An accurate $\mathbf{H}(\mathrm{div})$ flux reconstruction for discontinuous Galerkin approximations of elliptic problems

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Abstract

We introduce a new $\mathbf{H}(\mathrm{div})$ flux reconstruction for discontinuous Galerkin approximations of elliptic problems. The reconstructed flux is computed elementwise and its divergence equals the L^2-orthogonal projection of the source term onto the discrete space. Moreover, the energy-norm of the error in the flux is bounded by the discrete energy-norm of the error in the primal variable, independently of diffusion heterogeneities. *To cite this article: A. Ern, S. Nicaise, and M. Vohralík, C. R. Acad. Sci. Paris, Ser. I (2007).*

Résumé

Une reconstruction précise du flux dans $\mathbf{H}(\mathrm{div})$ pour des approximations par la méthode de Galerkine discontinue de problèmes elliptiques. On introduit une nouvelle reconstruction dans $\mathbf{H}(\mathrm{div})$ du flux pour des approximations par la méthode de Galerkine discontinue de problèmes elliptiques. Le flux reconstruit est calculé localement sur chaque maille et sa divergence est égale à la projection L^2-orthogonale du terme source sur l'espace discret. De plus, l'erreur en norme d'énergie sur le flux est bornée par l'erreur en norme d'énergie discrète sur la variable primale, indépendamment des hétérogénéités dans la diffusion. *Pour citer cet article : A. Ern, S. Nicaise, and M. Vohralík, C. R. Acad. Sci. Paris, Ser. I (2007).*

1. Introduction

The approximation of elliptic problems by the discontinuous Galerkin (dG) method has been introduced in the late 1970s and has been, more recently, the subject of extensive research; see, e.g., [2,6] and references

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therein. Advantages of dG methods include flexibility in the design of approximation spaces (allowing for nonmatching meshes and variable polynomial degree), compact discretization stencils amenable to parallelization, and, in the spirit of finite volumes, a local (elementwise) formulation in terms of numerical fluxes. An issue that still deserves further investigation is whether an accurate $\mathbf{H}(\mathrm{div})$ flux reconstruction can be performed using the discrete solution provided by the dG method. This type of postprocessing is important at least in two instances. Firstly, this flux can serve as input data in further calculations; this is for instance the case when solving contaminant transport problems in porous media where the flow velocity must be determined first by an approximation of Darcy's equation. Secondly, this flux can be used in a posteriori error estimates based on equilibrated fluxes. For conforming approximations, this type of estimates are explored, e.g., in [1,10] and references therein. Recent work where an $\mathbf{H}(\mathrm{div})$ flux reconstruction is used for a posteriori dG error estimates includes [5,8].

An $\mathbf{H}(\mathrm{div})$ flux reconstruction for the so-called nonsymmetric Interior Penalty (IP) Galerkin method has been proposed and analyzed in [3]. The idea therein is to reconstruct the flux in the Brezzi–Douglas–Marini finite element space and to use the mean values of the gradient of the dG solution at interfaces to specify the degrees of freedom. The reconstructed flux is proven to be accurate in the L^2-norm. In the present work, we propose a more accurate reconstruction, namely in the Raviart–Thomas finite element space, and we address a wider class of IP-like methods. The key improvement achieved when working with Raviart–Thomas finite element spaces is that the divergence of the reconstructed flux can be proven to be optimal, i.e., it is equal to the L^2-orthogonal projection of the data onto the dG approximation space. This is a key property when the reconstructed flux is further used as an advective flow velocity or as a tool for a posteriori error estimation. Moreover, as in [3], the reconstruction procedure can be performed elementwise; thus, it does not demand significant computational effort.

This Note is organized as follows. §2 introduces the model problem and its dG approximation. In particular, we treat recently introduced Weighted IP methods to cope satisfactorily with heterogeneities and anisotropies in the diffusion tensor [9]. §3 presents and analyzes the $\mathbf{H}(\mathrm{div})$ flux reconstruction. The main results are Theorems 3.1 and 3.2.

2. The model problem and its dG approximation

Consider the model elliptic problem

$$-\nabla\cdot(\mathbf{K}\nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

with (for simplicity) homogeneous Dirichlet boundary conditions. Here, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain, $\mathbf{K} \in [L^\infty(\Omega)]^{d,d}$ is the diffusion tensor, and $f \in L^2(\Omega)$ is the source term. The diffusion tensor is assumed to be symmetric and uniformly positive definite in Ω.

Let $\{\mathcal{T}_h\}_{h>0}$ be a conforming, shape-regular family of affine meshes of Ω consisting of simplices. The diffusion tensor is assumed to be piecewise constant on $\mathcal{T}_h$. On an element $T \in \mathcal{T}_h$, the maximal and minimal eigenvalues of $\mathbf{K}$ are denoted by $\Lambda_{\mathbf{K},T}$ and $\lambda_{\mathbf{K},T}$, respectively. For any integer $k \geq 0$, consider the usual dG approximation space $V_h^k = \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_k\}$, where $\mathbb{P}_k$ is the set of polynomials of total degree less than or equal to k. The L^2-scalar product and its associated norm on a subset $R \subset \Omega$ are indicated by the subscript $0, R$. The L^2-orthogonal projection from $L^2(\Omega)$ onto V_h^k is denoted by Π_h^k. Interior and boundary faces are collected in the sets $\mathcal{F}_h^{\mathrm{i}}$ and $\mathcal{F}_h^{\partial}$, respectively, and we set $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\partial}$. For $F \in \mathcal{F}_h^{\mathrm{i}}$, there are T^- and T^+ in $\mathcal{T}_h$ such that $F = T^- \cap T^+$. Let $\mathbf{n}_F$ be the unit normal vector to F pointing from T^- towards T^+. For a double-valued function v on F, its jump is defined as $[\![v]\!] = v^- - v^+$ with $v^\pm = v|_{T^\pm}$. Choosing non-negative weights $\omega_{T^-,F}$ and $\omega_{T^+,F}$ such that $\omega_{T^-,F} + \omega_{T^+,F} = 1$, the weighted average of v on F is $\{v\}_\omega = \omega_{T^-,F} v^- + \omega_{T^+,F} v^+$. The usual average consists of taking $\omega_{T^-,F} = \omega_{T^+,F} = \frac{1}{2}$. When the diffusion tensor is strongly heterogeneous, it is better [4,9] to consider diffusion-dependent weights defined as $\omega_{T^+,F} = (\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-})^{-1}\delta_{\mathbf{K},F-}$ and $\omega_{T^-,F} = (\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-})^{-1}\delta_{\mathbf{K},F+}$ where $\delta_{\mathbf{K},F\pm} = \mathbf{n}_F(\mathbf{K}|_{T^\pm})\mathbf{n}_F$. On boundary faces, we set $[\![v]\!] = v$, $\{v\}_\omega = v$, $\omega_{T,F} = 1$ (where T is the mesh element of which F is a face), and $\delta_{\mathbf{K},F} = \mathbf{n}_F(\mathbf{K}|_T)\mathbf{n}_F$ where $\mathbf{n}_F$ coincides with the outward unit normal of Ω.

Let $k \geq 1$. The dG approximation consists of finding $u_h \in V_h^k$ such that $B_h(u_h, v_h) = (f, v_h)_{0,\Omega}$ for all $v_h \in V_h^k$ with the bilinear form

$$B_h(v,w) = \sum_{T\in\mathcal{T}_h} (\mathbf{K}\nabla v, \nabla w)_{0,T} + \sum_{F\in\mathcal{F}_h} \alpha h_F^{-1}\gamma_{\mathbf{K},F}([\![v]\!], [\![w]\!])_{0,F} - \sum_{F\in\mathcal{F}_h} \big((\mathbf{n}_F^t\{\mathbf{K}\nabla v\}_\omega, [\![w]\!])_{0,F} + \theta(\mathbf{n}_F^t\{\mathbf{K}\nabla w\}_\omega, [\![v]\!])_{0,F}\big). \quad (3)$$

The penalty coefficient $\gamma_{\mathbf{K},F}$ is defined on interior faces as $\gamma_{\mathbf{K},F} = (\delta_{\mathbf{K},F+} + \delta_{\mathbf{K},F-})^{-1}\delta_{\mathbf{K},F+}\delta_{\mathbf{K},F-}$ (i.e., it depends on the diffusion tensor via the harmonic average of the normal diffusivity) and as $\gamma_{\mathbf{K},F} = \delta_{\mathbf{K},F}$ on boundary faces. Furthermore, h_F denotes the diameter of F, α is a positive parameter, and θ can take values in $\{-1, 0, +1\}$. As usual with IP-like methods, if $\theta \neq -1$, the parameter α must be chosen large enough to ensure that the bilinear form B_h is coercive. The threshold depends on the shape-regularity of the mesh family and the polynomial degree k, but not on the meshsize and the diffusion tensor if the penalty parameter is designed as above. An optimal (with respect to meshsize) a priori error estimate is proven in [9] in the discrete energy-norm

$$|\!|\!|v|\!|\!|_\Omega^2 = \sum_{T\in\mathcal{T}_h} |\!|\!|v|\!|\!|_T^2, \qquad |\!|\!|v|\!|\!|_T^2 = (\mathbf{K}\nabla v, \nabla v)_{0,T} + \sum_{F\subset\partial T} \alpha h_F^{-1}\gamma_{\mathbf{K},F}([\![v]\!], [\![v]\!])_{0,F}. \quad (4)$$

The estimate is robust with respect to diffusion heterogeneities and only depends on local diffusion anisotropies.

3. The accurate H(div) flux reconstruction

Consider the Raviart–Thomas spaces of vector functions $\mathbf{RT}_h^k = \{\mathbf{v}_h \in \mathbf{H}(\mathrm{div},\Omega)\,;\mathbf{v}_h|_T \in \mathbf{RT}_T^k\,;\forall T \in \mathcal{T}_h\}$ where $\mathbf{RT}_T^k = \mathbb{P}_k^d(T) + \mathbf{x}\mathbb{P}_k(T)$. The reconstructed flux introduced in this Note, $\mathbf{t}_h \in \mathbf{RT}_h^k$, is specified through its natural degrees of freedom, namely for all $F \in \mathcal{F}_h$ and $q_h \in \mathbb{P}_k(F)$,

$$(\mathbf{t}_h\cdot\mathbf{n}_F, q_h)_{0,F} = (-\mathbf{n}_F^t\{\mathbf{K}\nabla u_h\}_\omega + \alpha h_F^{-1}\gamma_{\mathbf{K},F}[\![u_h]\!], q_h)_{0,F}, \quad (5)$$

and for all $T \in \mathcal{T}_h$ and $\mathbf{r}_h \in \mathbb{P}_{k-1}^d(T)$,

$$(\mathbf{t}_h, \mathbf{r}_h)_{0,T} = -(\mathbf{K}\nabla u_h, \mathbf{r}_h)_{0,T} + \theta \sum_{F\subset\partial T} \omega_{T,F}(\mathbf{n}_F^t\mathbf{K}\mathbf{r}_h, [\![u_h]\!])_{0,F}. \quad (6)$$

Theorem 3.1 *There holds* $\nabla\cdot\mathbf{t}_h = \Pi_h^k f$.

Proof. For all $T \in \mathcal{T}_h$ and $\xi \in \mathbb{P}_k(T)$, $(f,\xi)_{0,T} = B_h(u_h, \xi \times 1_T) = -(\mathbf{t}_h, \nabla\xi)_{0,T} + \sum_{F\subset\partial T}(\mathbf{t}_h\cdot\mathbf{n}_T, \xi)_{0,F} = (\nabla\cdot\mathbf{t}_h, \xi)_{0,T}$ owing to (3), (5), and (6). $\square$

The following result estimates the energy-norm of the error in the diffusive flux in terms of the discrete energy-norm of the primal error $(u - u_h)$. In the sequel, $A \lesssim B$ denotes the inequality $A \leq cB$ with c independent of meshsize and of $\mathbf{K}$.

Theorem 3.2 *There holds* $\|\mathbf{K}^{\frac{1}{2}}\nabla u + \mathbf{K}^{-\frac{1}{2}}\mathbf{t}_h\|_{0,\Omega} \lesssim \max_{T\in\mathcal{T}_h}(\Lambda_{\mathbf{K},T}/\lambda_{\mathbf{K},T})|\!|\!| u - u_h |\!|\!|_\Omega$.

Proof. Clearly, it suffices to estimate $\|\mathbf{K}^{\frac{1}{2}}\nabla u_h + \mathbf{K}^{-\frac{1}{2}}\mathbf{t}_h\|_{0,\Omega}$. Using scaling arguments and the Piola transformation, one first shows that for all $T \in \mathcal{T}_h$ and $\mathbf{v}_h \in \mathbf{RT}_T^k$,

$$\|\mathbf{v}_h\|_{0,T}^2 \lesssim h_T \sum_{F\subset\partial T} \|\mathbf{v}_h\cdot\mathbf{n}_F\|_{0,F}^2 + \|\Pi_h^{k-1}\mathbf{v}_h\|_{0,T}^2. \tag{7}$$

We apply this estimate to $\mathbf{v}_h = (\mathbf{K}\nabla u_h + \mathbf{t}_h)|_T \in \mathbf{RT}_T^k$. Owing to (5)–(6) and using inverse inequalities, $\mathbf{v}_h\cdot\mathbf{n}_F = \bar{\omega}_{T,F}\mathbf{n}_F^t[\![\mathbf{K}\nabla u_h]\!] + \alpha h_F^{-1}\gamma_{\mathbf{K},F}[\![u_h]\!]$ and $\|\Pi_h^{k-1}\mathbf{v}_h\|_{0,T} \lesssim |\theta| h_T^{-1/2}\Lambda_{\mathbf{K},T}\sum_{F\subset\partial T}\omega_{T,F}\|[\![u_h]\!]\|_{0,F}$ where $\bar{\omega}_{T,F} := 1 - \omega_{T,F}$, so that

$$\lambda_{\mathbf{K},T}^{-1}\|\mathbf{v}_h\|_{0,T}^2 \lesssim \sum_{F\subset\partial T}\lambda_{\mathbf{K},T}^{-1}h_T\bar{\omega}_{T,F}^2\|\mathbf{n}_F^t[\![\mathbf{K}\nabla u_h]\!]\|_{0,F}^2 + \sum_{F\subset\partial T}\lambda_{\mathbf{K},T}^{-1}h_F^{-1}(\gamma_{\mathbf{K},F}^2 + \Lambda_{\mathbf{K},T}^2\omega_{T,F}^2)\|[\![u_h]\!]\|_{0,F}^2. \tag{8}$$

Let X and Y denote the two terms in the right-hand side. The first term is bounded using bubble functions, similarly to the a posteriori analysis of conforming finite elements; see [7] for details. The result is $X \lesssim \max_{T'\in\Delta_T}(\Lambda_{\mathbf{K},T'}/\lambda_{\mathbf{K},T'})^2\sum_{T'\in\Delta_T}|\!|\!| u - u_h|\!|\!|_{T'}^2$ where Δ_T denotes the set of mesh elements sharing at least a face with T. The second term is bounded observing that $\lambda_{\mathbf{K},T}^{-1}\gamma_{\mathbf{K},F}^2 \le (\Lambda_{\mathbf{K},T}/\lambda_{\mathbf{K},T})\gamma_{\mathbf{K},F}$ and $\lambda_{\mathbf{K},T}^{-1}\Lambda_{\mathbf{K},T}^2\omega_{T,F}^2 \le (\Lambda_{\mathbf{K},T}/\lambda_{\mathbf{K},T})^2\gamma_{\mathbf{K},F}$, leading to $Y \lesssim (\Lambda_{\mathbf{K},T}/\lambda_{\mathbf{K},T})^2\sum_{T'\in\Delta_T}|\!|\!| u - u_h|\!|\!|_{T'}^2$. Finally, summing over the mesh elements yields the desired result. $\square$

Because of the estimate for $|\!|\!| u - u_h|\!|\!|_\Omega$ established in [9], Theorem 3.2 implies the same bound for the reconstructed flux $\mathbf{t}_h$. A further consequence is that the computable quantity $\|\mathbf{K}^{\frac{1}{2}}\nabla u_h + \mathbf{K}^{-\frac{1}{2}}\mathbf{t}_h\|_{0,\Omega}$ is optimal for the purpose of a posteriori error estimation; see [5,8] for details including numerical experiments.

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