Polynomial-degree-robust liftings for potentials and fluxes

Alexandre Ern and Martin Vohralík

Université Paris-Est, CERMICS (ENPC)
and SERENA Project-Team, INRIA, France

RAFEM, Hong-Kong, March 2017

Outline

1. Motivation: a posteriori error estimates
2. Main results on H^1 and $\mathbf{H}(\mathrm{div})$ polynomial liftings
3. Main ingredients of proofs
4. Numerical results (with V. Dolejší)
5. Global $\mathbf{H}(\mathrm{div})$ polynomial liftings (with I. Smears)

A posteriori error estimates

- Model problem in $\Omega \subset \mathbb{R}^d$ with data $f \in L^2(\Omega)$

$$u \in H_0^1(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega,\ \forall v \in H_0^1(\Omega)$$

- H_0^1-conforming FEM solution u_h on a simplicial mesh $\mathcal{T}_h$
- **Fully computable** error upper bound

$$\|\nabla(u - u_h)\|_\Omega^2 \le \sum_{K \in \mathcal{T}_h} \eta_K^2$$

- Local error indicators η_K represent **local error lower bounds** up to data oscillation

$$\eta_K \le C_{\mathrm{eff}} \|\nabla(u - u_h)\|_{\omega_K} + \mathrm{osc}(f, \omega_K)$$

- Pioneered by [Babuška, Rheinbolt 78]; recent textbook [Verfürth 13]
  - classical technique to compute the η_K's is residual-based
  - drawback: upper bound features an undetermined constant

Equilibrated flux reconstruction

- Exact flux $\boldsymbol{\sigma} := -\nabla u \in \mathbf{H}(\mathrm{div}, \Omega)$ s.t. $\nabla{\cdot}\boldsymbol{\sigma} = f$ (equilibrium)
- An **equilibrated flux reconstruction** is s.t.

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega) \qquad (\nabla{\cdot}\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K, \ \forall K \in \mathcal{T}_h$$

- Setting $\eta_{\mathrm{F},K} := \|\nabla u_h + \boldsymbol{\sigma}_h\|_K$ and $\eta_{\mathrm{osc},K} := \frac{h_K}{\pi}\|f - \nabla{\cdot}\boldsymbol{\sigma}_h\|_K$,

$$\|\nabla(u - u_h)\|_\Omega^2 \leq \sum_{K\in\mathcal{T}_h} \Big(\underbrace{\eta_{\mathrm{F},K} + \eta_{\mathrm{osc},K}}_{=\eta_K}\Big)^2$$

- Literature: hypercircle method [Prager, Synge 47]; computational mechanics [Ladevèze et al. 75]; textbooks [Ainsworth, Oden 00; Repin 08]

A simple proof

- Residual $\rho(u_h) \in H^{-1}(\Omega)$ s.t.

$$\langle \rho(u_h), \varphi \rangle_\Omega := (f, \varphi)_\Omega - (\nabla u_h, \nabla \varphi)_\Omega, \ \forall \varphi \in H_0^1(\Omega)$$
$$\|\nabla(u - u_h)\|_\Omega = \|\rho(u_h)\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega);\, \|\nabla \varphi\|_\Omega = 1} \langle \rho(u_h), \varphi \rangle_\Omega$$

- Introduce $\mathbf{H}(\mathrm{div}, \Omega)$-flux and use Green's formula

$$\langle \rho(u_h), \varphi \rangle_\Omega = (f - \nabla \cdot \boldsymbol{\sigma}_h, \varphi)_\Omega - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla \varphi)_\Omega$$

- Cauchy–Schwarz and Poincaré–Steklov inequalities (equilibration)

$$|(f - \nabla \cdot \boldsymbol{\sigma}_h, \varphi)_\Omega| = \sum_{K \in \mathcal{T}_h} |(f - \nabla \cdot \boldsymbol{\sigma}_h, \varphi - \overline{\varphi}_K)_K| \le \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K \|\nabla \varphi\|_K$$
$$|(\nabla u_h + \boldsymbol{\sigma}_h, \nabla \varphi)_\Omega| \le \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \|\nabla \varphi\|_K$$

Poincaré (1894) [eigenvalue pb], Steklov (1897) [$d = 1$], Payne, Weinberger (60) [$d = 2$], Bebendorf (03) [$d \ge 3$]

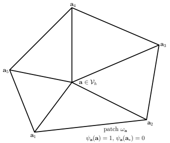
How to build $\boldsymbol{\sigma}_h$?

- Global flux equilibration (... expensive)

$$\boldsymbol{\sigma}_h := \underset{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{(\nabla\cdot\mathbf{v}_h)} f}{\arg\min} \|\nabla u_h + \mathbf{v}_h\|_\Omega$$

  for some global Raviart–Thomas spaces $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$

- Cheap local flux equilibration by working on **FE stars**
  - $\mathcal{T}_{\mathbf{a}} \subset \mathcal{T}_h$: patch of elements sharing vertex $\mathbf{a} \in \mathcal{V}_h$; domain $\omega_{\mathbf{a}}$



- Literature on a posteriori analysis on FE stars: [Babuška, Miller 87; Morin, Nochetto, Siebert 03; Cohen, DeVore, Nochetto 12; Veeser, Verfürth 12]

Local flux equilibration on FE stars

- Hat basis functions $\{\psi_{\mathbf{a}}\}_{\mathbf{a}\in\mathcal{V}_h}$ with local PU $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi_{\mathbf{a}}|_K = 1$
- We focus for simplicity on FE stars around **interior** vertices
  - flux equilibration for boundary vertices depends on the BC's for model problem, see [Dolejší, AE, MV, 16]
- Raviart–Thomas space $\mathbf{V}_h^{\mathbf{a}} \subset \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}})$ (Neumann BC's on $\partial\omega_{\mathbf{a}}$)

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \mathop{\arg\min}_{\mathbf{v}_h\in\mathbf{V}_h^{\mathbf{a}},\ \nabla\cdot\mathbf{v}_h = g_h^{\mathbf{a}}} \|\psi_{\mathbf{a}}\nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}$$

with data $g_h^{\mathbf{a}} := \Pi_{(\nabla\cdot\mathbf{V}_h^{\mathbf{a}})}(f\psi_{\mathbf{a}} - \nabla u_h\cdot\nabla\psi_{\mathbf{a}})$

- $\sum_{\mathbf{a}\in\mathcal{V}_K} g_h^{\mathbf{a}} = \Pi_{(\nabla\cdot\mathbf{V}_h^{\mathbf{a}})} f$ (PU), $(g_h^{\mathbf{a}}, 1)_{\omega_{\mathbf{a}}} = 0$ (Galerkin's orthogonality on hat basis functions)
- Literature [Destuynder, Métivet 99; Braess, Schöberl 08; MV 08; AE, MV 10]

p-robust local efficiency

- Let p be the polynomial degree used to compute u_h
- Local flux equilibration using RT spaces of order p
- **Key result:** Local lower error bound ($\omega_K = \cup_{\mathbf{a}\in\mathcal{V}_K}\omega_{\mathbf{a}}$)

$$\eta_{\mathrm{F},K} = \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \le C_{\mathrm{eff}}\|\nabla(u - u_h)\|_{\omega_K} + \mathrm{osc}(f, \omega_K)$$

- C_{eff} depends on patch geometry (mesh regularity)
  - C_{eff} is p-robust for $d = 2$ [Braess, Pillwein, Schöberl 09]
  - C_{eff} is p-robust for $d = 3$ [AE, MV 16]
- ... in contrast to residual-based estimators [Melenk, Wohlmuth 01]

A two-step proof

- Two-step proof using ∞-dimensional, local problems
  - replaces classical bubble-function argument by Verfürth
  - see [AE, MV 15]
- We have $C_{\mathrm{eff}} = C_{\mathrm{st}}\, C_{\mathrm{PS}}$
  - C_{st} results from p-robust stability properties of RT spaces [BPS 09]
  - C_{PS} results from Poincaré–Steklov inequalities on FE stars [Carstensen, Funken 00], values estimated in [Veeser, Verfürth 12]

Step 1

- Set $\boldsymbol{\tau}_h^{\mathbf{a}} := \psi_{\mathbf{a}} \nabla u_h$ and $\mathbf{V}_h^{\mathbf{a}} \subset \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) = \mathbf{V}^{\mathbf{a}}$
- Local PU yields $\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq \sum_{\mathbf{a} \in \mathcal{V}_K} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$
- Recall that local flux equilibration delivers

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \underset{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla \cdot \mathbf{v}_h = g_h^{\mathbf{a}}}{\arg\min} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

- Local ∞-dimensional constrained min. pb. with polynomial data

$$\boldsymbol{\sigma}^{\mathbf{a}} := \underset{\mathbf{v} \in \mathbf{V}^{\mathbf{a}},\, \nabla \cdot \mathbf{v} = g_h^{\mathbf{a}}}{\arg\min} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}\|_{\omega_{\mathbf{a}}}$$

- **Thm.** The following (nontrivial) result holds:

$$\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{st}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$$

Note the (trivial) converse bound $\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$

Step 2

- Evaluate $\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ using equivalent problem in primal form
  - $r^{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}}) = \{v \in H^1(\omega_{\mathbf{a}}) \,|\, (v,1)_{\omega_{\mathbf{a}}} = 0\}$ s.t.

$$(\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} + (g_h^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} \quad \forall v \in H^1_*(\omega_{\mathbf{a}})$$

  - Equivalence of primal/dual energies

$$\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \min_{\mathbf{v} \in \mathbf{V}^{\mathbf{a}},\, \nabla\cdot\mathbf{v} = g_h^{\mathbf{a}}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}\|_{\omega_{\mathbf{a}}} = \|\nabla r^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$$

- Since
  - $(\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = (\nabla(u - u_h), \nabla(\psi_{\mathbf{a}} v))_{\omega_{\mathbf{a}}} + \mathrm{osc}(f, \omega_{\mathbf{a}})$
  - $\|\nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}} \le (1 + C_{\mathrm{PS},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{L^\infty(\omega_{\mathbf{a}})})\|\nabla v\|_{\omega_{\mathbf{a}}}$

  we can conclude that

$$\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{PS}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} + \mathrm{osc}(f, \omega_{\mathbf{a}})$$

  with $C_{\mathrm{PS}} = \max_{\mathbf{a} \in \mathcal{V}_h}(1 + C_{\mathrm{PS},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{L^\infty(\omega_{\mathbf{a}})})$

Nonconforming case ($u_h \notin H^1_0(\Omega)$)

- Error measure w.r.t. broken gradient $\nabla_{\mathcal{T}}$ of discrete solution
- Additional **nonconformity estimator** in error upper bound
- H^1_0-potential reconstruction $s_h \in H^1_0(\Omega)$
- Setting $\eta_{\mathrm{NC},K} := \|\nabla_{\mathcal{T}}(u_h - s_h)\|$,

$$\|\nabla_{\mathcal{T}}(u - u_h)\|^2 \le \sum_{K\in\mathcal{T}_h} \left(\eta_{\mathrm{F},K} + \eta_{\mathrm{osc},K}\right)^2 + \sum_{K\in\mathcal{T}_h} \left(\eta_{\mathrm{NC},K}\right)^2$$

- Typically, s_h is built by prescribing its nodal values as averages
  - see [Achdou, Bernardi, Coquel 03; Karakashian, Pascal, 03] for Crouzeix–Raviart and IPDG residual-based estimates
  - ... not p-robust [Burman, AE 07; Houston, Schötzau, Wihler 07]

Potential reconstruction

- Local FEM solves of order $(p+1)$ on vertex-based patches
  - $s_h^{\mathbf{a}} \in P^{p+1}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})$
  - set $s_h = \sum_{\mathbf{a}\in\mathcal{V}_h} s_h^{\mathbf{a}}$
- *p*-robust local efficiency proved in [AE, MV 15] in 2D
  - assuming $\langle [\![u_h]\!], 1\rangle_F = 0$,

$$\eta_{\mathrm{NC},K} \le C_{\mathrm{eff}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla_{\mathcal{T}}(u-u_h)\|_{\omega_{\mathbf{a}}}, \qquad C_{\mathrm{eff}} = C_{\mathrm{st}}\, C_{\mathrm{bPS}}$$

  - two-step proof as above (no oscillation here)
  - 1st step: mixed RT solve of order p for rotated gradient of $s_h^{\mathbf{a}}$
  - 2nd step: broken PS inequalities, see also [Carstensen, Merdon 13]
  - jump seminorm added to error and estimator if $\langle [\![u_h]\!], 1\rangle_F \neq 0$
  - use discrete gradient for DG (instead of broken gdt); broken PS and jump seminorm can be avoided for symmetric IPDG [AE, MV 15]

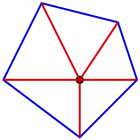
Main results [AE, MV 16]

- 3D, *p*-robust flux and potential reconstructions
  - **Thm. 1** *p*-robust, H^1-stable polynomial lifting for potentials
  - **Thm. 2** *p*-robust, $H(\mathrm{div})$-stable polynomial lifting for fluxes
- Our proofs are constructive (as 2D proofs)
  - e.g., build $\tilde{\boldsymbol{\sigma}}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ s.t. $\nabla{\cdot}\tilde{\boldsymbol{\sigma}}_h^{\mathbf{a}} = g_h^{\mathbf{a}}$, $\|\boldsymbol{\tau}_h^{\mathbf{a}} + \tilde{\boldsymbol{\sigma}}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{st}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$
  - then $\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le \|\boldsymbol{\tau}_h^{\mathbf{a}} + \tilde{\boldsymbol{\sigma}}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{st}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$
- **Main challenges**
  - how to enumerate tetrahedra in 3D star (triangles in 2D star are enumerated by circling around the vertex)
  - need to work with potentials (and not with rotated gradients)
- We focus for simplicity on FE stars around interior vertices

Some notation

- $\mathcal{T}_{\mathbf{a}} \subset \mathcal{T}_h$: FE star (elements sharing vertex $\mathbf{a} \in \mathcal{V}_h$); domain $\omega_{\mathbf{a}}$
- $\mathcal{F}_{\mathbf{a}} = \mathcal{F}_{\mathbf{a}}^{\mathrm{s}} \cup \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}$: faces of the elements in the star $\mathcal{T}_{\mathbf{a}}$

(2D) skeletal faces $\mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$
(2D) boundary faces $\mathcal{F}_{\mathbf{a}}^{\mathrm{b}}$



- Broken H^1- and $H(\mathrm{div})$-spaces (broken gradient $\nabla_{\mathcal{T}}$)

$$H^1(\mathcal{T}_{\mathbf{a}}) := \{v \in L^2(\omega_{\mathbf{a}}) \mid v|_K \in H^1(K)\ \forall K \in \mathcal{T}_{\mathbf{a}}\}$$

$$\mathbf{H}(\mathrm{div}, \mathcal{T}_{\mathbf{a}}) := \{\mathbf{v} \in \mathbf{L}^2(\omega_{\mathbf{a}}) \mid \mathbf{v}|_K \in \mathbf{H}(\mathrm{div}, K)\ \forall K \in \mathcal{T}_{\mathbf{a}}\}$$

- Broken polynomial subspaces

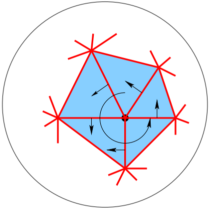
$$P^p(\mathcal{T}_{\mathbf{a}}) := \{v \in L^2(\omega_{\mathbf{a}}) \mid v|_K \in \mathbb{P}^p(K)\ \forall K \in \mathcal{T}_{\mathbf{a}}\}$$

$$\mathbf{RT}^p(\mathcal{T}_{\mathbf{a}}) := \{\mathbf{v} \in \mathbf{L}^2(\omega_{\mathbf{a}}) \mid \mathbf{v}|_K \in \mathbb{RT}^p K)\ \forall K \in \mathcal{T}_{\mathbf{a}}\}$$

Some notation (cont'd)

- In 3D, a FE star $\omega_{\mathbf{a}}$ is homeomorphic to a ball in $\mathbb{R}^3$
- We can look at the star boundary $\partial\omega_{\mathbf{a}}$
  - the traces of the tetrahedra in $\mathcal{T}_{\mathbf{a}}$ form a triangulation of $\partial\omega_{\mathbf{a}}$

every triangle is a boundary face $F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}$
every edge is the trace of a skeletal face $F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$
every point is the trace of a skeletal edge $e \in \mathcal{E}_{\mathbf{a}}$



- Orientation
  - every skeletal face is oriented so as to define a jump across it
  - every skeletal edge is oriented so as to circle around it
  - incidence coefficients $\iota_{F,e} = \pm 1$, for all $F \in \mathcal{F}_e$ and $e \in \mathcal{E}_{\mathbf{a}}$

H^1-stable polynomial lifting

- Let $p \geq 1$
- Let $r_F \in \mathbb{P}^p(F)$ for all $F \in \mathcal{F}_{\mathbf{a}}$
- Assume $r_F \equiv 0$ on boundary faces $F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}$, and assume on skeletal faces $F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$ the compatibility conditions

$$r_F|_{F\cap\partial\omega_{\mathbf{a}}} = 0, \qquad \sum_{F\in\mathcal{F}_e} \iota_{F,e} r_F|_e = 0 \ \forall e \in \mathcal{E}_{\mathbf{a}}$$

- **Then,**

$$\min_{\substack{v_h\in P^p(\mathcal{T}_{\mathbf{a}})\\ v_h=r_F\ \forall F\in\mathcal{F}_{\mathbf{a}}^{\mathrm{b}}\\ [\![v_h]\!]=r_F\ \forall F\in\mathcal{F}_{\mathbf{a}}^{\mathrm{s}}}} \|\nabla_{\mathcal{T}} v_h\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{st}} \min_{\substack{v\in H^1(\mathcal{T}_{\mathbf{a}})\\ v=r_F\ \forall F\in\mathcal{F}_{\mathbf{a}}^{\mathrm{b}}\\ [\![v]\!]=r_F\ \forall F\in\mathcal{F}_{\mathbf{a}}^{\mathrm{s}}}} \|\nabla_{\mathcal{T}} v\|_{\omega_{\mathbf{a}}}$$

with p-robust constant C_{st} only depending on mesh regularity

$\mathbf{H}(\mathrm{div})$-stable polynomial lifting

- Let $p \geq 0$
- Let $r_K \in \mathbb{P}^p(K)$ for all $K \in \mathcal{T}_{\mathbf{a}}$ and let $r_F \in \mathbb{P}^p(F)$ for all $F \in \mathcal{F}_{\mathbf{a}}$
- Assume the compatibility condition

$$\sum_{K\in\mathcal{T}_{\mathbf{a}}} (r_K, 1)_K - \sum_{F\in\mathcal{F}_{\mathbf{a}}} (r_F, 1)_F = 0$$

- **Then,**

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RT}^p(\mathcal{T}_{\mathbf{a}}) \\ \mathbf{v}_h\cdot\mathbf{n}_F = r_F \ \forall F\in\mathcal{F}^{\mathrm{b}}_{\mathbf{a}} \\ [\![\mathbf{v}_h\cdot\mathbf{n}_F]\!] = r_F \ \forall F\in\mathcal{F}^{\mathrm{s}}_{\mathbf{a}} \\ \nabla_{\mathcal{T}}\cdot\mathbf{v}_h|_K = r_K \ \forall K\in\mathcal{T}_{\mathbf{a}}}} \|\mathbf{v}_h\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div},\mathcal{T}_{\mathbf{a}}) \\ \mathbf{v}\cdot\mathbf{n}_F = r_F \ \forall F\in\mathcal{F}^{\mathrm{b}}_{\mathbf{a}} \\ [\![\mathbf{v}\cdot\mathbf{n}_F]\!] = r_F \ \forall F\in\mathcal{F}^{\mathrm{s}}_{\mathbf{a}} \\ \nabla_{\mathcal{T}}\cdot\mathbf{v}|_K = r_K \ \forall K\in\mathcal{T}_{\mathbf{a}}}} \|\mathbf{v}\|_{\omega_{\mathbf{a}}}$$

with p-robust constant C_{st} only depending on mesh regularity

Shifted reformulation: potential

- Let $\xi_h^{\mathbf{a}} \in P^p(\mathcal{T}_{\mathbf{a}})$ be any function from the minimization set

$$\xi_h^{\mathbf{a}} = r_F \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}, \qquad [\![\xi_h^{\mathbf{a}}]\!] = r_F \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$$

- An equivalent reformulation of Thm. 1 is

$$\min_{v_h \in P^p(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_{\mathcal{T}}(\xi_h^{\mathbf{a}} - v_h)\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{st}} \min_{v \in H_0^1(\omega_{\mathbf{a}})} \|\nabla_{\mathcal{T}}(\xi_h^{\mathbf{a}} - v)\|_{\omega_{\mathbf{a}}}$$

- Application to a posteriori error analysis: $\xi_h^{\mathbf{a}} = \psi_{\mathbf{a}} u_h$ and

$$r_F = 0 \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}, \qquad r_F = \psi_{\mathbf{a}} [\![u_h]\!] \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$$

  The compatibility conditions $\sum_{F \in \mathcal{F}_e} \iota_{F,e} r_F|_e = 0$, $\forall e \in \mathcal{E}_{\mathbf{a}}$, follow from algebraic properties of jump operator

Shifted reformulation: flux

- Let $\boldsymbol{\tau}_h^{\mathbf{a}} \in \mathbf{RT}^p(\mathcal{T}_{\mathbf{a}})$ be any function s.t.

$$\boldsymbol{\tau}_h^{\mathbf{a}}\cdot\mathbf{n}_F = r_F \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}, \qquad [\![\boldsymbol{\tau}_h^{\mathbf{a}}]\!]\cdot\mathbf{n}_F = r_F \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$$

- An equivalent reformulation of Thm. 2 is

$$\min_{\substack{\mathbf{v}_h\in\mathbf{RT}^p(\mathcal{T}_{\mathbf{a}})\cap\mathbf{H}_0(\mathrm{div},\omega_{\mathbf{a}})\\ \nabla\cdot\mathbf{v}_h|_K=r_K-\nabla\cdot\boldsymbol{\tau}_h^{\mathbf{a}}|_K\ \forall K\in\mathcal{T}_{\mathbf{a}}}} \|\boldsymbol{\tau}_h^{\mathbf{a}}+\mathbf{v}_h\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{st}} \min_{\substack{\mathbf{v}\in\mathbf{H}_0(\mathrm{div},\omega_{\mathbf{a}})\\ \nabla\cdot\mathbf{v}|_K=r_K-\nabla\cdot\boldsymbol{\tau}_h^{\mathbf{a}}|_K\ \forall K\in\mathcal{T}_{\mathbf{a}}}} \|\boldsymbol{\tau}_h^{\mathbf{a}}+\mathbf{v}\|_{\omega_{\mathbf{a}}}$$

- Application to a posteriori error analysis: $\boldsymbol{\tau}_h^{\mathbf{a}} = \psi_{\mathbf{a}}\nabla_{\mathcal{T}} u_h$ and

$$r_F = 0 \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{b}}, \qquad r_F = \psi_{\mathbf{a}}[\![\nabla_{\mathcal{T}} u_h]\!]\cdot\mathbf{n}_F \ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}$$
$$r_K = \psi_{\mathbf{a}}(f + \Delta_{\mathcal{T}} u_h) \ \forall K \in \mathcal{T}_{\mathbf{a}} \quad (f \text{ pcw. polynomial})$$

The compatibility condition $\sum_{K\in\mathcal{T}_{\mathbf{a}}}(r_K,1)_K - \sum_{F\in\mathcal{F}_{\mathbf{a}}}(r_F,1)_F = 0$ is nothing but Galerkin's orthogonality on the hat basis function $\psi_{\mathbf{a}}$

Main ingredients of proofs

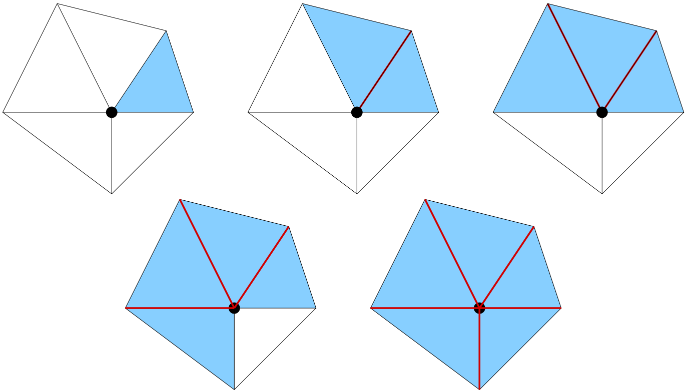
On a **fixed** tetrahedron $K \in \mathcal{T}_{\mathbf{a}}$, we can

- Lift the prescribed divergence of the flux using [Costabel, McIntosh 10] (valid in any space dimension)
- Lift prescribed polynomials for the flux normal component using [Demkowicz, Gopalakrishnan, Schöberl 12] (proved for $d = 3$) (a compatibility condition is required if the prescription is on all faces)
- Lift prescribed compatible polynomials for the potential trace using [DGS 09] (proved for $d = 3$)

We are left with the p-robust lifting of the prescribed jumps ... but this requires a careful **enumeration of the tetrahedra in the star**

2D enumeration

- Circle around the interior vertex **a**
  - K_1: do nothing
  - K_n: fix jump on face touching K_{n-1}, $n \in \{2...4\}$
  - K_5: fix last two jumps (possible owing to compatibility condition)



3D enumeration: shellability

- Let $\mathcal{T}_{\mathbf{a}}$ be a star of tetrahedra around the interior vertex $\mathbf{a}$
- Consider the triangulation of the sphere $S^{(2)} \subset \mathbb{R}^3$ with same connectivity
- Enumerate surface triangles so that $\cup_{j \leq i} T_j^{(2)}$ is connected for all i
- The notion of **shellability of polytopes** shows that this is possible [Ziegler, Lectures on Polytopes, Chap. 8, Springer, 2006]
- Enumerate patch tetrahedra following surface triangle enumeration and, for each $K_i \in \mathcal{T}_{\mathbf{a}}$, fix jump on the skeletal faces of K_i touching any tetrahedron K_j for $j < i$

Numerical results (with V. Dolejší)

- Smooth analytical solution in $\Omega = (0,1)^2$
  - $u(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2)$
- Uniformly refined, non-nested unstructured triangulations
- Discretization by symmetric IPDG method
  - asymptotic exactness observed for pol. degrees $p \in \{1...6\}$
  - similar results for incomplete version of IPDG
  - slightly larger effectivity indices for nonsymmetric version and even p

Errors, estimators, effectivity indices

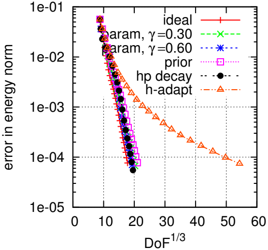
h/h_0	p	$\|\nabla e\|$	$j(e)$	$\|\nabla e\| + j(e)$	η_{F}	η_{osc}	η_{NC}	η	$\eta + j(u_h)$	I^{eff}	I_j^{eff}
1	1	1.07E-00	1.92E-01	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
1/2	1	5.56E-01	7.28E-02	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
1/4	1	2.92E-01	2.82E-02	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
1/8	1	1.39E-01	9.19E-03	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
1	2	1.54E-01	1.76E-02	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
1/2	2	4.07E-02	4.66E-03	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
1/4	2	1.10E-02	1.26E-03	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
1/8	2	2.50E-03	2.90E-04	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
1	3	1.37E-02	3.96E-04	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
1/2	3	1.85E-03	4.53E-05	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
1/4	3	2.60E-04	4.79E-06	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
1/8	3	2.75E-05	3.75E-07	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
1	4	9.87E-04	2.95E-05	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
1/2	4	6.92E-05	2.06E-06	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
1/4	4	5.04E-06	1.42E-07	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
1/8	4	2.58E-07	7.61E-09	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
1	5	5.64E-05	6.76E-07	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
1/2	5	2.01E-06	2.18E-08	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
1/4	5	7.74E-08	6.04E-10	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
1/8	5	1.86E-09	1.18E-11	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
1	6	2.85E-06	3.70E-08	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
1/2	6	5.42E-08	6.78E-10	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
1/4	6	1.07E-09	1.20E-11	1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.08E-09	1.01	1.01

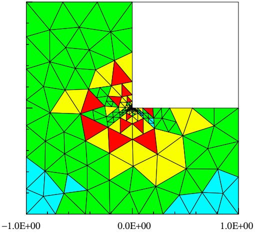
hp-adaptivity

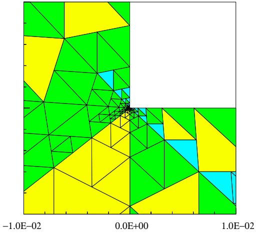
- Nested simplicial meshes allowing for **hanging nodes**
- Extension of reconstruction procedures to hanging nodes
  - only matching refinement of individual patches is needed
- Bulk chasing criterion based on local *p*-robust estimators
- *hp*-refinement decision criteria inspired from [Mitchell, McClain 14]
- Algebraic convergence w.r.t. dof's observed on several 2D benchmark problems from [Mitchell 13]

Numerical examples

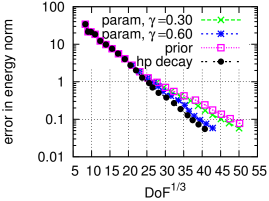
- **Re-entrant corner singularity** ($u \in H^{1+t}(\Omega)$, $t < \frac{2}{3}$)

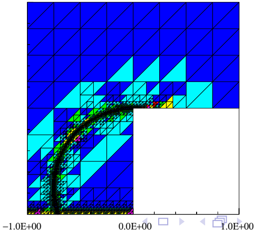






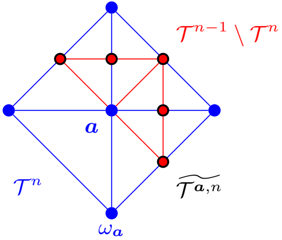
- **Multiple difficulties** (re-entrant corner, point sing, circular wave)

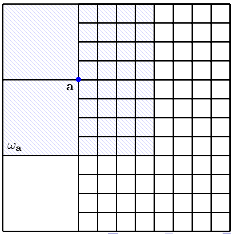




Global $\mathbf{H}(\mathrm{div})$-liftings (with I. Smears)

- So far, we devised **local** liftings of polynomial data on **FE stars**
- We now consider global $\mathbf{H}(\mathrm{div})$-liftings of polynomial data
- One important application is to devise liftings on patches of mesh cells that are **(much) larger than a star**
- This allows us to remove some theoretical barriers on
  - number of hanging nodes for elliptic problems
  - level of mesh coarsening between time-steps in parabolic problems





Main result

- Lipschitz domain $\Omega \subset \mathbb{R}^d$; boundary partition $\Gamma = \Gamma_\mathrm{D} \cup \Gamma_\mathrm{N}$
- Let $\mathcal{T}_h$ be a simplicial mesh of Ω, **possibly locally refined**
- **Thm.** Let $p \geq 1$, $f \in P^{p-1}(\mathcal{T}_h)$, $\boldsymbol{\xi} \in RT^{p-1}(\mathcal{T}_h)$ (broken spaces) (with $(f,1)_\Omega = 0$ if $\Gamma_\mathrm{N} = \Gamma$). Then,

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RT}^p(\mathcal{T}_h) \cap \mathbf{H}(\mathrm{div},\Omega) \\ \nabla\cdot\mathbf{v}_h = f \text{ in } \Omega \\ \mathbf{v}_h\cdot\mathbf{n} = 0 \text{ on } \Gamma_\mathrm{N}}} \|\boldsymbol{\xi} + \mathbf{v}_h\|_\Omega \leq C_\mathrm{st} \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div},\Omega) \\ \nabla\cdot\mathbf{v} = f \text{ in } \Omega \\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_\mathrm{N}}} \|\boldsymbol{\xi} + \mathbf{v}\|_\Omega$$

  with p-robust constant C_st only depending on mesh regularity
  (Note that the discrete minimizer is one polynomial order higher than the data)
- See [Ainsworth, Guzman, Sayas 16] for zero interior source terms, nonzero boundary traces, and fixed polynomial degree

Main idea in proof

- Primal problem with H^{-1} data

$$u \in H^1_*(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega - (\boldsymbol{\xi}, \nabla v)_\Omega,\ \forall v \in H^1_*(\Omega)$$

$H^1_*(\Omega) = \{v \in H^1(\Omega) \mid (v,1)_\Omega = 0\}$ if $\Gamma_{\mathrm{N}} = \Gamma$
$H^1_*(\Omega) = \{v \in H^1(\Omega) \mid v_{|\Gamma_{\mathrm{D}}} = 0\}$ otherwise

- Equivalence of primal/dual energies

$$\min_{\substack{\mathbf{v}\in\mathbf{H}(\mathrm{div},\Omega)\\ \nabla\cdot\mathbf{v}=f \text{ in } \Omega\\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma_{\mathrm{N}}}} \|\boldsymbol{\xi} + \mathbf{v}\|_\Omega = \max_{v\in H^1_*(\Omega)} \frac{(f, v)_\Omega - (\boldsymbol{\xi}, \nabla v)_\Omega}{\|\nabla v\|_\Omega} = \|\nabla u\|_\Omega$$

- Need to construct $\boldsymbol{\sigma}_h \in \mathbf{RT}^p(\mathcal{T}_h) \cap \mathbf{H}(\mathrm{div}, \Omega)$ s.t.

$$\|\boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_\Omega \le C_{\mathrm{st}} \|\nabla u\|_\Omega$$

A posteriori error estimate with H^{-1} data

- Consider general data $f \in L^2(\Omega)$ and $\boldsymbol{\xi} \in \mathbf{L}^2(\Omega)$

$$u \in H^1_*(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega - (\boldsymbol{\xi}, \nabla v)_\Omega, \ \forall v \in H^1_*(\Omega)$$

- H^1_*-conforming approximation $u_h \in P^{p'}(\mathcal{T}_h) \cap H^1_*(\Omega)$, $p' \geq 1$
- Equilibrated flux reconstruction $\boldsymbol{\sigma}_h \in \mathbf{RT}^p(\mathcal{T}_h) \cap \mathbf{H}(\mathrm{div}, \Omega)$, $p \geq p'$

$$\|\nabla(u - u_h)\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \left(\|\nabla u_h + \boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_K + \mathrm{osc}(f, K) \right)^2$$

$$\|\nabla u_h + \boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_K \leq C_{\mathrm{eff}} \|\nabla(u - u_h)\|_{\omega_K} + \mathrm{osc}(f, \boldsymbol{\xi}, \omega_K)$$

- $\boldsymbol{\sigma}_h$ constructed locally from local **shifted** flux equilibration in $\mathbf{V}_h^{\mathbf{a}} = \mathbf{RT}^p(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ (+ Neumann BC's)

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \mathop{\arg\min}_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\ \nabla \cdot \mathbf{v}_h = g_h^{\mathbf{a}}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

with data $\boldsymbol{\tau}_h^{\mathbf{a}} := \psi_{\mathbf{a}}(\boldsymbol{\xi} + \nabla u_h)$, $g_h^{\mathbf{a}} := \Pi_{(\nabla \cdot \mathbf{V}_h^{\mathbf{a}})}(f\psi_{\mathbf{a}} - (\boldsymbol{\xi} + \nabla u_h) \cdot \nabla \psi_{\mathbf{a}})$

Conclusion of proof

- Consider polynomial data $f \in P^{p-1}(\mathcal{T}_h)$, $\boldsymbol{\xi} \in RT^{p-1}(\mathcal{T}_h)$, $p \geq 1$
- Consider H^1_*-conforming FEM approximation with $1 = p' \leq p$
- The **data oscillation term osc**(f, ξ, ω_K) **vanishes** so that

$$\|\nabla u_h + \boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_\Omega \leq C_{\text{eff}} \|\nabla(u - u_h)\|_\Omega$$

- Combined with the basic bound $\|\nabla u_h\|_\Omega \leq \|\nabla u\|_\Omega$, we conclude that

$$\|\boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_\Omega \leq \|\nabla u_h + \boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_\Omega + \|\nabla u_h\|_\Omega \leq (2C_{\text{eff}} + 1)\|\nabla u\|_\Omega$$

Sharper stability result for special data

- Let $\psi_\dagger$ be continuous, pcw. affine on $\mathcal{T}_h$
  - $\psi_\dagger$ can be viewed as a **large-scale hat basis function**
- Let Γ_{N} be the subset of Γ where $\psi_\dagger$ vanishes
- Let $p \geq 1$, $f \in P^{p-1}(\mathcal{T}_h)$, $\boldsymbol{\xi} \in RT^{p-1}(\mathcal{T}_h)$ (with $(f,1)_\Omega = 0$ if $\Gamma_{\mathrm{N}} = \Gamma$)
- **Thm.** The following holds:

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RT}^p(\mathcal{T}_h)\cap \mathbf{H}(\mathrm{div},\Omega) \\ \nabla\cdot\mathbf{v}_h = \psi_\dagger f - \nabla\psi_\dagger\cdot\boldsymbol{\xi} \text{ in } \Omega \\ \mathbf{v}_h\cdot\mathbf{n}=0 \text{ on } \Gamma_{\mathrm{N}}}} \|\boldsymbol{\xi}+\mathbf{v}_h\|_\Omega \leq C_{\mathrm{st}}\, C(\psi_\dagger) \min_{\substack{\mathbf{v}\in\mathbf{H}(\mathrm{div},\Omega) \\ \nabla\cdot\mathbf{v}=f \text{ in } \Omega \\ \text{no BC}}} \|\boldsymbol{\xi}+\mathbf{v}\|_\Omega$$

with *p*-robust constant C_{st} only depending on mesh regularity and $C(\psi_\dagger) = \|\psi_\dagger\|_{L^\infty(\Omega)} + h_\Omega \|\nabla\psi_\dagger\|_{\mathbf{L}^\infty(\Omega)}$

  - lifting and data have **matching** polynomial order
  - the function $\psi_\dagger$ is "factored out" from RHS

Conclusions

- Equilibrated-flux estimates offer several benefits
  - **guaranteed** (fully computable) upper bounds
  - *p*-**robust** local efficiency
  - **adaptive inexact Newton solvers** [AE, MV 13]
- Unified analysis for p-robust H^1- and $\mathbf{H}(\mathrm{div})$-polynomial liftings
- New local efficiency proofs for arbitrary-level of hanging nodes (elliptic PDEs) and no coarsening restriction (parabolic PDEs)

Thank you for your attention

1. AE, MV, SINUM (2015), **53**, 1058–1081
2. V. Dolejší, AE, MV, SISC (2016), **38** A3220–A3246
3. AE, MV (2016), arXiv 1701.02161
4. AE, I. Smears, MV, Calcolo (2017); DOI 10.1007/s10092-017-0217-4