# Flux reconstruction and a posteriori error estimation for discontinuous Galerkin methods on general nonmatching grids

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**Abstract**

Discontinuous Galerkin methods handle very well general polygonal and nonmatching meshes. We present in this Note a **H**(div)-conforming reconstruction of the flux on such meshes in the setting of an elliptic problem. We exploit the local conservation property of discontinuous Galerkin methods and solve local Neumann problems by means of the Raviart–Thomas–Nédélec mixed finite element method. Our reconstruction can be used in a guaranteed a posteriori error estimate and it is also of an independent interest when the approximate flux is to be used subsequently in a transport problem. *To cite this article: A. Ern, M. Vohralík, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

**Résumé**

**Reconstruction du flux et estimations a posteriori pour la méthode de Galerkine discontinue sur des maillages non-coïncidants avec mailles polygonales** Les méthodes de Galerkine discontinues sont bien adaptées pour traiter des maillages non-coïncidants avec mailles polygonales. Nous présentons dans cette Note une reconstruction **H**(div)-conforme du flux sur de tels maillages pour un problème elliptique. Nous exploitons la propriété de conservativité locale des méthodes de Galerkine discontinues afin de résoudre des problèmes locaux de Neumann approchés par des éléments finis mixtes de Raviart–Thomas–Nédélec. Notre reconstruction peut être utilisée pour une estimation garantie d'erreur a posteriori et également afin d'évaluer une vitesse approchée pour un problème de transport. *Pour citer cet article : A. Ern, M. Vohralík, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## 1. Introduction

We have presented in [4] a **H**(div)-conforming flux reconstruction for discontinuous Galerkin (DG) approximation of elliptic problems on matching (containing no hanging nodes) simplicial meshes. We have extended the reconstruction to nonmatching simplicial meshes in [6]. Both of these reconstructions are based on a direct prescription of the local degrees of freedom of the flux. The purpose of the present Note is to improve these reconstructions by solving instead local minimization problems and to extend them to general polygonal nonmatching meshes. The local problems correspond more precisely to the Raviart–Thomas–Nédélec (RTN) mixed finite element approximation of a Neumann problem on each mesh element, where the Neumann boundary condition is prescribed by the conservative face fluxes available in DG methods. A conforming simplicial submesh of each original element needs to be constructed, and, as an advantage of the present procedure, the divergence of the reconstructed flux on each submesh element is an orthogonal projection of order $l$ of the source term, where $l \in \{k-1, k\}$ and $k$ is the scheme order.

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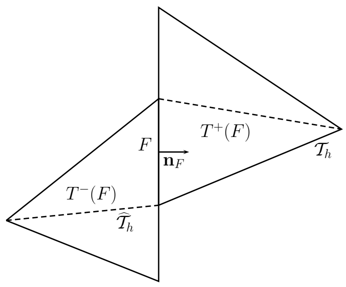


Figure 1. Notation for nonmatching meshes

$\mathbf{H}(\mathrm{div})$-conforming flux reconstructions on matching simplicial meshes have been used previously in the framework of a posteriori error estimation in [9] and in [3,5,8] for DG methods, and a similar technique has been devised in [1]. First results on nonmatching simplicial meshes are presented in [2,6]. Here we use the present reconstruction to obtain an a posteriori error estimate in the spirit of [6], with the improvements described in Section 4 below. We also remark that the derived reconstruction is of independent interest as it can be used in a subsequent transport model, where the fact that it belongs to $\mathbf{H}(\mathrm{div})$ is often crucial.

## 2. Elliptic problem and its DG approximation

Consider the elliptic problem

$$-\nabla \cdot (\mathbf{K}\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal domain, $\mathbf{K} \in [L^\infty(\Omega)]^{d,d}$ is the diffusion tensor, and $f \in L^2(\Omega)$ is the source term. The diffusion tensor is assumed to be symmetric and uniformly positive definite in $\Omega$. These assumptions are sufficient for the existence and uniqueness of a weak solution.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations of the domain $\Omega$ on which the diffusion tensor $\mathbf{K}$ is assumed to be piecewise constant. A generic element in $\mathcal{T}_h$ is denoted by $T$, $h_T$ stands for its diameter, and $\mathbf{n}_T$ for its unit outward normal. The mesh elements are polygonal, possibly nonconvex subsets of $\Omega$. Meshes can possess hanging nodes, as depicted in Figure 1 in the particular case of two triangular elements. The DG approximation space is $V^k(\mathcal{T}_h) := \{v_h \in L^2(\Omega); v_h|_T \in \mathbb{P}_k(T)\ \forall T \in \mathcal{T}_h\}$, where $\mathbb{P}_k(T)$, $k \geq 0$, is the set of polynomials of degree less than or equal to $k$ on an element $T$. For all $l \geq 0$, $\Pi_l$ denotes the $L^2$-orthogonal projection onto $V^l(\mathcal{T}_h)$.

We say that $F$ is an interior face if it has positive $(d{-}1)$-measure and if there are distinct $T^-(F)$ and $T^+(F)$ in $\mathcal{T}_h$ such that $F = \partial T^-(F) \cap \partial T^+(F)$ and we define $\mathbf{n}_F$ as the unit normal vector to $F$ pointing from $T^-(F)$ towards $T^+(F)$, *cf.* Figure 1. Similarly, we say that $F$ is a boundary face of the mesh if it has positive $(d{-}1)$-measure and there is $T(F) \in \mathcal{T}_h$ such that $F = \partial T(F) \cap \partial\Omega$ and we define $\mathbf{n}_F$ as the unit outward normal to $\partial\Omega$. The faces of the mesh are collected into the set $\mathcal{F}_h$. For $F \in \mathcal{F}_h$, $h_F$ denotes its diameter. For a function $v$ that is double-valued on an interior face, its jump and weighted average on $F$ are defined as $[\![v]\!] := v|_{T^-(F)} - v|_{T^+(F)}$ and $\{\!\{v\}\!\}_\omega := \omega_{T^-(F),F} v|_{T^-(F)} + \omega_{T^+(F),F} v|_{T^+(F)}$. Here, the nonnegative weights have to satisfy $\omega_{T^-(F),F} + \omega_{T^+(F),F} = 1$; for their choice leading to robustness with respect to inhomogeneities in $\mathbf{K}$, we refer to [7]. On boundary faces, we set $[\![v]\!] := v|_F$ and $\{\!\{v\}\!\}_\omega := v|_F$ and $\omega_{T(F),F} := 1$.

For each $\mathcal{T}_h$, there exists a matching simplicial submesh $\widehat{\mathcal{T}}_h$ of $\mathcal{T}_h$ such that $\widehat{\mathcal{T}}_h = \mathcal{T}_h$ if $\mathcal{T}_h$ is itself matching and simplicial. If $\mathcal{T}_h$ is nonmatching, $\widehat{\mathcal{T}}_h$ can lead to a subdivision of the faces in $\mathcal{F}_h$. We denote by $\widehat{\Pi}_l$ the $L^2$-orthogonal projection onto $V^l(\widehat{\mathcal{T}}_h)$, for all $l \geq 0$. Faces of $\widehat{\mathcal{T}}_h$ are collected into the set $\widehat{\mathcal{F}}_h$ and we define $\widehat{\Pi}_{l,F}$, for $l \geq 0$ and $F \in \widehat{\mathcal{F}}_h$, as the $L^2$-orthogonal projection onto $\mathbb{P}_l(F)$. For all $F \in \widehat{\mathcal{F}}_h$ such that there is $F' \in \mathcal{F}_h$ with $F \subset F'$, $\mathbf{n}_F$ is chosen with the same orientation as that of $\mathbf{n}_{F'}$. For all $T \in \widehat{\mathcal{T}}_h$, we define $\widehat{\mathcal{F}}_T$ as the set of all the faces of $T$ in $\widehat{\mathcal{F}}_h$. Finally, for all $T \in \mathcal{T}_h$, we consider the refinement of $T$ by $\widehat{\mathcal{T}}_h$, namely $\mathfrak{R}_T = \{T' \in \widehat{\mathcal{T}}_h;\ T' \subset T\}$. Clearly, $\mathfrak{R}_T = \{T\}$ if $\mathcal{T}_h$ is matching and simplicial.

The interior-penalty DG methods considered herein consists of finding $u_h \in V^k(\mathcal{T}_h)$ such that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (\mathbf{K}\nabla u_h, \nabla v_h)_{0,T} - \sum_{F\in\mathcal{F}_h} \{(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega, [\![v_h]\!])_{0,F} + \theta(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla v_h\}\!\}_\omega, [\![u_h]\!])_{0,F}\} \\ + \sum_{F\in\mathcal{F}_h} (\alpha_F\gamma_{\mathbf{K},F} h_F^{-1}[\![u_h]\!], [\![v_h]\!])_{0,F} = (f, v_h) \qquad \forall v_h \in V^k(\mathcal{T}_h),\end{aligned} \tag{2}$$

where $\alpha_F$ is a positive parameter, $k \geq 1$, $\gamma_{\mathbf{K},F}$ a $\mathbf{K}$-dependent weight, and $\theta \in \{-1, 0, 1\}$.

## 3. Flux reconstruction on nonmatching meshes

We present in this section our reconstructed flux. It belongs to the RTN space of vector functions on the mesh $\widehat{\mathcal{T}}_h$, $\mathbf{RTN}^l(\widehat{\mathcal{T}}_h) = \{\mathbf{v}_h \in \mathbf{H}(\mathrm{div}, \Omega)\,; \mathbf{v}_h|_T \in \mathbf{RTN}^l_T \quad \forall T \in \widehat{\mathcal{T}}_h\}$, where $l \in \{k-1, k\}$ and $\mathbf{RTN}^l_T = [\mathbb{P}_l(T)]^d + \mathbf{x}\mathbb{P}_l(T)$. In particular, $\mathbf{v}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ is such that for all $T \in \widehat{\mathcal{T}}_h$, $\nabla\cdot\mathbf{v}_h \in \mathbb{P}_l(T)$ and for all $F \in \widehat{\mathcal{F}}_T$, $\mathbf{v}_h\cdot\mathbf{n}_F \in \mathbb{P}_l(F)$ is continuous.

The reconstruction presented in [6, Appendix A] is very cheap, is only based on a construction of $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ by prescribing its local degrees of freedom, and in particular requires no local linear system solution. However, it only satisfies

$$(\nabla\cdot\mathbf{t}_h, \xi_h)_{0,T} = (f, \xi_h)_{0,T} \qquad \forall T \in \mathcal{T}_h,\ \forall \xi_h \in \mathbb{P}_l(T), \tag{3}$$

see [6, Lemma A.1]. Neither $\nabla\cdot\mathbf{t}_h = \Pi_l f$, nor $\nabla\cdot\mathbf{t}_h = \widehat{\Pi}_l f$ is satisfied. By solving local linear systems, the reconstruction of this Note improves on this point.

For a given $T \in \mathcal{T}_h$, let

$$\begin{aligned}\mathbf{RTN}^l_{\mathrm{N,DG}}(\mathfrak{R}_T) = \{&\mathbf{v}_h \in \mathbf{H}(\mathrm{div}, T);\ \mathbf{v}_h|_{T'} \in \mathbf{RTN}^l_{T'}\ \forall T' \in \mathfrak{R}_T, \\ &(\mathbf{v}_h\cdot\mathbf{n}_F, q_h)_{0,F} = (\phi_F(u_h), q_h)_{0,F}\ \forall F \in \widehat{\mathcal{F}}_h,\ F \subset \partial T,\ \forall q_h \in \mathbb{P}_l(F)\},\end{aligned} \tag{4}$$

where $\phi_F(u_h) = -\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega + \alpha_F\gamma_{\mathbf{K},F}h_F^{-1}[\![u_h]\!]$. This is a space of RTN vector functions on $\mathfrak{R}_T$ with the face fluxes on the boundary of $T$ imposed using the DG face fluxes. The main result of this section is:

**Definition** 3.1 (Flux reconstruction on nonmatching meshes) *We define $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ by solving on each $T \in \mathcal{T}_h$ the following minimization problem:*

$$\mathbf{t}_h|_T = \arg \inf_{\mathbf{v}_h \in \mathbf{RTN}^l_{\mathrm{N,DG}}(\mathfrak{R}_T),\ \nabla\cdot\mathbf{v}_h = \widehat{\Pi}_l f} \|\mathbf{K}^{\frac{1}{2}}\nabla u_h + \mathbf{K}^{-\frac{1}{2}}\mathbf{v}_h\|_{0,T}. \tag{5}$$

Indeed, $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ since the normal components on $\partial T$ for all $T$ are continuous. Moreover, the local conservation property $\nabla\cdot\mathbf{t}_h = \widehat{\Pi}_l f$ is directly enforced. Furthermore, the minimizing set in (5) is not empty since owing to (2), for all $\mathbf{v}_h \in \mathbf{RTN}^l_{\mathrm{N,DG}}(\mathfrak{R}_T)$,

$$(\widehat{\Pi}_l f, 1)_{0,T} = (f, 1)_{0,T} = \sum_{F\in\mathcal{F}_T} \mathbf{n}_T\cdot\mathbf{n}_F(\phi_F(u_h), 1)_{0,F} = \sum_{F\in\mathcal{F}_T} (\mathbf{v}_h\cdot\mathbf{n}_T, 1)_{0,F}. \tag{6}$$

Define $\mathbf{RTN}^l_{\mathrm{N},0}(\mathfrak{R}_T)$ as in (4) but with the normal flux condition $(\mathbf{v}_h\cdot\mathbf{n}_F, q_h)_{0,F} = 0$. Letting $\mathbb{P}^*_l(\mathfrak{R}_T)$ be spanned by piecewise $l$-th order polynomials on $\mathfrak{R}_T$ with zero mean on $T$, it is easy to show that (5) is equivalent to finding $\mathbf{t}_h \in \mathbf{RTN}^l_{\mathrm{N,DG}}(\mathfrak{R}_T)$ and $p_h \in \mathbb{P}^*_l(\mathfrak{R}_T)$ such that

$$\begin{aligned}(\mathbf{K}^{-1}\mathbf{t}_h + \nabla u_h, \mathbf{v}_h)_{0,T} - (p_h, \nabla\cdot\mathbf{v}_h)_{0,T} &= 0 \qquad &&\forall \mathbf{v}_h \in \mathbf{RTN}^l_{\mathrm{N},0}(\mathfrak{R}_T), \\ (\nabla\cdot\mathbf{t}_h, \phi_h)_{0,T} &= (f, \phi_h)_{0,T} &&\forall \phi_h \in \mathbb{P}^*_l(\mathfrak{R}_T),\end{aligned}$$

and that this problem has a unique solution. Letting $\mathbf{t}'_h := \mathbf{t}_h + \mathbf{K}\nabla u_h$, $(\mathbf{t}'_h, p_h)$ corresponds to the mixed finite element approximation to the local Neumann problem

$$\begin{aligned}-\nabla\cdot(\mathbf{K}\nabla p) &= f + \nabla\cdot(\mathbf{K}\nabla u_h) &&\text{in } T, \\ -\mathbf{K}\nabla p\cdot\mathbf{n}_T &= (\mathbf{n}_T\cdot\mathbf{n}_F)\phi_F(u_h) + \mathbf{K}\nabla u_h|_T\cdot\mathbf{n}_T &&\text{on all } F \in \widehat{\mathcal{F}}_T, \\ (p, 1)_T &= 0.\end{aligned}$$

The compatibility condition on the data of the above problem exactly amounts to (6). Finally, it is readily verified that if $\mathcal{T}_h$ is matching and simplicial, the reconstructions by direct prescription of [6] and the above one by solving local linear systems yield the same result if $l \in \{0,1\}$.

## 4. A posteriori error estimates

We present in this final section our a posteriori error estimates. Let $s \in H_0^1(\Omega)$ be arbitrary. In practice, $s$ is constructed from $u_h$ using the so-called Oswald interpolate on the mesh $\widehat{\mathcal{T}}_h$. For all $T \in \mathcal{T}_h$, we define the *nonconformity estimator* $\eta_{\mathrm{NC},T}$, the *diffusive flux estimator* $\eta_{\mathrm{DF},T}$, and the *residual estimator* $\eta_{\mathrm{R},T}$ as

$$\eta_{\mathrm{NC},T} := |||u_h - s|||_T, \qquad \eta_{\mathrm{DF},T} := \|\mathbf{K}^{\frac{1}{2}}\nabla u_h + \mathbf{K}^{-\frac{1}{2}}\mathbf{t}_h\|_{0,T}, \tag{7}$$

$$\eta_{\mathrm{R},T} := \left\{ \sum_{T' \in \mathfrak{R}_T} m_{T'}^2 \|f - \widehat{\Pi}_l f\|_{0,T'}^2 \right\}^{1/2}, \tag{8}$$

where $m_{T'} := C_{\mathrm{P},T'} h_{T'} c_{\mathbf{K},T'}^{-1/2}$. Here $c_{\mathbf{K},T'}$ is the minimum eigenvalue of $\mathbf{K}$ on $T'$, $C_{\mathrm{P},T'}$ is the Poincaré constant on $T'$, equal to $1/\pi$ owing to the convexity of $T'$, and $|||\cdot|||$ stands for the energy (semi-)norm associated with problem (1), $|||v|||^2 = \sum_{T \in \mathcal{T}_h} |||v|||_T^2$ with $|||v|||_T := \|\mathbf{K}^{1/2}\nabla v\|_{0,T}$.

**Theorem 4.1 (Guaranteed a posteriori error estimate)** *Let $u$ be the weak solution of* (1) *and let $u_h$ be its DG approximation given by* (2). *Then,*

$$|||u - u_h||| \le \left\{ \sum_{T \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},T}^2 + (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 \right\} \right\}^{1/2}.$$

This estimate is established by proceeding as in [6]. The advantage of the present result is twofold. Firstly, $\eta_{\mathrm{R},T}$ represents a (super-convergent) lower bound for the more classical estimator $m_T\|f - \nabla\cdot\mathbf{t}_h\|_{0,T}$ with $m_T = C_{\mathrm{P},T} h_T c_{\mathbf{K},T}^{-1/2}$, obtained by direct prescription of the reconstructed flux $\mathbf{t}_h$, since $\mathbf{t}_h$ only satisfies (3) and $h_{T'} \le h_T$. Secondly, whenever the mesh element $T$ is nonconvex, the Poincaré constant $C_{\mathrm{P},T}$ is no longer equal to $1/\pi$, and its evaluation is much more difficult leading to less sharp estimates. This issue is avoided in the present setting by working on the simplicial submesh of $T$.

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