Accurate velocity reconstruction for Discontinuous Galerkin approximations of two-phase porous media flows

A. Ern [a], I. Mozolevski [a,1], and L. Schuh [b]

[a] *Université Paris-Est, CERMICS, Ecole des Ponts, 6 & 8 Av. B. Pascal, 77455 Marne-la-Vallée cedex 2, France*
[b] *IME, Universidade de São Paulo, Rua do Matão, 1010, Cidade Universitária, 05508-090, São Paulo, SP, Brasil*

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Abstract

We consider Discontinuous Galerkin approximations of two-phase, immiscible porous media flows in the global pressure/fractional flow formulation with capillary pressure. A sequential approach is used with a backward Euler step for the saturation equation, equal-order interpolation for the pressure and the saturation, and without any limiters. An accurate total velocity field is recovered from the global pressure equation to be used in the saturation equation. Numerical experiments show the advantages of the proposed reconstruction. *To cite this article: A. Ern, I. Mozolevski and L. Schuh, C. R. Acad. Sci. Paris, Ser. I ??? (????).*

Résumé

Nous considérons une méthode de Galerkine discontinue pour approcher les écoulements diphasiques non-miscibles en milieu poreux dans la formulation en pression globale. Une approche séquentielle est utilisée avec un schéma d'Euler implicite pour l'équation de la saturation, le même ordre polynomial pour la pression et la saturation, et en l'absence de limiteurs. Nous montrons comment reconstruire à partir de l'équation en pression une vitesse totale précise pour l'équation de la saturation. Des exemples numériques illustrent les avantages de l'approche proposée. *Pour citer cet article : A. Ern, I. Mozolevski and L. Schuh, C. R. Acad. Sci. Paris, Ser. I ? ? ? (? ? ? ?).*

1. Introduction

One of the most popular formulations of the governing equations for two-phase immiscible porous media flows is the so-called global pressure/fractional flow approach introduced by Chavent and Jaffré

Email addresses: ern@cermics.enpc.fr (A. Ern), Igor.Mozolevski@mtm.ufsc.br (I. Mozolevski), lucianes@ime.usp.br (L. Schuh).

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[2]. Finite volume and finite element approximations to this formulation have been analyzed recently; see, e.g., [3,6]. In the global pressure/fractional flow formulation (for short, GP formulation), an elliptic equation of Darcy type for the global pressure is coupled to a nonlinear degenerate parabolic equation for the saturation of, say, the non-wetting phase by means of a total velocity that is calculated from the global pressure equation. Such weak coupling allows to develop and sequentially use efficient numerical methods for each type of equation and is considered as one of the important advantages of the GP formulation.

Discontinuous Galerkin (DG) methods are often advocated as a suitable approach to discretize in space both pressure and saturation equations. Advantages include the flexibility in using non-matching meshes and variable polynomial degrees and the potential to limit non-physical oscillations near singularities; see, e.g. [7]. However, one critical issue when approximating the global pressure equation by DG methods is the accurate recovery of the total velocity to be used sequentially in the saturation equation. One possibility suggested in [4] is to resort to a fully implicit DG method, since numerical experiments indicate that non-physical oscillations can be avoided, but this entails substantial computational costs to solve the full nonlinear system of discrete equations coupling the global pressure and saturation equations. Alternatively, a total velocity can be postprocessed from the broken gradient of the approximate global pressure. A first approach [1] considers Brezzi–Douglas–Marini finite element spaces and yields a postprocessed total velocity with continuous normal component at interelement faces. More recently, a reconstruction using Raviart–Thomas finite element spaces has been proposed [5], thereby improving the accuracy of the postprocessed velocity.

The aim of this Note is to present a DG method for two-phase, immiscible porous media flows in the GP formulation that uses an implicit scheme in time for the saturation and equal-order interpolation for the pressure and the saturation, while still avoiding non-physical oscillations near singularities. The key feature of the proposed method is to use the postprocessing of [5] to reconstruct accurately the total velocity from the global pressure distribution. We believe that the present formulation can be useful in petroleum reservoir and groundwater flow simulations. For simplicity, we consider herein a one-dimensional setting, but the methodology can be readily extended to multiple space dimensions.

2. The DG approximation of the global pressure/fractional flow formulation

Let P denote the global pressure, u the total velocity, and S the non-wetting phase saturation. The governing equations are for $x \in (0, L)$ and $t \in (0, T)$,

$$u = -\lambda(S)K\partial_x P, \qquad -\partial_x\left(\lambda(S)K\partial_x P\right) = 0, \tag{1}$$

$$\phi\partial_t S + \partial_x\left(uf(S)\right) - \partial_x\left(\epsilon(S)\partial_x\pi(S)\right) = 0, \tag{2}$$

where ϕ denotes the porosity and K the intrinsic (absolute) permeability, both parameters being taken constant for simplicity. Moreover, defining the total mobility $\lambda = \lambda_w + \lambda_n$ as the sum of the wetting and non-wetting phase mobilities, $f = \frac{\lambda_n}{\lambda}$ denotes the fractional flux, π the capillary pressure, and $\epsilon = \lambda_w f K$. The quantities λ_n, λ_w, π, and ϵ are smooth functions of S; examples of such functions are given below. The saturation S takes values in $[S_{rn}, 1 - S_{rw}]$, where S_{rn} and S_{rw} respectively denote the residual saturation of the non-wetting and wetting phases. Since we are concerned with accurate velocity recovery, we avoid additional difficulties when S approaches the limits of its admissible values; thus, we suppose here that S is uniformly bounded away from S_{rn} and $1 - S_{rw}$, so that the saturation equation is non-degenerate. Typical boundary and initial conditions are

$$P|_{x=0} = P_1,\ P|_{x=L} = P_2; \quad S|_{x=0} = S_1,\ -\epsilon(S)\partial_x\pi(S)|_{x=L} = 0; \quad S|_{t=0} = S_0. \tag{3}$$

Finally, we observe that owing to the first equation in (1), the total velocity u is constant in space and only depends on time, i.e., $\partial_x u = 0$. Owing to the choice of boundary conditions, its time evolution is a priori unknown.

To discretize in space, consider (for simplicity) a uniform partition of the domain $(0, L)$ with M elements $\mathcal{T} = \{T_i\}_{1\le i\le M}$ and nodes $\mathcal{N} = \{x_i\}_{1\le i\le M+1}$ with $T_i = (x_i, x_{i+1})$ of length $h = \frac{L}{M}$. Set $\mathcal{N}' = \mathcal{N}\backslash\{x_{M+1}\}$. Let an integer $p \ge 1$ and let V_h denote the space of piecewise polynomials of degree $\le p$ on each mesh element. For any node $x_i \in \mathcal{N}$, define "the unit normal vector" $\mathbf{n}_i$ as $\mathbf{n}_1 = -1$ and $\mathbf{n}_i = 1$ if $i > 1$. Moreover, for any smooth enough function v that is possibly two-valued at x_i, define its jump and mean-value at x_i respectively as $[\![v]\!]_i = v|_{T_{i-1}}(x_i) - v|_{T_i}(x_i)$ and $\{v\}_i = \frac{1}{2}(v|_{T_{i-1}}(x_i) + v|_{T_i}(x_i))$ if x_i is an interior node and as $[\![v]\!]_i = \{v\}_i = v(x_i)$ if x_i is a boundary node. Furthermore, to discretize in time, let N be an integer and let $\tau = \frac{T}{N}$ be the time step.

We consider a sequential scheme with implicit backward Euler time approximation of the saturation equation, symmetric interior penalty DG method for the global pressure equation and for the diffusion term in the saturation equation, and Godunov fluxes for the nonlinear hyperbolic term in the saturation equation. Namely, for $n = 0, 1, \cdots, N-1$, we solve for $P_h^{n+1} \in V_h$ such that $\forall z \in V_h$,

$$\begin{aligned}\sum_{T\in\mathcal{T}}\int_T \lambda(S_h^n)Kd_xP_h^{n+1}d_xz - \sum_{x_i\in\mathcal{N}}\left(\mathbf{n}_i\{\lambda(S_h^n)Kd_xP_h^{n+1}\}_i[\![z]\!]_i + \mathbf{n}_i\{\lambda(S_h^n)Kd_xz\}_i[\![P_h^{n+1}]\!]_i\right) \\ + \sum_{x_i\in\mathcal{N}}\gamma_i[\![P_h^{n+1}]\!]_i[\![z]\!]_i = (\lambda(S_h^n)Kd_xz + \gamma_1 z)|_{x=0}P_1 + (-\lambda(S_h^n)Kd_xz + \gamma_{M+1}z)|_{x=L}P_2,\end{aligned} \tag{4}$$

with $S_h^n \in V_h$ given from the previous step ($n \ge 1$) or by the initial data ($n = 0$), and then we solve for $S_h^{n+1} \in V_h$ such that $\forall v \in V_h$,

$$\begin{aligned}&\sum_{T\in\mathcal{T}}\int_T \phi\tau^{-1}S_h^{n+1}v - \sum_{T\in\mathcal{T}}\int_T u_h^{n+1}f(S_h^{n+1})d_xv + \sum_{x_i\in\mathcal{N}}\Phi_{hi}^{n+1}[\![v]\!]_i \\ &+ \sum_{T\in\mathcal{T}}\int_T \epsilon(S_h^n)\pi'(S_h^n)d_xS_h^{n+1}d_xv - \sum_{x_i\in\mathcal{N}'}\left(\mathbf{n}_i\{\epsilon(S_h^n)\pi'(S_h^n)d_xS_h^{n+1}\}[\![v]\!]_i + \mathbf{n}_i\{\epsilon(S_h^n)\pi'(S_h^n)d_xv\}[\![S_h^{n+1}]\!]_i\right) \\ &+ \sum_{x_i\in\mathcal{N}'}\delta_i[\![S_h^{n+1}]\!]_i[\![v_h]\!]_i = \sum_{T\in\mathcal{T}}\int_T \phi\tau^{-1}S_h^nv + (\epsilon(S_h^n)\pi'(S_h^n)d_xv + \delta_1 v)|_{x=0}S_1.\end{aligned} \tag{5}$$

The key point, namely the calculation of u_h^{n+1} from (4) to be used in (5), is discussed below. Furthermore, the penalty coefficients γ_i and δ_i are evaluated as $\gamma_i = \gamma_* p^2h^{-1}K\min_{T;x_i\in\partial T}\{\lambda(S_h^n)|_T(x_i)\}$ and $\delta_i = \delta_* p^2h^{-1}\min_{T;x_i\in\partial T}\{\epsilon(S_h^n)\pi'(S_h^n)|_T(x_i)\}$ with the numerical parameters γ_* and δ_* in the range $[5, 10]$. Godunov's flux for nondecreasing flux function f coincides with flux upwinding: $\Phi_{h1}^{n+1} = u_h^{n+1}(x_1)f(S_1)$ and $\Phi_{hi}^{n+1} = u_h^{n+1}(x_i)f(S_h^{n+1})|_{T_{i-1}}(x_i)$ for $i \ge 2$.

3. Total velocity reconstruction

In the present one-dimensional setting, the total velocity u_h^{n+1} belongs to the space W_h spanned by *continuous*, piecewise polynomials of degree $\le p+1$ on each mesh element. In a multi-dimensional setting, the total velocity belongs to the Raviart–Thomas finite element space of order p. Taking inspiration from [5], u_h^{n+1} is defined locally by setting

$$u_h^{n+1}(x_i) = -\{\lambda(S_h^n)Kd_xP_h^{n+1}\}_i + \mathbf{n}_i\gamma_i[\![P_h^{n+1}]\!]_i', \quad \forall x_i \in \mathcal{N}, \tag{6}$$

$$\int_T u_h^{n+1}w = -\int_T \lambda(S_h^n)Kd_xP_h^{n+1}w + \sum_{x_i\in\partial T}\mathbf{n}_i\{\lambda(S_h^n)Kw\}_i[\![P_h^{n+1}]\!]_i', \quad \forall T \in \mathcal{T}, \forall w \in \mathbb{P}_{p-1}(T), \tag{7}$$

where $[\![P_h^{n+1}]\!]_1' = P_h^{n+1}(x_1) - P_1$, $[\![P_h^{n+1}]\!]_{M+1}' = P_h^{n+1}(x_{M+1}) - P_2$, and $[\![P_h^{n+1}]\!]_i' = [\![P_h^{n+1}]\!]_i$ otherwise. To illustrate the impact of the above reconstruction, we present numerical results with the following data: $L = 300\text{m}$, $\phi = 0.2$, $K = 10^{-11}\text{m}^2$, $S_{rw} = 0.2$, and $S_{rn} = 0.15$. We use Brooks–Corey model for capillary pressure and mobilities, $\pi(S) = P_e(1-S_e)^{-\frac{1}{\theta}}$, $\lambda_w(S) = \frac{1}{\mu_w}(1-S_e)^{\frac{2+3\theta}{\theta}}$, $\lambda_n(S) = \frac{1}{\mu_n}(S_e)^2(1-(1-S_e)^{\frac{2+\theta}{\theta}})$, where $S_e = (S - S_{rn})(1 - S_{rw} - S_{rn})^{-1}$ denotes the effective saturation, $P_e = 10^3\text{Pa}$, $\theta = 2$, $\mu_w = 0.001\text{kg/ms}$, and $\mu_n = 0.01\text{kg/ms}$. The boundary condition for saturation is $S_1 = 0.25$ while the initial condition is $S_0 = 0.7$. The boundary condition for the wetting phase pressure is $p_{w1} = 0.3\text{MPa}$ and $p_{w2} = 0.15\text{MPa}$ whence the global pressure boundary data $P_1 = 0.30111\text{MPa}$ and $P_2 = 0.15217\text{MPa}$ are calculated from the global pressure formula $P = p_w + \int_{S_{rn}}^{S} f(\xi)\pi'(\xi)d\xi + \pi(S_{rn})$ at $S = S_1$ and $S = S_0$ respectively. First-order polynomial approximation is used both for the global pressure and for the saturation on a uniform grid with 32 elements; penalty parameter values are $\gamma_* = \delta_* = 10$. Final simulation time is $T = 360$ days and the time step is $\tau = 5$ days. Figure 1 compares the results obtained with the accurate velocity reconstruction (6)–(7) to two more simpler and less accurate velocity reconstructions, namely (i) taking the piecewise derivative of the global pressure which yields a discontinuous, piecewise constant velocity (labelled NoRec) and (ii) defining a continuous, piecewise affine velocity by setting its nodal values to the average of the global pressure fluxes (labelled NodAv). The NodAv-reconstruction actually amounts in the present setting to the postprocessing proposed in [1]; it accounts partly for the matching condition (6) (yet disregarding the jump contribution), but does not improve the velocity profile inside mesh elements by using (7). While the global pressure profile remains fairly insensitive to the velocity reconstruction, the saturation profile exhibits non-physical oscillations except in the case where the reconstruction defined by (6)–(7) is used. Moreover, oscillations in the phase pressure profiles can also occur (not shown). The origin of these instabilities is the oscillatory behavior in space of the NoRec- and NodAv-velocity reconstructions. Such reconstructions also produce an inaccurate description of the time evolution of the mean value of the total velocity. To conclude, we observe that the present velocity reconstruction, which can be extended to multiple dimensions by proceeding as in [5], yields accurate numerical solutions that are not polluted by spurious oscillations. We emphasize that no limiters were used in the present simulations.

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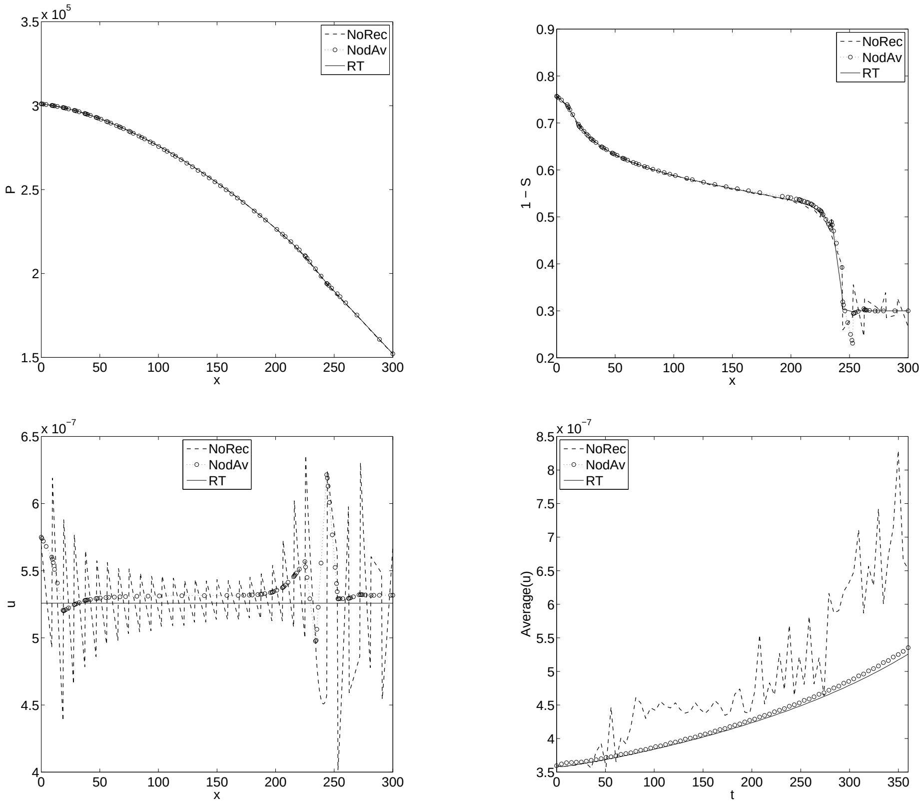

Figure 1. Top left: pressure at final time; top right: wetting phase saturation at final time; bottom left: total velocity at final time; bottom right: mean value of total velocity as a function of time