of the solution and related literature

The problem (P) is a system of transport equations with low regularity of the vector field, so that the uniqueness of the solution here is an open question. However, in the following we present some uniqueness results where the vector field has a better regularity.

From a technical point of view, (P) is related to other well known models, such as the transport equation with a low regularity vector field. This equation was studied in the work of DiPerna, and Lions [15] and Ambrosio [4], where the authors showed the existence and uniqueness of renormalized solutions by considering vector fields in $L^1((0, T); W^{1,1}_{\text{loc}}(\mathbb{R}^N))$ and $L^1((0, T); BV_{\text{loc}}(\mathbb{R}^N))$ respectively in both cases with bounded divergence. On the contrary in system (P), we are only able to prove that for the constructed solution, the vector field is in $L^2((0, T); W^{1,2}_{\text{loc}}(\mathbb{R}^2))$ without any better estimate on the divergence of the vector field.

More generally in the frame of symmetric hyperbolic systems, we refer to the book of Serre [34, Vol I, Th 3.6.1], for a typical result of local existence and uniqueness in $C([0, T); H^s(\mathbb{R}^N)) \cap C^1([0, T); H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$, by considering initial data in $H^s(\mathbb{R}^N)$. This result remains local in time, even in dimension $N = 2$.

We can also remark that in the case where we multiply the right side of the two equations in system (P) by -1, we get a quasi-geostrophic-like system. For those who are interested in quasi-geostrophic systems, we refer to CONSTANTIN et al. [11], and to [12] for certain 2D numerical results. We also refer to CÓRDOBA and CÓRDOBA [13], CHAE and CÓRDOBA [10] for blow-up results in finite time, in dimension one.

Let us also mention some related Vlasov-Poisson models (see NIETO et al. [30] for instance) and a related model in superconductivity studied by MASMOUDI et al. [28] and by AMBROSIO et al. [5]. These models were derived from some Vlasov-Poisson-Fokker-Planck models (see for instance GOUDON et al. [21] for an overview of similar models). It is also worth mentioning that this model is related to Vlasov-Navier-Stokes equation see GOUDON et al. [19,20].

1.4. Notation

In what follows, we are going to use the following notation:

1. $\rho = \rho^+ - \rho^-$,
2. $\rho^{\pm,\text{per}}(x_1, x_2, t) = \rho^{\pm}(x_1, x_2, t) - Lx_1$,
3. Let f be a function defined on $\mathbb{R}^2 \times (0, T)$ having values in $\mathbb{R}^2$, we denote by $f(t) = f(., t) : x \longmapsto f(x, t)$,
4. Throughout the paper, C is an arbitrary positive constant independent on T and C_T is an arbitrary positive constant depending on T.

1.5. Organization of the paper

First, in Section 2, we recall the physical derivation of system (P). In Section 3, we recall the definitions and properties of some useful fundamental spaces, and we give the proof of Proposition 1.3. We also prove that the bilinear term of our system has a better mathematical meaning (see Proposition 3.4). Next, in Section 4, we regularize the initial conditions and we show that the system (P), modified by a term ($\varepsilon \Delta \rho^{\pm}$), admits local solutions. Moreover, we show that these solutions are regular and increasing for all $t \in (0, T)$, for increasing initial data. In Section 5, we prove some ε-uniform *a priori* estimates for the regularized solution obtained in Section 4. Then, thanks to these *a priori* estimates, we extend the local in time solutions for the ε-problem constructed in Section 4, into global in time solution. Finally, in Section 6, we achieve the proof of our main theorem, passing to the limit in the equation as ε goes to 0, and using some compactness properties inherited from our *a priori* estimates.

2. Physical derivation of the model

In this section we explain how to derive physically the system (P). We consider a three-dimensional crystal, with displacement

$$u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}^3.$$

For $x = (x_1, x_2, x_3)$, and an orthogonal basis (e_1, e_2, e_3), we define the total strain by:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3.$$

This total strain is decomposed as

$$\varepsilon_{ij}(u) = \varepsilon^e_{ij} + \varepsilon^p_{ij},$$

with ε^e_{ij} is the elastic strain and ε^p_{ij} the plastic strain which is defined by:

$$\varepsilon^p_{ij} = \rho\varepsilon^0_{ij}, \tag{2.2}$$

with the fixed matrix $\varepsilon^0_{ij} = \frac{1}{2}(1 - \delta_{ij})$, where δ_{ij} is the Kronecker symbol, in the special case of a single slip system where dislocations move in the plane $\{x_2 = 0\}$ with Burgers vector $\mathbf{b} = e_1$. Here ρ is the resolved plastic strain, and will be clarified later. In the case of linear homogeneous and isotropic elasticity, the stress is given by

$$\sigma_{ij} = 2\mu\varepsilon^e_{ij} + \lambda\delta_{ij}\left(\sum_{k=1,2,3} \varepsilon^e_{kk}\right) \quad \text{for } i, j = 1, 2, 3, \tag{2.3}$$

where λ, μ are the constant LamÃľ coefficients of the crystal (satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$). Moreover the stress satisfies the equation of elasticity:

$$\sum_{j=1,2,3} \frac{\partial \sigma_{ij}}{\partial x_j} = 0.$$

We now assume that we are in a particular geometry where the dislocations are straight lines parallel to the direction e_3 and that the problem is invariant by translation in the x_3 direction. Moreover we assume that $u_3 = 0$ and $\sigma_{i3} = 0$ for $i = 1, 2, 3$. Then, this problem reduces to a two-dimensional problem with u_1, u_2 only depending on (x_1, x_2) and so we can express the resolved plastic strain ρ as

$$\rho = \rho^+ - \rho^-,$$

where $\frac{\partial \rho^+}{\partial x_1}$ and $\frac{\partial \rho^-}{\partial x_1}$ are respectively the densities of dislocations of Burgers vectors given by $\mathbf{b} = e_1$ and $\mathbf{b} = -e_1$.

Furthermore, these dislocation densities are transported in the direction of the Burgers vector at a given velocity. This velocity is indeed the resolved shear stress $\sum_{i,j=1,2,3} \sigma_{ij}\varepsilon^0_{ij} = \sigma_{12}$, up to sign of the Burgers vectors. More precisely, we have:

$$\frac{\partial \rho^\pm}{\partial t} = \pm(\sigma_{12})e_1.\nabla\rho^\pm.$$

Finally, the functions $\rho^{\pm}$ and $u = (u_1, u_2)$ are solutions of the coupled system (see GROMA [22] and GROMA and BALOGH [23]), on $\mathbb{R}^2 \times (0, T)$:

$$\begin{cases} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 & \text{for } i = 1, 2, \\ \sigma_{ij} = 2\mu\varepsilon^e_{ij} + \lambda\delta_{ij}\left(\displaystyle\sum_{k=1,2} \varepsilon^e_{kk}\right) & \text{for } i, j = 1, 2, \\ \varepsilon^e_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) - (\rho^+ - \rho^-)\varepsilon^0_{ij} & \text{for } i, j = 1, 2, \\ \varepsilon^0_{ij} = \dfrac{1}{2}\left(1 - \delta_{ij}\right) & \text{for } i, j = 1, 2, \\ \dfrac{\partial \rho^{\pm}}{\partial t} = \pm\sigma_{12}\dfrac{\partial \rho^{\pm}}{\partial x_1}. \end{cases} \tag{2.4}$$

Then the following lemma holds.

Lemma 2.1. (Computation of σ_{12}) *Assume that (u_1, u_2) and $\rho = \rho^+ - \rho^-$ are $\mathbb{Z}^2$-periodic functions. If (u_1, u_2), ρ^+, ρ^- are solutions of problem (2.4), then*

$$\sigma_{12} = -C_1\left(R_1^2 R_2^2 \rho\right), \tag{2.5}$$

where $C_1 = 4\frac{(\lambda+\mu)\mu}{\lambda+2\mu} > 0$.

Using this expression of σ_{12} and rescaling in time with the positive constant C_1 we obtain system (P), from the last equation (2.4).

Proof of Lemma 2.1. We can rewrite the first equation of (2.4) with div $u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}$

$$\mu\Delta u_1 + (\lambda + \mu)\frac{\partial}{\partial x_1}(\operatorname{div} u) = \mu\frac{\partial \rho}{\partial x_2}, \tag{2.6a}$$

$$\mu\Delta u_2 + (\lambda + \mu)\frac{\partial}{\partial x_2}(\operatorname{div} u) = \mu\frac{\partial \rho}{\partial x_1}. \tag{2.6b}$$

Considering $\frac{\partial}{\partial x_1}$(2.6a) + $\frac{\partial}{\partial x_2}$(2.6b), we get

$$(\lambda + 2\mu)\Delta(\operatorname{div} u) = 2\mu\frac{\partial^2 \rho}{\partial x_1 \partial x_2}.$$

Plugging the expression of div u into (2.6), we get

$$\Delta u_1 = \frac{\partial \rho}{\partial x_2} - 2\frac{(\lambda + \mu)}{(\lambda + 2\mu)}\frac{\partial}{\partial x_1}\Delta^{-1}\frac{\partial^2 \rho}{\partial x_1 \partial x_2}, \tag{2.7a}$$

$$\Delta u_2 = \frac{\partial \rho}{\partial x_1} - 2\frac{(\lambda + \mu)}{(\lambda + 2\mu)}\frac{\partial}{\partial x_2}\Delta^{-1}\frac{\partial^2 \rho}{\partial x_1 \partial x_2}. \tag{2.7b}$$

Considering now $\frac{\partial}{\partial x_2}$(2.7a) + $\frac{\partial}{\partial x_1}$(2.7b) , we obtain

$$\Delta\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)=\Delta(\rho^+-\rho^-)-4\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Delta^{-1}\frac{\partial^4}{\partial x_1^2\partial x_2^2}(\rho^+-\rho^-). \quad (2.8)$$

Recalling that

$$\sigma_{12}=\mu\left(\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)-(\rho^+-\rho^-)\right), \quad (2.9)$$

this yields $\sigma_{12}=-4\frac{(\lambda+\mu)\mu}{(\lambda+2\mu)}\Delta^{-2}\frac{\partial^4}{\partial x_1^2\partial x_2^2}(\rho^+-\rho^-)=-C_1(R_1^2R_2^2(\rho^+-\rho^-))$. □

Remark 2.2. (Property of the elastic energy) If we define the elastic energy by

$$E=\int_{\mathbb{R}^2/\mathbb{Z}^2}\mu\sum_{i,j=1,2}(\varepsilon_{ij}^e)^2+\frac{\lambda}{2}\left(\sum_{k=1,2}\varepsilon_{kk}^e\right)^2.$$

Using system (2.4) we can show formally that

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\int_{\mathbb{R}^2/\mathbb{Z}^2}(\sigma_{12})^2\left(\frac{\partial\rho^+}{\partial x_1}+\frac{\partial\rho^-}{\partial x_1}\right)\leqq 0.$$

where we have used the fact that $\frac{\partial\rho^+}{\partial x_1},\frac{\partial\rho^-}{\partial x_1}\geqq 0$ to see that the elastic energy is a non-increasing in time. Hence, the elastic energy E is a Lyapunov functional for our dissipative model.

3. Concerning the meaning of the solution of (P)

In this section we prove Proposition 1.3. This shows that if (P) admits solutions verifying the conditions of Theorem 1.4, then we can give a mathematical meaning to the bilinear term. In order to do this, we need to define some functional spaces and recall some of their properties, that will be used later in our work.

3.1. Properties of some useful Orlicz spaces

We recall the definition of Orlicz spaces and some of their properties. For details, we refer to Adams [1, Ch. 8] and Rao and Ren [33].

A real valued function $A:[0,+\infty)\to\mathbb{R}$ is called a Young function if it has the following properties (see O'Neil [31, Def 1.1]):

- A is a continuous, non-negative, non-decreasing and convex function.
- $A(0)=0$ and $\lim_{t\to+\infty}A(t)=+\infty$.

Let $A(\cdot)$ be a Young function. The Orlicz class $K_A(\mathbb{T}^2)$ is the set of (equivalence classes of) real-valued measurable function h on $\mathbb{T}^2$ satisfying

$$\int_{\mathbb{T}^2} A(|h(x)|) < +\infty.$$

The Orlicz space $L_A(\mathbb{T}^2)$ is the linear hull of $K_A(\mathbb{T}^2)$ supplemented with the Luxemburg norm

$$\|f\|_{L_A(\mathbb{T}^2)} = \inf\left\{\lambda > 0 : \int_{\mathbb{T}^2} A\left(\frac{|h(x)|}{\lambda}\right) \leqq 1\right\}.$$

Endowed with this norm, the Orlicz space $L_A(\mathbb{T}^2)$ is a Banach space. Moreover, for all $f \in L_A(\mathbb{T}^2)$, we have the following estimate

$$\|f\|_{L_A(\mathbb{T}^2)} \leqq 1 + \int_{\mathbb{T}^2} A(|f(x)|). \tag{3.10}$$

Definition 3.1. (Some Orlicz spaces)

- $EXP_\alpha(\mathbb{T}^2)$ denotes the Orlicz space defined by the function $A(t) = e^{t^\alpha} - 1$ for $\alpha \geqq 1$.
- $L\log^\beta L(\mathbb{T}^2)$ denotes the Orlicz space defined by the function $A(t) = t(\log(e+t))^\beta$, for $\beta \geqq 0$.

Observe that for $0 < \beta \leqq 1$ the space $EXP_{\frac{1}{\beta}}(\mathbb{T}^2)$ is the dual of the Zygmund space $L\log^\beta L(\mathbb{T}^2)$ (see Bennett and Sharpley [7, Def 6.11]). It is worth noticing that $L\log^1 L(\mathbb{T}^2) = L\log L(\mathbb{T}^2)$.

Let us recall some useful properties of these spaces. The first one is the generalized Hölder inequality.

Lemma 3.2. (Generalized Hölder inequality)

(i) *Let $f \in EXP_2(\mathbb{T}^2)$ and $g \in L\log^{\frac{1}{2}} L(\mathbb{T}^2)$. Then there exists a constant C such that (see* O'Neil [31, Th 2.3]*)*

$$\|fg\|_{L^1(\mathbb{T}^2)} \leqq C\|f\|_{EXP_2(\mathbb{T}^2)}\|g\|_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)}.$$

(ii) *Let $f \in EXP_2(\mathbb{T}^2)$ and $g \in L\log L(\mathbb{T}^2)$. Then there exists a constant C such that (see* O'Neil [31, Th 2.3]*)*

$$\|fg\|_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leqq C\|f\|_{EXP_2(\mathbb{T}^2)}\|g\|_{L\log L(\mathbb{T}^2)}.$$

The second property is the Trudinger inequality.

Lemma 3.3. (Trudinger inequality) *There exists a constant $\gamma > 0$ such that, for all $f \in W^{1,2}(\mathbb{T}^2)$, we have (see* Trudinger [37]*)*

$$\int_{\mathbb{T}^2} e^{\gamma\left(\frac{f}{\|f\|_{W^{1,2}(\mathbb{T}^2)}}\right)^2} \leqq 1.$$

In particular we have the following embedding

$$W^{1,2}(\mathbb{T}^2) \hookrightarrow EXP_2(\mathbb{T}^2).$$

3.2. Sharp estimate of the bilinear term

Now, we propose to verify with the help of the following proposition that the system (P) has indeed a sense. First we prove a better estimate than the one mentioned in Proposition 1.3. Namely, we have the following.

Proposition 3.4. (Estimate of the bilinear term) *Let $T > 0$, f and g be two functions defined on $\mathbb{T}^2 \times (0, T)$, such that*

(1) $f \in L^2((0, T); W^{1,2}(\mathbb{T}^2))$,
(2) $g \in L^\infty((0, T); L \log L(\mathbb{T}^2))$. *Then*

$$fg \in L^2((0, T); L \log^{\frac{1}{2}} L(\mathbb{T}^2))$$

and for a positive constant C, we have:

$$\|fg\|_{L^2((0,T); L \log^{\frac{1}{2}} L(\mathbb{T}^2))} \leqq C \|f\|_{L^2((0,T); W^{1,2}(\mathbb{T}^2))} \|g\|_{L^\infty((0,T); L \log L(\mathbb{T}^2))}.$$

For the proof of this Proposition , we use Lemma 3.2(ii), and integrate in time. Thanks to the Trudinger inequality (Lemma 3.3), we get the result. We proceed in the same way for the proof of the Proposition 1.3.

4. Local existence of solutions of a regularized system

In this section, we state a local in time existence result for system (P), modified by the term $\varepsilon \Delta \rho^{\pm}$, and for smoothed data. This modification brings us to study, for all $0 < \varepsilon \leqq 1$, the following regularized system:

$$\begin{cases} \dfrac{\partial \rho^{+,\varepsilon}}{\partial t} - \varepsilon \Delta \rho^{+,\varepsilon} = -(R_1^2 R_2^2 \rho^\varepsilon) \dfrac{\partial \rho^{+,\varepsilon}}{\partial x_1} \ \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0, T)), \\ \dfrac{\partial \rho^{-,\varepsilon}}{\partial t} - \varepsilon \Delta \rho^{-,\varepsilon} = (R_1^2 R_2^2 \rho^\varepsilon) \dfrac{\partial \rho^{-,\varepsilon}}{\partial x_1} \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0, T)), \end{cases} \tag{P_ε}$$

where $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$, with the following regular initial data:

$$\rho^{\pm,\varepsilon}(x, 0) = \rho_0^{\pm,\varepsilon}(x) = \rho_0^{\pm,\text{per}} * \eta_\varepsilon(x) + (L + \varepsilon)x_1 = \rho_0^{\pm,\varepsilon,\text{per}}(x) + L_\varepsilon x_1, \tag{IC_ε}$$

where $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon^2}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C_c^\infty(\mathbb{R}^2)$ is a non-negative function and $\int_{\mathbb{R}^2} \eta = 1$.

Remark 4.1. We consider $L_\varepsilon = L + \varepsilon$ to obtain strictly monotonous initial data $\rho_0^{\pm,\varepsilon}$. This condition will be useful in the proof of Lemma 5.4.

We also introduce, for later use, the analogue of condition $(H1)$ with L replaced by L_ε:
$(\text{H1})_\varepsilon$ $\quad \rho_0^{\pm,\varepsilon}(x_1 + 1, x_2) = \rho_0^{\pm,\varepsilon}(x_1, x_2) + L_\varepsilon$, almost everywhere on $\mathbb{R}^2$.

For the regularized system (P_ε) and (IC_ε) we have the following result.

Theorem 4.2. (Local existence result of monotone smooth solutions) *For all initial data* $\rho_0^{\pm} \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ *satisfying* $(H1)$, $(H2)$ *and* $(H3)$, *and all* $\varepsilon > 0$, *there exists* $T^\star > 0$, *which only depends on* L_ε *and on a bound on* $\|\rho_0^{\pm,\varepsilon,\mathrm{per}}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}$, *such that the system* (P_ε) *and* (IC_ε) *admits solutions* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0, T^\star))$. *Moreover* $\rho^{\pm,\varepsilon}(\cdot, t)$ *satisfy* $(H1)_\varepsilon$, $(H2)$ *and* $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} > 0$, *for all* $t \in [0, T^\star)$.

To prove Theorem 4.2, we need to recall the following result.

Lemma 4.3. (Decay estimate for the heat semi-group) *Let* $r, p, q \geqq 1$. *Then, for all functions* $f \in L^q(\mathbb{T}^2)$ *and* $g \in L^p(\mathbb{T}^2)$, *where* $\frac{1}{r} \leqq \frac{1}{q} + \frac{1}{p}$, *we have, for* $S_1(t) = e^{t\Delta}$, *the following estimates:*

(i) $\|S_1(t)(fg)\|_{L^r(\mathbb{T}^2)} \leqq Ct^{-\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$ *for all* $t > 0$,

(ii) $\|\nabla S_1(t)(fg)\|_{L^r(\mathbb{T}^2)} \leqq Ct^{-\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$ *for all* $t > 0$,

where C *is a positive constant depending only on* r, p, q.

The proof of this lemma is a direct application of the classical version of the L^r-L^p estimates for the heat semi-group (see PAZY [32, Lemma 1.1.8, Th 6.4.5]) and the Hölder inequality.

Proof of Theorem 4.2. We perform the proof in four steps.

Step 1. Rewriting the system

Introducing the periodic functions (because of $(H1)$–$(H2)$).

$$\rho^{\pm,\varepsilon,\mathrm{per}}(x_1, x_2, t) = \rho^{\pm,\varepsilon}(x_1, x_2, t) - L_\varepsilon x_1, \quad \text{and} \quad \rho_v^\varepsilon = (\rho^{+,\varepsilon,\mathrm{per}}, \rho^{-,\varepsilon,\mathrm{per}})$$

and similarly for the initial data

$$\rho_0^{\pm,\varepsilon,\mathrm{per}}(x_1, x_2) = \rho_0^{\pm,\varepsilon}(x_1, x_2) - L_\varepsilon x_1, \quad \text{and} \quad \rho_{0,v}^\varepsilon = (\rho_0^{+,\varepsilon,\mathrm{per}}, \rho_0^{-,\varepsilon,\mathrm{per}}),$$

we can rewrite system (IC_ε) and (P_ε) in its integral form as follows

$$\rho_v^\varepsilon(x, 0) = \rho_{0,v}^\varepsilon(x) \qquad (IC_\varepsilon^{\mathrm{per}})$$

and

$$\rho_v^\varepsilon(x, t) = S_\varepsilon(t)\rho_{0,v}^\varepsilon + B(\rho_v^\varepsilon, \rho_v^\varepsilon)(t) + A(\rho_v^\varepsilon)(t), \qquad (P_\varepsilon^{\mathrm{per}})$$

with the notation for general functions $u = (u_1, u_2)$ and $v = (v_1, v_2)$

$$\begin{cases} S_\varepsilon(t) = e^{\varepsilon t \Delta}, \\ B(u, v)(t) = \bar{I}_1 \displaystyle\int_0^t S_\varepsilon(t-s)\left(\left(R_1^2 R_2^2(u_1 - u_2)\right)\frac{\partial v}{\partial x_1}(s)\right) \mathrm{d}s, \\ \quad \text{with} \quad \bar{I}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ A(u)(t) \quad = L_\varepsilon \bar{J}_1 \displaystyle\int_0^t S_\varepsilon(t-s)\left(R_1^2 R_2^2(u_1 - u_2)(s)\right) \mathrm{d}s, \\ \quad \text{with} \quad \bar{J}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{cases}$$

In what follows, we want to apply a fixed point argument. In order to be specific, we choose to work with the Banach space $L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)$. Remark that other spaces $L^\infty((0,T);(W^{1,p}(\mathbb{T}^2))^2)$ could also be used for different and suitable values of p. To apply a fixed point argument, we need to get some estimates on B, A and $S_\varepsilon(t)\rho^\varepsilon_{0,v}$, which are presented in the next step.

Step 2. *A priori* estimates on B, A and $S_\varepsilon(t)\rho^\varepsilon_{0,v}$

We remark that

$$\begin{aligned}
&\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}\\
&\leqq \int_0^t \left\| S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)\mathrm{d}s\right\|_{(L^4(\mathbb{T}^2))^2}\\
&\quad+\int_0^t \left\| \nabla S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)\mathrm{d}s\right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2}
\end{aligned}\tag{4.11}$$

and can then use Lemma 4.3(i) with $r=4, q=3, p=\frac{3}{2}$ to estimate the first term and Lemma 4.3(ii) with $r=\frac{3}{2}, q=4, p=\frac{3}{2}$ to estimate the second term. Using moreover the fact that Riesz transforms preserve the L^p-spaces (see the result recalled later in Lemma 5.1(i)) and the classical Sobolev injection, we get

$$\begin{aligned}
\|B(u,v)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} &\leqq \eta(T)\|u\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}\\
&\quad\times\|v\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)},
\end{aligned}\tag{4.12}$$

with $\eta(T)=C_0T^{\frac{1}{4}}$ for some constant $C_0>0$. Similarly, we estimate the linear term and get

$$\|A(u)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leqq L_\varepsilon\eta(T)\|u\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}.\tag{4.13}$$

Finally, we know by classical properties of heat semi-group that

$$\|S_\varepsilon(t)\rho^\varepsilon_{0,v}\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leqq \|\rho^\varepsilon_{0,v}\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}.\tag{4.14}$$

Step 3. The fixed point argument

Using estimates given in Step 2, we can now apply the classical fixed point argument (see Cannone [9, Lemma 4.2.14]). We get the existence of a time $T^\star>0$, which only depends on L_ε and on a bound on $\|\rho_0^{\pm,\varepsilon,\mathrm{per}}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}$, such that there exists a solution ρ^ε_v to system ($IC^{\mathrm{per}}_\varepsilon$) and ($P^{\mathrm{per}}_\varepsilon$) with $\rho^\varepsilon_v\in L^\infty((0,T^\star);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)$.

Step 4. Monotone smooth solutions

Using the classical bootstrap arguments, we can show that ρ^ε_v is smooth. Moreover the monotonicity of the solution $\rho^{\pm,\varepsilon}$ in x_1 follows from assumption $(H3)$ and from the maximum principle (see Lieberman [27, Th 2.10]) for the scalar parabolic equations satisfied by $\frac{\partial\rho^{\pm,\varepsilon}}{\partial x_1}$. □

5. ε-Uniform estimates on the solution of the regularized system

In this section, we prove some fundamental ε-uniform estimates. In Section 5.1 we give some general estimates which are independent on the equation. In the second Section 5.2 we establish *a priori* estimates on the solutions of system (P_ε).

5.1. Useful estimates

Now we recall some well known properties of Riesz transforms, that will be used later in our work.

Lemma 5.1. (Properties of Riesz transforms)

(i) *For all $g \in L^p(\mathbb{T}^2)$, $1 < p < +\infty$, there exists a positive constant $C = C(p)$ such that*

$$\|R_i g\|_{L^p(\mathbb{T}^2)} \leqq C\|g\|_{L^p(\mathbb{T}^2)}.$$

(ii) *If $g \in L^2(\mathbb{T}^2)$, then $\int_{\mathbb{R}/\mathbb{Z}} R_1 g(x_1, x_2)dx_1 = 0$, for almost everywhere $x_2 \in \mathbb{R}/\mathbb{Z}$.*

(iii) *For all $g \in L^2(\mathbb{T}^2)$, we have $\frac{\partial}{\partial x_1} R_2 g = \frac{\partial}{\partial x_2} R_1 g$ and $R_1 R_2 g = R_2 R_1 g$.*

(iv) *For all $f, g \in L^2(\mathbb{T}^2)$, we have $\int_{\mathbb{T}^2} (R_i f) g = \int_{\mathbb{T}^2} f(R_i g)$.*

(v) *If $g \in L^2(\mathbb{T}^2)$ and does not depend on x_2, then $R_1 g = 0$.*

Proof of Lemma 5.1. For the proof of (i) (see Zygmund [40, Vol I, Page 254, (2.6)]). The proof of (iv) this is straightforward, using Fourier series. For the proof of (ii), it suffices to note that, if we denote by $f(x_2) = \int_{\mathbb{R}/\mathbb{Z}} R_1 g(x_1, x_2)\mathrm{d}x_1$, then we have $c_{k_2}(f) = c_{(0,k_2)}(R_1 g) = 0$ by definition of c_k for $k_1 = 0$. Finally, we prove (iii), checking simply that

$$c_k\left(\frac{\partial}{\partial x_1} R_2 g\right) = 2\pi i k_1 \frac{k_2}{|k|} c_k(g) = 2\pi i k_2 \frac{k_1}{|k|} c_k(g) = c_k\left(\frac{\partial}{\partial x_2} R_1 g\right),$$

and similar we prove second equality of (iii). The proof of (v) is a consequence of the fact:

$$c_{(k_1,k_2)}(R_1 g) = \frac{k_1}{|k|} \int_{\mathbb{T}^2} g(x_2) e^{-2\pi i(k_1 x_1 + k_2 x_2)} \mathrm{d}x_1 \mathrm{d}x_2 = 0.$$

□

Lemma 5.2. (L^∞ estimate) *If $f \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and f verifies $(H1)$, $(H2)$ and $(H3)$, then there exists a constant $C = C(L)$ such that*

$$\left\| f^{\mathrm{per}} - \int_0^1 f^{\mathrm{per}} \mathrm{d}x_1 \right\|_{L^\infty(\mathbb{T}^2)} \leqq C, \tag{5.15}$$

where $f^{\mathrm{per}} = f - Lx_1$.

Proof of Lemma 5.2. We compute

$$\int_0^1 \left|\frac{\partial f^{\mathrm{per}}}{\partial x_1}\right| \mathrm{d}x_1 = \int_0^1 \left|\frac{\partial f}{\partial x_1} - L\right| \mathrm{d}x_1 \leqq L + \int_0^1 \left|\frac{\partial f}{\partial x_1}\right| \mathrm{d}x_1$$
$$= L + \int_0^1 \frac{\partial f}{\partial x_1} \mathrm{d}x_1$$
$$= 2L,$$

where we use $(H3)$ in the second line and $(H1)$ in the last line. We next apply a "Poincaré-Wirtinger inequality" in x_1 and we deduce the result. □

We will also use the following technical result.

Lemma 5.3. ($L \log L$ Estimate) *Let $(\eta_\varepsilon)_\varepsilon$ be a non-negative mollifier, then for all $f \in L \log L(\mathbb{T}^2)$ and $f \geqq 0$, the function $f_\varepsilon = f * \eta_\varepsilon$ satisfies*

$$\int_{\mathbb{T}^2} f_\varepsilon \ln f_\varepsilon \to \int_{\mathbb{T}^2} f \ln f \quad \text{as} \quad \varepsilon \to 0.$$

For the proof see Adams [1, Th 8.20].

5.2. A priori estimates

In this subsection, we show some ε-uniform estimates on the solutions of the system (P_ε) and (IC_ε) obtained in Theorem 4.2. These estimates will be used, on the one hand to extend the solution in a global one and, on the other hand in Section 6.2, for ensuring by compactness the passage to the limit as ε tends to zero.

The first estimate concerns the physical entropy of the system, and is a key result. It shows that in our model, the dislocation densities cannot be so concentrated and then can be controlled.

Lemma 5.4. (Entropy estimate) *Let $\rho_0^\pm \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. If $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0, T))$ are solutions of the system (P_ε) and (IC_ε) and $\rho^{\pm,\varepsilon}(\cdot, t)$ satisfy $(H1)_\varepsilon$, $(H2)$, $(H3)$ and $(H4)$, then*

$$\int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} \ln\left(\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}\right) + \int_0^t \int_{\mathbb{T}^2} \left(R_1 R_2 \left(\frac{\partial \rho^\varepsilon}{\partial x_1}\right)\right)^2$$
$$\leqq \int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x_1} \ln\left(\frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x_1}\right), \tag{5.16}$$

where $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$.

In particular, there exists a constant C independent of $\varepsilon \in (0, 1]$ such that

$$\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}\right\|_{L^\infty((0,T); L \log L(\mathbb{T}^2))} + \left\|\frac{\partial}{\partial x_1}\left(R_1 R_2 \rho^\varepsilon\right)\right\|_{L^2(\mathbb{T}^2 \times (0,T))} \leqq C \tag{5.17}$$

with $C = C(\|\frac{\partial \rho_0{}^\pm}{\partial x_1}\|_{L \log L(\mathbb{T}^2)})$.

Proof of Lemma 5.4. First of all, we denote $\theta^{\pm,\varepsilon} = \frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}$ and $N^{\pm}(t) = \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t)\ln(\theta^{\pm,\varepsilon}(t))$.

Using the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$, we can derive $N(t) = N^+(t) + N^-(t)$ with respect to t, and since $\theta^{\pm,\varepsilon} > 0$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \int_{\mathbb{T}^2}\sum_{+,-}(\theta^{\pm,\varepsilon})_t \ln(\theta^{\pm,\varepsilon}) + \int_{\mathbb{T}^2}\sum_{+,-}(\theta^{\pm,\varepsilon})_t.$$

Using system (P_ε) we see that the second term is zero. Moreover, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \int_{\mathbb{T}^2}\sum_{+,-}\left[\mp\left((R_1^2R_2^2\rho^\varepsilon)\theta^{\pm,\varepsilon}\right)_{x_1} + \varepsilon\Delta\theta^{\pm,\varepsilon}\right]\ln(\theta^{\pm,\varepsilon}).$$

Integrating by parts in x_1, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}N(t) &= \int_{\mathbb{T}^2}\sum_{+,-}\left(\pm(R_1^2R_2^2\rho^\varepsilon)\theta^{\pm,\varepsilon}\right)\frac{\theta^{\pm,\varepsilon}_{x_1}}{\theta^{\pm,\varepsilon}} - \varepsilon\sum_{+,-}\int_{\mathbb{T}^2}\frac{\left|\nabla\theta^{\pm,\varepsilon}\right|^2}{\theta^{\pm,\varepsilon}}\\ &= \int_{\mathbb{T}^2}\left(R_1^2R_2^2\rho^\varepsilon\right)\frac{\partial\theta^\varepsilon}{\partial x_1} - \varepsilon\sum_{+,-}\int_{\mathbb{T}^2}\frac{\left|\nabla\theta^{\pm,\varepsilon}\right|^2}{\theta^{\pm,\varepsilon}}\end{aligned}$$

where $\theta^\varepsilon = \theta^{+,\varepsilon} - \theta^{-,\varepsilon}$. We integrate also the first term by parts in x_1, and we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}N(t) &= -\int_{\mathbb{T}^2}\left(R_1^2R_2^2\theta^\varepsilon\right)\theta^\varepsilon - \varepsilon\sum_{+,-}\int_{\mathbb{T}^2}\frac{\left|\nabla\theta^{\pm,\varepsilon}\right|^2}{\theta^{\pm,\varepsilon}}\\ &= -\int_{\mathbb{T}^2}\left(R_1R_2\theta^\varepsilon\right)^2 - \varepsilon\sum_{+,-}\int_{\mathbb{T}^2}\frac{\left|\nabla\theta^{\pm,\varepsilon}\right|^2}{\theta^{\pm,\varepsilon}} \leqq 0,\end{aligned}$$

where we have used Lemma 5.1(iii) and (iv) for the second line. Integrating in time, we get

$$N(t) + \int_0^t\int_{\mathbb{T}^2}\left(R_1R_2\theta^\varepsilon\right)^2 \leqq N(0).$$

Which proves (5.16). Moreover, we have

$$N(0) \leqq \int_{\mathbb{T}^2}\sum_{+,-}\theta^{\pm,\varepsilon}(0)\log(e + \theta^{\pm,\varepsilon}(0)).$$

Since the initial data (IC) satisfies ($H4$), we deduce by Lemma 5.3 that there exists a positive constant C independent of $\varepsilon \in (0,1]$ such that

$$N(t) + \int_0^t\int_{\mathbb{T}^2}\left(R_1R_2\theta^\varepsilon\right)^2 \leqq C.$$

Let us now consider

$$N_1^{\pm}(t) = \int_{\mathbb{T}^2}\theta^{\pm,\varepsilon}(t)\log(e + \theta^{\pm,\varepsilon}(t)).$$

We deduce, with another constant $C' > 0$, that

$$N_1^+(t) + N_1^-(t) + \int_0^t \int_{\mathbb{T}^2} \left(R_1 R_2 \theta^\varepsilon\right)^2 \leqq C'$$

which joint to (3.10) implies (5.17). □

Remark 5.5. (L^2 estimate on the gradient of the vector field) We want to bound $\nabla(R_1^2 R_2^2 \rho^\varepsilon)$. To this end, remark that by the property of Riesz transforms (see Lemma 5.1(iii)), we have

$$\frac{\partial}{\partial x_1} R_1^2 R_2^2 \rho^\varepsilon = R_1 R_2 \left(\frac{\partial}{\partial x_1} R_1 R_2 \rho^\varepsilon\right) \quad \text{and} \quad \frac{\partial}{\partial x_2} R_1^2 R_2^2 \rho^\varepsilon = R_2^2 \left(\frac{\partial}{\partial x_1} R_1 R_2 \rho^\varepsilon\right),$$

where those quantities involve $\frac{\partial}{\partial x_1} R_1 R_2 \rho^\varepsilon$ which is bounded in $L^2(\mathbb{T}^2 \times (0,T))$ by (5.17). Then using the fact the Riesz transforms are continuous from L^2 onto itself (see Lemma 5.1(i)), we deduce that

$$\left\| \nabla \left(R_1^2 R_2^2 \rho^\varepsilon\right) \right\|_{L^2(\mathbb{T}^2 \times (0,T))} \leqq C, \tag{5.18}$$

where the constant C is independent on ε.

We now present a second *a priori* estimate.

Lemma 5.6. (L^2 bound on the solutions) *Let $T > 0$. Under the condition $\rho_0^\pm \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. If $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of system (P_ε) and (IC_ε) and $\rho^{\pm,\varepsilon}(\cdot,t)$ satisfy $(H1)_\varepsilon$, $(H2)$, $(H3)$ and $(H4)$, then there exists a constant C_T independent of $\varepsilon \in (0,1]$, but depending on T, such that:*

$$\left\| \rho^{\pm,\varepsilon,\mathrm{per}} \right\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} \leqq C_T$$

with $\rho^{\pm,\varepsilon,\mathrm{per}} = \rho^{\pm,\varepsilon} - Lx_1$.

Proof of Lemma 5.6. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We want to bound $m^{\pm,\varepsilon}(x_2,t) = \int_{\mathbb{T}} \rho^{\pm,\varepsilon,\mathrm{per}}(x_1,x_2,t)\mathrm{d}x_1$. There is no problem of regularity since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$. We integrate equation (P_ε) with respect to x_1, and we get

$$\begin{aligned} \frac{\partial m^{\pm,\varepsilon}}{\partial t} - \varepsilon \frac{\partial^2 m^{\pm,\varepsilon}}{\partial x_2^2} &= \pm \int_{\mathbb{T}} \left(R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1}\right)(\rho^{\pm,\varepsilon,\mathrm{per}} - m^{\pm,\varepsilon})\mathrm{d}x_1 \\ &\quad \pm m^{\pm,\varepsilon} \int_{\mathbb{T}} \left(R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1}\right) \mathrm{d}x_1 \mp L_\varepsilon \int_{\mathbb{T}} (R_1^2 R_2^2 \rho^\varepsilon)\mathrm{d}x_1, \end{aligned} \tag{5.19}$$

where for the first line we have integrated by parts, and introduced the mean value $m^{\pm,\varepsilon}$. Therefore, using that ρ^ε is a 1-periodic function in x_1 and Lemma 5.1(ii) and (iii), we deduce that

$$\int_{\mathbb{T}} (R_1^2 R_2^2 \rho^\varepsilon)\mathrm{d}x_1 = 0 = \int_{\mathbb{T}} \left(R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1}\right) \mathrm{d}x_1,$$

Equation (5.19) is then equivalent to

$$\frac{\partial m^{\pm,\varepsilon}}{\partial t} - \varepsilon \frac{\partial^2 m^{\pm,\varepsilon}}{\partial x_2^2} = \pm \int_{\mathbb{T}} \left(R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1} \right) (\rho^{\pm,\varepsilon,\mathrm{per}} - m^{\pm,\varepsilon}) \mathrm{d}x_1 =: I^\pm(x_2, t). \tag{5.20}$$

We now show that $I^\pm \in L^2(\mathbb{T} \times (0, T))$. Indeed, we have

$$\begin{aligned} \left\| I^\pm \right\|_{L^2(\mathbb{T}\times(0,T))} &\leqq \left\| \int_{\mathbb{T}} \left(R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1} \right) (\rho^{\pm,\varepsilon,\mathrm{per}} - m^{\pm,\varepsilon}) \mathrm{d}x_1 \right\|_{L^2(\mathbb{T}\times(0,T))} \\ &\leqq \left\| \rho^{\pm,\varepsilon,\mathrm{per}} - m^{\pm,\varepsilon} \right\|_{L^\infty(\mathbb{T}^2\times(0,T))} \left\| R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1} \right\|_{L^2(\mathbb{T}^2\times(0,T))} \\ &\leqq C \end{aligned}$$

where for the last line we have used (5.18) and (Lemma 5.1(i)) to bound $\| R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1} \|_{L^2(\mathbb{T}^2\times(0,T))}$. Furthermore, the bound

$$\left\| \rho^{\pm,\varepsilon,\mathrm{per}} - m^{\pm,\varepsilon} \right\|_{L^\infty(\mathbb{T}^2\times(0,T))} \leqq C$$

follows from (5.15).

Multiplying (5.20) by $m^{\pm,\varepsilon}$ and integrating in x_2, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| m^{\pm,\varepsilon}(\cdot, t) \|_{L^2(\mathbb{T})}^2 + \varepsilon \left\| \frac{\partial}{\partial x_2} m^{\pm,\varepsilon}(\cdot, t) \right\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} (I^\pm m^{\pm,\varepsilon})(\cdot, t).$$

Using Cauchy-Schwarz inequality on the right hand side, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \| m^{\pm,\varepsilon}(\cdot, t) \|_{L^2(\mathbb{T})} \leqq \| I^\pm(\cdot, t) \|_{L^2(\mathbb{T})}.$$

We conclude to the result by integrating in time. □

Corollary 5.7. ($W^{1,2}$ estimate on the vector field) *Under the assumptions $\rho_0^\pm \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. If $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0, T))$ are solutions of the system (P_ε) and (IC_ε) and $\rho^{\pm,\varepsilon}(\cdot, t)$ satisfy $(H1)_\varepsilon$, $(H2)$, $(H3)$ and $(H4)$, then there exists a constant C independent of ε such that:*

$$\left\| R_1^2 R_2^2 \rho^\varepsilon \right\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \leqq C.$$

The proof of Corollary 5.7 follows from the application of "Poincaré-Wirtinger inequality" using (5.18) joined to the fact that $R_1^2 R_2^2 \rho^\varepsilon$ is of null average (see Lemma 5.1(ii)).

The following estimate will provide compactness in time of the solution, uniform with respect to ε.

Lemma 5.8. (Duality estimate for the time derivative of the solution) *Let $T > 0$. Under the assumptions $\rho_0^\pm \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. If $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0, T))$ are solutions of the system (P_ε) and (IC_ε) and $\rho^{\pm,\varepsilon}(\cdot, t)$ satisfy $(H1)_\varepsilon$, $(H2)$, $(H3)$ and $(H4)$, then*

(i) *For all* $\psi \in L^2((0,T); W^{1,2}(\mathbb{T}^2))$*, there exists a constant* C *independent of* $\varepsilon \in (0,1]$ *such that:*

$$\left|\int_{\mathbb{T}^2\times(0,T)} \psi R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t}\right)\right| \leqq C\|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}$$

where $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$.

(ii) *For all* $\psi \in L^2((0,T); W^{2,2}(\mathbb{T}^2))$*, there exists a constant* C_T *independent of* $\varepsilon \in (0,1]$ *such that:*

$$\left|\int_{\mathbb{T}^2\times(0,T)} \psi \left(\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right)\right| \leqq C_T\|\psi\|_{L^2((0,T);W^{2,2}(\mathbb{T}^2))}.$$

Proof of Lemma 5.8.

Proof of (i). The idea is somehow to bound $R_1^2 R_2^2\left(\frac{\partial\rho^\varepsilon}{\partial t}\right)$ using the available bounds on the right hand side of the equation (P_ε).

We will give a proof by duality. First of all, we subtract the two equations of system (P_ε) and we apply the Riesz transform $R_1^2 R_2^2$, to obtain that

$$R_1^2 R_2^2\left(\frac{\partial \rho^\varepsilon}{\partial t}\right) = -\overbrace{R_1^2 R_2^2\left((R_1^2 R_2^2\rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)}^{I_1} + \overbrace{\varepsilon R_1^2 R_2^2\left(\Delta\rho^\varepsilon\right)}^{I_2} \tag{5.21}$$

with $k^\varepsilon = \rho^{+,\varepsilon} + \rho^{-,\varepsilon}$. In what follows, we will prove that for a function $\psi \in L^2((0,T); W^{1,2}(\mathbb{T}^2))$, we can bound $J_i = \int_{\mathbb{T}^2\times(0,T)} \psi I_i$ for $i = 1, 2$.

Estimate of J_2: To estimate J_2, we integrate by parts, to get:

$$J_2 = -\varepsilon \int_{\mathbb{T}^2\times(0,T)} \nabla(R_1^2 R_2^2\rho^\varepsilon)\cdot\nabla\psi.$$

We deduce that for all $\varepsilon \in (0,1]$:

$$\begin{aligned}|J_2| &\leqq \left\|R_1^2 R_2^2\rho^\varepsilon\right\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\\ &\leqq C\|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))},\end{aligned} \tag{5.22}$$

where we have used Corollary 5.7 in the last line.

Estimate of J_1: To control J_1, we rewrite it under the following form:

$$\int_{\mathbb{T}^2\times(0,T)}\left[R_1^2 R_2^2\left((R_1^2 R_2^2\rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)\right]\psi = \int_{\mathbb{T}^2\times(0,T)}\left((R_1^2 R_2^2\rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)(R_1^2 R_2^2\psi).$$

We use the fact that

(i) $(R_1^2 R_2^2\rho^\varepsilon)$ is bounded in $L^2((0,T); W^{1,2}(\mathbb{T}^2))$ uniformly in ε (by Corollary 5.7),

(ii) $\frac{\partial k^\varepsilon}{\partial x_1}$ is bounded in $L^\infty((0,T); L\log L(\mathbb{T}^2))$, uniformly in ε (by Lemma 5.4).

We deduce from this and from Proposition 3.4, (with $f = R_1^2 R_2^2 \rho^\varepsilon$ and $g = \frac{\partial k^\varepsilon}{\partial x_1}$) the following estimate:

$$\begin{aligned}
&\left\| (R_1^2 R_2^2 \rho^\varepsilon) \frac{\partial k^\varepsilon}{\partial x_1} \right\|_{L^2((0,T); L \log^{\frac{1}{2}} L(\mathbb{T}^2))} \\
&\leqq C \| R_1^2 R_2^2 \rho^\varepsilon \|_{L^2((0,T); W^{1,2}(\mathbb{T}^2))} \left\| \frac{\partial k^\varepsilon}{\partial x_1} \right\|_{L^\infty((0,T); L \log L(\mathbb{T}^2))} \\
&\leqq C \left\| \frac{\partial k^\varepsilon}{\partial x_1} \right\|_{L^\infty((0,T); L \log L(\mathbb{T}^2))} \leqq C.
\end{aligned}$$

We use Lemma 3.2(i), to deduce that

$$\begin{aligned}
|J_1| &\leqq \left| \int_{\mathbb{T}^2 \times (0,T)} \left((R_1^2 R_2^2 \rho^\varepsilon) \frac{\partial k^\varepsilon}{\partial x_1} \right) (R_1^2 R_2^2 \psi) \right| \\
&\leqq \left\| (R_1^2 R_2^2 \rho^\varepsilon) \frac{\partial k^\varepsilon}{\partial x_1} \right\|_{L^2((0,T); L \log^{\frac{1}{2}} L(\mathbb{T}^2))} \left\| R_1^2 R_2^2 \psi \right\|_{L^2((0,T); EXP_2(\mathbb{T}^2))} \\
&\leqq C \left\| R_1^2 R_2^2 \psi \right\|_{L^2((0,T); W^{1,2}(\mathbb{T}^2))} \leqq C \|\psi\|_{L^2((0,T); W^{1,2}(\mathbb{T}^2))} \qquad (5.23)
\end{aligned}$$

where we have used the Trudinger inequality (see Lemma 3.3) in the third line and the fact that Riesz transforms are continuous from L^2 onto itself in the last line (see Lemma 5.1(i)).

Finally, collecting (5.23) and (5.22) together with (5.21) and the definitions of J_i, for $i = 1, 2$, we get that there exists a constant C independent of ε such that

$$\left| \int_{\mathbb{T}^2 \times (0,T)} \psi R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t} \right) \right| \leqq C \|\psi\|_{L^2((0,T); W^{1,2}(\mathbb{T}^2))}.$$

Proof of (ii). The proof of (ii) is similar to that of (i). The only difference is that we integrate by parts the viscosity term twice and use the estimate of Lemma 5.6. □

Remark 5.9. ($W^{-1,2}$ and $W^{-2,2}$ estimate) Let $W^{-1,2}(\mathbb{T}^2)$ be the dual space of $W^{1,2}(\mathbb{T}^2)$. By point (i) of the previous lemma, we deduce that there exists a constant C independent of ε, such that

$$\left\| R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t} \right) \right\|_{L^2((0,T); W^{-1,2}(\mathbb{T}^2))} \leqq C.$$

However, the point (ii) controls the time derivative of the solution in $L^2((0,T); W^{-2,2}(\mathbb{T}^2))$, where $W^{-2,2}(\mathbb{T}^2)$ is the dual space of $W^{2,2}(\mathbb{T}^2)$. This control will allow us later to recover the initial conditions in the limit as ε goes to zero.

Theorem 5.10. (Global existence) *For all $T > 0$, $\varepsilon \in (0, 1]$ and for all initial data $\rho_0^\pm \in L^2_{\text{loc}}(\mathbb{R}^2)$ satisfying $(H1)$, $(H2)$, $(H3)$ and $(H4)$, the system (P_ε) and (IC_ε) admits a solution $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0, T))$. Moreover, $\rho^{\pm,\varepsilon}(\cdot, t)$ satisfies $(H1)_\varepsilon$, $(H2)$ and $(H3)$ for all $t \in (0, T)$ and the estimates given in Lemmas 5.4, 5.6, 5.8 and Corollary 5.7.*

Before going into the proof, we need the following lemma.

Lemma 5.11. ($W^{1,\frac{3}{2}}$ estimate) *For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$ and $(H2)$, if $\rho^{\pm,\varepsilon,\text{per}} \in C^\infty(\mathbb{T}^2 \times [0,T))$, are solutions of the integral problem $(P_\varepsilon^{\text{per}})$, then there exists a constant $C = C(\varepsilon, L)$ such that*

$$\|\rho^{\pm,\varepsilon,\text{per}}\|_{L^\infty((0,T);W^{1,\frac{3}{2}}(\mathbb{T}^2))} \leqq B_0^\pm + CT^{\frac{1}{24}}\|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))} \times \left(\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial x_1}\right\|_{L^\infty((0,T);L^1(\mathbb{T}^2))} + 1\right), \quad (5.24)$$

where $B_0^\pm = \|\rho_0^{\pm,\varepsilon,\text{per}}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}$.

Proof of Lemma 5.11. We proceed as in Step 2 of the proof of Theorem 4.2. Here we apply Lemma 4.3(i) with $r = 4$, $q = \frac{24}{5}$, $p = 1$ to estimate the first term of the right hand side of (4.11), and Lemma 4.3(ii) with $r = \frac{3}{2}$, $q = 8$, $p = 1$ to estimate the second term. That leads

$$\|B(u,v)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leqq CT^{\frac{1}{24}}\|R_1^2R_2^2u\|_{L^\infty((0,T);(L^8(\mathbb{T}^2))^2)} \times \left\|\frac{\partial v}{\partial x_1}\right\|_{L^\infty((0,T);(L^1(\mathbb{T}^2))^2)}. \quad (5.25)$$

Similarly, we show that

$$\|A(u)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leqq CT^{\frac{1}{24}}\|R_1^2R_2^2u\|_{L^\infty((0,T);(L^8(\mathbb{T}^2))^2)}. \quad (5.26)$$

By using (4.14), (5.25) and (5.26) with $u = v = \rho_v^\varepsilon$ and equation $(P_\varepsilon^{\text{per}})$, we get the result. □

Proof of Theorem 5.10. We argue by contradiction. Suppose that there exists a maximum time $T_{\max}$ such that we have the existence of solutions of (P_ε) and (IC_ε) in $C^\infty(\mathbb{R}^2 \times [0, T_{\max}))$.

For $\delta > 0$, we consider the system (P_ε) with the new initial data

$$\rho_{\delta,\max}^{\pm,\varepsilon} = \rho^{\pm,\varepsilon}(x, T_{\max} - \delta).$$

Applying Theorem 4.2, we deduce the existence of time $T^\star_{\delta,\max}$ such that the system (P_ε) and (IC_ε) admits solutions defined until the time

$$T_0 = (T_{\max} - \delta) + T^\star_{\delta,\max}.$$

Then the contradiction $T_0 > T_{\max}$ follows from the fact that we can show that

$$\exists\, T_1 > 0 \quad \text{such that} \quad T^\star_{\delta,\max} \geqq T_1 > 0 \quad \text{for small enough} \quad \delta > 0. \quad (5.27)$$

To prove (5.27), we recall that for L_ε fixed, the time $T^\star_{\delta,\max}$ only depends on a bound on $\|\rho_{\delta,\max}^{\pm,\varepsilon,\text{per}}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}$ where $\rho_{\delta,\max}^{\pm,\varepsilon,\text{per}} = \rho_{\delta,\max}^{\pm,\varepsilon} - Lx_1$. This term is simply bounded by the right hand side of (5.24) with $T = T_{\max}$. The conclusion follows from the fact that there exists a constant $C > 0$ independent on T such that

$$\left\|\frac{\partial\rho^{\pm,\varepsilon}}{\partial x_1}\right\|_{L^\infty((0,T);L^1(\mathbb{T}^2))} + \|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))} \leqq C$$

where we have used (5.17) to bound the first term, and Lemmata 5.2 and 5.1(i)–(v) for the second term. □

6. Existence of solutions for the system (P) and (IC)

In this section, we will prove that the system (P) and (IC) admits solutions $\rho^{\pm}$ in the distributional sense. They are the limits when $\varepsilon \to 0$ of the solution $\rho^{\pm,\varepsilon}$ given in Theorem 5.10. To do this, we will justify the passage to the limit as ε tends to 0 in the system (P_ε) and (IC_ε), using some compactness arguments.

6.1. Preliminary results

Before proving the main theorem, let us recall some well known results.

Lemma 6.1. (Trudinger compact embedding) *The following injection (see* Trudinger [37]*):*

$$W^{1,2}(\mathbb{T}^2) \hookrightarrow EXP_\beta(\mathbb{T}^2),$$

is compact, for all $1 \leqq \beta < 2$.

For the proof of this lemma see also Adams [1, Th 8.32].

Lemma 6.2. (Simon's Lemma) *Let* X, B, Y *three Banach spaces, where* $X \hookrightarrow B$ *with compact embedding and* $B \hookrightarrow Y$ *with continuous embedding. If* $(\rho^n)_n$ *is a sequence such that*

$$\|\rho^n\|_{L^q((0,T);B)} + \|\rho^n\|_{L^1((0,T);X)} + \left\|\frac{\partial \rho^n}{\partial t}\right\|_{L^1((0,T);Y)} \leqq C,$$

where $q > 1$ *and* C *is a constant independent of* n*, then* $(\rho^n)_n$ *is relatively compact in* $L^p((0,T);B)$ *for all* $1 \leqq p < q$.

For the proof, see Simon [35, Th 6, Page 86].

In order to show the global existence of system (P) in Section 6.2, we will apply this lemma in the particular cases where $B = EXP_\beta(\mathbb{T}^2)$, $X = W^{1,2}(\mathbb{T}^2)$ and $Y = W^{-1,2}(\mathbb{T}^2)$, for $1 < \beta < 2$.

Lemma 6.3. (Weak star topology in $L\log L$) *Let* $E_{exp}(\mathbb{T}^2)$ *be the closure in* $EXP(\mathbb{T}^2)$ *of the space of functions bounded on* $\mathbb{T}^2$*. Then* $E_{exp}(\mathbb{T}^2)$ *is a separable Banach space which verifies*

(i) $L\log L(\mathbb{T}^2)$ *is the dual space of* $E_{exp}(\mathbb{T}^2)$.
(ii) $EXP_\beta(\mathbb{T}^2) \hookrightarrow E_{exp}(\mathbb{T}^2) \hookrightarrow EXP(\mathbb{T}^2)$ *for all* $\beta > 1$.

For the proof, see Adams [1, Th 8.16, 8.18, 8.20].

6.2. Proof of Theorem 1.4

Let us fix any $T > 0$. For any $\varepsilon \in (0, 1]$, we are considering the solution $\rho^{\pm,\varepsilon}$ of (P_ε) and (IC_ε) given in Theorem 5.10 on $\mathbb{R}^2 \times (0, T)$. First, by Lemma 5.6 we know that, the periodic part of the solutions, denoted by $\rho^{\pm,\varepsilon,\text{per}}$ are ε-uniformly bounded in $L^2(\mathbb{T}^2 \times (0, T))$. Hence, as ε goes to zero, we can extract a subsequence still denoted by $\rho^{\pm,\varepsilon,\text{per}}$, that converges weakly in $L^2(\mathbb{T}^2 \times (0, T))$ to some limit $\rho^{\pm,\text{per}}$. Then we want to prove that $\rho^{\pm} = \rho^{\pm,\text{per}} + Lx_1$ are solutions of the system (P) and (IC). Indeed, since the passage to the limit in the linear term is trivial in $\mathcal{D}'(\mathbb{T}^2 \times (0, T))$, it suffices to pass to the limit in the non-linear term

$$(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}. \tag{6.28}$$

Step 1. Compactness of $R_1^2 R_2^2 \rho^\varepsilon$

Now notice that:

- From Corollary 5.7 we know that the term $R_1^2 R_2^2 \rho^\varepsilon$ is ε-uniformly bounded in $L^2((0, T); W^{1,2}(\mathbb{T}^2))$. Then it is in particular ε-uniformly bounded in $L^1((0, T); W^{1,2}(\mathbb{T}^2))$.
- From the previous point and Lemma 6.1, we know that $R_1^2 R_2^2 \rho^\varepsilon$ is also ε-uniformly bounded in $L^2((0, T); EXP_\beta(\mathbb{T}^2))$ for all $1 \leqq \beta < 2$.
- From Lemma 5.8, the term $R_1^2 R_2^2(\frac{\partial \rho^\varepsilon}{\partial t})$ is ε-uniformly bounded in $L^2((0, T); W^{-1,2}(\mathbb{T}^2))$ and then in $L^1((0, T); W^{-1,2}(\mathbb{T}^2))$.

Collecting this, we get that there exists a constant C independent on ε such that $\bar{\rho}^\varepsilon = R_1^2 R_2^2 \rho^\varepsilon$ satisfies for some $1 < \beta < 2$

$$\left\|\bar{\rho}^\varepsilon\right\|_{L^2((0,T);EXP_\beta(\mathbb{T}^2))} + \left\|\bar{\rho}^\varepsilon\right\|_{L^1((0,T);W^{1,2}(\mathbb{T}^2))} + \left\|\frac{\partial \bar{\rho}^\varepsilon}{\partial t}\right\|_{L^1((0,T);W^{-1,2}(\mathbb{T}^2))} \leqq C.$$

Then Lemma 6.2 joint to Lemma 6.1, with $B = EXP_\beta(\mathbb{T}^2)$, $X = W^{1,2}(\mathbb{T}^2)$ and $Y = W^{-1,2}(\mathbb{T}^2)$, shows the relative compactness of $R_1^2 R_2^2 \rho^\varepsilon$ in $L^1((0, T); EXP_\beta(\mathbb{T}^2))$, and then using Lemma 6.3, we deduce the compactness in $L^1((0, T); E_{\exp}(\mathbb{T}^2))$.

Step 2. Weak-$\star$ convergence of $\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}$

By Lemma 5.4, we have that $\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}$ is ε-uniformly bounded in $L^\infty((0, T); L\log L(\mathbb{T}^2))$ which is the dual of $L^1((0, T); E_{\exp}(\mathbb{T}^2))$ by Lemma 6.3. Then, this term converges weakly-$\star$ in $L^\infty((0, T); L\log L(\mathbb{T}^2))$ toward $\frac{\partial \rho^\pm}{\partial x_1}$. That enables us to pass to the limit in the bilinear term (6.28) in the sense

$$L^1((0, T); E_{\exp}(\mathbb{T}^2)) - strong \times L^\infty((0, T); L\log L(\mathbb{T}^2)) - weak - \star$$

which shows that

$$(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} \to (R_1^2 R_2^2 \rho)\frac{\partial \rho^\pm}{\partial x_1} \quad \text{in } \mathcal{D}'(\mathbb{T}^2 \times (0, T)).$$

In what precedes, we have shown that $\rho^\pm$ are solutions of the system (P).

Step 3. Conclusion

Passing to the limit in the estimates of Lemma 5.4, 5.6, 5.8 and Corollary 5.7, we get in particular by Lemma 5.3, the entropy estimates (1.1) and $(E1)$, $(E2)$, $(E4)$, $(E5)$. At this stage we remark that, by Proposition 3.4 that

$$\frac{\partial \rho^{\pm}}{\partial t} = (R_1^2 R_2^2 \rho) \frac{\partial \rho^{\pm}}{\partial x_1} \in L^2((0,T); L \log^{\frac{1}{2}} L(\mathbb{T}^2)),$$

and then $\rho^{\pm,\mathrm{per}} \in C([0,T); L \log^{\frac{1}{2}} L(\mathbb{T}^2))$, which proves $(E3)$.

Since the functions $\rho^{\pm,\varepsilon}(\cdot,t)$ satisfy $(H1)_\varepsilon$, $(H2)$, $(H3)$, $(H4)$ (see Theorem 5.10) by passing in the limit $\varepsilon \to 0$, we can see that the limit functions $\rho^{\pm}(\cdot,t)$ satisfy the corresponding assumptions $(H1)$, $(H2)$, $(H3)$, $(H4)$.

It remains to prove that $\rho^{\pm}$ satisfies the initial conditions (IC). Indeed, from the estimates on $\rho^{\pm,\varepsilon,\mathrm{per}}$ given by Lemma 5.6 and $\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}$ given by Lemma 5.8(ii), we can prove easily, that

$$\|\rho^{\pm,\varepsilon,\mathrm{per}}(t) - \rho_0^{\pm,\varepsilon,\mathrm{per}}\|_{W^{-2,2}(\mathbb{T}^2)} \leqq C_T t^{\frac{1}{2}}.$$

where C_T is a constant independent of ε. Hence we can pass to the limit $\varepsilon \to 0$, which implies in particular that $\rho^{\pm,\mathrm{per}}(\cdot,0) = \rho_0^{\pm,\mathrm{per}}$ in $\mathcal{D}'(\mathbb{R}^2)$. □

Remark 6.4. In our proof, we have indirectly used a kind of compensated compactness technic for Hardy spaces. Nevertheless in our case, we do not have enough regularity to apply directly this technic.

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