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ANALYSIS OF THE MODIFIED MASS METHOD FOR THE DYNAMIC SIGNORINI PROBLEM WITH COULOMB FRICTION*

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Abstract. The aim of the present work is to analyze the modified mass method for the dynamic Signorini problem with Coulomb friction. We prove that the space semidiscrete problem is equivalent to an upper semicontinuous one-sided Lipschitz differential inclusion and is, therefore, well-posed. We derive an energy balance. Next, considering an implicit time-integration scheme, we prove that, under a CFL-type condition on the discretization parameters, the fully discrete problem is well-posed. For a fixed discretization in space, we also prove that the fully discrete solutions converge to the space semidiscrete solution when the time step tends to zero.

Key words. finite elements, time-integration scheme, elastodynamics, unilateral contact, Coulomb friction, differential inclusion, modified mass method

AMS subject classifications. 65M12, 65M60, 74H15, 74M10, 74M15, 74S05

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1. Introduction.

The modified mass method is a new approach for solving computationally dynamic problems with unilateral contact. Introduced in [20] for frictionless contact problems, it is based on a space semidiscrete formulation in which the mass matrix is modified: the mass associated with the (normal) displacements at the contact boundary is removed. This modified semidiscrete problem can then be discretized with various time-integration schemes. The modified mass method eliminates the large spurious oscillations on the contact pressure, which can appear with a standard mass matrix, while ensuring an exact enforcement of the contact condition. Moreover, with a suitable scheme such as the Newmark scheme (trapezoidal rule), a tight energy conservation and a good behavior in long-time are observed. In addition, the method does not require extra steps or extra parameters and can easily be implemented. Since its introduction, the modified mass method has been developed in several directions: alternative ways of building the modified mass matrix [13, 25, 16], use of semiexplicit time-integration schemes [9, 8], application to contact with friction [19, 13, 14], and application to thin structures [25]. For a comparison of the modified mass method with other popular methods, we refer to [19, 9, 22].

Currently no theoretical analysis has been carried out for the modified mass method in the frictional case. In the frictionless case with an elastic material, a number of results have been proven: the space semidiscrete problem is equivalent to a Lipschitz system of ordinary differential equations and is, therefore, well-posed [20]; the variation of energy is equal to the work of the external forces (the contact forces do not work) [20]; the semidiscrete solutions converge to a continuous solution in the case of viscoelastic materials [7]. Furthermore, an error analysis of the modified mass matrix has been performed in [14] for linear elastodynamics without contact.

The aim of the present work is to analyze the modified mass method for the dynamic Signorini problem with Coulomb friction. Implementation and numerical

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simulations are extensively discussed in [13, 14, 22]. We prove that the space semidiscrete problem is equivalent to an upper semicontinuous one-sided Lipschitz differential inclusion [6, 27] and is, therefore, well-posed (Theorem 3.4). Furthermore, the variation of energy is equal to the work of the external forces and friction forces (Theorem 3.5). For the time discretization, we consider an implicit scheme. Each time step requires solving a nonlinear problem similar to a static friction problem. It is well-known that such a problem can have several solutions [17]. Here we prove that, under a certain condition on the discretization parameters of CFL type, the fully discrete problem is well-posed (Theorem 4.2). For a fixed discretization in space, we also prove that the fully discrete solutions converge to the space semidiscrete solution when the time step tends to zero (Theorem 5.2).

With a standard mass term, proving the existence of a semidiscrete solution to a dynamic contact problem is quite delicate. It is necessary to add an impact law and to work with BV and measures spaces [3, 28, 2]. The modification of the mass term greatly simplifies the analysis. Indeed, the unilateral contact condition can be eliminated and replaced by a Lipschitz continuous term in the momentum equation [20]. Moreover, static and quasi-static Coulomb friction problems can have several solutions [17]. Uniqueness is only obtained for small friction coefficients (see [21, Theorem 11.4] for the static case and [15, Theorem 7.2.1] for the quasi-static case). It is worthwhile to notice that in the dynamic case uniqueness is recovered. Finally, we do not examine herein the convergence of the discrete solutions to a solution of the continuous problem. Nevertheless, it seems possible to extend the convergence result in [7] to the case of a nonlocal Coulomb friction (the nonlocal Coulomb friction is a regularization of Coulomb friction [21, 5]).

This paper is organized as follows. In section 2, we formulate the continuous problem. Sections 3 and 4 are devoted to the space semidiscrete and fully discrete problems, respectively. In section 5, we examine the convergence of the fully discrete solutions to the space semidiscrete solution. Conclusions are drawn in section 6.

2. Continuous problem. We consider the infinitesimal deformations of a body occupying a reference domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2,3\}$) during a time interval $[0,T]$. Let ν be the outward unit normal to Ω. The elasticity tensor is denoted by $\mathcal{A}$ and the mass density by ρ. An external load f is applied to the body. Let $u : (0,T) \times \Omega \to \mathbb{R}^d$, $\epsilon(u) : (0,T) \times \Omega \to \mathbb{R}^{d,d}$, and $\sigma(u) : (0,T) \times \Omega \to \mathbb{R}^{d,d}$ be the displacement field, the linearized strain tensor, and the stress tensor, respectively. Denoting time-derivatives by dots, the momentum conservation equation reads as

$$\rho\ddot{u} - \operatorname{div}\sigma = f, \qquad \sigma = \mathcal{A} : \epsilon, \qquad \epsilon = \frac{1}{2}(\nabla u +^{\mathrm{T}} \nabla u) \quad \text{in } \Omega \times (0,T). \tag{2.1}$$

The boundary $\partial\Omega$ is partitioned into three disjoint open subsets Γ^D, Γ^N, and Γ^{c}. Dirichlet and Neumann conditions are prescribed on Γ^D and Γ^N, respectively,

$$u = u_D \quad \text{on } \Gamma^D \times (0,T), \qquad \sigma \cdot \nu = f_N \quad \text{on } \Gamma^N \times (0,T). \tag{2.2}$$

In what follows, we assume $f \in W^{1,\infty}(0,T; L^2(\Omega)^d)$ and $f_N \in W^{1,\infty}(0,T; L^2(\Gamma^N)^d)$.

We let $u_\nu := u_{|\partial\Omega} \cdot \nu$ and $u_\tau := u_{|\partial\Omega} - u_\nu \nu$ be the normal and tangential displacements on $\partial\Omega$, respectively. We also let $\sigma_\nu(u) := \nu \cdot \sigma(u)_{|\partial\Omega} \cdot \nu$ and $\sigma_\tau(u) := \sigma(u)_{|\partial\Omega} \cdot \nu - \sigma(u)_\nu \nu$ be the normal and tangential stresses on $\partial\Omega$, respectively. Note that u_ν and $\sigma_\nu(u)$ are scalars while u_τ and $\sigma_\tau(u)$ are vectors in $\mathbb{R}^d$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^m$, $m \geq 1$. On Γ^{c}, a unilateral contact condition, also called

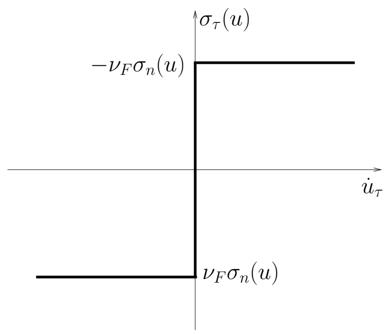

FIG. 2.1. *Coulomb condition ($d = 2$).*

the Signorini condition, and a Coulomb friction (see Figure 2.1) are enforced,

$$u_\nu \leq g, \ \sigma_\nu(u) \leq 0, \ \sigma_\nu(u)(u_\nu - g) = 0 \qquad \text{on } \Gamma^{\mathrm{c}} \times (0,T), \tag{2.3}$$

$$|\sigma_\tau(u)| \leq \mu|\sigma_\nu(u)| \qquad \text{on } \Gamma^{\mathrm{c}} \times (0,T), \tag{2.4}$$

$$\sigma_\tau(u) = -\mu|\sigma_\nu(u)|\frac{\dot{u}_\tau}{|\dot{u}_\tau|} \quad \text{if } \dot{u}_\tau \neq 0 \qquad \text{on } \Gamma^{\mathrm{c}} \times (0,T), \tag{2.5}$$

where $\mu > 0$ is the friction coefficient (taken to be constant for simplicity) and g is the initial gap. At the initial time, the displacement and velocity fields are prescribed,

$$u(0) = u^0, \quad \dot{u}(0) = v^0 \quad \text{in } \Omega. \tag{2.6}$$

The mathematical analysis of the above time-dependent problem entails substantial difficulties [10]. The existence of a weak solution is proven only for a viscoelastic material and a nonlocal Coulomb friction law [5].

3. Space semidiscrete formulation. In this section, we formulate the space semidiscrete problem and prove existence and uniqueness of a solution. We also establish an energy balance. In the frictionless case, the semidiscrete problem is equivalent to a Lipschitz system of ordinary differential equations, and existence and uniqueness are deduced from the Cauchy–Lipschitz theorem. With friction, the situation is more complicated. We choose to model the friction term as a set-valued map. The semidiscrete problem is then equivalent to a differential inclusion, for which generalizations of the Cauchy–Lipschitz theorem are available [6, 27].

3.1. Preliminaries. To begin with, we introduce some notions needed for the formulation of our problem as a differential inclusion.

- Given a set E, we define $\mathcal{P}(E)$ as the set of all subsets of E, and $\mathcal{P}^*(E) := \mathcal{P}(E) \setminus \{\emptyset\}$.
- A set-valued map is said to be closed convex if its images are closed convex sets.
- Various notions of continuity can be defined for set-valued maps. One of them is upper semicontinuity.[1] A set-valued map F is said to be upper semicontinuous at x if, for every open set V containing $F(x)$, there exists

[1] This notion of upper semicontinuity is distinct from the upper semicontinuity for single-valued functions.

a neighborhood U of x such that $F(U) \subset V$. Consider, for instance, the following set-valued maps:

$$F_1(x) = \begin{cases} [-1,1] & \text{if } x = 0, \\ \{0\} & \text{if } x \neq 0, \end{cases} \quad \text{and} \quad F_2(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ [-1,1] & \text{if } x \neq 0. \end{cases}$$

It is easy to verify that F_1 is upper semicontinuous for all $x \in \mathbb{R}$, whereas F_2 is not upper semicontinuous at $x = 0$. Here is another example of set-valued-map, closely related to the Coulomb friction term,

$$F_3(x,y) = \begin{cases} \{-|x|\} & \text{if } y < 0, \\ [-|x|, |x|] & \text{if } y = 0, \\ \{|x|\} & \text{if } y > 0. \end{cases}$$

It is easy to verify that this map is upper semicontinuous for all $(x,y) \in \mathbb{R}^2$. Finally, we observe that upper semicontinuity applied to single-valued functions is equivalent to continuity. For more details on set-valued maps, we refer to [1].

- The existence theorem for differential inclusions we use does not provide continuously differentiable solutions in time. The solutions are only absolutely continuous in time. For brevity, we do not define this concept and refer to [26]. For our purpose, it suffices to know that an absolutely continuous function y is continuous, is differentiable almost everywhere, and is equal to the integral of its derivative:

$$y(t_0) = y(0) + \int_0^{t_0} \dot{y}(t)dt.$$

Lipschitz continuous functions are absolutely continuous. In what follows, we denote by $AC([0,T];\mathbb{R}^m)$ the space spanned by absolutely continuous functions from $[0,T]$ to $\mathbb{R}^m$.

- The set-valued maps which appear in our space semidiscrete problem are subgradients and, for completeness, we define this notion. Let $J : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a convex function and $D(J) := \{v \in \mathbb{R}^m;\ J(v) < +\infty\}$ be its domain. We define the subgradient of J as the set-valued map $\partial J : D(J) \to \mathcal{P}^*(\mathbb{R}^m)$ such that

(3.1)
$$\forall v \in D(J), \quad \partial J(v) := \{\gamma \in \mathbb{R}^m;\ J(w) - J(v) \geq (\gamma, w - v)\ \forall w \in D(J)\},$$

where $(\cdot,\cdot)$ denotes the canonical inner product on $\mathbb{R}^m$. It is easy to prove that the subgradient of a convex function is well-defined and is a closed convex set-valued map. For more details on subgradients, we refer to [4, 18].

We can now state the main result we use for asserting the well-posedness of a problem posed in the form of a differential inclusion.

Theorem 3.1. *Let $P : [0,T] \times \mathbb{R}^m \to \mathcal{P}^*(\mathbb{R}^m)$ be a closed convex set-valued map. Let $x_0 \in \mathbb{R}^m$ and consider the following problem: Find $x \in AC([0,T];\mathbb{R}^m)$ such that*

$$\dot{x}(t) \in P(t, x(t)), \tag{3.2}$$

$$x(0) = x_0. \tag{3.3}$$

Assume that

1. *the set-valued map $P(t,\cdot)$ is upper semicontinuous for almost all $t \in [0,T]$;*
2. *for any $x \in \mathbb{R}^m$, there exists a measurable function $p(\cdot,x)$ satisfying $p(t,x) \in P(t,x)$ for almost all $t \in [0,T]$;*
3. *there exists a function $b \in L^1(0,T;\mathbb{R}^m)$ such that $|p(t,x)| \leq b(t)$ for almost all $t \in [0,T]$.*

Then, there exists a solution to (3.2)–(3.3)*. Furthermore, assume the following one-sided Lipschitz condition: there exists $K \in \mathbb{R}$ such that, for all $t \in [0,T]$, for all x_1, $x_2 \in \mathbb{R}^m$,*

$$(y_1 - y_2, x_1 - x_2) \leq K\|x_1 - x_2\|^2 \quad \forall y_1 \in P(t,x_1),\ \forall y_2 \in P(t,x_2). \tag{3.4}$$

Then, the solution is unique.

Proof. For the existence, see [27, Theorem 4.7] or [6, Theorem 5.2]. Uniqueness is straightforward owing to the one-sided Lipschitz condition since it implies that two solutions x_1 and x_2 satisfy $\frac{1}{2}\frac{d}{dt}(\|x_1 - x_2\|^2) \leq K\|x_1 - x_2\|^2$. ◻

Remark 3.1. In the single-valued case ($P : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$), the hypotheses of Theorem 3.1 become

1. $P(t,\cdot)$ is continuous for almost all $t \in [0,T]$;
2. for any $x \in \mathbb{R}$, $P(\cdot,x)$ is measurable;
3. there exists a function $b \in L^1(0,T;\mathbb{R})$ such that $|P(t,x)| \leq b(t)$ for almost all $t \in [0,T]$.

We recover Caratheodory's existence theorem for ordinary differential equations [12]. Furthermore, the one-sided Lipschitz condition means that $P(t,\cdot)$ is Lipschitz continuous for all $t \in [0,T]$ (uniformly).

Remark 3.2. If P is a monotone operator, i.e., for all $t \in [0,T]$, for all x_1, $x_2 \in \mathbb{R}^m$,

$$(y_1 - y_2, x_1 - x_2) \geq 0 \quad \forall y_1 \in P(t,x_1),\ \forall y_2 \in P(t,x_2),$$

then $-P$ satisfies the one-sided Lipschitz condition.

3.2. The discrete setting. For simplicity, we suppose that Ω is a polyhedron. Let $\mathcal{T}$ be a simplicial mesh of Ω (triangles in two dimensions and tetrahedra in three dimensions). Let $\{x_i\}_{i\in\mathcal{N}}$ and $\{\phi_i\}_{i\in\mathcal{N}}$ be the nodes of the mesh and the associated scalar basis functions (continuous and piecewise affine), respectively. We denote by $\mathcal{N}^D$ the set of indices where a Dirichlet condition is enforced, and we set $\tilde{\mathcal{N}} := \mathcal{N}\backslash\mathcal{N}^D$. The space of admissible displacements is approximated by the space

$$V = \{v \in C^0(\overline{\Omega})^d;\ v_{|T} \in (\mathbb{P}_1)^d\ \forall T \in \mathcal{T},\ \text{and}\ v(x_i) = 0\ \forall i \in \mathcal{N}^D\}.$$

The space V is spanned by $\{\phi_i e_\alpha\}_{i\in\tilde{\mathcal{N}},1\leq\alpha\leq d}$, where $\{e_\alpha\}_{1\leq\alpha\leq d}$ is the canonical basis of $\mathbb{R}^d$. Denote by $\mathcal{N}^{\mathrm{c}}$ the set of indices of contact nodes (that is, the nodes located on Γ^{c} which is fixed a priori) and by $\mathcal{N}^{\mathrm{i}} := \tilde{\mathcal{N}} \setminus \mathcal{N}^{\mathrm{c}}$ the set of indices of the remaining nodes (see Figure 3.1). Let $\{\nu_i\}_{i\in\mathcal{N}^{\mathrm{c}}}$ and $\{\tau_{i,\alpha}\}_{i\in\mathcal{N}^{\mathrm{c}},1\leq\alpha\leq d-1}$ be the contact normal vectors and tangential vectors, respectively. We set

$$V^{\mathrm{i}} = \mathrm{span}(\{\phi_i e_\alpha\}_{i\in\mathcal{N}^{\mathrm{i}},1\leq\alpha\leq d}),$$

$$V^{\mathrm{c}} = \mathrm{span}(\{\phi_i \nu_i\}_{i\in\mathcal{N}^{\mathrm{c}}}), \quad \text{and} \quad V^{\mathrm{f}} = \mathrm{span}(\{\phi_i \tau_{i,\alpha}\}_{i\in\mathcal{N}^{\mathrm{c}},1\leq\alpha\leq d-1}).$$

Clearly, $V = V^{\mathrm{i}} \oplus V^{\mathrm{c}} \oplus V^{\mathrm{f}}$, so that any discrete function $v \in V$ can be decomposed as

$$v = v_{\mathrm{i}} + v_{\mathrm{c}} + v_{\mathrm{f}} \quad \text{with} \quad v_{\mathrm{i}} \in V^{\mathrm{i}},\ v_{\mathrm{c}} \in V^{\mathrm{c}},\ v_{\mathrm{f}} \in V^{\mathrm{f}}.$$

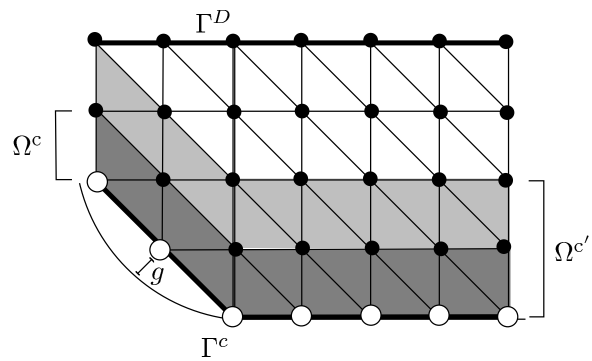

Fig. 3.1. *Decomposition of the domain Ω; bullets (resp., circles) indicate nodes indexed by elements of the set $\mathcal{N}^{\mathrm{i}}$ (resp., $\mathcal{N}^{\mathrm{c}}$). The open sets Ω^{c} and $\Omega^{\mathrm{c}'}$ are defined in section* 4.

We also introduce the space $V^* := V^{\mathrm{i}} \oplus V^{\mathrm{f}}$, so that any discrete function $v \in V$ can also be decomposed as

$$v = v_* + v_{\mathrm{c}} \quad \text{with} \quad v_* \in V^*,\ v_{\mathrm{c}} \in V^{\mathrm{c}}.$$

Let $(\cdot,\cdot)$ denote the L^2 inner product on V. Let $\|\cdot\|$ denote the norm associated with $(\cdot,\cdot)$. Herein, we always work in finite dimension on a fixed spatial mesh; the specific choice of the norm is therefore not critical. The present choice is made for simplicity.

The standard mass term stems from the bilinear form

$$m : L^2(\Omega)^d \times L^2(\Omega)^d \ni (v,w) \longmapsto \int_\Omega \rho v \cdot w \in \mathbb{R}.$$

The key idea in the modified mass method is to remove the mass associated with the normal components at the contact nodes. We consider an approximate mass term associated with the bilinear form m^* such that

$$m^*(\phi_i \nu_i, w) = m^*(w, \phi_i \nu_i) = 0 \quad \forall i \in \mathcal{N}^{\mathrm{c}},\ \forall w \in V. \tag{3.5}$$

Many choices are possible to build the rest of the mass term. In [13, 20], the authors devise various methods to preserve some features of the standard mass term (the total mass, the center of gravity, and the moments of inertia); see also [25] for further results. Here, we focus for simplicity on the choice

$$m^* : V \times V \ni (v,w) \longmapsto m(v_*, w_*) \in \mathbb{R}.$$

We define the associated operator $M^* : V^* \to V^*$ such that

$$(M^* v_*, w_*) = m^*(v_*, w_*) \quad \forall (v_*, w_*) \in V^* \times V^*.$$

We define the bilinear and linear forms

$$a : H^1(\Omega)^d \times H^1(\Omega)^d \ni (v,w) \longmapsto \int_\Omega \epsilon(v) : \mathcal{A} : \epsilon(w),$$

$$l : [0,T] \times H^1(\Omega)^d \ni (t,v) \longmapsto \int_\Omega f(t) \cdot v + \int_{\Gamma^N} f_N(t) \cdot v.$$

We define the linear operator $A : V \to V$ and the vector $L(t) \in V$ such that for all $v \in V$ and $w \in V$, and for all $t \in [0,T]$,

$$(Av, w) = a(v, w), \qquad (L(t), w) = l(t, w).$$

We also need $A^{\mathrm{c}} : V \to V^{\mathrm{c}}$ and $L^{\mathrm{c}}(t) \in V^{\mathrm{c}}$ such that for all $v \in V$ and all $w_{\mathrm{c}} \in V^{\mathrm{c}}$, and for all $t \in [0,T]$,

$$(A^{\mathrm{c}} v, w_{\mathrm{c}}) = a(v, w_{\mathrm{c}}), \qquad (L^{\mathrm{c}}(t), w_{\mathrm{c}}) = l(t, w_{\mathrm{c}}),$$

and, similarly, $A^* : V \to V^*$ and $L^*(t) \in V^*$ such that for all $v \in V$ and all $w_* \in V^*$, and for all $t \in [0,T]$,

$$(A^* v, w_*) = a(v, w_*), \qquad (L^*(t), w_*) = l(t, w_*).$$

We define the constraint set

$$K := \{v \in V;\ v(x_i) \cdot \nu_i \le g(x_i)\ \forall i \in \mathcal{N}^{\mathrm{c}}\},$$

and the unilateral contact term $I_K : V^{\mathrm{c}} \to \mathbb{R} \cup \{+\infty\}$ such that

$$I_K(v_{\mathrm{c}}) = \begin{cases} 0 & \text{if } v_{\mathrm{c}} \in K, \\ +\infty & \text{if } v_{\mathrm{c}} \notin K. \end{cases}$$

The function I_K is nondifferentiable, but convex since K is convex. Therefore, it is possible to define its subgradient $\partial I_K : V^{\mathrm{c}} \cap K \to \mathcal{P}^*(V^{\mathrm{c}})$,

$$\partial I_K(v_{\mathrm{c}}) := \{\gamma \in V^{\mathrm{c}};\ 0 \ge (\gamma, w_{\mathrm{c}} - v_{\mathrm{c}})\ \forall w_{\mathrm{c}} \in V^{\mathrm{c}} \cap K\}.$$

Now, we define the friction term $j : V \times V^{\mathrm{f}} \to \mathbb{R}$ such that

$$j(v, w_{\mathrm{f}}) = \int_{\Gamma^{\mathrm{c}}} \mu |\sigma_\nu(v)| |w_{\mathrm{f}}|. \tag{3.6}$$

The function j is nondifferentiable with respect to its second argument, but convex, and its domain is V^{f}. We can define its subgradient with respect to its second argument such that for all $z \in V$, $\partial_2 j(z, \cdot) : V^{\mathrm{f}} \to \mathcal{P}^*(V^{\mathrm{f}})$ with

$$\partial_2 j(z, v_{\mathrm{f}}) := \left\{\gamma \in V^{\mathrm{f}};\ j(z, w_{\mathrm{f}}) - j(z, v_{\mathrm{f}}) \ge (\gamma, w_{\mathrm{f}} - v_{\mathrm{f}})\ \forall w_{\mathrm{f}} \in V^{\mathrm{f}}\right\}. \tag{3.7}$$

3.3. Formulation of the semidiscrete problem. We can now formulate the semidiscrete problem. Let $u_*^0 \in V^*$ and $v_*^0 \in V^*$ be suitable approximations of the initial displacement and velocity u^0 and v^0, respectively.

PROBLEM 3.1. *Seek $u \in C^0([0,T]; K)$ such that $u_* \in C^1([0,T]; V^*)$, $\dot{u}_* \in AC([0,T]; V^*)$, and the following differential inclusion holds true:*

$$M^* \ddot{u}_* \in -Au - \partial_2 j(u, \dot{u}_{\mathrm{f}}) - \partial I_K(u_{\mathrm{c}}) + L(t) \quad \text{a.e. in } [0,T], \tag{3.8}$$

with the initial conditions $u_*(0) = u_*^0$ and $\dot{u}_*(0) = v_*^0$ in Ω.

Remark 3.3. The velocity $\dot{u}_*$ is absolutely continuous. Therefore, it is differentiable almost everywhere, and the acceleration $\ddot{u}_*$ in (3.8) is well-defined. Moreover, $u_{\mathrm{c}} \in K$ so that $\partial I_K(u_{\mathrm{c}})$ is well-defined.

To explicitate the link between the space semidiscrete Problem 3.1 and the continuous problem formulated in section 2, we observe that (3.8) means that, for almost all $t \in [0,T]$, there exist $\lambda_{\mathrm{c}} \in \partial I_K(u_{\mathrm{c}})$ and $\lambda_{\mathrm{f}} \in \partial_2 j(u, \dot{u}_{\mathrm{f}})$ such that

$$M^* \ddot{u}_* + Au + \lambda_{\mathrm{c}} + \lambda_{\mathrm{f}} = L(t).$$

Therefore, the vectors λ_{c} and λ_{f} are discrete counterparts of the normal and tangential contact stresses. Furthermore, lumping the mass matrices, it is easy to verify that the definitions of $\partial I_K(u_{\mathrm{c}})$ and $\partial_2 j(u, \dot{u}_{\mathrm{f}})$ imply that, for all $i \in \mathcal{N}^{\mathrm{c}}$,

$$u_{\mathrm{c}}(x_i) \cdot \nu_i \le g(x_i),\ \ \lambda_{\mathrm{c}}(x_i) \le 0,\ \ \lambda_{\mathrm{c}}(x_i)(u_{\mathrm{c}}(x_i) \cdot \nu_i - g(x_i)) = 0, \tag{3.9}$$

$$|\lambda_{\mathrm{f}}(x_i)| \le \mu |\sigma_\nu(u)(x_i)|, \tag{3.10}$$

$$\lambda_{\mathrm{f}}(x_i) = -\mu |\sigma_\nu(u)(x_i)| \frac{\dot{u}_{\mathrm{f}}(x_i)}{|\dot{u}_{\mathrm{f}}(x_i)|} \quad \text{if } \dot{u}_{\mathrm{f}}(x_i) \neq 0. \tag{3.11}$$

Thus, we recover the discrete counterpart of the contact and friction conditions (2.3)–(2.5).

3.4. Main results. This section contains our main results concerning the space semidiscrete problem. We define the map $q : [0,T] \times V^* \to V^{\mathrm{c}} \cap K$ such that for all $t \in [0,T]$ and for all $v_* \in V^*$, $v_{\mathrm{c}} = q(t, v_*) \in V^{\mathrm{c}} \cap K$ solves the following variational inequality:

$$a(v_{\mathrm{c}}, w_{\mathrm{c}} - v_{\mathrm{c}}) \ge l(t, w_{\mathrm{c}} - v_{\mathrm{c}}) - a(v_*, w_{\mathrm{c}} - v_{\mathrm{c}}) \quad \forall w_{\mathrm{c}} \in V^{\mathrm{c}} \cap K. \tag{3.12}$$

This variational inequality is well-posed since it is equivalent to the minimization of a strictly convex functional over a convex set. We first examine the properties of the map q.

LEMMA 3.2. *For all $v_* \in V^*$, the map $t \mapsto q(t, v_*)$ is Lipschitz continuous, and its Lipschitz constant is uniformly bounded in v_*. For all $t \in [0,T]$, the map $v_* \mapsto q(t, v_*)$ is Lipschitz continuous, and its Lipschitz constant is uniformly bounded in t.*

Proof. Let $t_1, t_2 \in [0,T]$ and $v_*, w_* \in V^*$. Set $v_{\mathrm{c}} = q(t_1, v_*)$ and $w_{\mathrm{c}} = q(t_2, w_*)$. Owing to (3.12),

$$a(v_{\mathrm{c}} - w_{\mathrm{c}}, v_{\mathrm{c}} - w_{\mathrm{c}}) \le a(v_* - w_*, w_{\mathrm{c}} - v_{\mathrm{c}}) + l(t_1, v_{\mathrm{c}} - w_{\mathrm{c}}) - l(t_2, v_{\mathrm{c}} - w_{\mathrm{c}}). \tag{3.13}$$

Since $l(t, v_{\mathrm{c}} - w_{\mathrm{c}}) = \int_\Omega f(t) \cdot (v_{\mathrm{c}} - w_{\mathrm{c}}) + \int_{\Gamma^N} f_N(t) \cdot (v_{\mathrm{c}} - w_{\mathrm{c}})$, $f \in W^{1,\infty}(0,T; L^2(\Omega)^d)$, and $f_N \in W^{1,\infty}(0,T; L^2(\Gamma^N)^d)$, there exists a constant c_l such that

$$l(t_1, v_{\mathrm{c}} - w_{\mathrm{c}}) - l(t_2, v_{\mathrm{c}} - w_{\mathrm{c}}) \le c_l |t_1 - t_2| \|v_{\mathrm{c}} - w_{\mathrm{c}}\|.$$

Moreover, the bilinear form a being continuous (with constant c_a) and elliptic (with constant α) for the norm $\|\cdot\|$, a straightforward calculation yields

$$\alpha \|v_{\mathrm{c}} - w_{\mathrm{c}}\| \le c_a \|v_* - w_*\| + c_l |t_1 - t_2|,$$

which proves the desired regularity for q. ◻

We now reformulate the differential inclusion (3.8) using the map q.

LEMMA 3.3. *The differential inclusion* (3.8) *is equivalent to*

$$M^* \ddot{u}_* \in -A^*(u_* + q(t, u_*)) - \partial_2 j(u_* + q(t, u_*), \dot{u}_{\mathrm{f}}) + L^*(t) \quad \textit{a.e. in } [0,T], \tag{3.14}$$

$$u_{\mathrm{c}} = q(t, u_*) \qquad \forall t \in [0,T]. \tag{3.15}$$

Proof. Distinguishing components in V^* and V^{c}, the inclusion (3.8) is equivalently split into the following inclusions:

$$M^*\ddot{u}_* \in -A^*u - \partial_2 j(u, \dot{u}_{\mathrm{f}}) + L^*(t) \qquad \text{a.e. in } [0,T], \tag{3.16}$$

$$0 \in -A^{\mathrm{c}}u - \partial I_K(u_{\mathrm{c}}) + L^{\mathrm{c}}(t) \qquad \text{a.e. in } [0,T]. \tag{3.17}$$

Consider (3.17). By continuity, the inclusion (3.17) is valid for all $t \in [0,T]$. For convenience, we recast it as a variational inequality,

$$a(u, v_{\mathrm{c}} - u_{\mathrm{c}}) \geq l(t, v_{\mathrm{c}} - u_{\mathrm{c}}) \quad \forall t \in [0,T],\ \forall v_{\mathrm{c}} \in V^{\mathrm{c}} \cap K, \tag{3.18}$$

or, equivalently,

$$a(u_{\mathrm{c}}, v_{\mathrm{c}} - u_{\mathrm{c}}) \geq l(t, v_{\mathrm{c}} - u_{\mathrm{c}}) - a(u_*, v_{\mathrm{c}} - u_{\mathrm{c}}) \quad \forall t \in [0,T],\ \forall v_{\mathrm{c}} \in V^{\mathrm{c}} \cap K. \tag{3.19}$$

Hence $u_{\mathrm{c}} = q(t, u_*)$ so that the system (3.16)–(3.17) is equivalent to the system (3.14)–(3.15). ∎

We can now state our main existence and uniqueness result for Problem 3.1.

THEOREM 3.4. *There exists a unique solution u to Problem* 3.1. *Furthermore, $u_{\mathrm{c}} \in W^{1,\infty}(0,T;V^{\mathrm{c}})$.*

Proof. (i) To prove the existence of a solution, we rewrite the second-order differential inclusion (3.8) as a first-order differential inclusion. We define the single-valued map $S : [0,T] \times V^* \times V^* \to V^* \times V^*$ such that, for all $t \in [0,T]$, for all $v_*, w_* \in V^*$,

$$S(t, v_*, w_*) = \begin{pmatrix} w_* \\ -A^*(v_* + q(t, v_*)) + L^*(t) \end{pmatrix},$$

and the set-valued map $P : [0,T] \times V^* \times V^* \to \{0\} \times \mathcal{P}^*(V^*)$ such that

$$P(t, v_*, w_*) = \begin{pmatrix} 0 \\ -\partial_2 j(v_* + q(t, v_*), w_{\mathrm{f}}) \end{pmatrix}.$$

We also define the linear single-valued map $D : V^* \times V^* \to V^* \times V^*$ such that

$$D(v_*, w_*) = \begin{pmatrix} v_* \\ M^* w_* \end{pmatrix}.$$

Setting $X(t) = \left(\begin{smallmatrix} u_*(t) \\ \dot{u}_*(t) \end{smallmatrix}\right) \in V^* \times V^*$, the differential inclusion (3.14) can be recast as

$$D\dot{X}(t) \in S(t, X(t)) + P(t, X(t)). \tag{3.20}$$

We equip the product space $V^* \times V^*$ with the product norm.

(ii) The operator S is a single-valued map. Since $q(t,\cdot)$ is continuous and $q(\cdot,x)$ is Lipschitz continuous, the operator S satisfies the hypotheses of Theorem 3.1 (see Remark 3.1).

(iii) We now examine the operator P. This operator is a closed convex set-valued map (owing to the properties of the subgradients of convex functions). Since $q(t,\cdot)$ is continuous and $\partial_2 j(\cdot,\cdot)$ is upper semicontinuous (see the example given by (3.1)), the map $P(t,\cdot)$ is upper semicontinuous. Hence, hypothesis 1 of Theorem 3.1 holds true. Since $q(\cdot,x)$ is Lipschitz continuous, hypotheses 2 and 3 of this theorem are also satisfied. Next, we check the one-sided Lipschitz condition (3.4). Let $(u_*^1, u_*^2, v_*^1, v_*^2) \in (V^*)^4$ and let $t \in [0,T]$. Set $u^1 = u_*^1 + q(t, u_*^1)$ and $u^2 = u_*^2 + q(t, u_*^2)$. Let $\gamma_1 \in$

$-\partial_2 j(u^1, v_{\rm f}^1)$ and let $\gamma_2 \in -\partial_2 j(u^2, v_{\rm f}^2)$. Using the definition of the subgradient, a reverse triangle inequality, norm equivalence in finite dimension, and the fact that $q(t,\cdot)$ is Lipschitz, we infer

$$\begin{aligned}
(\gamma_2 - \gamma_1, v_*^1 - v_*^2) &\le j(u^1, v_{\rm f}^1) - j(u^1, v_{\rm f}^2) + j(u^2, v_{\rm f}^2) - j(u^2, v_{\rm f}^1) \\
&\le \int_{\Gamma^{\rm c}} \mu \left(|\sigma_\nu(u^1)| - |\sigma_\nu(u^2)|\right)\left(|v_{\rm f}^1| - |v_{\rm f}^2|\right) \\
&\le \int_{\Gamma^{\rm c}} \mu \left||\sigma_\nu(u^1)| - |\sigma_\nu(u^2)|\right| \left||v_{\rm f}^1| - |v_{\rm f}^2|\right| \\
&\le \int_{\Gamma^{\rm c}} \mu |\sigma_\nu(u^1) - \sigma_\nu(u^2)||v_*^1 - v_*^2| \\
&\lesssim \|u^1 - u^2\| \|v_*^1 - v_*^2\| \\
&\lesssim \left(\|u_*^1 - u_*^2\| + \|q(t, u_*^1) - q(t, u_*^2)\|\right)\|v_*^1 - v_*^2\| \\
&\lesssim \|u_*^1 - u_*^2\|\|v_*^1 - v_*^2\| \lesssim \|u_*^1 - u_*^2\|^2 + \|v_*^1 - v_*^2\|^2.
\end{aligned}$$

Therefore, P satisfies the one-sided Lipschitz condition.

(iv) Owing to Theorem 3.1, there exists a unique $X \in AC([0,T]; V^* \times V^*)$ satisfying (3.20) with the initial condition $X(0) = \binom{u_*^0}{v_*^0}$. Therefore, there exists a unique $u_* \in C^1(0,T;V^*)$ such that $\dot u_* \in AC([0,T];V^*)$ satisfying (3.14) with the initial conditions $u_*(0) = u_*^0$ and $\dot u_*(0) = v_*^0$. Owing to (3.15), $u = u_* + u_{\rm c} = u_* + q(t, u_*)$. Therefore, Problem 3.1 has a unique solution, and it is clear that $u_{\rm c} = q(t, u_*) \in W^{1,\infty}(0,T;V^{\rm c})$. ∎

We conclude this section with the energy balance.

THEOREM 3.5. *For all $t_0 \in [0,T]$, the following energy balance holds true:*

$$E(u(t_0)) - E(u(0)) = \int_0^{t_0} \Big\{ l(t, \dot u(t)) - j(u(t), \dot u_{\rm f}(t)) \Big\} dt, \tag{3.21}$$

where $E(v) = \frac{1}{2}\left(m(\dot v_*, \dot v_*) + a(v,v)\right)$.

Proof. We recast the differential inclusion (3.14) as a variational inequality,

$$\begin{aligned}
m(\ddot u_*, v_* - \dot u_*) + a(u, v_* - \dot u_*) + j(u, v_{\rm f}) - j(u, \dot u_{\rm f}) \\
\ge l(t, v_* - \dot u_*) \quad \forall v_* \in V^*, \text{ a.e. in } [0,T].
\end{aligned} \tag{3.22}$$

Taking $v_* = 0$ and then $v_* = 2\dot u_*$ in the above inequality, we obtain

$$m(\ddot u_*, \dot u_*) + a(u, \dot u_*) + j(u, \dot u_{\rm f}) = l(t, \dot u_*) \quad \text{a.e. on } [0,T]. \tag{3.23}$$

Recalling that the family $\{\phi_i \nu_i\}_{i \in \mathcal{N}^{\rm c}}$ is a basis of $V^{\rm c}$, we decompose $u_{\rm c}$ on this basis yielding $u_{\rm c} = \sum_{i \in \mathcal{N}^{\rm c}} u_i \phi_i \nu_i$. Define $C_i^0 := \{t \in [0,T];\ u_i = 0\}$ and $C_i^- := \{t \in [0,T];\ u_i < 0\}$. The sets C_i^0 and C_i^- are, respectively, closed and open, and they form a partition of $[0,T]$. On $\mathrm{int}(C_i^0)$, $\dot u_i \phi_i \nu_i = 0$ so that $a(u, \dot u_i \phi_i \nu_i) = 0 = l(t, \dot u_i \phi_i \nu_i)$. On C_i^-, $a(u, \dot u_i \phi_i \nu_i) = l(t, \dot u_i \phi_i \nu_i)$ owing to (3.17). Finally, $a(u, \dot u_i \phi_i \nu_i) = l(t, \dot u_i \phi_i \nu_i)$ on $\mathrm{int}(C_i^0) \cup C_i^-$ and hence this equality is satisfied almost everywhere on $[0,T]$ (since an open set in $\mathbb{R}$ is a countable union of open intervals, so that its boundary has zero measure). Hence,

$$a(u, \dot u_{\rm c}) = l(t, \dot u_{\rm c}) \quad \text{a.e. on } [0,T]. \tag{3.24}$$

Using (3.24), we obtain

$$m(\ddot{u}_*, \dot{u}_*) + a(u, \dot{u}) + j(u, \dot{u}_{\mathrm{f}}) = l(t, \dot{u}) \quad \text{a.e. on } [0, T]. \tag{3.25}$$

Since $\dot{u}$ is absolutely continuous in time, by integrating in time (3.25), we obtain (3.21). ◻

4. Fully discrete formulation. In this section, we discretize the space semi-discrete problem with an implicit time scheme. We discretize the elastodynamic part with an implicit Newmark scheme (trapezoidal rule), while the unilateral contact and friction conditions are enforced in an implicit way. This choice of time discretization is very common. It is, for instance, employed in [13]. At each time step, we have thus to solve a nonlinear problem similar to a static friction problem. It is well-known that such a problem may have several solutions (see [17] and the references therein). Here we prove that, under a certain condition on the discretization parameters of CFL type, the fully discrete problem is well-posed. We also derive the energy balance of this time-integration scheme.

For simplicity, the interval $[0, T]$ is divided into N equal subintervals of length Δt. We set $t^n = n\Delta t$ and denote by u^n, v^n, and a^n the approximations of $u(t^n)$, $\dot{u}(t^n)$, and $\ddot{u}(t^n)$, respectively. We define the convex combination $\square^{n+\alpha} := (1-\alpha)\square^n + \alpha\square^{n+1}$, where $\square$ stands for u, v, a, or t, and $\alpha \in [0, 1]$. In this section, the notation $A \lesssim B$ means that $A \le cB$ with a constant c independent of h and Δt.

Let $\mathcal{T}^{\mathrm{c}} \subset \mathcal{T}$ be the set of simplices such that at least one vertex is a contact node. We set $\Omega^{\mathrm{c}} = \mathrm{int}\left(\cup_{T\in\mathcal{T}^{\mathrm{c}}}\overline{T}\right)$. Let $\mathcal{T}^{c'} \subset \mathcal{T}$ be the set of simplices such that at least one vertex belongs to $\overline{\Omega^{\mathrm{c}}}$. We set $\Omega^{c'} = \mathrm{int}\left(\cup_{T\in\mathcal{T}^{c'}}\overline{T}\right)$ (see Figure 3.1). We define

$$h_{\mathrm{c}} = \min_{T\in\mathcal{T}^{\mathrm{c}}} \mathrm{diam}(T) \quad \text{and} \quad h_{\mathrm{c}'} = \min_{T\in\mathcal{T}^{c'}} \mathrm{diam}(T),$$

where $\mathrm{diam}(T)$ denotes the diameter of the simplex T. Observe that h_{c} and $h_{\mathrm{c}'}$ are defined using a minimum.

Let us recall some classical discrete trace and inverse inequalities (see, e.g., [30] and [11]). For all $v_{\mathrm{c}} \in V^{\mathrm{c}}$,

$$\|v_{\mathrm{c}}\|_{L^2(\Gamma^{\mathrm{c}})^d} \le \frac{1}{\sqrt{h_{\mathrm{c}}}}\|v_{\mathrm{c}}\|_{L^2(\Omega^{\mathrm{c}})^d}, \tag{4.1}$$

$$|v_{\mathrm{c}}|_{H^1(\Omega^{\mathrm{c}})^d} = \|\nabla v_{\mathrm{c}}\|_{L^2(\Omega^{\mathrm{c}})^{d\times d}} \le \frac{1}{h_{\mathrm{c}}}\|v_{\mathrm{c}}\|_{L^2(\Omega^{\mathrm{c}})^d}. \tag{4.2}$$

The same inequalities hold when Ω^{c} is replaced by $\Omega^{c'}$, and h_{c} by $h_{\mathrm{c}'}$. We define the operator $q^n : V^* \to V^{\mathrm{c}}$, such that for all $0 \le n \le N$,

$$q^n(v_*) = q(t^n, v_*) \quad \forall v_* \in V^*, \tag{4.3}$$

where the map q is defined in section 3.4.

Lemma 4.1. *The function $q^n : V^* \to V^{\mathrm{c}}$ is Lipschitz continuous. More precisely,*

$$|q^n(v_*) - q^n(w_*)|_{H^1(\Omega^{\mathrm{c}})^d} \lesssim |v_* - w_*|_{H^1(\Omega^{c'})^d} \quad \forall v_*, w_* \in V^*. \tag{4.4}$$

Proof. Let $v_*, w_* \in V^*$. Set $v_{\mathrm{c}} = q^n(v_*)$ and $w_{\mathrm{c}} = q^n(w_*)$. Owing to (3.13),

$$a(v_{\mathrm{c}} - w_{\mathrm{c}}, v_{\mathrm{c}} - w_{\mathrm{c}}) \le a(v_* - w_*, w_{\mathrm{c}} - v_{\mathrm{c}}).$$

Since $v_{\rm c}$ and $w_{\rm c}$ are zero outside $\Omega^{\rm c}$, $a(v_{\rm c}-w_{\rm c},v_{\rm c}-w_{\rm c}) \gtrsim |v_{\rm c}-w_{\rm c}|^2_{H^1(\Omega^{\rm c})^d}$, and

$$a(v_*-w_*,w_{\rm c}-v_{\rm c}) = a((v_*-w_*)1_{\Omega^{c'}},w_{\rm c}-v_{\rm c}) \lesssim |v_*-w_*|_{H^1(\Omega^{c'})^d}|v_{\rm c}-w_{\rm c}|_{H^1(\Omega^{\rm c})^d},$$

whence the assertion. ∎

We can now formulate the fully discrete problem.

PROBLEM 4.1. *Seek* $u^{n+1} \in V$, $v_*^{n+1} \in V^*$, *and* $a_*^{n+1} \in V^*$ *such that*

$$M^* a_*^{n+1} \in -A^* u^{n+1} - \partial_2 j(u^{n+1}, v_{\rm f}^{n+1}) + L^*(t^{n+1}), \tag{4.5}$$

$$u_{\rm c}^{n+1} = q^{n+1}(u_*^{n+1}), \tag{4.6}$$

$$u_*^{n+1} = u_*^n + \Delta t \ v_*^n + \frac{\Delta t^2}{2} a_*^{n+\frac{1}{2}}, \tag{4.7}$$

$$v_*^{n+1} = v_*^n + \Delta t \ a_*^{n+\frac{1}{2}}. \tag{4.8}$$

To begin with, we reformulate Problem 4.1 by eliminating v_*^{n+1} and a_*^{n+1}. We set $\delta_*^n := -u_*^n - \frac{\Delta t}{2} v_*^n$ and $\varepsilon_*^n := -u_*^n - \Delta t v_*^n - \frac{\Delta t^2}{4} a_*^n$, and we rewrite v_*^{n+1} and a_*^{n+1} as

$$v_*^{n+1} = \frac{2}{\Delta t}(u_*^{n+1} + \delta_*^n),$$
$$a_*^{n+1} = \frac{4}{\Delta t^2}(u_*^{n+1} + \varepsilon_*^n).$$

Next, we define the linear operator $\tilde{A}^* : V^* \to V^*$ and the vector $\tilde{L}^{n+1} \in V^*$ such that, for all $v_* \in V^*$,

$$\tilde{A}^* v_* := A^* v_* + \frac{1}{4\Delta t^2} M^* v_*,$$
$$\tilde{L}^{n+1} := L^*(t^{n+1}) - \frac{1}{4\Delta t^2} M^* \varepsilon_*^n.$$

Then, using (4.6), it is straightforward to turn (4.5) into

$$0 \in \tilde{A}^* u_*^{n+1} + \partial_2 j\left(u_*^{n+1} + q^{n+1}(u_*^{n+1}), \frac{2}{\Delta t}(u_{\rm f}^{n+1} + \delta_{\rm f}^n)\right) - \tilde{L}^{n+1} + A^* q^{n+1}(u_*^{n+1}). \tag{4.9}$$

Observe that the last term on the right-hand side of (4.9) involves the operator A^* (and not $\tilde{A}^*$) owing to (3.5) and the fact that $q^{n+1}(u_*^{n+1}) \in V^{\rm c}$.

THEOREM 4.2. *Problem* 4.1 *has a unique solution under the CFL-type condition*

$$\frac{\Delta t}{h_{c'}} \lesssim 1. \tag{4.10}$$

Proof. Define the map $\Phi^n : V^* \to V^*$ such that for all $\hat{v}_* \in V^*$, $v_* = \Phi^n(\hat{v}_*)$ satisfies

$$0 \in \tilde{A}^* v_* + \partial_2 j\left(\hat{v}, \frac{2}{\Delta t}(v_{\rm f} + \delta_{\rm f}^n)\right) - \tilde{L}^{n+1} + A^* \hat{v}_{\rm c}, \tag{4.11}$$

where $\hat{v}_{\rm c} := q^{n+1}(\hat{v}_*)$ and $\hat{v} := \hat{v}_* + \hat{v}_{\rm c}$, so that (4.9) amounts to seeking a fixed-point for Φ^n. Setting $y_* := \frac{2}{\Delta t}(v_* + \delta_*^n)$, we rewrite the above inclusion as a variational inequality,

$$\tilde{a}(v_*, z_* - y_*) + j(\hat{v}, z_{\rm f}) - j(\hat{v}, y_{\rm f}) \geq l^{n+1}(z_* - y_*) - a(\hat{v}_{\rm c}, z_* - y_*) \ \forall z_* \in V^*, \tag{4.12}$$

where we have set $\tilde{a}(v_*, w_*) := (\tilde{A}^* v_*, w_*)$ and $l^{n+1}(v_*) := (L(t^{n+1}), v_*)$. Taking $z_* := \frac{2}{\Delta t}(w_* + \delta_*^n)$ in (4.12) and then dividing by $\frac{2}{\Delta t}$, we obtain for all $w_* \in V^*$,

$$\tilde{a}(v_*, w_* - v_*) + j(\hat{v}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{v}, v_\mathrm{f} + \delta_\mathrm{f}^n) \geq l^{n+1}(w_* - v_*) - a(\hat{v}_\mathrm{c}, w_* - v_*). \tag{4.13}$$

The variational inequality (4.13) has one and only one solution. Indeed, it is equivalent to the minimization of a strictly convex functional. The map Φ^n is thus well-defined. Now we shall prove that Φ^n is a contraction under the CFL condition (4.10). Let $\hat{v}_* \in V^*$ and $\hat{w}_* \in V^*$. Set $v_* := \Phi^n(\hat{v}_*)$ and $w_* := \Phi^n(\hat{w}_*)$. Using (4.13), a straightforward calculation yields

$$\begin{aligned}\tilde{a}(v_* - w_*, v_* - w_*) \leq\ & j(\hat{v}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{w}, w_\mathrm{f} + \delta_\mathrm{f}^n) \\ & - j(\hat{v}, v_\mathrm{f} + \delta_\mathrm{f}^n) + j(\hat{w}, v_\mathrm{f} + \delta_\mathrm{f}^n) - a(\hat{v}_\mathrm{c} - \hat{w}_\mathrm{c}, v_* - w_*).\end{aligned} \tag{4.14}$$

Using the ellipticity of m and a,

$$\tilde{a}(v_* - w_*, v_* - w_*) \gtrsim \frac{4}{\Delta t^2} \|v_* - w_*\|^2_{L^2(\Omega)^d} + |v_* - w_*|^2_{H^1(\Omega)^d}. \tag{4.15}$$

Using a reverse triangle inequality,

$$\begin{aligned} & j(\hat{v}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{w}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{v}, v_\mathrm{f} + \delta_\mathrm{f}^n) + j(\hat{w}, v_\mathrm{f} + \delta_\mathrm{f}^n) \\ & \quad \leq \int_{\Gamma^\mathrm{c}} \mu \, |\sigma_\nu(\hat{v}) - \sigma_\nu(\hat{w})| \left| |w_\mathrm{f} + \delta_\mathrm{f}^n| - |v_\mathrm{f} + \delta_\mathrm{f}^n| \right| \\ & \quad \leq \int_{\Gamma^\mathrm{c}} |\mu| |\sigma_\nu(\hat{v}) - \sigma_\nu(\hat{w})| |v_\mathrm{f} - w_\mathrm{f}| \\ & \quad \lesssim \int_{\Gamma^\mathrm{c}} |\sigma_\nu(\hat{v}) - \sigma_\nu(\hat{w})| |v_\mathrm{f} - w_\mathrm{f}|. \end{aligned}$$

Using the Cauchy–Schwarz inequality and the trace inequality (4.1),

$$\begin{aligned} & j(\hat{v}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{w}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{v}, v_\mathrm{f} + \delta_\mathrm{f}^n) + j(\hat{w}, v_\mathrm{f} + \delta_\mathrm{f}^n) \\ & \quad \lesssim \|\sigma_\nu(\hat{v}) - \sigma_\nu(\hat{w})\|_{L^2(\Gamma^\mathrm{c})} \|v_\mathrm{f} - w_\mathrm{f}\|_{L^2(\Gamma^\mathrm{c})^d} \\ & \quad \lesssim \frac{1}{h_\mathrm{c}} \|\sigma_\nu(\hat{v}) - \sigma_\nu(\hat{w})\|_{L^2(\Omega^\mathrm{c})} \|v_* - w_*\|_{L^2(\Omega^\mathrm{c})^d} \\ & \quad \lesssim \frac{1}{h_\mathrm{c}} |\hat{v} - \hat{w}|_{H^1(\Omega^\mathrm{c})^d} \|v_* - w_*\|_{L^2(\Omega^\mathrm{c})^d}. \end{aligned} \tag{4.16}$$

Furthermore, using (4.4) and the inverse inequality (4.2),

$$\begin{aligned} |\hat{v} - \hat{w}|_{H^1(\Omega^\mathrm{c})^d} &= |\hat{v}_\mathrm{c} - \hat{w}_\mathrm{c}|_{H^1(\Omega^\mathrm{c})^d} + |\hat{v}_* - \hat{w}_*|_{H^1(\Omega^\mathrm{c})^d} \\ &= |q^{n+1}(\hat{v}_*) - q^{n+1}(\hat{w}_*)|_{H^1(\Omega^\mathrm{c})^d} + |\hat{v}_* - \hat{w}_*|_{H^1(\Omega^\mathrm{c})^d} \\ &\lesssim |\hat{v}_* - \hat{w}_*|_{H^1(\Omega^{\mathrm{c}'})^d} + |\hat{v}_* - \hat{w}_*|_{H^1(\Omega^\mathrm{c})^d} \\ &\lesssim |\hat{v}_* - \hat{w}_*|_{H^1(\Omega^{\mathrm{c}'})^d} \lesssim \frac{1}{h_{\mathrm{c}'}} \|\hat{v}_* - \hat{w}_*\|_{L^2(\Omega^{\mathrm{c}'})^d}. \end{aligned} \tag{4.17}$$

Collecting inequalities (4.16) and (4.17), and since $h_{\mathrm{c}'} \leq h_\mathrm{c}$,

$$\begin{aligned} & j(\hat{v}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{w}, w_\mathrm{f} + \delta_\mathrm{f}^n) - j(\hat{v}, v_\mathrm{f} + \delta_\mathrm{f}^n) + j(\hat{w}, v_\mathrm{f} + \delta_\mathrm{f}^n) \\ & \qquad \leq \frac{1}{h^2_{\mathrm{c}'}} \|\hat{v}_* - \hat{w}_*\|_{L^2(\Omega)^d} \|v_* - w_*\|_{L^2(\Omega^\mathrm{c})^d}. \end{aligned}$$

Using the boundedness of a, Lemma 4.1, and the inverse inequality (4.2),

$$a(\hat{v}_{\mathrm{c}} - \hat{w}_{\mathrm{c}}, w_* - v_*) \lesssim \frac{1}{h_{c'}^2} \|\hat{w}_* - \hat{v}_*\|_{L^2(\Omega)^d} \|v_* - w_*\|_{L^2(\Omega)^d}.$$

Collecting these different estimates,

$$\|\Phi^n(\hat{v}_*) - \Phi^n(\hat{w}_*)\|_{L^2(\Omega)^d} = \|v_* - w_*\|_{L^2(\Omega)^d} \lesssim \left(\frac{\Delta t}{h_{c'}}\right)^2 \|\hat{v}_* - \hat{w}_*\|_{L^2(\Omega)^d}.$$

Hence, if the ratio $\frac{\Delta t}{h_{c'}}$ is sufficiently small, the mapping Φ^n is a contraction. The Banach fixed-point theorem guarantees that the problem has a unique solution. ∎

Remark 4.1. In the above proof, the inertial term plays a key role. By strengthening the coercivity of $\tilde{a}$, it allows one to prove that Φ^n is a contraction (for a time step sufficiently small). Without the help of the inertial term (as in the static case), this fixed-point argument works only for a certain range of physical parameters, for instance when the Young modulus is large compared with the friction coefficient [21, Theorem 11.4]. We also observe that the CFL-type restriction on the time step (4.10) is a sufficient condition for uniqueness, but not a necessary condition. However, this condition is relevant from a practical viewpoint since many algorithms for Coulomb friction problems rely on fixed-point strategies (see [24], for instance) and condition (4.10) ensures the convergence of these algorithms.

To conclude this part, we formulate the energy balance. We define the energy at time t^n as

$$E^n := \frac{1}{2}(Au^n, u^n) + \frac{1}{2}(M^* v^n, v^n). \tag{4.18}$$

At each time t^n, there exist $\lambda_{\mathrm{c}}^n \in \partial I_K(u_{\mathrm{c}}^n)$ and $\lambda_{\mathrm{f}}^n \in \partial_2 j(u^n, v_{\mathrm{f}}^n)$ such that

$$M^* a_*^n + Au^n + \lambda_{\mathrm{c}}^n + \lambda_{\mathrm{f}}^n = L(t^n). \tag{4.19}$$

Proceeding as in [23], it is readily shown that

$$\begin{aligned} E^{n+1} - E^n = &- \frac{1}{2}(\lambda_{\mathrm{c}}^n + \lambda_{\mathrm{c}}^{n+1}, u^{n+1} - u^n) - \frac{1}{2}(\lambda_{\mathrm{f}}^n + \lambda_{\mathrm{f}}^{n+1}, u^{n+1} - u^n) \\ &+ \frac{1}{2}(L^n + L^{n+1}, u^{n+1} - u^n). \end{aligned} \tag{4.20}$$

5. Convergence of the fully discrete solutions. We fix the space discretization and we build the approximate solutions $\omega^{\Delta t} : [0, T] \to V$ as follows:

$$\omega^{\Delta t}(t) := u^n + v_*^n(t - t^n) + \frac{1}{2} a_*^{n+\frac{1}{2}} (t - t^n)^2 \qquad \forall t \in [t^n, t^{n+1}), \tag{5.1}$$

$$\omega^{\Delta t}(T) := u^N. \tag{5.2}$$

It is readily verified that, by construction, $\omega^{\Delta t} \in C^0([0, T]; V)$ and $\omega_*^{\Delta t} \in C^1([0, T]; V^*)$. Furthermore, $\omega^{\Delta t} \in W^{1,\infty}(0, T; V)$. We are now going to prove the convergence of these approximate solutions to the semidiscrete solution u of Problem 3.1. In this section, the notation $A \lesssim B$ means that $A \leq cB$ with a constant c independent of Δt, but which can depend on h. We assume without loss of generality that $\Delta t \leq 1$.

LEMMA 5.1. *Let* (u^n, v^n, a^n) *solve, for all* $n \in \{0, \ldots, N\}$, *Problem* 4.1. *Then, for* Δt *small enough,*

$$\|u^n\| \lesssim 1, \quad \|v_*^n\| \lesssim 1, \quad \|a_*^n\| \lesssim 1. \tag{5.3}$$

Proof. (i) Let $n \in \{0, \dots, N\}$. From (4.19) we deduce $A^{\mathrm{c}} u^n + \lambda_{\mathrm{c}}^n = L^{\mathrm{c}}(t^n)$, and then, $\|\lambda_{\mathrm{c}}^n\| \lesssim \|u^n\| + \|L(t^n)\|$. Owing to the inequality (3.10), we obtain $\|\lambda_{\mathrm{f}}^n\| \lesssim \|u^n\|$. Hence, owing to the equilibrium equation (4.19), $\|\mathrm{a}_*^n\| \lesssim \|u^n\| + \|L(t^n)\|$.

(ii) Using the energy balance (4.20), it follows that

$$E^{n+1} - E^n \lesssim \left(\|u^n\| + \|u^{n+1}\| + \|L(t^n)\| + \|L(t^{n+1})\|\right) \|u^{n+1} - u^n\|.$$

Observing by (3.13) that $\|q^{n+1}(u_*^n) - q^n(u_*^n)\| \lesssim \|L(t^{n+1}) - L(t^n)\|$, we infer

$$\begin{aligned}\|u^{n+1} - u^n\| &\le \|u_*^{n+1} - u_*^n\| + \|q^{n+1}(u_*^{n+1}) - q^{n+1}(u_*^n)\| + \|q^{n+1}(u_*^n) - q^n(u_*^n)\| \\ &\lesssim \|u_*^{n+1} - u_*^n\| + \|L(t^{n+1}) - L(t^n)\| \\ &\lesssim \|u_*^{n+1} - u_*^n\| + \|L(t^{n+1})\| + \|L(t^n)\|,\end{aligned}$$

so that

$$\begin{aligned}E^{n+1} - E^n \lesssim {} & \left(\|u^n\| + \|u^{n+1}\| + \|L(t^n)\| + \|L(t^{n+1})\|\right) \\ & \left(\|u_*^{n+1} - u_*^n\| + \|L(t^{n+1})\| + \|L(t^n)\|\right).\end{aligned} \tag{5.4}$$

Using (4.7),

$$\begin{aligned}E^{n+1} - E^n \lesssim {} & \Delta t \left(\|u^n\| + \|u^{n+1}\| + \|L(t^n)\| + \|L(t^{n+1})\|\right) \\ & \left(\|\mathrm{v}_*^n\| + \frac{\Delta t}{2}\|\mathrm{a}_*^{n+\frac{1}{2}}\| + \|L(t^{n+1})\| + \|L(t^n)\|\right).\end{aligned} \tag{5.5}$$

Thus, using the previous bound on $\|\mathrm{a}_*^n\|$ and $\|\mathrm{a}_*^{n+1}\|$, and since $\Delta t \le 1$,

$$\begin{aligned}E^{n+1} - E^n \lesssim {} & \Delta t \left(\|u^n\| + \|u^{n+1}\| + \|L(t^n)\| + \|L(t^{n+1})\|\right) \\ & \left(\|\mathrm{v}_*^n\| + \|u^n\| + \|u^{n+1}\| + \|L(t^{n+1})\| + \|L(t^n)\|\right).\end{aligned} \tag{5.6}$$

Now, using Young's inequality and the coercivity of the energy E^n,

$$E^{n+1} - E^n \le C_1 \Delta t\ E^{n+1} + C_2 \Delta t\ E^n + C_3 \Delta t \left(\|L(t^n)\|^2 + \|L(t^{n+1})\|^2\right),$$

where C_1, C_2, C_3 are three constants independent of Δt. Next,

$$E^{n+1} - E^n \le C_1 \Delta t\ (E^{n+1} - E^n) + (C_1 + C_2) \Delta t\ E^n + C_3 \Delta t \left(\|L(t^n)\|^2 + \|L(t^{n+1})\|^2\right).$$

For $\Delta t \le 1/(2C_1)$,

$$\frac{1}{2}(E^{n+1} - E^n) \le (C_1 + C_2) \Delta t\ E^n + C_3 \Delta t \left(\|L(t^n)\|^2 + \|L(t^{n+1})\|^2\right),$$

so that

$$E^{n+1} - E^n \lesssim \Delta t \left(E^n + \|L(t^n)\|^2 + \|L(t^{n+1})\|^2\right).$$

Finally, using a discrete Gronwall lemma,

$$E^n \lesssim E^0 + \sum_{j=0}^{n} \Delta t \|L(t^j)\|^2 \lesssim 1.$$

Then, it is straightforward to obtain the estimates (5.3). ◻

THEOREM 5.2. *The following convergence results hold true as* $\Delta t \to 0$*:*

$$\begin{aligned}&\omega^{\Delta t} \to u \quad \textit{in } C^0([0,T];V),\\&\dot{\omega}_*^{\Delta t} \to \dot{u}_* \ \textit{in } C^0([0,T];V^*),\\&\ddot{\omega}_*^{\Delta t} \to \ddot{u}_* \ \textit{weakly} * \textit{in } L^\infty(0,T;V^*),\end{aligned}$$

where u *solves Problem* 3.1.

Proof. (i) From the estimates (5.3), we deduce that

$$\|\omega^{\Delta t}\|_{L^\infty(0,T;V)} \lesssim 1, \quad \|\dot{\omega}^{\Delta t}\|_{L^\infty(0,T;V)} \lesssim 1,$$

$$\|\dot{\omega}_*^{\Delta t}\|_{L^\infty(0,T;V^*)} \lesssim 1, \quad \|\ddot{\omega}_*^{\Delta t}\|_{L^\infty(0,T;V^*)} \lesssim 1.$$

(ii) Using standard compactness arguments [29], there exists $\omega \in C^0(0,T;V)$ such that $\dot{\omega}_* \in C^0(0,T;V^*)$, $\ddot{\omega}_* \in L^\infty(0,T;V^*)$, and, up to a subsequence,

$$\begin{aligned}&\omega^{\Delta t} \to \omega \quad \text{in } C^0([0,T];V),\\&\dot{\omega}_*^{\Delta t} \to \dot{\omega}_* \text{ in } C^0([0,T];V^*),\\&\ddot{\omega}_*^{\Delta t} \rightharpoonup \ddot{\omega}_* \text{weakly} * \text{in } L^\infty(0,T;V^*).\end{aligned}$$

(iii) Next, we introduce the auxiliary (piecewise constant in time) approximate solutions $\underline{\omega}^{\Delta t} : [0,T] \to V$ and $\zeta_*^{\Delta t} : [0,T] \to V^*$ such that

$$\begin{aligned}&\zeta_*^{\Delta t}(t) := v_*^{n+1} \quad \forall t \in [t^n, t^{n+1}), \qquad \zeta_*^{\Delta t}(T) := v_*^N,\\&\underline{\omega}^{\Delta t}(t) := u^{n+1} \quad \forall t \in [t^n, t^{n+1}), \qquad \underline{\omega}^{\Delta t}(T) := u^N.\end{aligned}$$

By definition of the approximate solutions $\omega^{\Delta t}$ and $\underline{\omega}^{\Delta t}$, and using relation (4.7), for all $n \in \{0, \dots, N\}$, for all $t \in [t^n, t^{n+1})$,

$$\begin{aligned}\|\omega^{\Delta t}(t) - \underline{\omega}^{\Delta t}(t)\| &\le \|u^{n+1} - u^n\| + \Delta t\|v_*^n\| + \frac{1}{2}\Delta t^2\|a_*^{n+\frac{1}{2}}\|\\&\le 2\Delta t\|v_*^n\| + \Delta t^2\|a_*^{n+\frac{1}{2}}\|.\end{aligned}$$

Hence, using estimates (5.3),

$$\|\omega^{\Delta t}(t) - \underline{\omega}^{\Delta t}(t)\| \lesssim \Delta t \quad \text{a.e. in } [0,T].$$

We deduce that $\underline{\omega}^{\Delta t} \to \omega$ in $L^\infty(0,T;V)$. In the same way, we prove that $\zeta_*^{\Delta t} \to \dot{\omega}_*$ in $L^\infty(0,T;V^*)$. We define an approximate external force vector,

$$\underline{L}^{\Delta t}(t) := L(t^{n+1}) \quad \forall t \in [t^n, t^{n+1}), \qquad \underline{L}^{\Delta t}(T) := L(t^N).$$

Since $t \mapsto L(t)$ is Lipschitz continuous, $\underline{L}^{\Delta t} \to L$ in $L^\infty(0,T;V)$.

(iv) Owing to (4.5), the approximate solutions satisfy

$$M^*\ddot{\omega}_*^{\Delta t} \in -A^*\underline{\omega}^{\Delta t} - \partial_2 j(\underline{\omega}^{\Delta t}, \zeta_{\rm f}^{\Delta t}) + \underline{L}^{\Delta t*}(t) \qquad \text{a.e. in } [0,T],$$

so that

$$\begin{aligned}m^*(\ddot{\omega}_*^{\Delta t}, v_* - \zeta_*^{\Delta t}) + a(\underline{\omega}^{\Delta t}, v_* - \zeta_*^{\Delta t}) + j(\underline{\omega}^{\Delta t}, v_{\rm f}) - j(\underline{\omega}^{\Delta t}, \zeta_{\rm f}^{\Delta t})&\\ \ge \left(\underline{L}^{\Delta t}, v_* - \zeta_*^{\Delta t}\right) \quad \forall v_* \in V^*, \text{ a.e. in } [0,T].&\end{aligned}$$

Passing to the limit,

$$m^*(\ddot{\omega}_*, v_* - \dot{\omega}_*) + a(\omega, v_* - \dot{\omega}_*) + j(\omega, v_{\mathrm{f}}) - j(\omega, \dot{\omega}_{\mathrm{f}}) \geq l(t, v_* - \dot{\omega}_*) \quad \forall v_* \in V^*, \text{ a.e. in } [0,T],$$

and hence

$$M^*\ddot{\omega}_* \in -A^*\omega - \partial_2 j(\omega, \dot{\omega}_{\mathrm{f}}) + L^*(t) \qquad \text{a.e. in } [0,T].$$

By uniqueness of the solution, we conclude that $\omega = u$. This uniqueness also implies that the whole sequence $(\omega^{\Delta t})$ converges, not only a subsequence. ∎

6. Conclusions. In this work, we have established three results on the numerical analysis of the modified mass method for dynamic Signorini problems with Coulomb friction: the well-posedness of the space semidiscrete problem, the well-posedness of the fully discrete problem under a CFL-type condition on the mesh size and the time step, and the convergence of the fully discrete solution to the space semidiscrete solution as the time step tends to zero. These results contribute to the theoretical foundations of a computationally attractive method for which various numerical results and implementation aspects are discussed in [13, 14, 22]. Further work can aim at examining the convergence of the space semidiscrete problem and deriving upper bounds on the approximation error.

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