UPMC–M2R Parcours ANEDP
Discontinuous Galerkin Methods and Applications
Exam – 13 April 2015

The exam lasts three hours. The only authorized document is, as announced in class, one A4 sheet of paper with hand-written annotations (recto and verso).

Important. Write on **separate sheets** your answers for Part A and Part B.

1 Part A

1.1 Error analysis (3 pts)

a) Consider the following problem :

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,w) = \ell(w), \quad \forall w \in W, \end{cases}$$

where V and W are Hilbert spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, a a bounded bilinear form on $V \times W$ and ℓ a bounded linear form on W. Specify the necessary and sufficient conditions for this problem to be well-posed.

b) Consider a finite-dimensional space V_h and the discrete problem

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a_h(u_h,w_h) = \ell(w_h), \quad \forall w_h \in V_h. \end{cases}$$

What does it mean that the approximation is consistent ?

c) Assume there is $C_{\mathrm{sta}} > 0$ such that

$$\forall v_h \in V_h, \qquad C_{\mathrm{sta}} |||v_h||| \le \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h,w_h)}{|||w_h|||},$$

where $|||\cdot|||$ is a norm defined on $V_{(h)} := V + V_h$, and there is C_{bnd} such that

$$\forall (v,w_h) \in V_{(h)} \times V_h, \qquad |a_h(v,w_h)| \le C_{\mathrm{bnd}} |||v||| \, |||w_h|||.$$

Assuming consistency, state and prove an error estimate.

1.2 Advection-reaction (3 pts)

Consider the steady advection-reaction equation

$$\begin{aligned} \mu u + \beta\cdot\nabla u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega^-, \end{aligned}$$

where $\mu \in L^\infty(\Omega)$, $\beta \in W^{1,\infty}(\Omega)^d$ with $\mu - \frac{1}{2}\nabla\cdot\beta \ge \mu_0 > 0$ and $\partial\Omega^- = \{x \in \partial\Omega;\ \beta\cdot n(x) < 0\}$.

a) Consider the discontinuous Galerkin approximation using upwind fluxes and the broken polynomial space $\mathbb{P}_d^k(\mathcal{T}_h)$. Specify the discrete bilinear form a_h, the right-hand side of the discrete problem, and formulate the discrete problem in local form using fluxes (no proof is required).

b) Prove that the bilinear form a_h is consistent and coercive for a norm to be specified (assuming the exact solution is smooth enough so that $(\beta{\cdot}n_F)[\![u]\!]_F = 0$ for all $F \in \mathcal{F}_h^i$).

c) Give (without proof) a norm (stronger than the coercivity norm) for which the discrete bilinear form a_h is inf-sup stable.

1.3 Diffusion : Symmetric Interior Penalty (4 pts)

a) Consider the following problem :

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla u{\cdot}\nabla w = \int_\Omega f w, \quad \forall w \in H_0^1(\Omega). \end{cases}$$

Write the Symmetric Interior Penalty bilinear form to be used to approximate this problem on the broken polynomial space $\mathbb{P}_d^k(\mathcal{T}_h)$, $k \geq 1$.

b) In which norm do we have coercivity and what is the condition on the penalty coefficient (a proof is expected) ?

c) Devise the local formulation using fluxes (introduce the discrete gradient $G_h^l(u_h)$ with $l \in \{k-1, k\}$).

d) Consider the discrete flux $\sigma_h = G_h^l(u_h)$. Prove that the pair (σ_h, u_h) solves the following local formulation, for all $T \in \mathcal{T}_h$,

$$\int_T \sigma_h{\cdot}\zeta - \int_T u_h \nabla{\cdot}\zeta + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \hat{u}_F(\zeta{\cdot}n_F) = 0,$$

$$-\int_T \sigma_h{\cdot}\nabla\xi + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F (\hat{\sigma}_F{\cdot}n_F)\xi = \int_T f\xi,$$

for all $\zeta \in [\mathbb{P}_d^k(T)]^d$ and all $\xi \in \mathbb{P}_d^k(T)$, where the fluxes $\hat{u}_F$ and $\hat{\sigma}_F$ have to be specified.

2 Part B

2.1 Linear transport (2 pts)

We consider the ordinary differential equation

$$\begin{aligned} \mathrm{d}_t u_h(t) + A_h^{\mathrm{up}} u_h(t) &= 0, \\ u_h(0) &= u_{0,h}, \end{aligned}$$

where A_h^{up} denotes the operator associated to the DG-upwind linear form and $u_{0,h}$ is a given function. Find a set of coefficients

$$A = (a_{ij}) \in \mathbb{R}^{3\times 3}, \qquad b = (b_i) \in \mathbb{R}^3,$$

such that the Runge–Kutta method

$$\begin{aligned} k_i &= -A_h^{\mathrm{up}}\big(u_h^n + \delta t \sum_{j=1}^{3} a_{ij} k_j\big), \qquad i = 1, 2, 3, \\ u_h^{n+1} &= u_h^n + \delta t \sum_{i=1}^{3} b_i k_i, \end{aligned}$$

is explicit and of order 3. Justify your answer.

2.2 Nonlinear conservation laws (2 pts)

a) Recall the definition of a monotone numerical flux and show that the Rusanov flux

$$\Phi_{\mathrm{Rusanov}}(n, u^-, u^+) = \frac{f_n(u^-) + f_n(u^+)}{2} + \sup_{v \in \mathcal{U}} |f_n'(v)| \frac{(u^- - u^+)}{2}$$

is monotone. Give its precise form in the particular case of the linear equation with $f(u) = \beta u$.

b) Recall the definition of the Roe numerical flux and show that it can be written in an equivalent way as

$$\Phi_{\mathrm{Roe}}(n, u^-, u^+) = \begin{cases} f_n(u^+) & \text{if } \frac{f_n(u^+) - f_n(u^-)}{u^+ - u^-} \geq 0, \\ f_n(u^-) & \text{if } \frac{f_n(u^+) - f_n(u^-)}{u^+ - u^-} < 0. \end{cases}$$

2.3 Implementation (2 pts)

a) Explain the advantages and disadvantages of a high-order method with respect to a low-order method.

b) What are the consequences of the choice of the basis functions in the approximation space for an unsteady and a steady problem ?

2.4 Stokes equation (4 pts)

a) Let

$$b_h(v_h, q_h) := -\int_\Omega q_h \nabla_h \cdot v_h + \sum_{F\in\mathcal{F}_h} \int_F [\![v_h]\!] \cdot n_F \{q_h\}$$

Show the following equality

$$b_h(v_h, q_h) = \int_\Omega \nabla_h q_h \cdot v_h - \sum_{F\in\mathcal{F}_h^i} \int_F \{v_h\} \cdot n_F [\![q_h]\!],$$

for all $v_h \in [\mathbb{P}_d^k(\mathcal{T}_h)]^d$ and $q_h \in \mathbb{P}_d^k(\mathcal{T}_h)$.

b) Let $q_h \in P_h$. We know that there exists $v_{q_h} \in U$ such that $\nabla \cdot v_{q_h} = q_h$ and $\beta_\Omega \|v_{q_h}\|_U \leq \|q_h\|_P$ with $\beta_\Omega > 0$. Show that

$$\|q_h\|_P^2 = -\int_\Omega \nabla_h q_h \cdot v_{q_h} + \sum_{F\in\mathcal{F}_h^i} \int_F [\![q_h]\!] v_{q_h} \cdot n_F.$$

c) Then, show that

$$\|q_h\|_P^2 = -b_h(\Pi_h v_{q_h}, q_h) + \sum_{F\in\mathcal{F}_h^i} \int_F [\![q_h]\!] \{v_{q_h} - \Pi_h v_{q_h}\} \cdot n_F,$$

where $\Pi_h : [L^2(\Omega)]^d \to U_h$ is the component-wise $L^2(\Omega)$-projection such that

$$|\!|\!| \Pi_h v |\!|\!|_{\text{vel}} \leq C_\Pi \|v\|_U, \qquad \forall v \in U,$$

$$\|v - \Pi_h v\|_{[L^2(F)]^d} \leq C h_T^{\frac{1}{2}} \|v\|_{[L^2(T)]^{d\times d}}, \qquad \forall v \in U, T \in \mathcal{T}_h, F \in \mathcal{F}_T.$$

d) Finally, show that there exists $\beta > 0$ such that

$$\forall q_h \in P_h, \qquad \beta \|q_h\|_P \leq \sup_{w_h \in U_h \setminus \{0\}} \frac{b_h(w_h, q_h)}{|\!|\!| w_h |\!|\!|_{\text{vel}}} + |q_h|_P.$$

Reminder on the notation :

$$U = [H_0^1(\Omega)]^d, \quad P = \Big\{q \in L^2(\Omega) \mid \int_\Omega q = 0\Big\}, \quad U_h = [\mathbb{P}_d^k(\mathcal{T}_h)]^d, \quad P_h = \mathbb{P}_d^k(\mathcal{T}_h) \cap P,$$

$$\|v\|_U^2 = \sum_{i=1}^d \|v_i\|_{H^1(\Omega)}^2, \qquad \|q\|_P = \|q\|_{L^2(\Omega)},$$

$$|\!|\!| v_h |\!|\!|_{\text{vel}}^2 = \|\nabla_h v_h\|_{[L^2(\Omega)]^{d\times d}}^2 + \sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]\|_{[L^2(F)]^d}^2, \qquad |q_h|_P^2 = \sum_{F\in\mathcal{F}_h^i} h_F \|[\![v_h]\!]\|_{L^2(F)}^2$$