Hybrid high-order methods for the wave equation

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Hybrid high-order (HHO) methods ...

- in a nutshell
- for wave propagation
- on unfitted meshes (curved interfaces/boundary)
HHO in a nutshell
Basic ideas

- Degrees of freedom (dofs) located on mesh cells and faces
- Let us start with polynomials of the same degree \( k \geq 0 \) on cells and faces

```
\[
\begin{align*}
\text{mesh} & & k = 0 & & k = 1 & & k = 2 \\
\end{align*}
\]```

- In each cell, one devises a local gradient reconstruction operator
- One adds a local stabilization to weakly enforce the matching of cell dofs trace with face dofs
- The global problem is assembled cellwise as in FEM
Basic ideas

- Degrees of freedom (dofs) located on mesh cells and faces.
- Let us start with polynomials of the same degree $k \geq 0$ on cells and faces.

![Diagram of mesh and $k=0$, $k=1$, $k=2$](image)

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- One adds a local stabilization to weakly enforce the matching of cell dofs trace with face dofs.
- The global problem is assembled cellwise as in FEM.
Mesh cell \( T \in \mathcal{T} \), cell dofs \( u_T \in P^k(T) \), face dofs \( u_{\partial T} \in P^k(\mathcal{F}_{\partial T}) \)

\[
\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := P^k(T) \times P^k(\mathcal{F}_{\partial T})
\]

Local potential reconstruction \( R_T : \hat{U}_T \rightarrow P^{k+1}(T) \) s.t.

\[
(\nabla R_T(\hat{u}_T), \nabla q)_T = -(u_T, \Delta q)_T + (u_{\partial T}, \nabla q \cdot n_T)_{\partial T}, \quad \forall q \in P^{k+1}(T)/\mathbb{R}
\]

together with \( (R_T(\hat{u}_T), 1)_T = (u_T, 1)_T \)
Gradient reconstruction and stabilization

- Mesh cell $T \in \mathcal{T}$, cell dofs $u_T \in \mathbb{P}^k(T)$, face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

- Local potential reconstruction $R_T : \hat{U}_T \to \mathbb{P}^{k+1}(T)$ s.t.

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together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$

- Local gradient reconstruction $G_T(\hat{u}_T) := \nabla R_T(\hat{u}_T) \in \nabla \mathbb{P}^{k+1}(T)$

- Local stabilization operator acting on $\delta := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\hat{u}_T) := \Pi_{\partial T}^k \left( \delta - \left( I - \Pi_T^k \right) R_T(0, \delta) \right)|_{\partial T}$$

high-order correction
Local bilinear form

- Local bilinear form for Poisson model problem

\[ a_T(\hat{u}_T, \hat{w}_T) := (G_T(\hat{u}_T), G_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T} \]

- Stability and boundedness

\[ \alpha \| \hat{u}_T \|_{\hat{U}_T}^2 \leq a_T(\hat{u}_T, \hat{u}_T) \leq \omega \| \hat{u}_T \|_{\hat{U}_T}^2, \quad \forall \hat{u}_T \in \hat{U}_T \]

with \( \| \hat{u}_T \|_{\hat{U}_T}^2 := \| \nabla u_T \|_T^2 + h_T^{-1} \| u_T|_{\partial T} - u_{\partial T} \|_{\partial T}^2 \)
Local bilinear form

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  with
  \[ \|\hat{u}_T\|_{\hat{U}_T}^2 := \|\nabla u_T\|_T^2 + h_T^{-1}\|u_T|_{\partial T} - u_{\partial T}\|_{\partial T}^2 \]

- Reduction operator
  \[ \hat{I}_T(v) := (\Pi^k_T(v), \Pi^k_{\partial T}(v|_{\partial T})) \in \hat{U}_T, \forall v \in H^1(T) \]

- Main consistency properties
  - \[ h_T^{-1}\|v - R_T(\hat{I}_T(v))\|_T + \|\nabla(v - R_T(\hat{I}_T(v)))\|_T \leq h_T^{k+1}|v|_{H^{k+2}(T)} \]
  - \[ h_T^{-\frac{1}{2}}\|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \leq h_T^{k+1}|v|_{H^{k+2}(T)} \]
assembly and static condensation

- Global dofs $\hat{u}_h = (u_T, u_F)$ ($T := \{\text{mesh cells}\}, F := \{\text{mesh faces}\}$)
  
  $\hat{U}_h := P^k(T) \times P^k(F)$, $P^k(T) := \bigotimes_{T \in T} P^k(T)$, $P^k(F) := \bigotimes_{F \in F} P^k(F)$

- Global assembly: $\sum_{T \in T} a_T(\hat{u}_T, \hat{w}_T) = \sum_{T \in T} (f, w_T)_T$

- Dirichlet conditions can be directly enforced on the face boundary dofs

- Cell dofs are eliminated locally by **static condensation**
  
  - global problem couples only face dofs
  - cell dofs recovered by local post-processing
Main characteristics

- **General meshes**: polytopal cells, hanging nodes

- **Optimal error estimates** (smooth solutions)
  - $O(h^{k+1}) H^1$-error estimate (face dofs of order $k \geq 0$)
  - $O(h^{k+2}) L^2$-error estimate (with full elliptic regularity)
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  - more generally, $O(h^t)$ $H^1$-error estimate if $u \in H^{1+t}(\Omega)$, $t \in (\frac{1}{2}, k + 1]$
  - for $t \in (0, \frac{1}{2})$, see [AE, Guermond 21 (FoCM)]
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- **Local conservation**
  - optimally convergent and algebraically balanced fluxes on faces
  - as any face-based method, balance at cell level

- **Attractive computational costs**
  - only face dofs are globally coupled
  - compact stencil
Variants

- Variant on gradient reconstruction $G_T : \hat{U}_T \rightarrow P^k(T; \mathbb{R}^d)$ s.t.

$$
(G_T(\hat{u}_T), q)_T = -(u_T, \text{div } q)_T + (u_{\partial T}, q \cdot n_T)_{\partial T}, \quad \forall q \in P^k(T; \mathbb{R}^d)
$$

- same scalar mass matrix for each component of $G_T(\hat{u}_T)$
- useful for nonlinear problems

[Di Pietro, Droniou 17; Botti, Di Pietro, Sochala 17; Abbas, AE, Pignet 18]
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- Variants on cell dofs and stabilization

  - mixed-order setting: $k \geq 0$ for face dofs and $(k + 1)$ for cell dofs
  - this variant allows for the simpler Lehrenfeld–Schöberl HDG stabilization

    $$S_{\partial T}(\hat{u}_T) := \Pi_{\partial T}^k(\delta)$$

  - another variant is $k \geq 1$ for face dofs and $(k - 1)$ for cell dofs
Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

HHO fits into HDG setting [Cockburn, Di Pietro, AE 16]

equal-order HHO uses reconstruction in the stabilization

HHO allows for a simpler analysis based on $L_2$-projections: avoids invoking the special HDG projection

Similar devising of HHO and weak Galerkin methods [Wang, Ye 13]

weak gradient $\leftrightarrow$ HHO grad. rec.

WG often uses plain LS stabilization (can be suboptimal)

HHO equivalent (up to stab.) to ncVEM [Ayuso, Manzini, Lipnikov 16]

HHO dof space $\hat{U}_T$ isomorphic to virtual space $V_T$

$P_{k+1}(T) \subset V_T: \{ v \in H_1(T) | \Delta v \in P_k(T), n \cdot \nabla v \mid \partial T \in P_k(F_{\partial T}) \}$

HHO grad. rec. $\leftrightarrow$ computable gradient projection

stabilization controls energy-norm of noncomputable remainder see [Cockburn, Di Pietro, AE 16; Di Pietro, Droniou, Manzini 18; Lemaire 21]

Different devising viewpoints should be mutually enriching
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- Different devising viewpoints should be mutually enriching
Applications, libraries, textbooks

- **Broad area of applications** (non-exhaustive list...)
  - **solid mechanics**: nonlinear elasticity, hyperelasticity and plasticity, contact, Tresca friction, obstacle pb
  - **fluid mechanics/porous media**: Stokes, NS, poroelasticity, fractures
  - Leray-Lions, spectral pb, $H^{-1}$-loads, magnetostatics, de Rham complexes

- **Libraries**
  - industry (**code_aster**, **code_saturne**, EDF R&D), ongoing developments at CEA
  - academia: **diskpp** (C++) (**ENPC/INRIA github.com/wareHH0use**), **HArD::Core** (**Monash/Montpellier github.com/jdroniou/HArDCore**)

- **Textbooks**
  - **Cicuttin, AE, Pignet**, *HHO methods. A primer with application to solid mechanics* (Springer Briefs, 2021)
HHO for wave propagation

- Second-order formulation in time: Newmark schemes
- First-order formulation in time: RK schemes

[Burman, Duran, AE 21 (CAMC)], [Burman, Duran, AE, Steins 21 (JSC)]
- Domain $\Omega \subset \mathbb{R}^d$, time interval $J := (0, T_f)$, $T_f > 0$

- Acoustic wave equation with wave speed $c := \sqrt{\kappa / \rho}$

\[
\frac{1}{\kappa} \partial_{tt} p - \text{div} \left( \frac{1}{\rho} \nabla p \right) = f \quad \text{in } J \times \Omega
\]

Everything can be extended to elastodynamics

- Weak form: Under mild regularity assumptions on the data,

\[
(\partial_{tt} p(t), w)_{\kappa;\Omega} + (\nabla p(t), \nabla w)_{\rho;\Omega} = (f(t), w)_{\Omega}, \quad \forall w \in H^1_0(\Omega) \forall t \in J
\]

- Energy balance: $\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t (f(s), \partial_t p(s))_{\Omega} ds$ with

\[
\mathcal{E}(t) := \frac{1}{2} \| \partial_t p(t) \|_{\kappa;\Omega}^2 + \frac{1}{2} \| \nabla p(t) \|_{\rho;\Omega}^2
\]
• Local cell dofs in $\mathbb{P}^{k'}(T)$, $k' \in \{k, k + 1\}$, and local face dofs in $\mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^{k'}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

• Local gradient reconstruction $G_T(\hat{u}_T) \in \mathbb{P}^k(T; \mathbb{R}^d)$ (or in $\nabla \mathbb{P}^{k+1}(T)$)

• Local stabilization acting on $\delta := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\hat{u}_T) := \begin{cases} 
\Pi_{\partial T}^k(\delta - ((I - \Pi_T^k)R_T(0, \delta))|_{\partial T}) & \text{if } k' = k \\
\Pi_{\partial T}^k(\delta) & \text{if } k' = k + 1
\end{cases}$$

• Local bilinear form

$$a_T(\hat{u}_T, \hat{w}_T) := (G_T(\hat{u}_T), G_T(\hat{w}_T))_{1/2; T} + \tau_{\partial T}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}$$

with $\tau_{\partial T} := (\rho|_T h_T)^{-1}$
Global dofs $\hat{u}_h = (u_T, u_F) \in \hat{U}_h := P^k(T) \times P^k(F)$

Global assembly leading to

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in T} a_T(\hat{u}_T, \hat{w}_T) := (G_T(\hat{u}_h), G_T(\hat{w}_h))_{\frac{1}{\rho};\Omega} + s_h(\hat{u}_h, \hat{w}_h)$$

Dirichlet conditions can be directly enforced on the face boundary dofs

$$\hat{U}_{h0} := P^{k'}(T) \times P^k(F^\circ)$$

with $F^\circ := \{\text{mesh interfaces}\}$
• Wave equation in space semi-discrete form: $\hat{p}_h \in C^2(\bar{J}; \hat{U}_{h0})$ s.t.

$$
\left( \partial_{tt} p_T(t), w_T \right)_{\frac{1}{\kappa}; \Omega} + a_h(\hat{p}_h(t), \hat{w}_h) = (f(t), w_T)_{\Omega}, \quad \forall \hat{w}_h \in \hat{U}_{h0} \forall t \in J
$$
Wave equation in space semi-discrete form: \( \hat{p}_h \in C^2(\bar{J}; \hat{U}_{h0}) \) s.t.

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(\partial_{tt} p_T(t), w_T)_{\frac{1}{\kappa},\Omega} + a_h(\hat{p}_h(t), \hat{w}_h) = (f(t), w_T)_\Omega, \quad \forall \hat{w}_h \in \hat{U}_{h0} \quad \forall t \in J
\]

Energy balance: \( \mathcal{E}_h(t) = \mathcal{E}_h(0) + \int_0^t (f(s), \partial_t p_T(s))_\Omega ds \) with

\[
\mathcal{E}_h(t) := \frac{1}{2} \| \partial_t p_T(t) \|_{\frac{1}{\kappa},\Omega}^2 + \frac{1}{2} \| G_T(\hat{p}_h(t)) \|_{\frac{1}{\rho},\Omega}^2 + \frac{1}{2} s_h(\hat{p}_h(t), \hat{p}_h(t))
\]

Stabilization is taken into account in the energy definition

HDG methods for wave equation in second-order form [Cockburn, Fu, Hungria, Ji, Sanchez, Sayas 18]
Algebraic realization

- Bases for $\mathbb{P}^k(T)$ and $\mathbb{P}^k(F)$, component vector $(P_T(t), P_F(t)) \in \mathbb{R}^{N_T \times N_F}$

\[
\begin{bmatrix}
M_{\mathcal{T}\mathcal{T}} \partial_{tt} P_T(t) \\
0
\end{bmatrix} + \begin{bmatrix}
K_{\mathcal{T}\mathcal{T}} & K_{\mathcal{T}\mathcal{F}} \\
K_{\mathcal{F}\mathcal{T}} & K_{\mathcal{F}\mathcal{F}}
\end{bmatrix}
\begin{bmatrix}
P_T(t) \\
P_F(t)
\end{bmatrix} = \begin{bmatrix}
F_T(t) \\
0
\end{bmatrix}
\]

- Mass matrix $M_{\mathcal{T}\mathcal{T}}$ and stiffness submatrix $K_{\mathcal{T}\mathcal{T}}$ are block-diagonal

- Stiffness submatrix $K_{\mathcal{F}\mathcal{F}}$ is only sparse: face dofs from the same cell are coupled together owing to reconstruction
Error analysis

- Assuming a smooth solution,
  \[ \| \partial_t p - \partial_t p_T \|_{L^\infty(J;L^2(\frac{1}{\kappa};\Omega))} + \| \nabla p - G_T(\hat{p}_h) \|_{L^2(J;L^2(\frac{1}{\rho};\Omega))} \text{ decays as } O(h^{k+1}) \]
  \[ \| \Pi_{T}^{k'}(p) - p_T \|_{L^\infty(J;L^2(\frac{1}{\rho};\Omega))} \text{ decays as } O(h^{k+2}) \text{ under (full) elliptic reg.} \]

- Some comments on proofs
  - adapt ideas for FEM analysis from [Dupont 73; Wheeler 73; Baker 76]
  - simpler than for HDG (avoids HDG projection which needs a special initialization in HDG scheme)
  - could be re-used in DG setting using discrete gradients (revisiting [Grote, Schneebeli, Schötzau 06])
Newmark scheme

- Newmark scheme with parameters \((\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})\)
  - implicit, second-order, unconditionally stable
  - \(p, \partial_t p, \partial_{tt} p\) are approximated by hybrid pairs \(\hat{p}_h^n, \hat{v}_h^n, \hat{a}_h^n \in \hat{U}_h, \forall n \geq 0\)

- Each time-step implemented as usual
Newmark scheme with parameters \((\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})\)

- implicit, second-order, unconditionally stable
- \(p, \partial_t p, \partial_{tt} p\) are approximated by hybrid pairs \(\hat{p}^n_h, \hat{v}^n_h, \hat{a}^n_h \in \mathring{U}_h, \forall n \geq 0\)

- Each time-step implemented as usual

- Discrete energy is **exactly conserved**

- Central FD scheme is not efficient: inversion of stiffness submatrix \(K_{FF}\)
Introduce velocity $v := \partial_t p$ and dual variable $\sigma := \frac{1}{\rho} \nabla p$

$$\begin{aligned}
\rho \partial_t \sigma - \nabla v &= 0 \\
\frac{1}{\kappa} \partial_t v - \text{div} \sigma &= f
\end{aligned}$$ in $J \times \Omega$

Weak form: $\forall (\tau, w) \in L^2(\Omega; \mathbb{R}^d) \times H^1_0(\Omega), \forall t \in J$,

$$\begin{aligned}
(\partial_t \sigma(t), \tau)_{\rho;\Omega} - (\nabla v(t), \tau)_{\Omega} &= 0 \\
(\partial_t v(t), w)_{\frac{1}{\kappa};\Omega} + (\sigma(t), \nabla w)_{\Omega} &= (f(t), w)_{\Omega}
\end{aligned}$$

Energy balance: $\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t (f(s), v(s))_{\Omega} ds$ with

$$\mathcal{E}(t) := \frac{1}{2} \| v(t) \|_{\frac{1}{\kappa};\Omega}^2 + \frac{1}{2} \| \sigma(t) \|_{\rho;\Omega}^2$$
HHO space semi-discretization

- $\hat{v}_h \in C^1(\overline{J}; \hat{U}_{h0})$ and $\sigma_T \in C^1(\overline{J}; S_T)$ with $S_T := \mathbb{P}^k(T; \mathbb{R}^d)$

- Space semi-discrete form:

\[
\begin{aligned}
& ((\partial_t \sigma_T(t), \tau_T)_{\rho;\Omega} - (G_T(\hat{v}_h(t)), \tau_T)_\Omega = 0 \\
& ((\partial_t v_T(t), w_T)_{\frac{1}{\kappa};\Omega} + (\sigma_T(t), G_T(\hat{w}_h))_\Omega + \tilde{s}_h(\hat{v}_h(t), \hat{w}_h) = (f(t), w_T)_\Omega
\end{aligned}
\]

- Stabilization $\tilde{s}_h(\cdot, \cdot)$ with weight $\tilde{\tau}_{\partial T} = (\rho c)_{\mid T}^{-1}$, i.e., $\tilde{\tau}_{\partial T} = O(1)$
HHO space semi-discretization

- \( \hat{v}_h \in C^1(\bar{J}; \hat{U}_{h0}) \) and \( \sigma_\mathcal{T} \in C^1(\bar{J}; S_\mathcal{T}) \) with \( S_\mathcal{T} := \mathbb{P}^k(\mathcal{T}; \mathbb{R}^d) \)

- Space semi-discrete form:

\[
\begin{align*}
&\left(\partial_t \sigma_\mathcal{T}(t), \mathbf{T}_\mathcal{T}\right)_\rho;\Omega - \left(G_\mathcal{T}(\hat{v}_h(t)), \mathbf{T}_\mathcal{T}\right)_\Omega = 0 \\
&\left(\partial_t \mathbf{v}_\mathcal{T}(t), \mathbf{w}_\mathcal{T}\right)_{1/\kappa};\Omega + \left(\sigma_\mathcal{T}(t), G_\mathcal{T}(\hat{\mathbf{w}}_h)\right)_\Omega + \tilde{s}_h(\hat{v}_h(t), \hat{\mathbf{w}}_h) = (f(t), \mathbf{w}_\mathcal{T})_\Omega
\end{align*}
\]

- Stabilization \( \tilde{s}_h(\cdot, \cdot) \) with weight \( \tilde{\tau}_{\partial T} = (\rho c)^{-1} \mid_{T} \), i.e., \( \tilde{\tau}_{\partial T} = O(1) \)

- Energy balance: \( \mathcal{E}_h(t) + \int_0^t \tilde{s}_h(\hat{v}_h(s), \hat{\mathbf{w}}_h(s)) ds = \mathcal{E}_h(0) + \int_0^t (f(s), \mathbf{v}_\mathcal{T}(s))_\Omega ds \)

\[
\mathcal{E}_h(t) := \frac{1}{2} \left\| \mathbf{v}_\mathcal{T}(t) \right\|_{1/\kappa};\Omega^2 + \frac{1}{2} \left\| \sigma_\mathcal{T}(t) \right\|_\rho;\Omega^2
\]

Stabilization acts as a dissipative mechanism

- HDG methods for wave equation in first-order form [Nguyen, Peraire, Cockburn 11; Stranglmeier, Nguyen, Peraire, Cockburn 16]
Component vectors $Z_\mathcal{T}(t) \in \mathbb{R}^{M_\mathcal{T}}$ and $(V_\mathcal{T}(t), V_\mathcal{F}(t)) \in \mathbb{R}^{N_\mathcal{T} \times N_\mathcal{F}}$

$$
\begin{bmatrix}
M^\sigma_\mathcal{T}\mathcal{T} \partial_t Z_\mathcal{T}(t) \\
M_\mathcal{T}\mathcal{T} \partial_t V_\mathcal{T}(t) \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 & -G_\mathcal{T} & -G_\mathcal{F} \\
G^\dagger_\mathcal{T} & S_{\mathcal{T}\mathcal{T}} & S_{\mathcal{T}\mathcal{F}} \\
G^\dagger_\mathcal{F} & S_{\mathcal{F}\mathcal{T}} & S_{\mathcal{F}\mathcal{F}}
\end{bmatrix}
\begin{bmatrix}
Z_\mathcal{T}(t) \\
V_\mathcal{T}(t) \\
V_\mathcal{F}(t)
\end{bmatrix}
= \begin{bmatrix}
0 \\
F_\mathcal{T}(t) \\
0
\end{bmatrix}
$$

Mass matrices $M^\sigma_\mathcal{T}\mathcal{T}$ and $M_\mathcal{T}\mathcal{T}$ are block-diagonal

Key point: stab. submatrix $S_{\mathcal{F}\mathcal{F}}$ block-diagonal only if $k' = k + 1$

- for $k' = k$, high-order HHO correction in stabilization destroys this property (couples all faces of the same cell!)
Runge–Kutta (RK) schemes

- Natural choice for first-order formulation in time
  - single diagonally implicit RK: SDIRK\((s, s + 1)\) (\(s\) stages, order \((s + 1)\))
  - explicit RK: ERK\((s)\) (\(s\) stages, order \(s\))

- ERK schemes subject to CFL stability condition \(\frac{c\Delta t}{h} \leq \beta(s)\mu(k)\)
  - \(\beta(s)\) slightly increases with \(s \in \{2, 3, 4\}\)
  - \(\mu(k)\) essentially behaves as \((k + 1)^{-1}\) w.r.t. polynomial degree
Numerical results: homogeneous media (1/2)

- Smooth solution
- Newmark scheme (equal-order, quadrilateral mesh)

**cv. in space**

**cv. in time**

**energy cons.**
Numerical results: homogeneous media (2/2)

- SDIRK(3,4) and ERK(4) schemes (mixed-order, quad/poly meshes)
  - recall that $\tilde{\tau}_T = O(1)$
  - we also consider over-penalty with $\tilde{\tau}_T = O(h_T^{-1})$

- Energy dissipation strongly tempered by increasing polynomial degree
Numerical results: homogeneous media (2/2)

- SDIRK(3,4) and ERK(4) schemes (mixed-order, quad/poly meshes)
  - recall that $\tilde{\tau}_\partial T = O(1)$
  - we also consider over-penalty with $\tilde{\tau}_\partial T = O(h_T^{-1})$

- Energy dissipation strongly tempered by increasing polynomial degree

- Discussion on $\tilde{\tau}_\partial T$
  - energy-error decays optimally as $O(h^{k+1})$ for both $\tilde{\tau}_\partial T$
    ⇒ proof for (HHO, $O(h_T^{-1})$) and HDG, but using different tools
  - $L^2$-error decays optimally as $O(h^{k+2})$ only for $\tilde{\tau}_\partial T = O(h_T^{-1})$
    ⇒ HDG, $\tilde{\tau}_\partial T = O(1)$, special post-proc. [Cockburn, Quenneville-Bélair 12]
  - $\tilde{\tau}_\partial T = O(h_T^{-1})$ worsens CFL condition for ERK schemes
Numerical results: heterogeneous media (1/3)

- 1D test case, $\Omega_1 = (0, 0.5), \Omega_2 = (0.5, 1), c_1/c_2 = 10$
  - initial Gaussian profile in $\Omega_1$
  - analytical solution available (series)

- Benefits of increasing polynomial degree
  - Newmark scheme, equal-order, $k \in \{1, 2, 3\}$, $h = 0.1 \times 2^{-8}$, $\Delta t = 0.1 \times 2^{-9}$
  - HHO-Newmark solution at $t = \frac{1}{2}$ (after reflection/transmission at $x = \frac{1}{2}$)
Numerical results: heterogeneous media (2/3)

- 2D test case, Ricker (Mexican hat) wavelet
  - $\Omega_1 = (0, 1) \times (0, \frac{1}{2})$, $\Omega_2 = (0, 1) \times (\frac{1}{2}, 1)$, $c_1/c_2 = 5$
  - $p_0 = 0$, $v_0 = -\frac{4}{10} \sqrt{\frac{10}{3}} \left( 1600 \ r^2 - 1 \right) \pi^{-\frac{1}{4}} \exp \left( -800r^2 \right)$,
    $$r^2 = (x - x_c)^2 + (y - y_c)^2, \ (x_c, y_c) = (\frac{1}{2}, \frac{1}{4}) \in \Omega_1$$
  - semi-analytical solution (infinite media): gar6more2d software (INRIA)

- HHO-SDIRK(3,4) velocity profiles
  - mixed-order, $k = 5$, polygonal meshes
  - $\Delta t = 0.025 \times 2^{-6}$ (four times larger than Newmark for similar accuracy)
Comparison of computational efficiency

- all schemes tuned to comparable max. rel. error on a sensor at \( \left( \frac{1}{2}, \frac{2}{3} \right) \)
- very preliminary results! (on-the-shelf solvers)
- if no direct solvers allowed, \( \text{ERK}(4) \) wins despite CFL restriction
- with direct solvers, \( \text{SDIRK}(3,4) \) wins
- RK schemes more efficient than Newmark scheme
- for \( \text{SDIRK}(3,4) \), \( \tilde{\tau}_{\partial T} = O(h^{-1}) \) more accurate/expensive than \( \tilde{\tau}_{\partial T} = O(1) \)

<table>
<thead>
<tr>
<th>scheme</th>
<th>((k', k))</th>
<th>stab</th>
<th>solver</th>
<th>t/step</th>
<th>steps</th>
<th>time</th>
<th>err</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ERK}(4) )</td>
<td>(6, 5)</td>
<td>( O(1) )</td>
<td>n/a</td>
<td>0.410</td>
<td>5,120</td>
<td>2,099</td>
<td>2.23</td>
</tr>
<tr>
<td>( \text{Newmark} )</td>
<td>(7, 6)</td>
<td>( O(h^{-1}) )</td>
<td>iter</td>
<td>56.74</td>
<td>2,560</td>
<td>58,265</td>
<td>2.15</td>
</tr>
<tr>
<td>( \text{SDIRK}(3,4) )</td>
<td>(6, 5)</td>
<td>( O(h^{-1}) )</td>
<td>iter</td>
<td>31.24</td>
<td>640</td>
<td>5,639</td>
<td>2.21</td>
</tr>
<tr>
<td>( \text{SDIRK}(3,4) )</td>
<td>(6, 5)</td>
<td>( O(1) )</td>
<td>iter</td>
<td>22.52</td>
<td>640</td>
<td>2,200</td>
<td>4.45</td>
</tr>
<tr>
<td>( \text{Newmark} )</td>
<td>(7, 6)</td>
<td>( O(h^{-1}) )</td>
<td>direct</td>
<td>0.515</td>
<td>2,560</td>
<td>1,318</td>
<td>2.15</td>
</tr>
<tr>
<td>( \text{SDIRK}(3,4) )</td>
<td>(6, 5)</td>
<td>( O(h^{-1}) )</td>
<td>direct</td>
<td>1.579</td>
<td>640</td>
<td>1,010</td>
<td>2.21</td>
</tr>
</tbody>
</table>
Unfitted meshes
Elliptic interface problem

- Polytopal domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$
- Subdomains $\Omega_1, \Omega_2 \subset \Omega$ with different (contrasted) material properties
- Curved interface $\Gamma$, jump $[a]_{\Gamma} = a_{|\Omega_1} - a_{|\Omega_2}$
- Model problem

\[-\text{div} (\kappa \nabla u) = f \quad \text{in } \Omega_1 \cup \Omega_2\]
\[[u]_{\Gamma} = g_D, \ [\kappa \nabla u]_{\Gamma} \cdot n_{\Gamma} = g_N \quad \text{on } \Gamma\]
\[u = 0 \quad \text{on } \partial\Omega\]

- Everything can be adapted to a **single domain with curved boundary**
Motivation for unfitted meshes

- **Use of unfitted meshes for interface problems**
  - curved interface can cut arbitrarily through mesh cells
  - numerical method must deal with **badly cut cells**

- **Classical FEM on unfitted meshes**
  - double unknowns in cut cells and use a consistent Nitsche’s penalty technique to enforce jump conditions [Hansbo, Hansbo 02]
  - **ghost penalty** [Burman 10] to counter bad cuts (gradient jump penalty across faces near curved boundary/interface)
Motivation for unfitted meshes

- Use of unfitted meshes for interface problems
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  - numerical method must deal with badly cut cells

- Classical FEM on unfitted meshes
  - double unknowns in cut cells and use a consistent Nitsche’s penalty technique to enforce jump conditions [Hansbo, Hansbo 02]
  - ghost penalty [Burman 10] to counter bad cuts (gradient jump penalty across faces near curved boundary/interface)

- An alternative to ghost penalty: local cell agglomeration
  - natural for polytopal methods as dG [Sollie, Bokhove, van der Vegt 11; Johansson, Larson 13]
  - cG agglomeration procedure in [Badia, Verdugo, Martín 18]
Main ideas [Burman, AE 18 (SINUM)]
- double cell and face dofs in cut cells, no dofs on curved boundary/interface
- local cell agglomeration to counter bad cuts
- mixed-order setting: $k \geq 0$ for face dofs and $(k + 1)$ for cell dofs
Main ideas [Burman, AE 18 (SINUM)]
- double cell and face dofs in cut cells, no dofs on curved boundary/interface
- local cell agglomeration to counter bad cuts
- mixed-order setting: $k \geq 0$ for face dofs and $(k + 1)$ for cell dofs

Improvements in [Burman, Cicuttin, Delay, AE 21 (SISC)]
- novel gradient reconstruction, avoiding that the penalty parameter in Nitsche’s method is large enough
- robust cell agglomeration procedure (guaranteeing locality)

Stokes interface problems [Burman, Delay, AE 20 (IMANUM)]

Wave propagation [Burman, Duran, AE 21] hal-03086432
Local dofs

Mesh still composed of polytopal cells (with planar faces)

- Decomposition of cut cells: $\overline{T} = \overline{T_1} \cup \overline{T_2}$, $T^\Gamma = T \cap \Gamma$

- Decomposition of cut faces: $\partial(T_i) = (\partial T)^i \cup T^\Gamma$, $i \in \{1, 2\}$

- Local dofs (no dofs on $T^\Gamma$!)

$$\hat{u}_T = (u_{T_1}, u_{T_2}, u_{(\partial T)^1}, u_{(\partial T)^2}) \in \mathbb{P}^{k+1}(T_1) \times \mathbb{P}^{k+1}(T_2) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^1}) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^2})$$
Gradient reconstruction in cut cells

- Gradient reconstruction $G_{T_i}(\hat{u}_T) \in \mathbb{P}^k(T_i; \mathbb{R}^d)$ in each subcell
  
  - **(Option 1)** Independent reconstruction in each subcell
    
    $$(G_{T_i}(\hat{u}_T), q)_{T_i} = -(u_{T_i}, \text{div } q)_{T_i} + (u_{(\partial T)^i}, q \cdot n_{T})_{(\partial T)^i} + (u_{T_i}, q \cdot n_{T_i})_{T\Gamma}$$

  - **(Option 2)** Reconstruction mixing data from both subcells
    
    $$(G_{T_i}(\hat{u}_T), q)_{T_i} = -(u_{T_i}, \text{div } q)_{T_i} + (u_{(\partial T)^i}, q \cdot n_{T})_{(\partial T)^i} + (u_{T_{3-i}}, q \cdot n_{T_i})_{T\Gamma}$$

- Both options avoid Nitsche’s consistency terms
  - no penalty parameter needs to be taken large enough!
Local bilinear form in cut cells

• Local bilinear form

\[ a_T(\hat{u}_T, \hat{w}_T) := \sum_{i \in \{1, 2\}} \left\{ \kappa_i (G_{T_i}(\hat{u}_T), G_{T_i}(\hat{w}_T))_{T_i} + s_{T_i}(\hat{u}_T, \hat{w}_T) \right\} + s_T^\Gamma(u_T, w_T) \]

• LS stabilization inside each subdomain

\[ s_{T_i}(\hat{u}_T, \hat{w}_T) := \kappa_i h_{T_i}^{-1} (\Pi_k (\partial T_i)(u_{T_i}|_{\partial T_i} - u(\partial T_i)^i), w_{T_i}|(\partial T_i) - w(\partial T_i)^i)(\partial T_i)^i \]

• Interface bilinear form

\[ s_T^\Gamma(u_T, w_T) := \eta \kappa_1 h_T^{-1} ([u_T]_\Gamma, [w_T]_\Gamma)_{T\Gamma} \text{ with } \eta = O(1) \]

• The use of two gradient reconstructions allows for robustness w.r.t. contrast \((\kappa_1 \ll \kappa_2)\)
  - use option 1 in \(\Omega_1\) and option 2 in \(\Omega_2\)
  - \(a_T\) is symmetric, but \(\Omega_1/\Omega_2\) do not play symmetric roles
The global dofs are in

\[ \hat{u}_h \in \hat{U}_h := \bigotimes_{T \in \mathcal{T}^1} \mathbb{P}^{k+1}(T_1) \times \bigotimes_{T \in \mathcal{T}^2} \mathbb{P}^{k+1}(T_2) \times \bigotimes_{F \in \mathcal{F}^1} \mathbb{P}^k(F_1) \times \bigotimes_{F \in \mathcal{F}^2} \mathbb{P}^k(F_2) \]

- We set to zero all the face components attached to \( \partial \Omega \)
- We collect in \( \hat{u}_T \) all the global unknowns related to a mesh cell \( T \)
Global problem: Find $\hat{u}_h \in \hat{U}_h$ such that

$$a_h(\hat{u}_h, \hat{w}_h) = \ell_h(\hat{w}_h), \quad \forall \hat{w}_h \in \hat{U}_h$$

with $a_h(\hat{u}_h, \hat{w}_h) = \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T)$ and $\ell_h(\hat{w}_h) = \sum_{T \in \mathcal{T}} \ell_T(\hat{w}_T)$ with the consistent rhs

$$\ell_T(\hat{w}_T) := (f, w_{T_1})_{T_1} + (f, w_{T_2})_{T_2} + (g_N, w_{T_2})_{T \Gamma}$$

$$- \kappa_1 (g_D, G_{T_1}(\hat{w}_T) \cdot n_{T \Gamma} + \eta h_T^{-1} [\! [w_T] \! ]_{T \Gamma})$$

- All the cell dofs are eliminated locally by static condensation
- Only the face dofs are globally coupled
Error analysis

- Multiplicative and discrete trace inequalities [Burman, AE 18]
  - for any cut cell $T$, there is a ball $T^{\dagger}$ of size $O(h_T)$ containing $T$ and a finite number of its neighbors, and s.t. all $T \cap \Gamma$ is visible from a point in $T^{\dagger}$
  - small ball with diameter $O(h_T)$ present on both sides of interface
  - achievable using local cell agglomeration if mesh fine enough

Error estimate

Assuming that $u|_{\Omega_i} \in H^{1+t}(\Omega_i)$ with $t \in (\frac{1}{2}, k + 1]$, 

$$
\sum_{T} \sum_{i \in \{1,2\}} \kappa_i \|\nabla(u - u_{T_i})\|_{T_i}^2 \leq C h^{2t} \sum_{i \in \{1,2\}} \kappa_i |u|_{H^{t+1}(\Omega_i)}^2
$$

Convergence order $O(h^{k+1})$ if $u|_{\Omega_i} \in H^{k+2}(\Omega_i)$
Agglomeration procedure (1/3)

- Three-stage procedure with proven locality in the agglomeration
  - for any cell KO in $\Omega_1$, find matching partner OK in $\Omega_2$
  - for any cell KO in $\Omega_2$ not matched, find matching partner OK in $\Omega_1$
  - rearrange locally partnerships to avoid propagation
Agglomeration procedure (2/3)

- A 16x16 mesh with circular interface
A 16x16 mesh with flower-like interface
Test case with contrast

- \( \kappa_1 = 1, \kappa_2 = 10^4, g_D = g_N = 0, \eta = 1 \)

- Circular interface \((r^2 = (x_1 - 0.5)^2 + (x_2 - 0.5)^2)\)

- Exact solution: \(u_1 := \frac{r^6}{\kappa_1}, u_2 := \frac{r^6}{\kappa_2} + R^6(\frac{1}{\kappa_1} - \frac{1}{\kappa_2})\)
Test case with jump

- Flower-like interface, $\kappa_1 = \kappa_2 = 1$
- Exact solution with jump

$$u(x_1, x_2) := \begin{cases} 
\sin(\pi x_1) \sin(\pi x_2) & \text{in } \Omega_1 \\
\sin(\pi x_1) \sin(\pi x_2) + 2 + x^3 y^3 & \text{in } \Omega_2 
\end{cases}$$
Wave propagation

- Subdomains $\Omega_1, \Omega_2 \subset \Omega$, interface $\Gamma$, jump $[a]_\Gamma = a|_{\Omega_1} - a|_{\Omega_2}$
- Acoustic wave propagation across interface

\[
\begin{cases}
\frac{1}{k} \partial_{tt} p - \text{div} \left( \frac{1}{\rho} \nabla p \right) = f \\
[p]_\Gamma = 0, \quad \left[ \frac{1}{\rho} \nabla p \right]_\Gamma \cdot \mathbf{n}_\Gamma = 0
\end{cases}
\text{in } J \times (\Omega_1 \cup \Omega_2)
\text{on } J \times \Gamma
\]

- Main ideas as for elliptic interface problems
  - mixed-order setting $k' = k + 1$
  - distinct gradient reconstructions $G_{T_i}$ in $\mathbb{P}^k(T_i; \mathbb{R}^d)$, $i \in \{1, 2\}$
  - LS stabilization on $(\partial T)^i, i \in \{1, 2\} \Rightarrow s_{T_i}(\cdot, \cdot)$
Unfitted HHO discretization

- **Second-order formulation**

\[
\partial_{tt} p_T(t, w_T) + \frac{1}{\kappa} \; \Omega + (G_T(\hat{p}_h(t)), G_T(\hat{w}_h)) + \frac{1}{\rho} \; \Omega + s_h^{1,2}(\hat{p}_h(t), \hat{w}_h) + s_h^\Gamma(p_T(t), w_T) = (f(t), w_T)\Omega
\]

- \( s_h^\Gamma(p_T(t), w_T) := (\rho h_T)^{-1}(\left[p_T\right]\Gamma, \left[w_T\right]\Gamma)\Gamma \)

- Algebraic realization and **Newmark time-stepping** as in fitted case
Unfitted HHO discretization

- Second-order formulation

\[(\partial_{tt} p_T(t), w_T)_{\frac{1}{\kappa} \Omega} + (G_T(\hat{p}_h(t)), G_T(\hat{w}_h))_{\frac{1}{\rho} \Omega} + s^1_{hT}(\hat{p}_h(t), \hat{w}_h) + s^\Gamma_h(p_T(t), w_T) = (f(t), w_T)_{\Omega}\]

\[s^\Gamma_h(p_T(t), w_T) := (\rho_1 h_T)^{-1}([p_T] \Gamma, [w_T] \Gamma)_T\Gamma\]

- Algebraic realization and Newmark time-stepping as in fitted case

- First-order formulation \((v := \partial_t p, \sigma := \frac{1}{\rho} \nabla p)\)

\[
\begin{cases}
(\partial_t \sigma_T(t), \tau_T)_{\rho;\Omega} - (G_T(\hat{v}_h(t)), \tau_T)_{\Omega} = 0 \\
(\partial_t v_T(t), w_T)_{\frac{1}{\kappa} \Omega} + (\sigma_T(t), G_T(\hat{w}_h))_{\Omega} + \tilde{s}^1_{h}(\hat{v}_h(t), \hat{w}_h) + \tilde{s}^\Gamma_h(v_T(t), w_T) = (f(t), w_T)_{\Omega}
\end{cases}
\]

\[\tilde{s}^\Gamma_h(v_T(t), w_T) := \tilde{\tau}^\Gamma_{\partial T}([v_T] \Gamma, [w_T] \Gamma)_T\Gamma\]

\[\tilde{\tau}^\Gamma_{\partial T} = (\rho_1 c_1)^{-1} = O(1) \text{ for ERK, and } \tilde{\tau}^\Gamma_{\partial T} = O(h_T^{-1}) \text{ for SDIRK}\]

- Algebraic realization and RK time-stepping as in fitted case
Fitted-unfitted comparison

- 2D heterogeneous test case with flat interface
  - $\Omega_1 := (-\frac{3}{2}, \frac{3}{2}) \times (-\frac{3}{2}, 0)$, $\Omega_2 := (-\frac{3}{2}, \frac{3}{2}) \times (0, \frac{3}{2})$
  - Ricker wavelet centered at $(0, \frac{2}{3}) \in \Omega_2$, sensor $S_1 = (\frac{3}{4}, -\frac{1}{3}) \in \Omega_1$
  - fitted and unfitted HHO behave similarly, both benefit from increasing $k$
Fitted-unfitted comparison

- 2D heterogeneous test case with flat interface
  - $\Omega_1 := (-\frac{3}{2}, \frac{3}{2}) \times (-\frac{3}{2}, 0)$, $\Omega_2 := (-\frac{3}{2}, \frac{3}{2}) \times (0, \frac{3}{2})$
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  - fitted and unfitted HHO behave similarly, both benefit from increasing $k$

- HHO-Newmark, $\sigma_x$ signals
  - comparison of semi-analytical and HHO (fitted or unfitted) solutions
  - $k = 1$ (top) and $k = 3$ (bottom)
  - $c_2/c_1 = \sqrt{3}$ (low contrast, left) or $c_2/c_1 = 8\sqrt{3}$ (high contrast, right)
Homogeneous test case, flat interface

CFL condition for ERK(s): \( \frac{c\Delta t}{h} \leq \beta(s)\mu(k) \)
- \( \beta(s) \) mildly depends on the number of stages
- \( \mu(k) \) behaves as \((k + 1)^{-1}\) and is quantified by solving a generalized eigenvalue problem with the mass and stiffness matrices

Additional jump penalties in unfitted HHO only mildly impact \( \mu(k) \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fitted-HHO</td>
<td>0.118</td>
<td>0.0522</td>
<td>0.0338</td>
<td>0.0229</td>
</tr>
<tr>
<td>Unfitted-HHO</td>
<td>0.0765</td>
<td>0.0373</td>
<td>0.0232</td>
<td>0.0159</td>
</tr>
<tr>
<td>Ratio</td>
<td>1.5</td>
<td>1.4</td>
<td>1.5</td>
<td>1.4</td>
</tr>
</tbody>
</table>
CFL condition for ERK (2/2)

- Homogeneous test case, circular interface
  - study of impact of agglomeration parameter $\theta_{agg}$ on $\mu(k)$
  - “badly cut” cell flagged if relative area of any subcell falls below $\theta_{agg}$
- Agglomerated cells for $\theta_{agg} = 0.3$ on a sequence of refined quad meshes
Homogeneous test case, circular interface

- study of impact of agglomeration parameter $\theta_{\text{agg}}$ on $\mu(k)$
- “badly cut” cell flagged if relative area of any subcell falls below $\theta_{\text{agg}}$

- Agglomerated cells for $\theta_{\text{agg}} = 0.3$ on a sequence of refined quad meshes

Behavior of $h\mu(k)$ and impact of $\theta_{\text{agg}}$ on $\mu(k)$

- tolerating badly cut cells deteriorates the CFL condition

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{\text{agg}} = 0.5$</td>
<td>0.042</td>
<td>0.022</td>
<td>0.014</td>
<td>0.0099</td>
</tr>
<tr>
<td>$\theta_{\text{agg}} = 0.3$</td>
<td>0.030</td>
<td>0.015</td>
<td>0.0094</td>
<td>0.0065</td>
</tr>
<tr>
<td>Ratio</td>
<td>1.4</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>$\theta_{\text{agg}} = 0.1$</td>
<td>0.017</td>
<td>0.0087</td>
<td>0.0055</td>
<td>0.0039</td>
</tr>
<tr>
<td>Ratio</td>
<td>2.5</td>
<td>2.6</td>
<td>2.6</td>
<td>2.5</td>
</tr>
</tbody>
</table>
Flower-like interface

- Agglomerated cells for a flower-like interface (quad mesh, $h = 2^{-5}$), HHO-SDIRK(3,4) signal for $\sigma_x$ at two sensors, $k \in \{1, 2, 3\}$, $c_2/c_1 = \sqrt{3}$

- Pressure isovalues, SDIRK(3,4), $k = 3$, $h = 0.1 \times 2^{-8}$, $\Delta t = 2^{-6}$
Some references

- **HHO**
  - textbooks [Di Pietro, Droniou, 20; Cicuttin, AE, Pignet, 21]

- **HHO for wave propagation**
  - [Burman, Duran, AE 21 (CAMC)], [Burman, Duran, AE, Steins 21 (JSC)]

- **Unfitted HHO**
  - [Burman, AE 18 (SINUM)], [Burman, Cicuttin, Delay, AE 21 (SISC)]

Thank you for your attention!
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- **Unfitted HHO**
  - [Burman, AE 18 (SINUM)], [Burman, Cicuttin, Delay, AE 21 (SISC)]

- **New Finite Element book(s) (Springer, TAM vols. 72-74, 2021)**
  - with J.-L. Guermond, 83 chapters of 12/14 pages plus about 500 exercises
Some references

- **HHO**
  - textbooks [Di Pietro, Droniou, 20; Cicuttin, AE, Pignet, 21]

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  - [Burman, Duran, AE 21 (CAMC)], [Burman, Duran, AE, Steins 21 (JSC)]

- **Unfitted HHO**
  - [Burman, AE 18 (SINUM)], [Burman, Cicuttin, Delay, AE 21 (SISC)]

- **New Finite Element book(s) (Springer, TAM vols. 72-74, 2021)**
  with J.-L. Guermond, 83 chapters of 12/14 pages plus about 500 exercises

Thank you for your attention!