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THE DISCONTINUOUS GALERKIN APPROXIMATION OF THE GRAD-DIV AND CURL-CURL OPERATORS IN FIRST-ORDER FORM IS INVOLUTION-PRESERVING AND SPECTRALLY CORRECT*

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Abstract. The discontinuous Galerkin approximation of the grad-div and curl-curl problems formulated in conservative first-order form is investigated. It is shown that the approximation is spectrally correct, thereby confirming numerical observations made by various authors in the literature. This result hinges on the existence of discrete involutions which are formulated as discrete orthogonality properties. The involutions are crucial to establish discrete versions of weak Poincaré–Steklov inequalities that hold true at the continuous level.

Key words. grad-div problem, curl-curl problem, first-order system, Friedrichs system, involution, spectrum approximation

MSC codes. 65M60, 65M12, 65N30, 35L02, 35L05, 35Q61

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1. Introduction. Many conservation equations generate involutions, e.g., the elastodynamics equations, Maxwell's equations, the magnetohydrodynamics equations, the wave equation, etc. For instance, on domains with trivial topology, if one considers the wave equation (which is a linearized version of the compressible Euler equations), the involution on the velocity, $\boldsymbol{v}$, is $\nabla\times\boldsymbol{v}=\boldsymbol{0}$, and if one considers Maxwell's equations, the involutions on the electric and magnetic fields, $\boldsymbol{E}$, $\boldsymbol{B}$, are Gauss's laws, i.e., $\nabla\cdot\boldsymbol{E}=\boldsymbol{0}$ (in the absence of free charges) and $\nabla\cdot\boldsymbol{B}=\boldsymbol{0}$. An algebraic characterization of involutions for general conservation equations is given in Boillat [7]. Involutions are important to prove compactness and entropy inequalities and to establish well-posedness. The question addressed in this paper is whether using the discontinuous Galerkin (dG) method to approximate in space conservation equations endowed with involutions generates discrete involutions that are strong enough to establish compactness. Since the problem is very difficult for generic nonlinear conservation equations, we restrict ourselves to the grad-div and curl-curl operators written in first-order conservation form, as they are good representatives of the problems encountered with the wave equation and Maxwell's equations. For both operators, the involutions are formulated as orthogonality properties (see, e.g., Hiptmair [22, sect. 4.1] for handling Gauss's law involutions). We restrict ourselves to the investi-

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gation of the spectral properties of the dG approximation, leaving the study of other discretization methods to future work.

The spectral correctness of the dG method for approximating the grad-div operator written in second-order form is proved in Antonietti, Buffa, and Perugia [3, Thm. 4.1]. The spectral correctness of the curl-curl problem written in second-order form has been established in Buffa and Perugia [9, Prop. 7.3] (this result is prefigured in Houston et al. [23, Cor. 3.6] for constant coefficients). We recall that spectral correctness means that the point spectrum of the approximation does not contain spurious eigenvalues. Whether this is also the case when the equations are written in first-order form has not yet been established at the time of this writing and to the best of the authors' knowledge. The reader is referred to Boffi, Brezzi, and Gastaldi [6] for examples of (conforming) approximations of the grad-div problem written in first-order form that are not spectrally correct. The dG approximation of the time-dependent Maxwell's equations written in first-order conservation form has been investigated in Hesthaven and Warburton [20]. Using energy arguments, it is shown in [20, Thm. 4.2] that the approximation is convergent, and an observation is made in [20, Thm. 4.3] regarding the involution properties of the scheme. This observation is, however, not sufficient to establish that the associated eigenvalue problem is spectrally correct. The eigenvalue problem is numerically investigated in Alvarez et al. [1], Cohen and Duruflé [13, sect. 3.1], and Hesthaven and Warburton [21], and the authors observe that the eigenvalue problem is pollution free provided the numerical flux is dissipative, i.e., includes penalty terms on the jump of the tangential components of both fields. Furthermore, spectral correctness has been established for a combination of conforming and dG approximations of Maxwell's equations by Campos Pinto and Sonnendrücker [10]. It is also observed therein that the full dG approximation is compatible with Gauss's laws.

In the paper, we prove that indeed the dG method yields approximations of the first-order form of the grad-div and curl-curl operators that are spectrally correct. In particular, we show that it is essential to invoke discrete counterparts of the involutions associated with the continuous problem to establish this result. The discrete involutions, which are formulated as (topology-blind) discrete orthogonality properties, are crucial to establish discrete (weak) Poincaré–Steklov inequalities. These inequalities are, in turn, pivotal to gain full L^2-stability of the dG approximation. The discrete Poincaré–Steklov inequalities usually available in the literature involve the L^2-norm of the gradient, curl, or divergence when using conforming spaces (see, e.g., Hiptmair [22, Thm. 4.7] or Monk and Demkowicz [25, Cor. 3.2]) or reconstructions thereof when using dG approximations (see Buffa and Perugia [9, Lem. 7.6]). The route followed here consists instead of extending weak Poincaré–Steklov (in)equalities that hold true at the continuous level which involve a dual norm of the gradient, curl, or divergence.

The material is organized as follows. We present in section 2 the continuous operators we want to approximate (see Definitions 2.13 and 2.17). The involutions mentioned above are formalized as orthogonality properties (see Remarks 2.1 and 2.5). The finite element setting is introduced in section 3. Lemma 3.2, which establishes discrete Poincaré–Steklov inequalities hinging on discrete involutions, is the key result of this section. As in the continuous setting, the discrete involutions are formalized as orthogonality properties. The dG approximation of the grad-div and curl-curl operators is analyzed in sections 4 and 5, respectively. The main results of these sections are Theorems 4.11 and 5.10, which prove the spectral correctness of the dG approximation. For completeness, standard results on Helmholtz decompositions are

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collected in Appendix A. Although everything that is said in the paper could be written using the unified formalism of finite element exterior calculus (see Arnold, Falk, and Winther [4]), we prefer to use the formalism of vector calculus to be more explicit even though the structure of the grad-div and curl-curl operators is similar. Finally, the main compactness result invoked for both operators is the consequence of the regularity of the solution in fractional-order Sobolev spaces with regularity index $s > \frac{1}{2}$ (see Lemma 2.14 and 2.18). This regularity is known in the literature to hold true in the case of homogeneous materials. The case of heterogeneous materials is left to future work.

2. Continuous setting.

Let D be an open, bounded, Lipschitz polyhedron of $\mathbb{R}^d$, $d \in \{2,3\}$, with unit outward normal vector $\boldsymbol{n}_D$. Additional topological assumptions on D are collected in section A.1. We implicitly assume that $d = 3$ whenever working with the curl operator. To be dimensionally consistent, we introduce a length scale, ℓ_D, associated with D (it could be, for instance, the diameter of D).

2.1. Functional spaces.

We use standard notation for Lebesgue and Sobolev spaces. We use boldface fonts for $\mathbb{R}^d$-valued vectors, vector fields, and functional spaces composed of such fields. The spaces $L^2(D)$ and $\boldsymbol{L}^2(D)$ are composed of Lebesgue integrable scalar-valued functions and vector fields that are square integrable, respectively. The canonical inner products in these two spaces are denoted $(\cdot,\cdot)_{L^2(D)}$ and $(\cdot,\cdot)_{\boldsymbol{L}^2(D)}$, respectively. Depending on the context, the symbol $^\perp$ denotes the orthogonality in $L^2(D)$ or $\boldsymbol{L}^2(D)$. We define

$$H^1(D) := H(\mathbf{grad}; D) := \{p \in L^2(D) \mid \nabla p \in \boldsymbol{L}^2(D)\}, \tag{2.1a}$$

$$\boldsymbol{H}(\mathbf{curl}; D) := \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(D)\}, \tag{2.1b}$$

$$\boldsymbol{H}(\mathrm{div}; D) := \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla\cdot\boldsymbol{v} \in L^2(D)\}. \tag{2.1c}$$

These Hilbert spaces are equipped with their natural graph norms

$$\|p\|^2_{H^1(D)} := \|p\|^2_{L^2(D)} + \ell_D^2 \|\nabla p\|^2_{\boldsymbol{L}^2(D)}, \tag{2.2a}$$

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl};D)} := \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2 \|\nabla\times\boldsymbol{v}\|^2_{\boldsymbol{L}^2(D)}, \tag{2.2b}$$

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div};D)} := \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2 \|\nabla\cdot\boldsymbol{v}\|^2_{L^2(D)}. \tag{2.2c}$$

We also consider the following closed subspaces:

$$H^1_0(D) := H_0(\mathbf{grad}; D) := \{p \in H^1(D) \mid \gamma^{\mathrm{g}}_{\partial D}(p) = 0\}, \tag{2.3a}$$

$$\boldsymbol{H}_0(\mathbf{curl}; D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; D) \mid \gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{v}) = \mathbf{0}\}, \tag{2.3b}$$

$$\boldsymbol{H}_0(\mathrm{div}; D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}; D) \mid \gamma^{\mathrm{d}}_{\partial D}(\boldsymbol{v}) = 0\}, \tag{2.3c}$$

where $\gamma^{\mathrm{g}}_{\partial D} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is the extension by density of the usual trace operator such that $\gamma^{\mathrm{g}}_{\partial D}(p) = p|_{\partial D}$ for every smooth function $p \in H^s(D)$, $s > \frac{1}{2}$, and the tangential and normal trace operators $\gamma^{\mathrm{c}}_{\partial D} : \boldsymbol{H}(\mathbf{curl}; D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ and $\gamma^{\mathrm{d}}_{\partial D} : \boldsymbol{H}(\mathrm{div}; D) \to H^{-\frac{1}{2}}(\partial D)$ are the extensions by density of the tangent and normal trace operators such that $\gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{v}) = \boldsymbol{v}|_{\partial D}\times\boldsymbol{n}_D$ and $\gamma^{\mathrm{d}}_{\partial D}(\boldsymbol{v}) = \boldsymbol{v}|_{\partial D}\cdot\boldsymbol{n}_D$ for every smooth field $\boldsymbol{v} \in \boldsymbol{H}^s(D)$, $s > \frac{1}{2}$ (see, e.g., [17, sect. 4.3]). We are also going to make use of the following closed subspaces:

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$$\mathbb{P}_0 := H(\mathbf{grad}=\mathbf{0};D) := \{p \in H^1(D) \mid \nabla p = \mathbf{0}\}, \tag{2.4a}$$

$$\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};D) \mid \nabla\times\boldsymbol{v} = \mathbf{0}\}, \tag{2.4b}$$

$$\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};D) \mid \nabla\times\boldsymbol{v} = \mathbf{0}\}, \tag{2.4c}$$

$$\boldsymbol{H}(\mathrm{div}=0;D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};D) \mid \nabla\cdot\boldsymbol{v} = 0\}, \tag{2.4d}$$

$$\boldsymbol{H}_0(\mathrm{div}=0;D) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};D) \mid \nabla\cdot\boldsymbol{v} = 0\}. \tag{2.4e}$$

Finally, we recall the following result (see Amrouche et al. [2, Prop. 7]) which is a consequence of the elliptic regularity theory: There exists $s \in (\frac{1}{2}, 1]$ so that for all $\boldsymbol{e} \in \{\boldsymbol{H}(\mathbf{curl};D) \cap \boldsymbol{H}_0(\mathrm{div};D), \boldsymbol{H}_0(\mathbf{curl};D) \cap \boldsymbol{H}(\mathrm{div};D)\}$,

$$\|\boldsymbol{e}\|_{\boldsymbol{H}^s(D)} \le C_D(\|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl};D)} + \|\boldsymbol{e}\|_{\boldsymbol{H}(\mathrm{div};D)}). \tag{2.5}$$

Here and in what follows, C_D denotes a generic constant that only depends on D and whose value can change at each occurrence.

2.2. Heuristics for the involutions. We introduce in this section the differential operators associated with two model eigenvalue and boundary value problems. We identify the associated involutions and interpret them as orthogonality properties.

2.2.1. Grad-div eigenvalue problem. Given a scaling factor $\mathfrak{c} > 0$ (see Remark 2.3), we consider the following eigenvalue problem: Find $\lambda \in \mathbb{C}$ and a nonzero pair $(\boldsymbol{v}, p) \in \boldsymbol{H}_0(\mathrm{div};D) \times H^1(D)$ so that

$$\nabla p = \lambda \boldsymbol{v}, \quad \mathfrak{c}^2 \nabla_0\cdot\boldsymbol{v} = \lambda p \tag{2.6}$$

with the operators $\nabla : H^1(D) \ni q \longmapsto \nabla q \in \boldsymbol{L}^2(D)$ and $\nabla_0\cdot : \boldsymbol{H}_0(\mathrm{div};D) \ni \boldsymbol{w} \longmapsto \nabla\cdot\boldsymbol{w} \in L^2(D)$. Notice that $-\nabla$ and $\nabla_0\cdot$ are adjoint to each other since $(q, \nabla_0\cdot\boldsymbol{w})_{L^2(D)} = -(\nabla q, \boldsymbol{w})_{\boldsymbol{L}^2(D)}$ for all $q \in H^1(D)$ and all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div};D)$. Moreover, we will see in section 2.3 that both operators have a closed range.

We are only interested in the case $\lambda \neq 0$. The assumption $\lambda \neq 0$ implies that $p = \nabla_0\cdot(\lambda^{-1}\mathfrak{c}^2\boldsymbol{v})$ and $\boldsymbol{v} = \nabla(\lambda^{-1}p)$. This means that

$$p \in \mathrm{im}(\nabla_0\cdot) = \ker(\nabla)^\perp = \mathbb{P}_0^\perp, \tag{2.7a}$$

$$\boldsymbol{v} \in \mathrm{im}(\nabla) = \ker(\nabla_0\cdot)^\perp = \boldsymbol{H}_0(\mathrm{div}=0;D)^\perp, \tag{2.7b}$$

where we recall that the symbol "im" means "image of" or "range of" and the symbol "ker" means "kernel of" or "nullspace of." (Notice that $p \in \mathbb{P}_0^\perp$ simply means that $(p,1)_{L^2(D)} = 0$, i.e., p has zero mean-value over D.) We call *involutions* of the eigenvalue problem (2.6) the properties $p \in \mathbb{P}_0^\perp$ and $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}=0;D)^\perp$. One objective of the paper is to prove that the dG approximation of (2.6) preserves discrete versions of the involutions (2.7) and that the approximation of the spectrum is pollution free.

Remark 2.1 (involutions and topology of D). The involution property (2.7b) implies that $\nabla\times\boldsymbol{v} = \mathbf{0}$ since $\boldsymbol{H}_0(\mathrm{div}=0;D)^\perp = \nabla H^1(D)$ (see (A.5c)). However, $\nabla\times\boldsymbol{v} = \mathbf{0}$ fully describes the involution only if D is simply connected. Indeed, the decomposition $\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) = \nabla H^1(D) \overset{\perp}{\oplus} \boldsymbol{K}_{\mathrm{T}}(D)$ (see (A.4a)) implies that $\boldsymbol{H}_0(\mathrm{div}=0;D)^\perp \subset \boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)$ with equality iff D is simply connected.

Remark 2.2 (time domain). Consider the wave equation in the time domain:

$$\partial_t\boldsymbol{v} + \nabla p = \mathbf{0}, \quad \partial_t p + \mathfrak{c}^2\nabla_0\cdot\boldsymbol{v} = 0 \tag{2.8}$$

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with initial conditions $p(\cdot,0)=p^0(\cdot)$, $\boldsymbol{v}(\cdot,0)=\boldsymbol{v}^0(\cdot)$ and boundary condition $\gamma^{\rm d}_{\partial D}(\boldsymbol{v})=0$. Arguing as above, we observe that if $p^0 \in \mathbb{P}_0^{\perp}$, $\boldsymbol{v}^0 \in \boldsymbol{H}_0(\mathrm{div}=0;D)^{\perp}$, then the involutions $p(\cdot,t) \in \mathbb{P}_0^{\perp}$, $\boldsymbol{v}(\cdot,t) \in \boldsymbol{H}_0(\mathrm{div}=0;D)^{\perp}$ hold true at all times. That the dG method satisfies discrete counterparts of these involutions guarantees that the (semidiscrete) system behaves properly over long times.

Remark 2.3 (scaling factor). The scaling factor $\mathfrak{c}$ is introduced to remind us that the field $\boldsymbol{v}$ and the function p can have different units. Here, $\mathfrak{c}$ has the same units as the ratio of p to $\|\boldsymbol{v}\|$. In applications, one often thinks of $\mathfrak{c}$ as a wave speed. With this scaling, the eigenvalue λ scales as a frequency. The reader can assume that $\mathfrak{c}=1$ without loosing anything essential in what follows.

Remark 2.4 (other boundary conditions). The problems (2.6) and (2.8) can be equipped with other boundary conditions. For instance, one can enforce $\gamma^{\rm g}_{\partial D}(p)=0$ instead of $\gamma^{\rm d}_{\partial D}(\boldsymbol{v})=0$. In this case, the involutions are $p\in\{0\}^{\perp}$ and $\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div}=0;D)^{\perp}$. (Notice that the involution $p\in\{0\}^{\perp}$ is trivial.) The present analysis extends to this situation; see Remarks 2.11 and 4.1.

2.2.2. Curl-curl eigenvalue problem. Given $\mathfrak{c}>0$, the eigenvalue problem is to find $\lambda\in\mathbb{C}$ and a nonzero pair $(\boldsymbol{B},\boldsymbol{E})\in\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D)$ s.t.

$$-\nabla\times\boldsymbol{E}=\lambda\boldsymbol{B},\quad \mathfrak{c}^2\nabla_0\times\boldsymbol{B}=\lambda\boldsymbol{E} \tag{2.9}$$

with the operators $\nabla\times:\boldsymbol{H}(\mathbf{curl};D)\ni\boldsymbol{e}\longmapsto\nabla\times\boldsymbol{e}\in\boldsymbol{L}^2(D)$ and $\nabla_0\times:\boldsymbol{H}_0(\mathbf{curl};D)\ni\boldsymbol{b}\longmapsto\nabla\times\boldsymbol{b}\in\boldsymbol{L}^2(D)$. These operators are adjoint to each other since we have $(\boldsymbol{e},\nabla_0\times\boldsymbol{b})_{\boldsymbol{L}^2(D)}=(\boldsymbol{b},\nabla\times\boldsymbol{e})_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl};D)$ and all $\boldsymbol{b}\in\boldsymbol{H}_0(\mathbf{curl};D)$. Moreover, we will see in section 2.3 that both operators have a closed range.

We are only interested in the case $\lambda\neq0$. The assumption $\lambda\neq0$ implies that $\boldsymbol{E}=\nabla_0\times(\lambda^{-1}\mathfrak{c}^2\boldsymbol{B})$ and $\boldsymbol{B}=-\nabla\times(\lambda^{-1}\boldsymbol{E})$. This means that

$$\boldsymbol{E}\in\mathrm{im}(\nabla_0\times)=\ker(\nabla\times)^{\perp}=\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}, \tag{2.10a}$$

$$\boldsymbol{B}\in\mathrm{im}(\nabla\times)=\ker(\nabla_0\times)^{\perp}=\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}. \tag{2.10b}$$

We call *involutions* of the eigenvalue problem (2.9) the fact that $\boldsymbol{E}\in\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}$ and $\boldsymbol{B}\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}$. One objective of the paper is to prove that the dG approximation of (2.9) preserves discrete versions of the involutions (2.10) and that the approximation of the spectrum is pollution free.

Remark 2.5 (involutions and topology of D). Owing to (A.4a) and (A.5c), we have $\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}\subset\boldsymbol{H}_0(\mathrm{div}=0;D)$, and equality holds true iff D is simply connected. Thus, $\nabla\cdot\boldsymbol{E}=0$ and $\gamma^{\rm d}_{\partial D}(\boldsymbol{E})=0$ fully describe the involution satisfied by $\boldsymbol{E}$ only if D is simply connected. Similarly, owing to (A.4b) and (A.5d), we have $\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}\subset\boldsymbol{H}(\mathrm{div}=0;D)$, and equality holds true iff ∂D is connected. Thus, $\nabla\cdot\boldsymbol{B}=0$ fully describes the involution satisfied by $\boldsymbol{B}$ only if ∂D is connected.

Remark 2.6 (time domain). Consider Maxwell's equations in the time domain:

$$\partial_t\boldsymbol{B}+\nabla\times\boldsymbol{E}=\mathbf{0},\quad \partial_t\boldsymbol{E}-\mathfrak{c}^2\nabla_0\times\boldsymbol{B}=\mathbf{0} \tag{2.11}$$

with initial conditions $\boldsymbol{B}(\cdot,0)=\boldsymbol{B}^0(\cdot)$, $\boldsymbol{E}(\cdot,0)=\boldsymbol{E}^0(\cdot)$ and boundary condition $\gamma^{\rm c}_{\partial D}(\boldsymbol{B})=\mathbf{0}$. Assuming $\boldsymbol{B}^0\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}$, $\boldsymbol{E}^0\in\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}$, then the involutions $\boldsymbol{B}(\cdot,t)\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}$, $\boldsymbol{E}(\cdot,t)\in\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}$ hold true at all times. That the dG method satisfies discrete counterparts of these involutions guarantees that the (semidiscrete) system behaves properly over long times.

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Remark 2.7 (other boundary conditions). The problems (2.9) and (2.11) can also be equipped with the boundary condition $\gamma^{c}_{\partial D}(\boldsymbol{E}) = \boldsymbol{0}$. The analysis in the paper readily extends to this case (it suffices to swap the roles of $\boldsymbol{E}$ and $\boldsymbol{B}$).

2.3. Weak Poincaré–Steklov (in)equalities. In this section, we identify important properties of the differential operators involved in the eigenvalue problems (2.6) and (2.9) and in the time-evolution problems (2.8) and (2.11). All these results are consequences of well-known Helmholtz decompositions which we recall in Appendix A. Recall that we defined $H^1(D) := H(\mathbf{grad}; D)$ and $\mathbb{P}_0 := H(\mathbf{grad}=\mathbf{0}; D)$ and that orthogonality is meant in L^2 and $\boldsymbol{L}^2$ depending on the context. We consider the following subspaces:

$$X^{\mathrm{g}} := H(\mathbf{grad}; D) \cap H(\mathbf{grad}=\mathbf{0}; D)^{\perp} = H^1(D) \cap \mathbb{P}_0^{\perp}, \tag{2.12a}$$

$$\boldsymbol{X}_0^{\mathrm{d}} := \boldsymbol{H}_0(\mathrm{div}; D) \cap \boldsymbol{H}_0(\mathrm{div}=0; D)^{\perp}, \tag{2.12b}$$

$$\boldsymbol{X}^{\mathrm{c}} := \boldsymbol{H}(\mathbf{curl}; D) \cap \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)^{\perp}, \tag{2.12c}$$

$$\boldsymbol{X}_0^{\mathrm{c}} := \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^{\perp}. \tag{2.12d}$$

As these spaces are closed in $H^1(D)$, $\boldsymbol{H}(\mathrm{div}; D)$, $\boldsymbol{H}(\mathbf{curl}; D)$, and $\boldsymbol{H}_0(\mathbf{curl}; D)$, respectively, they are Hilbert spaces when equipped with the inherited inner products.

Lemma 2.8 (isomorphisms). *The following operators are isomorphims:*

$$\nabla : X^{\mathrm{g}} \to \boldsymbol{H}_0(\mathbf{div}=0; D)^{\perp}, \qquad \nabla_0\cdot : \boldsymbol{X}_0^{\mathrm{d}} \to H(\mathbf{grad}=\mathbf{0}; D)^{\perp} = \mathbb{P}_0^{\perp}. \tag{2.13a}$$

$$\nabla\times : \boldsymbol{X}^{\mathrm{c}} \to \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^{\perp}, \qquad \nabla_0\times : \boldsymbol{X}_0^{\mathrm{c}} \to \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)^{\perp}. \tag{2.13b}$$

Proof. (1) These operators are well defined since $\nabla(H^1(D)) \subset \boldsymbol{H}_0(\mathrm{div}=0; D)^{\perp}$, $\nabla_0\cdot(\boldsymbol{H}_0(\mathrm{div}; D)) \subset \mathbb{P}_0^{\perp}$, $\nabla\times(\boldsymbol{H}(\mathbf{curl}; D)) \subset \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^{\perp}$, $\nabla_0\times(\boldsymbol{H}_0(\mathbf{curl}; D)) \subset \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)^{\perp}$. They are also bounded.

(2) Injectivity. Let $p \in X^{\mathrm{g}} = H(\mathbf{grad}; D) \cap H(\mathbf{grad}=\mathbf{0}; D)^{\perp}$ be such that $\nabla p = \mathbf{0}$. Then $p \in H(\mathbf{grad}=\mathbf{0}; D) \cap H(\mathbf{grad}=\mathbf{0}; D)^{\perp} = \{0\}$. A similar argument shows that the other operators are injective as well.

(3) Surjectivity. The Helmholtz decompositions (A.5c) and (A.5b) show that the operators ∇ and $\nabla_0\cdot$ are surjective. The surjectivity of the $\nabla\times$ operator follows from the Helmholtz decomposition (A.6a) and $\nabla(H^1_{\Gamma}(D)) \subset \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)$. Similarly, the surjectivity of the $\nabla_0\times$ operator follows from the Helmholtz decomposition (A.6b) and $\nabla_{\Sigma}(H^1_{\Sigma}(D)) \subset \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)$ (see Amrouche et al. [2, Lem. 3.11, p. 840]). ∎

Remark 2.9 (literature). Referring to Appendix A for the notation and to Dautray and Lions [14, Table I, p. 314], we have $\boldsymbol{H}^{\Gamma}(\mathrm{div}=0; D) = \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^{\perp}$ and $\boldsymbol{H}_0^{\Sigma}(\mathrm{div}=0; D) = \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)^{\perp}$. Hence, (2.13b) is a topology-blind restatement of Theorems 3.2 and 3.17 in Amrouche et al. [2].

Lemma 2.8 implies that the images of the operators ∇, $\nabla_0\cdot$, $\nabla\times$, $\nabla_0\times$ are closed. Therefore, there is $C_D > 0$ so that $C_D\|p\|_{H^1(D)} \le \ell_D\|\nabla p\|_{\boldsymbol{L}^2(D)}$ for all $p \in X^{\mathrm{g}}$. Hence, we can equip X^{g} with the norm $\|p\|_{X^{\mathrm{g}}} := \ell_D\|\nabla p\|_{\boldsymbol{L}^2(D)}$ for all $p \in X^{\mathrm{g}}$. By using a similar argument, we equip $\boldsymbol{X}_0^{\mathrm{d}}$, $\boldsymbol{X}_0^{\mathrm{c}}$, $\boldsymbol{X}^{\mathrm{c}}$ with the norms $\|\boldsymbol{v}\|_{\boldsymbol{X}_0^{\mathrm{d}}} := \ell_D\|\nabla_0\cdot\boldsymbol{v}\|_{L^2(D)}$, $\|\boldsymbol{b}\|_{\boldsymbol{X}_0^{\mathrm{c}}} := \ell_D\|\nabla_0\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}$, $\|\boldsymbol{e}\|_{\boldsymbol{X}^{\mathrm{c}}} := \ell_D\|\nabla\times\boldsymbol{e}\|_{\boldsymbol{L}^2(D)}$, respectively. We extend by density the above operators to $\nabla : L^2(D) \to (\boldsymbol{X}_0^{\mathrm{d}})'$, $\nabla_0\cdot : \boldsymbol{L}^2(D) \to (X^{\mathrm{g}})'$, $\nabla\times : \boldsymbol{L}^2(D) \to (\boldsymbol{X}_0^{\mathrm{c}})'$, $\nabla_0\times : \boldsymbol{L}^2(D) \to (\boldsymbol{X}^{\mathrm{c}})'$. For all $p \in L^2(D)$ and all $\boldsymbol{v}, \boldsymbol{b}, \boldsymbol{e} \in \boldsymbol{L}^2(D)$, we set

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(2.14a)
$$\|\nabla p\|_{(\boldsymbol{X}_0^{\rm d})'} := \sup_{\boldsymbol{v}\in \boldsymbol{X}_0^{\rm d}} \frac{|(p,\nabla_0\cdot\boldsymbol{v})_{L^2(D)}|}{\ell_D\|\nabla_0\cdot\boldsymbol{v}\|_{L^2(D)}}, \qquad \|\nabla_0\cdot\boldsymbol{v}\|_{(X^{\rm g})'} := \sup_{p\in X^{\rm g}} \frac{|(\boldsymbol{v},\nabla p)_{\boldsymbol{L}^2(D)}|}{\ell_D\|\nabla p\|_{\boldsymbol{L}^2(D)}},$$

(2.14b)
$$\|\nabla_0\times\boldsymbol{b}\|_{(\boldsymbol{X}^{\rm c})'} := \sup_{\boldsymbol{e}\in \boldsymbol{X}^{\rm c}} \frac{|(\boldsymbol{b},\nabla\times\boldsymbol{e})_{\boldsymbol{L}^2(D)}|}{\ell_D\|\nabla\times\boldsymbol{e}\|_{\boldsymbol{L}^2(D)}}, \qquad \|\nabla\times\boldsymbol{e}\|_{(\boldsymbol{X}_0^{\rm c})'} := \sup_{\boldsymbol{b}\in \boldsymbol{X}_0^{\rm c}} \frac{|(\boldsymbol{e},\nabla_0\times\boldsymbol{b})_{\boldsymbol{L}^2(D)}|}{\ell_D\|\nabla_0\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}}.$$

COROLLARY 2.10 (weak Poincaré–Steklov (in)equalities). *The following holds:*

(2.15a) $$\|p\|_{L^2(D)} = \ell_D\|\nabla p\|_{(\boldsymbol{X}_0^{\rm d})'} \qquad \forall p\in H(\mathbf{grad}=\mathbf{0};D)^\perp = \mathbb{P}_0^\perp,$$

(2.15b) $$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} = \ell_D\|\nabla_0\cdot\boldsymbol{v}\|_{(X^{\rm g})'} \qquad \forall \boldsymbol{v}\in \boldsymbol{H}_0(\mathrm{div}=0;D)^\perp,$$

(2.15c) $$\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} = \ell_D\|\nabla_0\times\boldsymbol{b}\|_{(\boldsymbol{X}^{\rm c})'} \qquad \forall \boldsymbol{b}\in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp,$$

(2.15d) $$\|\boldsymbol{e}\|_{\boldsymbol{L}^2(D)} = \ell_D\|\nabla\times\boldsymbol{e}\|_{(\boldsymbol{X}_0^{\rm c})'} \qquad \forall \boldsymbol{e}\in \boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^\perp.$$

Proof. Owing to (2.13a) in Lemma 2.8, for all $q\in\mathbb{P}_0^\perp$, there exists a unique $\boldsymbol{v}(q)\in\boldsymbol{X}_0^{\rm d}$ so that $\nabla_0\cdot\boldsymbol{v}(q)=q$. Let $p\in\mathbb{P}_0^\perp$, $p\neq 0$. We have

$$\begin{aligned}\|p\|_{L^2(D)} &= \sup_{q\in\mathbb{P}_0^\perp}\frac{|(p,q)_{L^2(D)}|}{\|q\|_{L^2(D)}} = \sup_{q\in\mathbb{P}_0^\perp}\frac{|(p,\nabla_0\cdot\boldsymbol{v}(q))_{L^2(D)}|}{\|\nabla_0\cdot\boldsymbol{v}(q)\|_{L^2(D)}}\\ &= \sup_{\boldsymbol{v}\in\boldsymbol{X}_0^{\rm d}}\frac{|(p,\nabla_0\cdot\boldsymbol{v})_{L^2(D)}|}{\|\nabla_0\cdot\boldsymbol{v}\|_{L^2(D)}} = \ell_D\|\nabla p\|_{(\boldsymbol{X}_0^{\rm d})'}.\end{aligned}$$

The proof of the other identities is similar. ∎

Remark 2.11 (other boundary conditions). Setting $X_0^{\rm g} := H_0(\mathbf{grad};D)\cap H_0(\mathbf{grad}=\mathbf{0};D)^\perp = H_0^1(D)\cap\{0\}^\perp = H_0^1(D)$ and $\boldsymbol{X}^{\rm d} := \boldsymbol{H}(\mathrm{div};D)\cap\boldsymbol{H}(\mathrm{div}=0;D)^\perp$, the operators $\nabla_0 : X_0^{\rm g}\to\boldsymbol{H}(\mathrm{div}=0;D)^\perp$ and $\nabla\cdot : \boldsymbol{X}^{\rm d}\to H_0(\mathbf{grad}=\mathbf{0};D)^\perp = L^2(D)$ are isomorphisms owing, in particular, to the Helmholtz decompositions (A.5d) and (A.5a). Moreover, equipping $X_0^{\rm g}$ and $\boldsymbol{X}^{\rm d}$ with the norms $\|p\|_{X_0^{\rm g}} := \ell_D\|\nabla_0 p\|_{\boldsymbol{L}^2(D)}$ and $\|\boldsymbol{v}\|_{\boldsymbol{X}^{\rm d}} := \ell_D\|\nabla\cdot\boldsymbol{v}\|_{L^2(D)}$, respectively, and considering the extended operators $\nabla_0 : \boldsymbol{L}^2(D)\to(\boldsymbol{X}^{\rm d})'$ and $\nabla\cdot : \boldsymbol{L}^2(D)\to(X_0^{\rm g})'$, the following weak Poincaré–Steklov (in)equalities hold true for all $(p,\boldsymbol{v})\in L^2(D)\times\boldsymbol{H}(\mathrm{div}=0;D)^\perp$:

$$\|p\|_{L^2(D)} = \ell_D\|\nabla_0 p\|_{(\boldsymbol{X}^{\rm d})'}, \qquad \|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} = \ell_D\|\nabla\cdot\boldsymbol{v}\|_{(X_0^{\rm g})'},$$

with dual norms defined as in (2.14). (Notice that $\|\cdot\|_{(X_0^{\rm g})'} = \|\cdot\|_{H^{-1}(D)}$.)

2.4. Eigenvalue problems. We are now ready to give a precise definition of the operators involved in the eigenvalue problems (2.6) and (2.9). The main difficulty we address is to get rid of the eigenspace associated with the 0 eigenvalue. For this purpose, we consider the following L^2- or $\boldsymbol{L}^2$-orthogonal projections:

(2.16a) $$\Pi^{\rm g} : L^2(D)\to H(\mathbf{grad}=\mathbf{0};D) = \mathbb{P}_0 = \ker(\nabla) = \mathrm{im}(\nabla_0\cdot)^\perp,$$

(2.16b) $$\boldsymbol{\Pi}_0^{\rm d} : \boldsymbol{L}^2(D)\to\boldsymbol{H}_0(\mathrm{div}=0;D) = \ker(\nabla_0\cdot) = \mathrm{im}(\nabla)^\perp,$$

(2.16c) $$\boldsymbol{\Pi}_0^{\rm c} : \boldsymbol{L}^2(D)\to\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) = \ker(\nabla_0\times) = \mathrm{im}(\nabla\times)^\perp,$$

(2.16d) $$\boldsymbol{\Pi}^{\rm c} : \boldsymbol{L}^2(D)\to\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) = \ker(\nabla\times) = \mathrm{im}(\nabla_0\times)^\perp.$$

Recall that ℓ_D is a length scale associated with D and that $\mathfrak{c}$ is a scaling factor typically representing a wave speed; in what follows, we use the time scale $\tau_D := \mathfrak{c}^{-1}\ell_D$.

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2.4.1. Grad-div problem. Let us first address the grad-div operator.

THEOREM 2.12 (well-posedness). *For all* $(\boldsymbol{f},g)\in \boldsymbol{L}^2(D)\times L^2(D) =: L^{\rm d}$, *there exists a unique pair* $(\boldsymbol{v},p)\in \boldsymbol{H}_0(\mathrm{div};D)\times H^1(D)$ *such that*

$$\tau_D^{-1}\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v})+\nabla p=(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\rm d})(\boldsymbol{f}), \tag{2.17a}$$

$$\tau_D^{-1}\Pi^{\rm g}(p)+\mathfrak{c}^2\nabla_0{\cdot}\boldsymbol{v}=(I-\Pi^{\rm g})(g). \tag{2.17b}$$

This pair is in $\boldsymbol{X}_0^{\rm d}\times X^{\rm g}$ *and continuously depends on* $(\boldsymbol{f},g)$, *i.e.*, $\|\boldsymbol{v}\|_{\boldsymbol{X}_0^{\rm d}}=\ell_D\mathfrak{c}^{-2}\|(I-\Pi^{\rm g})(g)\|_{L^2(D)}$, $\|p\|_{X^{\rm g}}=\ell_D\|(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\rm d})(\boldsymbol{f})\|_{\boldsymbol{L}^2(D)}$.

Proof. Lemma 2.8 implies that there there exists a unique pair $(\boldsymbol{v},p)\in\boldsymbol{X}_0^{\rm d}\times X^{\rm g}$ verifying $\nabla p=(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\rm d})(\boldsymbol{f})$ and $\nabla_0{\cdot}\boldsymbol{v}=(I-\Pi^{\rm g})(g)$. Since $\Pi^{\rm g}(q)=0$ for all $q\in X^{\rm g}$ and $\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{w})=\boldsymbol{0}$ for all $\boldsymbol{w}\in\boldsymbol{X}_0^{\rm d}$, the pair $(\boldsymbol{v},p)$ solves (2.17). This proves existence. Let now $(\boldsymbol{v},p)\in\boldsymbol{H}_0(\mathrm{div};D)\times H^1(D)$ be a solution to (2.17) with zero right-hand side. Taking the inner product of (2.17a) with $\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v})$, we conclude that $\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v})=\boldsymbol{0}$. This, in turn, implies that $\boldsymbol{v}\in\boldsymbol{X}_0^{\rm d}$. Similarly, taking the inner product of (2.17b) with $\Pi^{\rm g}(p)$, we conclude that $p\in X^{\rm g}$. Finally, $\nabla_0{\cdot}\boldsymbol{v}=0$ and $\boldsymbol{v}\in\boldsymbol{X}_0^{\rm d}$ imply that $\boldsymbol{v}=\boldsymbol{0}$, and $\nabla p=\boldsymbol{0}$ and $p\in X^{\rm g}$ imply that $p=0$. This proves uniqueness. The boundedness assertion is a consequence of the definition of the norms $\|{\cdot}\|_{\boldsymbol{X}_0^{\rm d}}$ and $\|{\cdot}\|_{X^{\rm g}}$. ∎

DEFINITION 2.13. *We define the operator* $T:L^{\rm d}\to L^{\rm d}$ *so that, for all* $(\boldsymbol{f},g)\in L^{\rm d}$, *the pair* $(\boldsymbol{v},p):=T(\boldsymbol{f},g)$ *solves* (2.17).

LEMMA 2.14 (compactness). (i) *There is* $s\in(\frac12,1]$ *s.t. for all* $(\boldsymbol{v},p)\in\boldsymbol{X}_0^{\rm d}\times X^{\rm g}$,

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^s(D)}\le C_D\ell_D\|\nabla_0{\cdot}\boldsymbol{v}\|_{L^2(D)},\qquad \|p\|_{H^1(D)}\le C_D\ell_D\|\nabla p\|_{\boldsymbol{L}^2(D)}. \tag{2.18}$$

(ii) *The operator* $T:L^{\rm d}\to L^{\rm d}$ *is compact.*

Proof. The decomposition (A.5c) implies that $\boldsymbol{v}\in\nabla(H^1(D))$ for all $\boldsymbol{v}\in\boldsymbol{X}_0^{\rm d}$; hence, $\nabla{\times}\boldsymbol{v}=\boldsymbol{0}$. Then, the inequality (2.5) implies that, for all $\boldsymbol{v}\in\boldsymbol{X}_0^{\rm d}$,

$$\begin{aligned}\|\boldsymbol{v}\|_{\boldsymbol{H}^s(D)}&\le C_D\big(\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div};D)}+\ell_D\|\nabla{\times}\boldsymbol{v}\|_{\boldsymbol{L}^2(D)}\big)\\&=C_D\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div};D)}\le C'_D\|\boldsymbol{v}\|_{\boldsymbol{X}_0^{\rm d}}=C'_D\ell_D\|\nabla_0{\cdot}\boldsymbol{v}\|_{L^2(D)}.\end{aligned}$$

Moreover, we have already seen that $\|p\|_{H^1(D)}\le C_D\ell_D\|\nabla p\|_{\boldsymbol{L}^2(D)}=C_D\|p\|_{X^{\rm g}}$ for all $p\in X^{\rm g}$. This proves the first assertion. The second assertion follows from the Rellich–Kondrachov compactness theorem in fractional-order Sobolev spaces implying that the embedding $\boldsymbol{X}_0^{\rm d}\times X^{\rm g}\to L^{\rm d}$ is compact. ∎

Let $\sigma(T)$ be the spectrum of T and $\sigma_{\rm p}(T)$ be the point spectrum of T. Since T is compact, the spectrum of T reduces to its point spectrum away from 0, and 0 is an accumulation point. Let $\sigma_{\rm p}(-\Delta_N^{-1})$ be the point spectrum of the operator $-\Delta_N^{-1}:L^2(D)\cap\mathbb{P}_0^{\perp}\to L^2(D)\cap\mathbb{P}_0^{\perp}$, where $-\Delta(-\Delta_N^{-1}(f))=f$ and $\partial_n(-\Delta_N^{-1}(f))|_{\partial D}=0$. Recall that $\sigma_{\rm p}(-\Delta_N^{-1})\subset\mathbb{R}_{>0}$. Notice also that $\alpha\in\sigma_{\rm p}(-\Delta_N^{-1})$ iff $\mathrm{i}\sqrt{\alpha}$ and $-\mathrm{i}\sqrt{\alpha}$ are both members of $\sigma_{\rm p}(T)$. Hence, the point spectrum of T is purely imaginary. Let us now relate the operator T to the spectral problem (2.6).

LEMMA 2.15. (i) *Let* $\mu\neq0$, $(\boldsymbol{v},p)\in\boldsymbol{X}_0^{\rm d}\times X^{\rm g}$ *be an eigenpair of* T. *Then* $\frac1\mu$, $(\boldsymbol{v},p)$ *is an eigenpair of* (2.6). (ii) *Let* $\lambda\neq0$, $(\boldsymbol{v},p)\in\boldsymbol{H}_0(\mathrm{div};D)\times H^1(D)$ *be an eigenpair of* (2.6). *Then* $\frac1\lambda$, $(\boldsymbol{v},p)$ *is an eigenpair of* T.

Proof. (i) Let $\mu\neq0$, $(\boldsymbol{v},p)\in\boldsymbol{X}_0^{\rm d}\times X^{\rm g}$ be an eigenpair of T. Then $\boldsymbol{\Pi}_0^{\rm d}(\mu\boldsymbol{v})+\nabla(\mu p)=(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\rm d})(\boldsymbol{v})$ and $\Pi^{\rm g}(\mu p)+\nabla_0{\cdot}(\mu\boldsymbol{v})=(I-\Pi^{\rm g})(p)$. Since $\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v})=\boldsymbol{0}$ and $\Pi^{\rm g}(p)=0$, we conclude that $\frac1\mu$, $(\boldsymbol{v},p)$ is an eigenpair of (2.6).

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(ii) Let $\lambda \neq 0$, $(\boldsymbol{v},p)\in \boldsymbol{H}_0(\mathrm{div};D)\times H^1(D)$ be an eigenpair of (2.6). Then $\nabla(\frac{1}{\lambda}p) = \boldsymbol{v}$ and $\nabla_0\cdot(\frac{1}{\lambda}\mathfrak{c}^2\boldsymbol{v}) = p$. This implies that $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}=0;D)^{\perp}$ and $p \in \mathbb{P}_0^{\perp}$. Hence, $\boldsymbol{\Pi}_0^{\mathrm{d}}(\boldsymbol{v}) = \boldsymbol{0}$ and $\Pi^{\mathrm{g}}(p) = 0$, i.e., $\frac{1}{\lambda}$, $(\boldsymbol{v},p)\in \boldsymbol{X}_0^{\mathrm{d}}\times X^{\mathrm{g}}$ is an eigenpair of T. ∎

Lemma 2.15 shows that the eigenstructure of T is the same as that of (2.6) for the nonzero eigenvalues. Hence, it suffices to study the spectrum of T to have full knowledge of the eigenstructure of (2.6). In the paper, we prove that the dG approximation of T is spectrally correct, i.e., pollution free.

2.4.2. Curl-curl problem. We proceed as in section 2.4.1 for the analysis of the eigenvalue problem associated with the curl-curl operator. The reader is referred to Monk [24, Chap. 1] for an introduction to Maxwell's equations.

THEOREM 2.16 (well-posedness). *For all* $(\boldsymbol{f},\boldsymbol{g}) \in \boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D) =: L^{\mathrm{c}}$, *there exists a unique pair* $(\boldsymbol{B},\boldsymbol{E})\in \boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$ *such that*

$$\tau_D^{-1}\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{B}) - \nabla\times\boldsymbol{E} = (\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{f}), \tag{2.19a}$$

$$\tau_D^{-1}\boldsymbol{\Pi}^{\mathrm{c}}(\boldsymbol{E}) + \mathfrak{c}^2\nabla_0\times\boldsymbol{B} = (\boldsymbol{I}-\boldsymbol{\Pi}^{\mathrm{c}})(\boldsymbol{g}). \tag{2.19b}$$

This pair is in $\boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$ *and continuously depends on* $(\boldsymbol{f},\boldsymbol{g})$, *i.e., we have* $\|\boldsymbol{B}\|_{\boldsymbol{X}_0^{\mathrm{c}}} = \ell_D\mathfrak{c}^{-2}\|(\boldsymbol{I}-\boldsymbol{\Pi}^{\mathrm{c}})(\boldsymbol{g})\|_{\boldsymbol{L}^2(D)}$, $\|\boldsymbol{E}\|_{\boldsymbol{X}^{\mathrm{c}}} = \ell_D\|(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{f})\|_{\boldsymbol{L}^2(D)}$.

Proof. The proof is similar to that of Theorem 2.12. ∎

DEFINITION 2.17. *We define the operator* $T: L^{\mathrm{c}} \to L^{\mathrm{c}}$ *so that, for all* $(\boldsymbol{f},\boldsymbol{g})\in L^{\mathrm{c}}$, *the pair* $(\boldsymbol{B},\boldsymbol{E}) := T(\boldsymbol{f},\boldsymbol{g})$ *solves* (2.19).

LEMMA 2.18 (compactness). (i) *There exists* $s\in(\frac{1}{2},1]$ *so that, for all* $(\boldsymbol{B},\boldsymbol{E})\in \boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$,

$$\|\boldsymbol{B}\|_{\boldsymbol{H}^s(D)} \le C_D\ell_D\|\nabla_0\times\boldsymbol{B}\|_{\boldsymbol{L}^2(D)}, \qquad \|\boldsymbol{E}\|_{\boldsymbol{H}^s(D)} \le C_D\ell_D\|\nabla\times\boldsymbol{E}\|_{\boldsymbol{L}^2(D)}. \tag{2.20}$$

(ii) *The operator* $T: L^{\mathrm{c}}\to L^{\mathrm{c}}$ *is compact.*

Proof. By definition, we have $\boldsymbol{X}^{\mathrm{c}} \subset \boldsymbol{H}(\mathbf{curl};D)$. Moreover, the identity (A.4f) implies that $\boldsymbol{H}(\mathbf{curl}=\boldsymbol{0};D)^{\perp} = \boldsymbol{H}_0^{\Sigma}(\mathrm{div}=0;D)$. Hence, $\boldsymbol{X}^{\mathrm{c}} \subset \boldsymbol{H}(\mathbf{curl};D)\cap \boldsymbol{H}(\mathrm{div}=0;D)$. Then, the inequality (2.5) implies that, for all $\boldsymbol{E}\in\boldsymbol{X}^{\mathrm{c}}$,

$$\begin{aligned}\|\boldsymbol{E}\|_{\boldsymbol{H}^s(D)} &\le C_D\big(\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl};D)} + \ell_D\|\nabla\cdot\boldsymbol{E}\|_{L^2(D)}\big)\\ &= C_D\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl};D)} \le C_D'\|\boldsymbol{E}\|_{\boldsymbol{X}^{\mathrm{c}}} = C_D'\ell_D\|\nabla\times\boldsymbol{E}\|_{\boldsymbol{L}^2(D)}.\end{aligned}$$

A similar argument is used to prove the first inequality in (2.20). This proves the first assertion. The second assertion follows from the Rellich–Kondrachov compactness theorem in fractional-order Sobolev spaces. ∎

Since T is compact, we have $\sigma(T)\backslash\{0\} = \sigma_{\mathrm{p}}(T)\backslash\{0\}$ and 0 is an accumulation point of $\sigma(T)$. Let us now relate the operator T to the spectral problem (2.9).

LEMMA 2.19. (i) *Let* $\mu\neq 0$, $(\boldsymbol{B},\boldsymbol{E})\in\boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$ *be an eigenpair of* T. *Then* $\frac{1}{\mu}$, $(\boldsymbol{B},\boldsymbol{E})$ *is an eigenpair of* (2.9). (ii) *Let* $\lambda\neq 0$, $(\boldsymbol{B},\boldsymbol{E})\in\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D)$ *be an eigenpair of* (2.9). *Then* $\frac{1}{\lambda}$, $(\boldsymbol{B},\boldsymbol{E})$ *is an eigenpair of* T.

Proof. The proof is similar to that of Lemma 2.15. ∎

Lemma 2.19 implies that the eigenstructure of T is the same as that of the spectral problem (2.11). In the paper, we prove that the dG approximation of T is spectrally correct, i.e., pollution free.

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3. Discrete setting. In this section, we introduce the discrete setting used in the paper. The main result of this section is Lemma 3.2 which establishes discrete counterparts of the weak Poincaré–Steklov (in)equalities from Corollary 2.10.

3.1. Broken polynomial spaces, jumps, and averages. Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular family of affine simplicial meshes such that each mesh covers D exactly. More general meshes can be considered provided suitable polynomial spaces composing the corresponding discrete de Rham sequence are available. A generic mesh cell is denoted K, its diameter h_K, and its outward unit normal $\boldsymbol{n}_K$. We define $\tilde{h}$ as the piecewise constant function on $\mathcal{T}_h$ such that $\tilde{h}|_K = h_K$ for all $K \in \mathcal{T}_h$; we set $h := \|\tilde{h}\|_{L^\infty(D)}$. The set of mesh faces, $\mathcal{F}_h$, is split into the subset of mesh interfaces (shared by two distinct mesh cells which we denote K_l, K_r), say $\mathcal{F}_h^\circ$, and the subset of mesh boundary faces (shared by one mesh cell, K_l, and the boundary, ∂D), say $\mathcal{F}_h^\partial$. For every mesh face $F\in\mathcal{F}_h$, h_F denotes the diameter of F. Every mesh interface $F\in\mathcal{F}_h^\circ$ is oriented by the unit normal, $\boldsymbol{n}_F$, pointing from K_l to K_r. Every boundary face $F\in\mathcal{F}_h^\partial$ is oriented by the unit normal $\boldsymbol{n}_F := \boldsymbol{n}_D$. For all $K\in\mathcal{T}_h$, $\mathcal{F}_K$ denotes the collection of the mesh faces composing the boundary of K, and we set $\mathcal{F}_K^\circ := \mathcal{F}_K \cap \mathcal{F}_h^\circ$.

In what follows, for positive real numbers A, B, we abbreviate as $A \lesssim B$ the inequality $A \le CB$, where C is a generic constant, independent of $h\in\mathcal{H}$ and the fields involved in the inequality, and whose value can change at each occurrence.

Let $k\ge 0$ be the polynomial degree. Let $\mathbb{P}_{k,d}$ be the space composed of d-variate polynomials of total degree at most k, and set $\boldsymbol{\mathbb{P}}_{k,d} := [\mathbb{P}_{k,d}]^d$. Consider the scalar- and vector-valued broken polynomial spaces

$$P_k^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in L^\infty(D) \mid v_{h|K} \in \mathbb{P}_{k,d} \ \ \forall K\in\mathcal{T}_h\}, \tag{3.1a}$$

$$\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^\infty(D) \mid \boldsymbol{v}_{h|K} \in \boldsymbol{\mathbb{P}}_{k,d} \ \ \forall K\in\mathcal{T}_h\}. \tag{3.1b}$$

We introduce the L^2- and $\boldsymbol{L}^2$-orthogonal projections

$$\Pi_h^{\mathrm{b}} : L^2(D) \to P_k^{\mathrm{b}}(\mathcal{T}_h), \qquad \boldsymbol{\Pi}_h^{\mathrm{b}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{3.2}$$

For every $p_h \in P_k^{\mathrm{b}}(\mathcal{T}_h)$, $\nabla_h p_h$ denotes the broken gradient of p_h (evaluated piecewise over each mesh cell). For every $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, $\nabla_h{\times}\boldsymbol{v}_h$ and $\nabla_h{\cdot}\boldsymbol{v}_h$ denote the broken curl and divergence of $\boldsymbol{v}_h$, respectively.

For all $K\in\mathcal{T}_h$, all $F\in\mathcal{F}_K$, all $p_h\in P_k^{\mathrm{b}}(\mathcal{T}_h)$, and all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we define the local trace operators such that $\gamma_{K,F}^{\mathrm{g}}(p_h)(\boldsymbol{x}) := p_h|_K(\boldsymbol{x})$, $\gamma_{K,F}^{\mathrm{g}}(\boldsymbol{v}_h)(\boldsymbol{x}) := \boldsymbol{v}_h|_K(\boldsymbol{x})$, $\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{v}_h)(\boldsymbol{x}) := \boldsymbol{v}_h|_K(\boldsymbol{x}){\times}\boldsymbol{n}_F$, and $\gamma_{K,F}^{\mathrm{d}}(\boldsymbol{v}_h)(\boldsymbol{x}) := \boldsymbol{v}_h|_K(\boldsymbol{x}){\cdot}\boldsymbol{n}_F$ for a.e. $\boldsymbol{x}\in F$. Then, for all $F\in\mathcal{F}_h^\circ$ and $\mathrm{x}\in\{\mathrm{g},\mathrm{c},\mathrm{d}\}$, we define the jump and average operators such that

$$[\![p_h]\!]_F^{\mathrm{g}} := \gamma_{K_l,F}^{\mathrm{g}}(p_h) - \gamma_{K_r,F}^{\mathrm{g}}(p_h), \qquad \{\!\{p_h\}\!\}_F^{\mathrm{g}} := \frac{1}{2}\big(\gamma_{K_l,F}^{\mathrm{g}}(p_h) + \gamma_{K_r,F}^{\mathrm{g}}(p_h)\big), \tag{3.3a}$$

$$[\![\boldsymbol{v}_h]\!]_F^{\mathrm{x}} := \gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{v}_h) - \gamma_{K_r,F}^{\mathrm{x}}(\boldsymbol{v}_h), \qquad \{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{x}} := \frac{1}{2}\big(\gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{v}_h) + \gamma_{K_r,F}^{\mathrm{x}}(\boldsymbol{v}_h)\big). \tag{3.3b}$$

To allow for more compact expressions, we also set $[\![p_h]\!]_F^{\mathrm{g}} := \{\!\{p_h\}\!\}_F^{\mathrm{g}} := \gamma_{K_l,F}^{\mathrm{g}}(p_h)$, $[\![\boldsymbol{v}_h]\!]_F^{\mathrm{x}} := \{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{x}} := \gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{v}_h)$ for all $F\in\mathcal{F}_h^\partial$. Finally, we define the jump sesquilinear forms such that for all $p_h, q_h \in P_k^{\mathrm{b}}(\mathcal{T}_h)$ and all $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

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$$s_h^{\mathrm{g}}(p_h,q_h) := \sum_{F\in\mathcal{F}_h} ([\![p_h]\!]_F^{\mathrm{g}}, [\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)}, \qquad s_h^{\mathrm{g},\circ}(p_h,q_h) := \sum_{F\in\mathcal{F}_h^\circ} ([\![p_h]\!]_F^{\mathrm{g}}, [\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)},$$
$$s_h^{\mathrm{c}}(\boldsymbol{v}_h,\boldsymbol{w}_h) := \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}, \quad s_h^{\mathrm{c},\circ}(\boldsymbol{v}_h,\boldsymbol{w}_h) := \sum_{F\in\mathcal{F}_h^\circ} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)},$$
$$s_h^{\mathrm{d}}(\boldsymbol{v}_h,\boldsymbol{w}_h) := \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)}, \quad s_h^{\mathrm{d},\circ}(\boldsymbol{v}_h,\boldsymbol{w}_h) := \sum_{F\in\mathcal{F}_h^\circ} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)},$$

and the seminorms $|q_h|_h^{\mathrm{g}} := s_h^{\mathrm{g}}(q_h,q_h)^{\frac12}$, $|q_h|_h^{\mathrm{g},\circ} := s_h^{\mathrm{g},\circ}(q_h,q_h)^{\frac12}$, $|\boldsymbol{v}_h|_h^{\mathrm{c}} := s_h^{\mathrm{c}}(\boldsymbol{v}_h,\boldsymbol{v}_h)^{\frac12}$, $|\boldsymbol{v}_h|_h^{\mathrm{c},\circ} := s_h^{\mathrm{c},\circ}(\boldsymbol{v}_h,\boldsymbol{v}_h)^{\frac12}$, $|\boldsymbol{v}_h|_h^{\mathrm{d}} := s_h^{\mathrm{d}}(\boldsymbol{v}_h,\boldsymbol{v}_h)^{\frac12}$, and $|\boldsymbol{v}_h|_h^{\mathrm{d},\circ} := s_h^{\mathrm{d},\circ}(\boldsymbol{v}_h,\boldsymbol{v}_h)^{\frac12}$.

3.2. Discrete Poincaré–Steklov inequalities. In this section, we prove discrete counterparts to the Poincaré–Steklov (in)equalities established in Corollary 2.10. We first define the (broken) polynomial subspaces

$$\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{H}_0(\mathrm{div}=0;D), \tag{3.4a}$$
$$\boldsymbol{P}_{k}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{H}(\mathrm{div}=0;D), \tag{3.4b}$$
$$\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D), \tag{3.4c}$$
$$\boldsymbol{P}_{k}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{H}(\mathbf{curl}=\mathbf{0};D), \tag{3.4d}$$

and we consider the following L^2- or $\boldsymbol{L}^2$-orthogonal complements:

$$X_h^{\mathrm{g}} := P_k^{\mathrm{b}}(\mathcal{T}_h)\cap \mathbb{P}_0^{\perp}, \tag{3.5a}$$
$$\boldsymbol{X}_{h0}^{\mathrm{d}} := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h)^{\perp}, \tag{3.5b}$$
$$\boldsymbol{X}_{h}^{\mathrm{d}} := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{P}_{k}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h)^{\perp}, \tag{3.5c}$$
$$\boldsymbol{X}_{h0}^{\mathrm{c}} := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)^{\perp}, \tag{3.5d}$$
$$\boldsymbol{X}_{h}^{\mathrm{c}} := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap \boldsymbol{P}_{k}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)^{\perp}. \tag{3.5e}$$

The broken polynomial spaces X_h^{g}, $\boldsymbol{X}_{h0}^{\mathrm{d}}$, $\boldsymbol{X}_h^{\mathrm{d}}$, $\boldsymbol{X}_{h0}^{\mathrm{c}}$, and $\boldsymbol{X}_h^{\mathrm{c}}$ are nonconforming approximations of X^{g}, $\boldsymbol{X}_0^{\mathrm{d}}$, $\boldsymbol{X}^{\mathrm{d}}$, $\boldsymbol{X}_0^{\mathrm{c}}$, and $\boldsymbol{X}^{\mathrm{c}}$, respectively. We could have set $P_k^{\mathrm{g}}(\mathbf{grad}=\mathbf{0};\mathcal{T}_h) := P_k^{\mathrm{b}}(\mathcal{T}_h)\cap H(\mathbf{grad}=\mathbf{0};D)$ and $X_h^{\mathrm{g}} = P_k^{\mathrm{b}}(\mathcal{T}_h)\cap P_k^{\mathrm{g}}(\mathbf{grad}=\mathbf{0};\mathcal{T}_h)^{\perp}$, but this would have led to the same definition of X_h^{g} as above because $P_k^{\mathrm{g}}(\mathbf{grad}=\mathbf{0};\mathcal{T}_h) = \mathbb{P}_0$.

Lemma 3.1 (discrete Poincaré–Steklov inequalities for scalars). *The following holds:*

$$\|p_h\|_{L^2(D)} \le \ell_D \|\nabla p_h\|_{(\boldsymbol{X}_0^{\mathrm{d}})'} \qquad \forall p_h\in X_h^{\mathrm{g}}. \tag{3.6}$$

Proof. Since $X_h^{\mathrm{g}}\subset \mathbb{P}_0^{\perp}$, (3.6) is a consequence of (2.15a). ∎

The situation is not as simple for discrete vector fields because $\boldsymbol{X}_{h0}^{\mathrm{d}}$ is not a subspace of $\boldsymbol{H}_0(\mathrm{div}=0;D)^{\perp}$, $\boldsymbol{X}_h^{\mathrm{d}}$ is not a subspace of $\boldsymbol{H}(\mathrm{div}=0;D)^{\perp}$, $\boldsymbol{X}_{h0}^{\mathrm{c}}$ is not a subspace of $\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}$, and $\boldsymbol{X}_h^{\mathrm{c}}$ is not a subspace of $\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}$. To establish discrete Poincaré–Steklov inequalities for vector fields, we use that the broken polynomial spaces contain polynomial spaces from the entire discrete de Rham sequence based on curl-free Nédélec or divergence-free Raviart–Thomas polynomials.

Lemma 3.2 (discrete Poincaré–Steklov inequalities for fields). *The following holds:*

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(3.7a) $$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla_0{\cdot}\boldsymbol{v}_h\|_{(X^{\rm g})'} + h^{\frac{1}{2}}|\boldsymbol{v}_h|_h^{\rm d} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0}^{\rm d},$$

(3.7b) $$\|\boldsymbol{w}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla{\cdot}\boldsymbol{w}_h\|_{(X_0^{\rm g})'} + h^{\frac{1}{2}}|\boldsymbol{w}_h|_h^{\rm d,0} \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_{h}^{\rm d},$$

(3.7c) $$\|\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla_0{\times}\boldsymbol{b}_h\|_{(\boldsymbol{X}^{\rm c})'} + h^{\frac{1}{2}}|\boldsymbol{b}_h|_h^{\rm c} \qquad \forall \boldsymbol{b}_h \in \boldsymbol{X}_{h0}^{\rm c},$$

(3.7d) $$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla{\times}\boldsymbol{e}_h\|_{(\boldsymbol{X}_0^{\rm c})'} + h^{\frac{1}{2}}|\boldsymbol{e}_h|_h^{\rm c,0} \qquad \forall \boldsymbol{e}_h \in \boldsymbol{X}_{h}^{\rm c}.$$

Proof. (1) Proof of (3.7a). Here, the $\boldsymbol{H}_0(\text{div};D)$-conforming space composed of piecewise Raviart–Thomas polynomials of order $k \geq 0$, $\boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$, plays a central role. Notice that $\boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$ is not a subspace of $\boldsymbol{P}_{k}^{\rm b}(\mathcal{T}_h)$, but we have

(3.8) $$\boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\text{div}=0;D) = \boldsymbol{P}_{k}^{\rm b}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\text{div}=0;D) =: \boldsymbol{P}_{k0}^{\rm d}(\text{div}=0;\mathcal{T}_h).$$

Let $\mathcal{I}_{h0}^{\rm d,av} : \boldsymbol{P}_{k}^{\rm b}(\mathcal{T}_h) \to \boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$ be the $\boldsymbol{H}_0(\text{div};D)$-conforming averaging operator with zero normal boundary prescription constructed in [16, sect. 6]. Let $\boldsymbol{v}_h \in \boldsymbol{X}_{h0}^{\rm d}$, and define

$$\boldsymbol{v}_h^{\rm d} := \mathcal{I}_{h0}^{\rm d,av}(\boldsymbol{v}_h), \qquad \boldsymbol{\xi} := \boldsymbol{v}_h^{\rm d} - \boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v}_h^{\rm d}).$$

Since $\nabla_0{\cdot}(\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v}_h^{\rm d})) = 0$ (because $\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v}_h^{\rm d}) \in \boldsymbol{H}_0(\text{div}=0;D)$) and $\boldsymbol{\xi} \in \boldsymbol{H}_0(\text{div}=0;D)^{\perp}$, the weak Poincaré–Steklov (in)equality (2.15b) from Corollary 2.10 gives

(3.9) $$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2(D)} = \ell_D \|\nabla_0{\cdot}\boldsymbol{\xi}\|_{(X^{\rm g})'} = \ell_D \|\nabla_0{\cdot}\mathcal{I}_{h0}^{\rm d,av}(\boldsymbol{v}_h)\|_{(X^{\rm g})'}.$$

Let $\mathcal{J}_{h0}^{\rm d} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$ and $\mathcal{J}_{h}^{\rm b} : L^2(D) \to P_{k}^{\rm b}(\mathcal{T}_h)$ be the commuting approximation operators devised in [17, sect. 23.3] (see also Arnold, Falk, and Winther [4], Christiansen [11], Christiansen and Winther [12], Schöberl [27]). Since $\boldsymbol{v}_h^{\rm d} \in \boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$, we have

$$\boldsymbol{v}_h^{\rm d} - \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\xi}) = \mathcal{J}_{h0}^{\rm d}(\boldsymbol{v}_h^{\rm d} - \boldsymbol{\xi}) = \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v}_h^{\rm d})).$$

The commuting property of $\mathcal{J}_{h0}^{\rm d}$ implies that

$$\nabla_0{\cdot}(\boldsymbol{v}_h^{\rm d} - \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\xi})) = \nabla_0{\cdot}(\mathcal{J}_{h0}^{\rm d}(\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v}_h^{\rm d}))) = \mathcal{J}_{h}^{\rm b}(\nabla_0{\cdot}(\boldsymbol{\Pi}_0^{\rm d}(\boldsymbol{v}_h^{\rm d}))) = \mathcal{J}_{h}^{\rm b}(0) = 0.$$

Hence, $\boldsymbol{v}_h^{\rm d} - \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\xi}) \in \boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\text{div}=0;D) = \boldsymbol{P}_{k0}^{\rm d}(\text{div}=0;\mathcal{T}_h)$ owing to (3.8). Using that $\boldsymbol{v}_h \in \boldsymbol{P}_{k0}^{\rm d}(\text{div}=0;\mathcal{T}_h)^{\perp}$ then gives

$$\begin{aligned}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)}^2 &= (\boldsymbol{v}_h, \boldsymbol{v}_h - \boldsymbol{v}_h^{\rm d})_{\boldsymbol{L}^2(D)} + (\boldsymbol{v}_h, \boldsymbol{v}_h^{\rm d} - \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\xi}))_{\boldsymbol{L}^2(D)} + (\boldsymbol{v}_h, \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\xi}))_{\boldsymbol{L}^2(D)} \\ &= (\boldsymbol{v}_h, \boldsymbol{v}_h - \boldsymbol{v}_h^{\rm d})_{\boldsymbol{L}^2(D)} + (\boldsymbol{v}_h, \mathcal{J}_{h0}^{\rm d}(\boldsymbol{\xi}))_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Invoking the Cauchy–Schwarz inequality and the $\boldsymbol{L}^2$-stability of $\mathcal{J}_{h0}^{\rm d}$ yields

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \lesssim \|\boldsymbol{v}_h - \boldsymbol{v}_h^{\rm d}\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\xi}\|_{\boldsymbol{L}^2(D)}.$$

Recalling the definition of $\boldsymbol{v}_h^{\rm d}$ and the bound (3.9) on $\boldsymbol{\xi}$ then gives

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \lesssim \|\boldsymbol{v}_h - \mathcal{I}_{h0}^{\rm d,av}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \ell_D \|\nabla_0{\cdot}\mathcal{I}_{h0}^{\rm d,av}(\boldsymbol{v}_h)\|_{(X^{\rm g})'}.$$

Adding and subtracting $\nabla{\cdot}\boldsymbol{v}_h$ to the second term on the right-hand side, using the triangle inequality, and since $\ell_D \|\nabla_0{\cdot}\boldsymbol{\phi}\|_{(X^{\rm g})'} \leq \|\boldsymbol{\phi}\|_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{\phi} \in \boldsymbol{L}^2(D)$, we obtain

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \lesssim \|\boldsymbol{v}_h - \mathcal{I}_{h0}^{\rm d,av}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \ell_D \|\nabla_0{\cdot}\boldsymbol{v}_h\|_{(X^{\rm g})'}.$$

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Finally, we invoke the approximation properties of $\mathcal{I}_{h0}^{\mathrm{d,av}}$. For all $K\in\mathcal{T}_h$, we have

$$\|\boldsymbol{v}_h-\mathcal{I}_{h0}^{\mathrm{d,av}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(K)}\lesssim h_K^{\frac12}\left\{\sum_{F\in\mathcal{F}_K}\|[\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}}\|_{\boldsymbol{L}^2(F)}^2\right\}^{\frac12}.$$

The assertion follows from the shape-regularity of the mesh sequence.

(2) The proof of the other inequalities proceeds similarly. Here, one considers the $\boldsymbol{H}(\mathrm{div};D)$-conforming space composed of piecewise Raviart–Thomas polynomials of order $k\ge 0$, $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$, or the $\boldsymbol{H}(\mathbf{curl};D)$- and $\boldsymbol{H}_0(\mathbf{curl};D)$-conforming spaces composed of piecewise Nédélec polynomials of order $k\ge 0$, $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ and $\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathcal{T}_h)$. The corresponding averaging operators and commuting approximation operators are constructed in, e.g., [17, Chaps. 22–23]. ∎

3.3. Discrete projection operators. The (broken) polynomial subspaces introduced in (3.4) naturally lead to the following $\boldsymbol{L}^2$-orthogonal projections:

$$\text{(3.10a)}\quad \boldsymbol{\Pi}_{h0}^{\mathrm{d}}:\boldsymbol{L}^2(D)\to\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h),\qquad \boldsymbol{\Pi}_{h}^{\mathrm{d}}:\boldsymbol{L}^2(D)\to\boldsymbol{P}_{k}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h),$$
$$\text{(3.10b)}\quad \boldsymbol{\Pi}_{h0}^{\mathrm{c}}:\boldsymbol{L}^2(D)\to\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h),\qquad \boldsymbol{\Pi}_{h}^{\mathrm{c}}:\boldsymbol{L}^2(D)\to\boldsymbol{P}_{k}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h).$$

We can define the L^2-orthogonal projection $\Pi_h^{\mathrm{g}}:L^2(D)\to P_k^{\mathrm{g}}(\mathbf{grad}=\mathbf{0};\mathcal{T}_h)$, but Π_h^{g} coincides with Π^{g} since $P_k^{\mathrm{g}}(\mathbf{grad}=\mathbf{0};\mathcal{T}_h)=\mathbb{P}_0$. We now record instrumental properties of the above operators to be used in the analysis of the dG approximation. Recall that the continuous projection operators are defined in (2.16) and that the projection operators onto the broken polynomial spaces are defined in (3.2).

LEMMA 3.3 (discrete projections). *The following holds:*

$$\text{(3.11a)}\quad \boldsymbol{\Pi}_{h0}^{\mathrm{d}}\circ\boldsymbol{\Pi}_{0}^{\mathrm{d}}=\boldsymbol{\Pi}_{h0}^{\mathrm{d}},\quad \boldsymbol{\Pi}_{h}^{\mathrm{d}}\circ\boldsymbol{\Pi}^{\mathrm{d}}=\boldsymbol{\Pi}_{h}^{\mathrm{d}},\quad \boldsymbol{\Pi}_{h0}^{\mathrm{c}}\circ\boldsymbol{\Pi}_{0}^{\mathrm{c}}=\boldsymbol{\Pi}_{h0}^{\mathrm{c}},\quad \boldsymbol{\Pi}_{h}^{\mathrm{c}}\circ\boldsymbol{\Pi}^{\mathrm{c}}=\boldsymbol{\Pi}_{h}^{\mathrm{c}},$$
$$\text{(3.11b)}\quad \boldsymbol{\Pi}_{h0}^{\mathrm{d}}\circ\boldsymbol{\Pi}_{h}^{\mathrm{b}}=\boldsymbol{\Pi}_{h0}^{\mathrm{d}},\quad \boldsymbol{\Pi}_{h}^{\mathrm{d}}\circ\boldsymbol{\Pi}_{h}^{\mathrm{b}}=\boldsymbol{\Pi}_{h}^{\mathrm{d}},\quad \boldsymbol{\Pi}_{h0}^{\mathrm{c}}\circ\boldsymbol{\Pi}_{h}^{\mathrm{b}}=\boldsymbol{\Pi}_{h0}^{\mathrm{c}},\quad \boldsymbol{\Pi}_{h}^{\mathrm{c}}\circ\boldsymbol{\Pi}_{h}^{\mathrm{b}}=\boldsymbol{\Pi}_{h}^{\mathrm{c}}.$$

Remark 3.4 (Π^{g}). The identities in (3.11) also hold for Π^{g} and Π_h^{g}. We additionally have $\Pi^{\mathrm{g}}\circ\Pi_h^{\mathrm{b}}=\Pi_h^{\mathrm{b}}\circ\Pi^{\mathrm{g}}=\Pi^{\mathrm{g}}$ because $\mathbb{P}_0\subset P_k^{\mathrm{b}}(\mathcal{T}_h)$. Finally, if the boundary condition is $\gamma_{\partial D}^{\mathrm{g}}(p)=0$ for the grad-div problem, we simply set $\Pi_0^{\mathrm{g}}=\Pi_{h0}^{\mathrm{g}}=0$.

4. dG approximation of grad-div operator. This section deals with the analysis of the dG approximation of the grad-div operator. The main result is Theorem 4.11, which implies that the approximation is spectrally correct. For simplicity, we set the scaling coefficient to $\mathfrak{c}:=1$.

4.1. Definitions. We define the discrete space $L_h^{\mathrm{d}}:=\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\times P_k^{\mathrm{b}}(\mathcal{T}_h)$. The sesquilinear form $a_h:L_h^{\mathrm{d}}\times L_h^{\mathrm{d}}\to\mathbb{C}$ associated with the problem (2.17) is

$$\begin{aligned}a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big):=\;&\ell_D^{-1}(\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}+\ell_D^{-1}(\Pi^{\mathrm{g}}(p_h),q_h)_{L^2(D)}\\&-(p_h,\nabla_h\cdot\boldsymbol{w}_h)_{L^2(D)}-(\boldsymbol{v}_h,\nabla_h q_h)_{\boldsymbol{L}^2(D)}\\&+\sum_{F\in\mathcal{F}_h}(\{\!\{p_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)}+\sum_{F\in\mathcal{F}_h^{\circ}}(\{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{d}},[\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)}\\&+s_h^{\mathrm{d}}(\boldsymbol{v}_h,\boldsymbol{w}_h)+s_h^{\mathrm{g},\circ}(p_h,q_h).\end{aligned}\tag{4.1}$$

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Integrating by parts the broken divergence and gradient operators also gives

$$\begin{aligned} a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big) &= \ell_D^{-1}(\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + \ell_D^{-1}(\Pi^{\mathrm{g}}(p_h),q_h)_{L^2(D)} \\ &\quad + (\nabla_h p_h,\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + (\nabla_h{\cdot}\boldsymbol{v}_h,q_h)_{L^2(D)} \\ &\quad - \sum_{F\in\mathcal{F}_h^\circ}([\![p_h]\!]_F^{\mathrm{g}},\{\!\{\boldsymbol{w}_h\}\!\}_F^{\mathrm{d}})_{L^2(F)} - \sum_{F\in\mathcal{F}_h}([\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}},\{\!\{q_h\}\!\}_F^{\mathrm{g}})_{L^2(F)} \\ &\quad + s_h^{\mathrm{d}}(\boldsymbol{v}_h,\boldsymbol{w}_h) + s_h^{\mathrm{g},\circ}(p_h,q_h). \end{aligned} \tag{4.2}$$

Notice that the stabilization sesquilinear forms s_h^{d} and $s_h^{\mathrm{g},\circ}$ could be scaled by $O(1)$ positive weights; for simplicity, we choose these weights to be equal to 1 here.

We now define $T_h : L^{\mathrm{d}} \to L_h^{\mathrm{d}} \subset L^{\mathrm{d}}$, the discrete counterpart of the operator $T : L^{\mathrm{d}} \to L^{\mathrm{d}}$ introduced in Definition 2.13 (recall that $L^{\mathrm{d}} := \boldsymbol{L}^2(D)\times L^2(D)$). For all $(\boldsymbol{f},g)\in L^{\mathrm{d}}$, $T_h(\boldsymbol{f},g) := (\boldsymbol{v}_h,p_h)$ is the unique pair in L_h^{d} so that, for all $(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}$,

$$a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big) = ((\boldsymbol{I}-\boldsymbol{\Pi}_{h0}^{\mathrm{d}})(\boldsymbol{f}),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + ((I-\Pi^{\mathrm{g}})(g),q_h)_{L^2(D)}. \tag{4.3}$$

The definition of T_h makes sense owing to the stability result established in Lemma 4.6.

Our goal is to prove that $\lim_{h\in\mathcal{H}\to 0}\|T-T_h\|_{\mathcal{L}(L^{\mathrm{d}};L^{\mathrm{d}})} = 0$. This is done in two steps. First we prove an inf-sup condition which establishes stability. Then we prove a consistency/boundedness result. Convergence follows by combing these two results.

Remark 4.1 (other boundary conditions). To approximate the grad-div operator with the boundary condition $\gamma_{\partial D}^{\mathrm{g}}(p) = 0$ (see Remarks 2.4 and 2.11), we use the sesquilinear form

$$\begin{aligned} a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big) &:= \ell_D^{-1}(\boldsymbol{\Pi}_h^{\mathrm{d}}(\boldsymbol{v}_h),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} - (p_h,\nabla_h{\cdot}\boldsymbol{w}_h)_{L^2(D)} - (\boldsymbol{v}_h,\nabla_h q_h)_{\boldsymbol{L}^2(D)} \\ &\quad + \sum_{F\in\mathcal{F}_h^\circ}(\{\!\{p_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)} + \sum_{F\in\mathcal{F}_h}(\{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{d}},[\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)} \\ &\quad + s_h^{\mathrm{d},\circ}(\boldsymbol{v}_h,\boldsymbol{w}_h) + s_h^{\mathrm{g}}(p_h,q_h). \end{aligned}$$

Notice that there is no projection operator acting on p_h as $\Pi_{h0}^{\mathrm{g}} = 0$.

4.2. Discrete involutions and other comments. The projection operators $\boldsymbol{\Pi}_{h0}^{\mathrm{d}}$ and Π^{g} are only invoked for theoretical purposes. One does not need to construct these operators in practice when one wants to approximate the eigenvalue problem (2.6) or when one wants to approximate the wave equation in the time domain (2.8). Indeed, let us consider the following sesquilinear form:

$$\begin{aligned} \hat{a}_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big) &:= -(p_h,\nabla_h{\cdot}\boldsymbol{w}_h)_{L^2(D)} - (\boldsymbol{v}_h,\nabla_h q_h)_{\boldsymbol{L}^2(D)} \\ &\quad + \sum_{F\in\mathcal{F}_h}(\{\!\{p_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)} + \sum_{F\in\mathcal{F}_h^\circ}(\{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{d}},[\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)} \\ &\quad + s_h^{\mathrm{d}}(\boldsymbol{v}_h,\boldsymbol{w}_h) + s_h^{\mathrm{g},\circ}(p_h,q_h). \end{aligned} \tag{4.4}$$

Notice that $\hat{a}_h((\cdot,\cdot),(\boldsymbol{w}_h,q_h)) = 0$ for all $(\boldsymbol{w}_h,q_h)\in\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h)\times\mathbb{P}_0$ because every field $\boldsymbol{w}_h\in\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div}=0;\mathcal{T}_h)$ satisfies $\nabla_h{\cdot}\boldsymbol{w}_h = 0$ and $[\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}} = 0$ for all $F\in\mathcal{F}_h$, and every function $q_h\in\mathbb{P}_0$ satisfies $\nabla_h q_h = \boldsymbol{0}$ and $[\![q_h]\!]_F^{\mathrm{g}} = 0$ for all $F\in\mathcal{F}_h^\circ$.

LEMMA 4.2 (eigenvalue problems for a_h and $\hat{a}_h$). *Let $\lambda\neq 0$, $(\boldsymbol{v}_h,p_h)\in L_h^{\mathrm{d}}$. Then $T_h(\boldsymbol{v}_h,p_h) = \frac{1}{\lambda}(\boldsymbol{v}_h,p_h)$ iff $\hat{a}_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big) = \lambda\big((\boldsymbol{v}_h,\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + (p_h,q_h)_{L^2(D)}\big)$ for all $(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}$.*

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Proof. (1) Let $(\boldsymbol{v}_h, p_h) \in L_h^{\mathrm{d}}$ be so that $T_h(\boldsymbol{v}_h, p_h) = \frac{1}{\lambda}(\boldsymbol{v}_h, p_h)$. This means that $a_h\big((\boldsymbol{v}_h, p_h), (\boldsymbol{w}_h, q_h)\big) = \lambda\big(((\boldsymbol{I} - \boldsymbol{\Pi}_{h0}^{\mathrm{d}})(\boldsymbol{v}_h), \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + ((I - \Pi^{\mathrm{g}})(p_h), q_h)_{L^2(D)}\big)$ for all $(\boldsymbol{w}_h, q_h) \in L_h^{\mathrm{d}}$. Using the test functions $\boldsymbol{w}_h = \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)$ and $q_h = \Pi^{\mathrm{g}}(p_h)$, we obtain $\ell_D^{-1}\|\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)}^2 + \ell_D^{-1}\|\Pi^{\mathrm{g}}(p_h)\|_{L^2(D)}^2 = 0$, which then gives

$$\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h) = \boldsymbol{0}, \qquad \Pi^{\mathrm{g}}(p_h) = 0. \tag{4.5}$$

This implies that $\hat{a}_h\big((\boldsymbol{v}_h, p_h), (\boldsymbol{w}_h, q_h)\big) = \lambda\big((\boldsymbol{v}_h, \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + (p_h, q_h)_{L^2(D)}\big)$ for all $(\boldsymbol{w}_h, q_h) \in L_h^{\mathrm{d}}$, whence the assertion.

(2) Assume now that $\hat{a}_h\big((\boldsymbol{v}_h, p_h), (\boldsymbol{w}_h, q_h)\big) = \lambda\big((\boldsymbol{v}_h, \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + (p_h, q_h)_{L^2(D)}\big)$ for all $(\boldsymbol{w}_h, q_h) \in L_h^{\mathrm{d}}$. Using the test functions $\boldsymbol{w}_h = \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)$ and $q_h = \Pi^{\mathrm{g}}(p_h)$, we observe that $\hat{a}_h\big((\boldsymbol{v}_h, p_h), (\boldsymbol{w}_h, q_h)\big) = 0$, which, in turn, implies that (4.5) holds true. The assertion readily follows. ∎

Remark 4.3 (discrete involutions). The proof of Lemma 4.2 shows that the involutions enforced by a_h and $\hat{a}_h$ are (4.5). Notice that the projections $\boldsymbol{\Pi}_{h0}^{\mathrm{d}}$ and Π^{g} are not involved in the construction of $\hat{a}_h$. As shown in Lemma 3.2, these involutions are essential to prove the discrete Poincaré–Steklov inequalities (3.7). These inequalities play a pivotal in the proof of the spectral correctness of T_h, which owing to Lemma 4.2 implies the spectral correctness of the dG approximation realized by $\hat{a}_h$.

Let us now consider the approximation in time and space of the wave equation (2.8). For simplicity, we use the backward Euler time-stepping. Letting $(\boldsymbol{v}_h^n, p_h^n) \in L_h^{\mathrm{d}}$ be the approximation at time t^n and letting τ be the time step, $(\boldsymbol{v}_h^{n+1}, p_h^{n+1}) \in L_h^{\mathrm{d}}$ is the unique pair that solves, for all $(\boldsymbol{w}_h, q_h) \in L_h^{\mathrm{d}}$,

$$\begin{aligned}(\boldsymbol{v}_h^{n+1}, \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + (p_h^{n+1}, q_h)_{L^2(D)} + \tau\hat{a}_h((\boldsymbol{v}_h^{n+1}, p_h^{n+1}), (\boldsymbol{w}_h, q_h)) &= \\ (\boldsymbol{v}_h^n, \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + (p_h^n, q_h)_{L^2(D)}&.\end{aligned} \tag{4.6}$$

LEMMA 4.4 (time involution). *Assume that the pair $(\boldsymbol{v}_h^n, p_h^n)$ satisfies the involutions* (4.5)*. Then the pair $(\boldsymbol{v}_h^{n+1}, p_h^{n+1})$ satisfies* (4.5) *as well.*

Proof. Using $\boldsymbol{w}_h = \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h^{n+1})$ and $q_h = \Pi^{\mathrm{g}}(p_h^{n+1})$ yields the assertion. ∎

Lemma 4.4 shows that if $\boldsymbol{v}_h^0$ is orthogonal to $\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div}=0; \mathcal{T}_h)$ and the mean of p_h^0 over D is zero, then this is also the case for $(\boldsymbol{v}_h^n, p_h^n)$ for all $n \geq 0$.

4.3. Stability. We equip the discrete space L_h^{d} with the mesh-dependent norm

$$\begin{aligned}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h} := {}& \ell_D^{-\frac{1}{2}}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + \ell_D^{-\frac{1}{2}}\|p_h\|_{L^2(D)} \\ &+ \|\tilde{h}^{\frac{1}{2}}\nabla_h{\cdot}\boldsymbol{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}}\nabla_h p_h\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{v}_h|_h^{\mathrm{d}} + |p_h|_h^{\mathrm{g},\circ}.\end{aligned} \tag{4.7}$$

Recall that $\boldsymbol{\Pi}_h^{\mathrm{b}}$ and Π_h^{b} are defined in (3.2), and the spaces $\boldsymbol{X}_0^{\mathrm{d}}$ are X^{g} are equipped with the norms $\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}} := \ell_D\|\nabla_0{\cdot}\boldsymbol{w}\|_{L^2(D)}$ and $\|q\|_{X^{\mathrm{g}}} := \ell_D\|\nabla q\|_{\boldsymbol{L}^2(D)}$, respectively.

LEMMA 4.5 (stability of broken projections). *The following holds:*

$$\|(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}), \Pi_h^{\mathrm{b}}(q))\|_{\flat,h} \lesssim \ell_D^{-\frac{1}{2}}\big(\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}} + \|q\|_{X^{\mathrm{g}}}\big) \qquad \forall(\boldsymbol{w}, q) \in \boldsymbol{X}_0^{\mathrm{d}} \times X^{\mathrm{g}}. \tag{4.8}$$

Proof. (1) Bound on $\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})$. Let $\boldsymbol{w} \in \boldsymbol{X}_0^{\mathrm{d}}$. The $\boldsymbol{L}^2$-stability of $\boldsymbol{\Pi}_h^{\mathrm{b}}$ together with the closedness of the image of $\nabla_0{\cdot}$ (see Lemma 2.8) implies that

$$\ell_D^{-\frac{1}{2}}\|\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})\|_{\boldsymbol{L}^2(D)} \leq \ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}}.$$

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Moreover, we observe that, for all $K \in \mathcal{T}_h$,

$$\|\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))\|_{L^2(K)}^2 = (\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}) - \boldsymbol{w}), \nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))_{L^2(K)} + (\nabla\cdot\boldsymbol{w}, \nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))_{L^2(K)}.$$

Letting A be the first term on the right-hand side, integration by parts gives

$$A = -\big(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}) - \boldsymbol{w}, \nabla(\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})))\big)_{\boldsymbol{L}^2(K)} + \big(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}) - \boldsymbol{w}, \boldsymbol{n}_K\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))\big)_{\boldsymbol{L}^2(\partial K)}.$$

Invoking the Cauchy–Schwarz inequality, the approximation properties of $\mathbf{\Pi}_h^{\mathrm{b}}$, the boundedness of the embedding $\boldsymbol{X}_0^{\mathrm{d}} \hookrightarrow \boldsymbol{H}^s(D)$ with $s > \frac{1}{2}$ (see Lemma 2.14), and inverse and trace inequalities for the polynomial $\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))$, we obtain

$$|A| \lesssim h_K^{s-1}|\boldsymbol{w}|_{\boldsymbol{H}^s(K)}\|\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))\|_{L^2(K)}.$$

For the second term on the right-hand side, we apply the Cauchy–Schwarz inequality. After simplifying by $\|\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))\|_{L^2(K)}$ and multiplying the result by $h_K^{\frac{1}{2}}$, we obtain

$$h_K^{\frac{1}{2}}\|\nabla\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))\|_{L^2(K)} \lesssim h_K^{s-\frac{1}{2}}|\boldsymbol{w}|_{\boldsymbol{H}^s(K)} + h_K^{\frac{1}{2}}\|\nabla\cdot\boldsymbol{w}\|_{L^2(K)}.$$

Summing over the mesh cells, we infer that

$$\begin{aligned}\|\tilde{h}^{\frac{1}{2}}\nabla_h\cdot(\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}))\|_{L^2(D)} &\lesssim (h/\ell_D)^{s-\frac{1}{2}}\ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}} + (h/\ell_D)^{\frac{1}{2}}\ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}} \\ &\le 2\ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}},\end{aligned}$$

where the last bound follows from $h \le \ell_D$. Finally, since $\boldsymbol{w} \in \boldsymbol{H}^s(D)$ with $s > \frac{1}{2}$, it is legitimate to assert that $\boldsymbol{w}$ has zero normal jump across every mesh interface and zero normal trace at every mesh boundary face. This implies that

$$|\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})|_h^{\mathrm{d}} = |\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}) - \boldsymbol{w}|_h^{\mathrm{d}} \lesssim h^{s-\frac{1}{2}}|\boldsymbol{w}|_{\boldsymbol{H}^s(D)} \lesssim (h/\ell_D)^{s-\frac{1}{2}}\ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}}.$$

Hence $|\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})|_h^{\mathrm{d}} \lesssim \ell_D^{-\frac{1}{2}}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}}$ because $h \le \ell_D$. This completes the bound on $\mathbf{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})$.

(2) The arguments for $\Pi^{\mathrm{g}}(q)$ are similar (and simpler) and are therefore omitted. ∎

We are now ready to establish our main stability result.

LEMMA 4.6 (stability). *The following holds:*

$$\|(\boldsymbol{v}_h, p_h)\|_{\flat,h} \lesssim \sup_{(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}} \frac{\big|a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big)\big|}{\|(\boldsymbol{w}_h,q_h)\|_{\flat,h}} \qquad \forall(\boldsymbol{v}_h,p_h) \in L_h^{\mathrm{d}}. \tag{4.9}$$

Proof. Let $(\boldsymbol{v}_h, p_h) \in L_h^{\mathrm{d}}$, and let $\mathbb{S}$ denote the right-hand side of (4.9). We need to show that $\|(\boldsymbol{v}_h, p_h)\|_{\flat,h} \lesssim \mathbb{S}$.

(1) The first step of the proof is classical (see, e.g., [15]). We observe that

$$\begin{aligned}&\ell_D^{-1}\|\mathbf{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)}^2 + \ell_D^{-1}\|\Pi^{\mathrm{g}}(p_h)\|_{L^2(D)}^2 + (|\boldsymbol{v}_h|_h^{\mathrm{d}})^2 + (|p_h|_h^{\mathrm{g},\circ})^2 \\ &\qquad = a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{v}_h,p_h)\big) \le \mathbb{S}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}.\end{aligned} \tag{4.10}$$

Moreover, using $\boldsymbol{w}_h := \tilde{h}\nabla_h p_h$ and $q_h := \tilde{h}\nabla_h\cdot\boldsymbol{v}_h$ in the expression (4.2) for a_h gives

$$\|\tilde{h}^{\frac{1}{2}}\nabla_h\cdot\boldsymbol{v}_h\|_{L^2(D)}^2 + \|\tilde{h}^{\frac{1}{2}}\nabla_h q_h\|_{\boldsymbol{L}^2(D)}^2 = a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big) - \Delta_1,$$

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with

$$\Delta_1 := \ell_D^{-1}(\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h), \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} + \ell_D^{-1}(\Pi^{\mathrm{g}}(p_h), q_h)_{L^2(D)} + s_h^{\mathrm{d}}(\boldsymbol{v}_h, \boldsymbol{w}_h) + s_h^{\mathrm{g},\circ}(p_h, q_h) - \sum_{F\in\mathcal{F}_h^\circ} ([\![p_h]\!]_F^{\mathrm{g}}, \{\!\{\boldsymbol{w}_h\}\!\}_F^{\mathrm{d}})_{L^2(F)} - \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}}, \{\!\{q_h\}\!\}_F^{\mathrm{g}})_{L^2(F)}.$$

Using inverse inequalities and $h \le \ell_D$ shows that

$$\begin{aligned}\|(\boldsymbol{w}_h, q_h)\|_{\flat,h} &\lesssim \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{w}_h\|_{\boldsymbol{L}^2(D)} + \|\tilde{h}^{-\frac{1}{2}}q_h\|_{L^2(D)} \\ &= \|\tilde{h}^{\frac{1}{2}}\nabla_h p_h\|_{\boldsymbol{L}^2(D)} + \|\tilde{h}^{\frac{1}{2}}\nabla_h\cdot\boldsymbol{v}_h\|_{L^2(D)} \le \|(\boldsymbol{v}_h, p_h)\|_{\flat,h}.\end{aligned}$$

This proves $\big|a_h\big((\boldsymbol{v}_h, p_h), (\boldsymbol{w}_h, q_h)\big)\big| \lesssim \mathbb{S}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h}$. Moreover, invoking the Cauchy–Schwarz inequality, $h \le \ell_D$, inverse inequalities, and the bound from step (1) gives

$$\begin{aligned}|\Delta_1| &\lesssim \big(\ell_D^{-1}\|\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)}^2 + \ell_D^{-1}\|\Pi^{\mathrm{g}}(p_h)\|_{L^2(D)}^2 + (|\boldsymbol{v}_h|_h^{\mathrm{d}})^2 + (|p_h|_h^{\mathrm{g},\circ})^2\big)^{\frac{1}{2}} \\ &\quad\times (\|\tilde{h}^{-\frac{1}{2}}\boldsymbol{w}_h\|_{\boldsymbol{L}^2(D)} + \|\tilde{h}^{-\frac{1}{2}}q_h\|_{L^2(D)}) \le \mathbb{S}^{\frac{1}{2}}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h}^{\frac{3}{2}}.\end{aligned}$$

Putting the above bounds together yields

$$\|\tilde{h}^{\frac{1}{2}}\nabla_h\cdot\boldsymbol{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}}\nabla_h p_h\|_{\boldsymbol{L}^2(D)} \lesssim \mathbb{S}^{\frac{1}{2}}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h}^{\frac{3}{4}}. \tag{4.11}$$

Combining (4.10) with (4.11) shows that

$$\begin{aligned}&\ell_D^{-\frac{1}{2}}\|\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \ell_D^{-\frac{1}{2}}\|\Pi^{\mathrm{g}}(p_h)\|_{L^2(D)} + |\boldsymbol{v}_h|_h^{\mathrm{d}} + |p_h|_h^{\mathrm{g},\circ} \\ &\quad + \|\tilde{h}^{\frac{1}{2}}\nabla_h\cdot\boldsymbol{v}_h\|_{L^2(D)} + \|\tilde{h}^{\frac{1}{2}}\nabla_h p_h\|_{\boldsymbol{L}^2(D)} \lesssim \mathbb{S}^{\frac{1}{2}}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h}^{\frac{1}{2}} + \mathbb{S}^{\frac{1}{4}}\|(\boldsymbol{v}_h, p_h)\|_{\flat,h}^{\frac{3}{4}}.\end{aligned} \tag{4.12}$$

(2) In the second step, we prove that, for all $(\boldsymbol{v}_h', p_h') \in \boldsymbol{X}_{h0}^{\mathrm{d}} \times X_h^{\mathrm{g}}$ (see (3.5)),

$$\ell_D^{-\frac{1}{2}}\big(\|\boldsymbol{v}_h'\|_{\boldsymbol{L}^2(D)} + \|p_h'\|_{L^2(D)}\big) \lesssim \ell_D^{\frac{1}{2}}\mathbb{S}^{\pi} + (h/\ell_D)^{s-\frac{1}{2}}\big(|\boldsymbol{v}_h'|_h^{\mathrm{d}} + |p_h'|_h^{\mathrm{g},\circ}\big), \tag{4.13}$$

with

$$\mathbb{S}^{\pi} := \sup_{(\boldsymbol{w},q)\in\boldsymbol{X}_0^{\mathrm{d}}\times X^{\mathrm{g}}} \frac{\big|a_h\big((\boldsymbol{v}_h', p_h'), (\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{w}), \Pi_h^{\mathrm{b}}(q))\big)\big|}{\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}} + \|q\|_{X^{\mathrm{g}}}}.$$

The proof of (4.13) heavily relies on the discrete Poincaré–Steklov inequalities (3.6) and (3.7a). Let $q \in X^{\mathrm{g}}$ with $\ell_D\|\nabla q\|_{\boldsymbol{L}^2(D)} =: \|q\|_{X^{\mathrm{g}}} = 1$, and set $q_h := \Pi_h^{\mathrm{b}}(q)$. Notice that $\Pi^{\mathrm{g}}(q_h) = \Pi^{\mathrm{g}}(q) = 0$ since $\Pi^{\mathrm{g}} \circ \Pi_h^{\mathrm{b}} = \Pi^{\mathrm{g}}$ (see Remark 3.4). Using the expression (4.2) for a_h, and since q has zero jumps across the mesh interfaces, we infer that

$$\begin{aligned}(\boldsymbol{v}_h', \nabla q)_{\boldsymbol{L}^2(D)} &= -(\nabla_h\cdot\boldsymbol{v}_h', q)_{L^2(D)} + \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h']\!]_F^{\mathrm{d}}, \{\!\{q\}\!\}_F^{\mathrm{g}})_{L^2(F)} \\ &= -a_h\big((\boldsymbol{v}_h', p_h'), (\boldsymbol{0}, q_h)\big) - \Delta_2,\end{aligned}$$

with

$$\Delta_2 := (\nabla_h\cdot\boldsymbol{v}_h', q - q_h)_{L^2(D)} - \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h']\!]_F^{\mathrm{d}}, \{\!\{q - q_h\}\!\}_F^{\mathrm{g}})_{L^2(F)} + s_h^{\mathrm{g},\circ}(p_h', q - q_h).$$

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Noticing that the first term on the right-hand side vanishes, invoking the Cauchy–Schwarz inequality, the approximation properties of Π_h^{b}, and the continuous embedding $X^{\mathrm{g}} \hookrightarrow H^s(D)$ with $s > \frac{1}{2}$ gives

$$|\Delta_2| \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} \big(|\boldsymbol{v}_h'|_h^{\mathrm{d}} + |p_h'|_h^{\mathrm{g},\circ}\big).$$

Similarly, let $\boldsymbol{w} \in \boldsymbol{X}_0^{\mathrm{d}}$ with $\ell_D \|\nabla_0{\cdot}\boldsymbol{w}\|_{L^2(D)} =: \|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\mathrm{d}}} = 1$, and set $\boldsymbol{w}_h := \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})$. Notice that $\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{w}_h) = \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{w})) = \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{w}) = \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\Pi}_0^{\mathrm{d}}(\boldsymbol{w})) = \boldsymbol{0}$ owing to Lemma 3.3. Using (4.2), and since $\boldsymbol{w}$ has zero normal jumps across the mesh interfaces and zero normal component at the mesh boundary faces, we infer that

$$\begin{aligned}(p_h', \nabla_0{\cdot}\boldsymbol{w})_{L^2(D)} &= -(\nabla_h p_h', \boldsymbol{w})_{\boldsymbol{L}^2(D)} + \sum_{F\in\mathcal{F}_h^\circ} ([\![p_h']\!]_F^{\mathrm{g}}, \{\!\{\boldsymbol{w}\}\!\}_F^{\mathrm{d}})_{L^2(F)} \\ &= -a_h\big((\boldsymbol{v}_h', p_h'), (\boldsymbol{w}_h, 0)\big) - \Delta_3,\end{aligned}$$

with

$$\Delta_3 := (\nabla_h p_h', \boldsymbol{w} - \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} - \sum_{F\in\mathcal{F}_h^\circ} ([\![p_h']\!]_F^{\mathrm{g}}, \{\!\{\boldsymbol{w} - \boldsymbol{w}_h\}\!\}_F^{\mathrm{d}})_{L^2(F)} + s_h^{\mathrm{d}}(\boldsymbol{v}_h', \boldsymbol{w} - \boldsymbol{w}_h).$$

Invoking the Cauchy–Schwarz inequality, the approximation properties of $\boldsymbol{\Pi}_h^{\mathrm{b}}$, and the continuous embedding $\boldsymbol{X}_0^{\mathrm{d}} \hookrightarrow \boldsymbol{H}^s(D)$ with $s > \frac{1}{2}$ gives

$$|\Delta_3| \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} \big(|\boldsymbol{v}_h'|_h^{\mathrm{d}} + |p_h'|_h^{\mathrm{g},\circ}\big).$$

Using the above identities for $(\boldsymbol{v}_h', \nabla q)_{\boldsymbol{L}^2(D)}$ and $(p_h', \nabla_0{\cdot}\boldsymbol{w})_{L^2(D)}$ together with the above bounds on Δ_2 and Δ_3 shows that

$$\|\nabla_0{\cdot}\boldsymbol{v}_h'\|_{(X^{\mathrm{g}})'} + \|\nabla p_h'\|_{(\boldsymbol{X}_0^{\mathrm{d}})'} \lesssim \mathbb{S}^\pi + (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} \big(|\boldsymbol{v}_h'|_h^{\mathrm{d}} + |p_h'|_h^{\mathrm{g},\circ}\big).$$

Finally, using the discrete Poincaré–Steklov inequalities (3.6) and (3.7a) gives

$$\begin{aligned}&\ell_D^{-\frac{1}{2}} \big(\|\boldsymbol{v}_h'\|_{\boldsymbol{L}^2(D)} + \|p_h'\|_{L^2(D)}\big) \\ &\lesssim \ell_D^{\frac{1}{2}} \big(\|\nabla_0{\cdot}\boldsymbol{v}_h'\|_{(X^{\mathrm{g}})'} + \|\nabla p_h'\|_{(\boldsymbol{X}_0^{\mathrm{d}})'}\big) + (h/\ell_D)^{\frac{1}{2}} \big(|\boldsymbol{v}_h'|_h^{\mathrm{d}} + |p_h'|_h^{\mathrm{g},\circ}\big) \\ &\lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi + (h/\ell_D)^{s-\frac{1}{2}} \big(|\boldsymbol{v}_h'|_h^{\mathrm{d}} + |p_h'|_h^{\mathrm{g},\circ}\big),\end{aligned}$$

where we used that $h \le \ell_D$. This completes the proof of (4.13).

(3) In this last step, we prove (4.9). Let $(\boldsymbol{v}_h, p_h) \in L_h^{\mathrm{d}}$, and set $\boldsymbol{v}_h' := \boldsymbol{v}_h - \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)$, $p_h' := p_h - \Pi^{\mathrm{g}}(p_h)$. Notice that $(\boldsymbol{v}_h', p_h') \in \boldsymbol{X}_{h0}^{\mathrm{g}} \times X_h^{\mathrm{g}}$. Then, using (4.13) yields

$$\ell_D^{-\frac{1}{2}} \big(\|\boldsymbol{v}_h - \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \|p_h - \Pi^{\mathrm{g}}(p_h)\|_{L^2(D)}\big) \lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi + (h/\ell_D)^{s-\frac{1}{2}} \big(|\boldsymbol{v}_h|_h^{\mathrm{d}} + |p_h|_h^{\mathrm{g},\circ}\big).$$

Using $h \le \ell_D$ and invoking (4.12) gives

$$\ell_D^{-\frac{1}{2}} \big(\|\boldsymbol{v}_h - \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \|p_h - \Pi^{\mathrm{g}}(p_h)\|_{L^2(D)}\big) \lesssim \ell_D^{\frac{1}{2}} \mathbb{S}^\pi + \mathbb{S}^{\frac{1}{2}} \|(\boldsymbol{v}_h, p_h)\|_{\flat,h}^{\frac{1}{2}}.$$

Owing to Lemma 4.5, we infer that

$$\ell_D^{\frac{1}{2}} \mathbb{S}^\pi \lesssim \sup_{(\boldsymbol{w}_h, q_h)\in L_h^{\mathrm{d}}} \frac{\big|a_h\big((\boldsymbol{v}_h - \boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v}_h), p_h - \Pi^{\mathrm{g}}(p_h)), (\boldsymbol{w}_h, q_h)\big)\big|}{\|(\boldsymbol{w}_h, q_h)\|_{\flat,h}}.$$

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We observe that

$$a_h\big((\boldsymbol{v}_h-\boldsymbol{\Pi}^{\rm d}_{h0}(\boldsymbol{v}_h),p_h-\Pi^{\rm g}(p_h)),(\boldsymbol{w}_h,q_h)\big)\\ =a_h\big((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)\big)-\ell_D^{-1}(\boldsymbol{\Pi}^{\rm d}_{h0}(\boldsymbol{v}_h),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}-\ell_D^{-1}(\Pi^{\rm g}(p_h),q_h)_{L^2(D)}.$$

Bounding the last two terms on the right-hand side by using the Cauchy–Schwarz inequality, invoking the estimate (4.10), and using the above bounds gives

$$\ell_D^{-\frac12}\big(\|\boldsymbol{v}_h-\boldsymbol{\Pi}^{\rm d}_{h0}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)}+\|p_h-\Pi^{\rm g}(p_h)\|_{L^2(D)}\big)\\ \lesssim \mathbb{S}+\mathbb{S}^{\frac12}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}^{\frac12}+\mathbb{S}^{\frac14}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}^{\frac34}.$$

Invoking the triangle inequality and the estimate (4.10) yields

$$\ell_D^{-\frac12}\big(\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)}+\|p_h\|_{L^2(D)}\big)\lesssim \mathbb{S}+\mathbb{S}^{\frac12}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}^{\frac12}+\mathbb{S}^{\frac14}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}^{\frac34}.$$

Combining this bound with (4.12) finally gives

$$\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}^2\lesssim \mathbb{S}^2+\mathbb{S}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}+\mathbb{S}^{\frac12}\|(\boldsymbol{v}_h,p_h)\|_{\flat,h}^{\frac32}.$$

The inf-sup condition (4.9) follows by repeated applications of Young's inequality. ◻

Introducing $\mathbb{S}^\pi$ in the proof of Lemma 4.6 may seem surprising as $\mathbb{S}^\pi$ is not used in the final result (4.9). The inequality (4.13) finds its justification in the following sharper stability estimate which will be instrumental to bound the consistency error.

COROLLARY 4.7 (sharper $L^{\rm d}$-stability). *The following inequality holds true for all* $(\boldsymbol{v}'_h,p'_h)\in \boldsymbol{X}^{\rm d}_{h0}\times X^{\rm g}_h$*:*

$$\ell_D^{-\frac12}\big(\|\boldsymbol{v}'_h\|_{\boldsymbol{L}^2(D)}+\|p'_h\|_{L^2(D)}\big)\lesssim (h/\ell_D)^{s-\frac12}\sup_{(\boldsymbol{w}_h,q_h)\in L^{\rm d}_h}\frac{\big|a_h\big((\boldsymbol{v}'_h,p'_h),(\boldsymbol{w}_h,q_h)\big)\big|}{\|(\boldsymbol{w}_h,q_h)\|_{\flat,h}}\\ +\ell_D^{\frac12}\sup_{(\boldsymbol{w},q)\in\boldsymbol{X}^{\rm d}_0\times X^{\rm g}}\frac{\big|a_h\big((\boldsymbol{v}'_h,p'_h),(\boldsymbol{\Pi}^{\rm b}_h(\boldsymbol{w}),\Pi^{\rm b}_h(q))\big)\big|}{\|\boldsymbol{w}\|_{\boldsymbol{X}^{\rm d}_0}+\|q\|_{X^{\rm g}}}. \tag{4.14}$$

Proof. Combine (4.9) with (4.13). ◻

4.4. Consistency and boundedness. The second step of our program consists of proving a consistency/boundedness result. This is done by first considering the discrete operator $\tilde T_h: L^{\rm d}\to L^{\rm d}_h\subset L^{\rm d}$ so that, for all $(\boldsymbol{f},g)\in L^{\rm d}$, $\tilde T_h(\boldsymbol{f},g):=(\tilde{\boldsymbol{v}}_h,\tilde p_h)$ is the unique pair in $L^{\rm d}_h$ so that, for all $(\boldsymbol{w}_h,q_h)\in L^{\rm d}_h$,

$$a_h\big((\tilde{\boldsymbol{v}}_h,\tilde p_h),(\boldsymbol{w}_h,q_h)\big)=((\boldsymbol{I}-\boldsymbol{\Pi}^{\rm d}_0)(\boldsymbol{f}),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}+((I-\Pi^{\rm g})(g),q_h)_{L^2(D)}. \tag{4.15}$$

The definition of $\tilde T_h$ is meaningful owing to Lemma 4.6. The difference between the operators T_h and $\tilde T_h$ lies in the way $\boldsymbol{f}$ is projected on the right-hand sides of (4.3) and (4.15) (see also Remark 4.10 below). The operator $\tilde T_h$ is introduced since it is easier to bound the associated consistency error. We postpone the control on $\tilde T_h - T_h$ to a second step (see Lemma 4.9 below). We augment the stability norm $\|\cdot\|_{\flat,h}$ by defining, for all $s\in(\frac12,1]$, the following mesh-dependent norm on $\boldsymbol{H}^s(D)\times H^s(D)+L^{\rm d}_h$:

$$\|(\boldsymbol{w},q)\|_{\sharp,h}:=\|(\boldsymbol{w},q)\|_{\flat,h}+\|\tilde h^{-\frac12}\boldsymbol{w}\|_{\boldsymbol{L}^2(D)}+\|\tilde h^{-\frac12}q\|_{L^2(D)}\\ +\Big\{\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_K}\|\gamma^{\rm d}_{K,F}(\boldsymbol{w})\|^2_{L^2(F)}+\|\gamma^{\rm g}_{K,F}(q)\|^2_{L^2(F)}\Big\}^{\frac12}. \tag{4.16}$$

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LEMMA 4.8 (consistency/boundedness). *Let* $(\boldsymbol{f},g)\in L^{\mathrm{d}}$. *Set* $(\boldsymbol{v},p):=T(\boldsymbol{f},g)$, $(\tilde{\boldsymbol{v}}_h,\tilde{p}_h):=\tilde{T}_h(\boldsymbol{f},g)$, $\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}}:=\tilde{\boldsymbol{v}}_h-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v})$, $\boldsymbol{\xi}^{\boldsymbol{v}}:=\boldsymbol{v}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v})$, $\tilde{e}_h^p=\tilde{p}_h-\Pi_h^{\mathrm{b}}(p)$, *and* $\xi^p:=p-\Pi_h^{\mathrm{b}}(p)$. *The following holds for all* $(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}$*:*

$$\big|a_h\big((\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde{e}_h^p),(\boldsymbol{w}_h,q_h)\big)\big|\lesssim\|(\boldsymbol{\xi}^{\boldsymbol{v}},\xi^p)\|_{\sharp,h}\|(\boldsymbol{w}_h,q_h)\|_{\flat,h}. \tag{4.17}$$

Proof. Let $(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}$. By definition of $\tilde{T}_h$, we obtain

$$\begin{aligned}a_h\big((\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde{e}_h^p),(\boldsymbol{w}_h,q_h)\big)&=((I-\boldsymbol{\Pi}_0^{\mathrm{d}})(\boldsymbol{f}),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}+((I-\Pi^{\mathrm{g}})(g),q_h)_{L^2(D)}\\&\quad-a_h\big((\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v}),\Pi_h^{\mathrm{b}}(p)),(\boldsymbol{w}_h,q_h)\big).\end{aligned}$$

Since $\nabla p=(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{d}})(\boldsymbol{f})$ and $\nabla_0{\cdot}\boldsymbol{v}=(I-\Pi^{\mathrm{g}})(g)$ by definition of T, we infer that

$$\begin{aligned}a_h\big((\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde{e}_h^p),(\boldsymbol{w}_h,q_h)\big)&=(\nabla p,\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}+(\nabla_0{\cdot}\boldsymbol{v},q_h)_{L^2(D)}\\&\quad-a_h\big((\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v}),\Pi_h^{\mathrm{b}}(p)),(\boldsymbol{w}_h,q_h)\big).\end{aligned}$$

We integrate by parts the first two terms on the right-hand side (this is legitimate owing to Lemma 2.14 since $s>\frac{1}{2}$). Using the expression (4.1) for a_h, this gives

$$\begin{aligned}a_h\big((\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde{e}_h^p),(\boldsymbol{w}_h,q_h)\big)&=-\ell_D^{-1}(\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v})),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}-\ell_D^{-1}(\Pi^{\mathrm{g}}(\Pi_h^{\mathrm{b}}(p)),q_h)_{L^2(D)}\\&\quad+(\Pi_h^{\mathrm{b}}(p)-p,\nabla_h{\cdot}\boldsymbol{w}_h)_{L^2(D)}+(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v})-\boldsymbol{v},\nabla_h q_h)_{\boldsymbol{L}^2(D)}\\&\quad+\sum_{F\in\mathcal{F}_h}(\{\!\{p-\Pi_h^{\mathrm{b}}(p)\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)}+\sum_{F\in\mathcal{F}_h^\circ}(\{\!\{\boldsymbol{v}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v})\}\!\}_F^{\mathrm{d}},[\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)}\\&\quad-s_h^{\mathrm{d}}(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v}),\boldsymbol{w}_h)-s_h^{\mathrm{g},\circ}(\Pi_h^{\mathrm{b}}(p),q_h).\end{aligned}$$

Recall from Lemma 2.12 that $\boldsymbol{v}\in\boldsymbol{X}_0^{\mathrm{d}}$ and $p\in X^{\mathrm{g}}$, i.e., $\boldsymbol{\Pi}_0^{\mathrm{d}}(\boldsymbol{v})=\boldsymbol{0}$ and $\Pi^{\mathrm{g}}(p)=0$. We then observe that $\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v}))=\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{v})=\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\Pi}_0^{\mathrm{d}}(\boldsymbol{v}))=\boldsymbol{0}$ (owing to Lemma 3.3), $\Pi^{\mathrm{g}}(\Pi_h^{\mathrm{b}}(p))=\Pi^{\mathrm{g}}(p)=0$, $[\![\boldsymbol{v}]\!]_F^{\mathrm{d}}=0$ for all $F\in\mathcal{F}_h$, and $[\![p]\!]_F^{\mathrm{g}}=0$ for all $F\in\mathcal{F}_h^\circ$. Recalling the notation $\boldsymbol{\xi}^{\boldsymbol{v}}:=\boldsymbol{v}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{v})$, $\xi^p:=p-\Pi_h^{\mathrm{b}}(p)$, we infer that

$$\begin{aligned}a_h\big((\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde{e}_h^p),(\boldsymbol{w}_h,q_h)\big)=&-(\xi^p,\nabla_h{\cdot}\boldsymbol{w}_h)_{L^2(D)}-(\boldsymbol{\xi}^{\boldsymbol{v}},\nabla_h q_h)_{\boldsymbol{L}^2(D)}\\&+\sum_{F\in\mathcal{F}_h}(\{\!\{\xi^p\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)}+\sum_{F\in\mathcal{F}_h^\circ}(\{\!\{\boldsymbol{\xi}^{\boldsymbol{v}}\}\!\}_F^{\mathrm{d}},[\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)}\\&+s_h^{\mathrm{d}}(\boldsymbol{\xi}^{\boldsymbol{v}},\boldsymbol{w}_h)+s_h^{\mathrm{g},\circ}(\xi^p,q_h).\end{aligned}$$

The assertion follows from the Cauchy–Schwarz inequality and $h\le\ell_D$. ∎

The second step of the consistency error analysis is to estimate $T_h-\tilde{T}_h$.

LEMMA 4.9 (bound on $(\tilde{T}_h-T_h)$). *We have* $\lim_{\mathcal{H}\ni h\to 0}\|\tilde{T}_h-T_h\|_{\mathcal{L}(L^{\mathrm{d}};L^{\mathrm{d}})}=0$.

Proof. Let $(\boldsymbol{f},g)\in L^{\mathrm{d}}$, and let us set $(\boldsymbol{v}_h,p_h):=T_h(\boldsymbol{f},g)$, $(\tilde{\boldsymbol{v}}_h,\tilde{p}_h):=\tilde{T}_h(\boldsymbol{f},g)$, $\boldsymbol{\eta}_h^{\boldsymbol{v}}:=\boldsymbol{v}_h-\tilde{\boldsymbol{v}}_h$, and $\eta_h^p:=p_h-\tilde{p}_h$. We have, for all $(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}$,

$$a_h\big((\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p),(\boldsymbol{w}_h,q_h)\big)=((\boldsymbol{\Pi}_0^{\mathrm{d}}-\boldsymbol{\Pi}_{h0}^{\mathrm{d}})(\boldsymbol{f}),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}. \tag{4.18}$$

Invoking Lemma 3.3 gives $((\boldsymbol{\Pi}_0^{\mathrm{d}}-\boldsymbol{\Pi}_{h0}^{\mathrm{d}})(\boldsymbol{f}),\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\eta}_h^{\boldsymbol{v}}))_{\boldsymbol{L}^2(D)}=0$. Then testing (4.18) with $\boldsymbol{w}_h=\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\eta}_h^{\boldsymbol{v}})$ yields $\boldsymbol{\Pi}_{h0}^{\mathrm{d}}(\boldsymbol{\eta}_h^{\boldsymbol{v}})=\boldsymbol{0}$. Moreover, testing (4.18) with $q_h=\Pi^{\mathrm{g}}(\eta_h^p)$ readily gives $\Pi^{\mathrm{g}}(\eta_h^p)=0$. Hence, $(\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p)\in\boldsymbol{X}_{h0}^{\mathrm{d}}\times X_h^{\mathrm{g}}$. We can then invoke Corollary 4.7 to bound $(\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p)$. Owing to (4.18), we infer that

$$\sup_{(\boldsymbol{w}_h,q_h)\in L_h^{\mathrm{d}}}\frac{\big|a_h\big((\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p),(\boldsymbol{w}_h,q_h)\big)\big|}{\|(\boldsymbol{w}_h,q_h)\|_{\flat,h}}\le\ell_D^{\frac{1}{2}}\|(\boldsymbol{\Pi}_0^{\mathrm{d}}-\boldsymbol{\Pi}_{h0}^{\mathrm{d}})(\boldsymbol{f})\|_{\boldsymbol{L}^2(D)}\le\ell_D^{\frac{1}{2}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}.$$

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Moreover, we have, for all $(\boldsymbol{w},q)\in \boldsymbol{X}_0^{\rm d}\times X^{\rm g}$,

$$\begin{aligned} a_h\big((\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p),(\boldsymbol{\Pi}_h^{\rm b}(\boldsymbol{w}),\Pi_h^{\rm b}(q))\big) &= ((\boldsymbol{\Pi}_0^{\rm d}-\boldsymbol{\Pi}_{h0}^{\rm d})(\boldsymbol{f}),\boldsymbol{\Pi}_h^{\rm b}(\boldsymbol{w}))_{\boldsymbol{L}^2(D)} \\ &= ((\boldsymbol{\Pi}_0^{\rm d}-\boldsymbol{\Pi}_{h0}^{\rm d})(\boldsymbol{f}),\boldsymbol{\Pi}_h^{\rm b}(\boldsymbol{w})-\boldsymbol{w})_{\boldsymbol{L}^2(D)}, \end{aligned}$$

where the last equality follows from $\boldsymbol{w}\in \boldsymbol{H}_0(\mathrm{div}=0;D)^\perp$ and $(\boldsymbol{\Pi}_0^{\rm d}-\boldsymbol{\Pi}_{h0}^{\rm d})(\boldsymbol{f})\in \boldsymbol{H}_0(\mathrm{div}=0;D)$. Invoking the Cauchy–Schwarz inequality, the approximation properties of $\boldsymbol{\Pi}_h^{\rm b}$, and the embedding $\boldsymbol{X}_0^{\rm d}\hookrightarrow \boldsymbol{H}^s(D)$ with $s>\frac12$ (see Lemma 2.14) gives

$$\big|a_h\big((\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p),(\boldsymbol{\Pi}_h^{\rm b}(\boldsymbol{w}),\Pi_h^{\rm b}(q))\big)\big| \lesssim h^s\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}|\boldsymbol{w}|_{\boldsymbol{H}^s(D)} \lesssim (h/\ell_D)^s\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\rm d}}.$$

Hence, we have

$$\sup_{(\boldsymbol{w},q)\in \boldsymbol{X}_0^{\rm d}\times X^{\rm g}} \frac{\big|a_h\big((\boldsymbol{\eta}_h^{\boldsymbol{v}},\eta_h^p),(\boldsymbol{\Pi}_h^{\rm b}(\boldsymbol{w}),\Pi_h^{\rm b}(q))\big)\big|}{\|\boldsymbol{w}\|_{\boldsymbol{X}_0^{\rm d}}+\|q\|_{X^{\rm g}}} \lesssim (h/\ell_D)^s\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}.$$

Putting the above two bounds together and invoking Corollary 4.7 finally gives

$$\ell_D^{-\frac12}\big(\|\boldsymbol{\eta}_h^{\boldsymbol{v}}\|_{\boldsymbol{L}^2(D)}+\|\eta_h^p\|_{L^2(D)}\big) \lesssim (h/\ell_D)^{s-\frac12}\ell_D^{\frac12}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)} + (h/\ell_D)^s\ell_D^{\frac12}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}.$$

Since $h\le \ell_D$, this proves that

$$\ell_D^{-\frac12}\|\tilde T_h(\boldsymbol{f},g)-T_h(\boldsymbol{f},g)\|_{L^{\rm d}} \lesssim (h/\ell_D)^{s-\frac12}\ell_D^{\frac12}\|(\boldsymbol{f},g)\|_{L^{\rm d}},$$

whence we conclude that $\lim_{\mathcal{H}\ni h\to 0}\|\tilde T_h - T_h\|_{\mathcal{L}(L^{\rm d};L^{\rm d})}=0$. ∎

Remark 4.10 (T_h vs. $\tilde T_h$). Although we have $\lim_{\mathcal{H}\ni h\to 0}\|T-\tilde T_h\|_{\mathcal{L}(L^{\rm d},L^{\rm d})}=0$ (see the proof of Theorem 4.11), this does not prove the spectral correctness of the dG approximation induced by the sesquilinear form $\hat a_h$ defined in (4.4) because Lemma 4.2 does not hold true for $\tilde T_h$. More precisely, let $\lambda\neq 0$, and assume that $\frac{1}{\lambda}$, $(\boldsymbol{v}_h,p_h)\in L_h^{\rm d}$ is an eigenpair of $\tilde T_h$. Then $\boldsymbol{\Pi}_{h0}^{\rm d}(\boldsymbol{v}_h)=\boldsymbol{0}$ and $\Pi^{\rm g}(p_h)=0$, but $\hat a((\boldsymbol{v}_h,p_h),(\boldsymbol{w}_h,q_h)) = \lambda(\boldsymbol{v}_h,\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}+\lambda(p_h,q_h)_{L^2(D)}-\lambda((\boldsymbol{\Pi}_0^{\rm d}-\boldsymbol{\Pi}_{h0}^{\rm d})(\boldsymbol{v}_h),\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}$ for all $(\boldsymbol{w}_h,q_h)\in L_h^{\rm d}$. Hence λ, $(\boldsymbol{v}_h,p_h)$ is not an eigenpair of the dG approximation associated with $\hat a_h$.

4.5. Conclusion. We are now ready to state the main result of this section. Owing to standard spectral approximation results (see, e.g., Bramble and Osborn [8, Lem. 2.2], Osborn [26, Thms. 3 and 4], Boffi [5, Prop. 7.4]), this theorem proves the spectral correctness of the dG approximation.

THEOREM 4.11 (convergence). *We have* $\lim_{\mathcal{H}\ni h\to 0}\|T-T_h\|_{\mathcal{L}(L^{\rm d};L^{\rm d})}=0$.

Proof. Since we have already established in Lemma 4.9 that $\lim_{\mathcal{H}\ni h\to 0}\|\tilde T_h - T_h\|_{\mathcal{L}(L^{\rm d};L^{\rm d})}=0$, it suffices to prove that $\lim_{\mathcal{H}\ni h\to 0}\|T-\tilde T_h\|_{\mathcal{L}(L^{\rm d};L^{\rm d})}=0$ and invoke the triangle inequality. Let $(\boldsymbol{f},g)\in L^{\rm d}$. Recalling the notation introduced in Lemma 4.8, and using Lemma 4.6 and Lemma 4.8, we infer that

$$\|(\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde e_h^p)\|_{\flat,h} \lesssim \sup_{(\boldsymbol{w}_h,q_h)\in L_h^{\rm d}} \frac{\big|a_h\big((\tilde{\boldsymbol{e}}_h^{\boldsymbol{v}},\tilde e_h^p),(\boldsymbol{w}_h,q_h)\big)\big|}{\|(\boldsymbol{w}_h,q_h)\|_{\flat,h}} \lesssim \|(\boldsymbol{\xi}^{\boldsymbol{v}},\xi^p)\|_{\sharp,h}.$$

This estimate combined with the triangle inequality, $\|\cdot\|_{\flat,h}\le\|\cdot\|_{\sharp,h}$, standard approximation properties of $\boldsymbol{\Pi}_h^{\rm b}$ and $\Pi_h^{\rm b}$, and $h\le\ell_D$ gives

$$\begin{aligned} \|(\boldsymbol{v}-\tilde{\boldsymbol{v}}_h,p-\tilde p_h)\|_{\flat,h} \lesssim \|(\boldsymbol{\xi}^{\boldsymbol{v}},\xi^p)\|_{\sharp,h} &\lesssim \Bigg\{\sum_{K\in\mathcal{T}_h} h_K^{2s-1}|\boldsymbol{v}|^2_{\boldsymbol{H}^s(K)} + h_K|p|^2_{H^1(K)}\Bigg\}^{\frac12} \\ &\lesssim h^{s-\frac12}\big(|\boldsymbol{v}|_{\boldsymbol{H}^s(D)}+\ell_D^{1-s}|p|_{H^1(D)}\big). \end{aligned}$$

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(Recall that we have set $h := \max_{K\in\mathcal{T}_h} h_K$.) Lemma 2.14 gives

$$\ell_D^s|\boldsymbol{v}|_{\boldsymbol{H}^s(D)} \le \|\boldsymbol{v}\|_{\boldsymbol{H}^s(D)} \lesssim \ell_D\|\nabla_0{\cdot}\boldsymbol{v}\|_{L^2(D)} = \ell_D\|(I-\Pi^{\mathrm{g}})(g)\|_{L^2(D)} \le \ell_D\|g\|_{L^2(D)},$$
$$|p|_{H^1(D)} = \|\nabla p\|_{\boldsymbol{L}^2(D)} = \|(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{d}})(\boldsymbol{f})\|_{\boldsymbol{L}^2(D)} \le \|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}.$$

Hence,

$$\ell_D^{-\frac12}\|T(\boldsymbol{f},g)-\tilde{T}_h(\boldsymbol{f},g)\|_{L^{\mathrm{d}}} \le \|(\boldsymbol{v}-\tilde{\boldsymbol{v}}_h, p-\tilde{p}_h)\|_{\flat,h} \lesssim (h/\ell_D)^{s-\frac12}\ell_D^{\frac12}\|(\boldsymbol{f},g)\|_{L^{\mathrm{d}}}.$$

This proves that $\|T-\tilde{T}_h\|_{\mathcal{L}(L^{\mathrm{d}};L^{\mathrm{d}})} \lesssim \ell_D(h/\ell_D)^{s-\frac12}$. The proof is complete. ∎

5. dG approximation of curl-curl operator.

This section deals with the analysis of the dG approximation of the curl-curl operator. The main result is Theorem 5.10, which implies that the approximation is spectrally correct. As most of the arguments are similar to those in section 4, most details are omitted. For simplicity, we set the scaling coefficient to $\mathfrak{c} := 1$.

5.1. Definitions. We define the discrete space $L_h^{\mathrm{c}} := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\times\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$. The sesquilinear form $a_h : L_h^{\mathrm{c}}\times L_h^{\mathrm{c}} \to \mathbb{C}$ associated with the problem (2.19) is

$$\begin{aligned} a_h\big((\boldsymbol{B}_h,\boldsymbol{E}_h),(\boldsymbol{b}_h,\boldsymbol{e}_h)\big) := {}& \ell_D^{-1}(\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{B}_h),\boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + \ell_D^{-1}(\boldsymbol{\Pi}_h^{\mathrm{c}}(\boldsymbol{E}_h),\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ & - (\boldsymbol{E}_h,\nabla_h{\times}\boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{B}_h,\nabla_h{\times}\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ & - \sum_{F\in\mathcal{F}_h}(\{\!\{\boldsymbol{E}_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{b}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} + \sum_{F\in\mathcal{F}_h^\circ}(\{\!\{\boldsymbol{B}_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} \\ & + s_h^{\mathrm{c}}(\boldsymbol{B}_h,\boldsymbol{b}_h) + s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h,\boldsymbol{e}_h). \end{aligned} \tag{5.1}$$

Integrating by parts the broken curl operators gives

$$\begin{aligned} a_h\big((\boldsymbol{B}_h,\boldsymbol{E}_h),(\boldsymbol{b}_h,\boldsymbol{e}_h)\big) = {}& \ell_D^{-1}(\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{B}_h),\boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + \ell_D^{-1}(\boldsymbol{\Pi}_h^{\mathrm{c}}(\boldsymbol{E}_h),\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ & - (\nabla_h{\times}\boldsymbol{E}_h,\boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\nabla_h{\times}\boldsymbol{B}_h,\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ & - \sum_{F\in\mathcal{F}_h^\circ}([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{b}_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)} + \sum_{F\in\mathcal{F}_h}([\![\boldsymbol{B}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{e}_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)} \\ & + s_h^{\mathrm{c}}(\boldsymbol{B}_h,\boldsymbol{b}_h) + s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h,\boldsymbol{e}_h). \end{aligned} \tag{5.2}$$

We now define $T_h : L^{\mathrm{c}} \to L_h^{\mathrm{c}} \subset L^{\mathrm{c}}$, the discrete counterpart of the operator $T : L^{\mathrm{c}} \to L^{\mathrm{c}}$ introduced in Definition 2.17 (recall that $L^{\mathrm{c}} := \boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)$). For all $(\boldsymbol{f},\boldsymbol{g}) \in L^{\mathrm{c}}$, $T_h(\boldsymbol{f},\boldsymbol{g}) := (\boldsymbol{B}_h,\boldsymbol{E}_h)$ is the unique pair in L_h^{c} so that, for all $(\boldsymbol{b}_h,\boldsymbol{e}_h) \in L_h^{\mathrm{c}}$,

$$a_h\big((\boldsymbol{B}_h,\boldsymbol{E}_h),(\boldsymbol{b}_h,\boldsymbol{e}_h)\big) = \big((\boldsymbol{I}-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\boldsymbol{f}),\boldsymbol{b}_h\big)_{\boldsymbol{L}^2(D)} + \big((\boldsymbol{I}-\boldsymbol{\Pi}_h^{\mathrm{c}})(\boldsymbol{g}),\boldsymbol{e}_h\big)_{\boldsymbol{L}^2(D)}. \tag{5.3}$$

The definition of T_h makes sense owing to the stability result established in Lemma 5.5. We prove that $\lim_{h\in\mathcal{H}\to 0}\|T-T_h\|_{\mathcal{L}(L^{\mathrm{c}};L^{\mathrm{c}})} = 0$ by proceeding as in section 4.

5.2. Discrete involutions and other comments. The projections $\boldsymbol{\Pi}_{h0}^{\mathrm{c}}$ and $\boldsymbol{\Pi}_h^{\mathrm{c}}$ are only invoked for theoretical purposes. They are not needed when one wants to approximate (2.9) or (2.11). Indeed, let us consider the following sesquilinear form:

$$\begin{aligned} \hat{a}_h\big((\boldsymbol{B}_h,\boldsymbol{E}_h),(\boldsymbol{b}_h,\boldsymbol{e}_h)\big) := {}& -(\boldsymbol{E}_h,\nabla_h{\times}\boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{B}_h,\nabla_h{\times}\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ & - \sum_{F\in\mathcal{F}_h}(\{\!\{\boldsymbol{E}_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{b}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} + \sum_{F\in\mathcal{F}_h^\circ}(\{\!\{\boldsymbol{B}_h\}\!\}_F^{\mathrm{g}},[\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} \\ & + s_h^{\mathrm{c}}(\boldsymbol{B}_h,\boldsymbol{b}_h) + s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h,\boldsymbol{e}_h). \end{aligned} \tag{5.4}$$

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LEMMA 5.1 (eigenvalue problems for a_h and $\hat{a}_h$). *Let $\lambda \neq 0$, $(\boldsymbol{B}_h, \boldsymbol{E}_h) \in L_h^{\mathrm{c}}$. Then $T_h(\boldsymbol{B}_h, \boldsymbol{E}_h) = \frac{1}{\lambda}(\boldsymbol{B}_h, \boldsymbol{E}_h)$ iff $\hat{a}_h\big((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)\big) = \lambda\big((\boldsymbol{B}_h, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}\big)$ for all $(\boldsymbol{b}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}$.*

Remark 5.2 (discrete involutions). Lemma 5.1 reveals that the involutions enforced by a_h and $\hat{a}_h$ are

$$\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{B}_h) = \boldsymbol{0}, \qquad \boldsymbol{\Pi}_h^{\mathrm{c}}(\boldsymbol{E}_h) = \boldsymbol{0}. \tag{5.5}$$

These involutions are essential to prove discrete Poincaré–Steklov inequalities which play pivotal roles in the proof of the spectral correctness of T_h, which in turn implies spectral correctness of the dG approximation realized by $\hat{a}_h$ owing to Lemma 5.1.

Let us now consider the approximation in time and space of (2.11) using the backward Euler time-stepping. Let $(\boldsymbol{B}_h^n, \boldsymbol{E}_h^n) \in L_h^{\mathrm{c}}$ be the approximation at t^n, and let τ be the time step. Let $(\boldsymbol{B}_h^{n+1}, \boldsymbol{E}_h^{n+1}) \in L_h^{\mathrm{c}}$ be s.t., for all $(\boldsymbol{b}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}$,

$$\begin{aligned}(\boldsymbol{B}_h^{n+1}, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h^{n+1}, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} + \tau \hat{a}_h((\boldsymbol{B}_h^{n+1}, \boldsymbol{E}_h^{n+1}), (\boldsymbol{b}_h, \boldsymbol{e}_h))\\ = (\boldsymbol{B}_h^n, \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{E}_h^n, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}.\end{aligned} \tag{5.6}$$

LEMMA 5.3 (time involution). *Assume that the pair $(\boldsymbol{B}_h^n, \boldsymbol{E}_h^n)$ satisfies the involutions* (5.5). *Then the pair $(\boldsymbol{B}_h^{n+1}, \boldsymbol{E}_h^{n+1})$ satisfies* (5.5) *as well.*

5.3. Stability. We equip the discrete space L_h^{c} with the mesh-dependent norm

$$\begin{aligned}\|(\boldsymbol{e}_h, \boldsymbol{b}_h)\|_{\flat,h} := \ell_D^{-\frac12}\|\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} + \ell_D^{-\frac12}\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)}\\ + \|\tilde{h}^{\frac12}\nabla_h{\times}\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + \|\tilde{h}^{\frac12}\nabla_h{\times}\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{b}_h|_h^{\mathrm{c}} + |\boldsymbol{e}_h|_h^{\mathrm{c},\circ}.\end{aligned} \tag{5.7}$$

Recall that $\boldsymbol{\Pi}_h^{\mathrm{b}}$ is defined in (3.2), and the spaces $\boldsymbol{X}_0^{\mathrm{c}}$ are $\boldsymbol{X}^{\mathrm{c}}$ are equipped with the norms $\|\boldsymbol{b}\|_{\boldsymbol{X}_0^{\mathrm{c}}} := \ell_D\|\nabla_0{\times}\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}$ and $\|\boldsymbol{e}\|_{\boldsymbol{X}^{\mathrm{c}}} := \ell_D\|\nabla{\times}\boldsymbol{e}\|_{\boldsymbol{L}^2(D)}$, respectively.

LEMMA 5.4 (stability of broken projections). *The following holds:*

$$\|(\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{b}), \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\flat,h} \lesssim \ell_D^{-\frac12}\big(\|\boldsymbol{b}\|_{\boldsymbol{X}_0^{\mathrm{c}}} + \|\boldsymbol{e}\|_{\boldsymbol{X}^{\mathrm{c}}}\big) \qquad \forall (\boldsymbol{b}, \boldsymbol{e}) \in \boldsymbol{X}_0^{\mathrm{c}}{\times}\boldsymbol{X}^{\mathrm{c}}. \tag{5.8}$$

We can now state our main stability results.

LEMMA 5.5 (stability). *The following holds:*

$$\|(\boldsymbol{B}_h, \boldsymbol{E}_h)\|_{\flat,h} \lesssim \sup_{(\boldsymbol{b}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}} \frac{\big|a_h\big((\boldsymbol{B}_h, \boldsymbol{E}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)\big)\big|}{\|(\boldsymbol{b}_h, \boldsymbol{e}_h)\|_{\flat,h}} \qquad \forall (\boldsymbol{B}_h, \boldsymbol{E}_h) \in L_h^{\mathrm{c}}. \tag{5.9}$$

Proof. Proceed as in the proof of Lemma 4.6. In particular, the second step of the proof establishes that, for all $(\boldsymbol{b}_h', \boldsymbol{e}_h') \in \boldsymbol{X}_{h0}^{\mathrm{c}}{\times}\boldsymbol{X}_h^{\mathrm{c}}$ (these spaces are defined in (3.5)),

$$\ell_D^{-\frac12}\big(\|\boldsymbol{b}_h'\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{e}_h'\|_{\boldsymbol{L}^2(D)}\big) \lesssim \ell_D^{\frac12}\mathbb{S}^\pi + (h/\ell_D)^{s-\frac12}\big(|\boldsymbol{b}_h'|_h^{\mathrm{c}} + |\boldsymbol{e}_h'|_h^{\mathrm{c},\circ}\big), \tag{5.10}$$

with $\mathbb{S}^\pi := \sup_{(\boldsymbol{b},\boldsymbol{e}) \in \boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}} \frac{|a((\boldsymbol{b}_h', \boldsymbol{e}_h'), (\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{b}), \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{e})))|}{\|\boldsymbol{b}\|_{\boldsymbol{X}_0^{\mathrm{c}}} + \|\boldsymbol{e}\|_{\boldsymbol{X}^{\mathrm{c}}}}$. ∎

COROLLARY 5.6 (sharper L^{c}-stability). *The following inequality holds true for all $(\boldsymbol{b}_h', \boldsymbol{e}_h') \in \boldsymbol{X}_{h0}^{\mathrm{c}}{\times}\boldsymbol{X}_h^{\mathrm{c}}$:*

$$\begin{aligned}\ell_D^{-\frac12}\big(\|\boldsymbol{b}_h'\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{e}_h'\|_{\boldsymbol{L}^2(D)}\big) \lesssim (h/\ell_D)^{s-\frac12} \sup_{(\boldsymbol{b}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}} \frac{\big|a_h\big((\boldsymbol{b}_h', \boldsymbol{e}_h'), (\boldsymbol{b}_h, \boldsymbol{e}_h)\big)\big|}{\|(\boldsymbol{b}_h, \boldsymbol{e}_h)\|_{\flat,h}}\\ + \ell_D^{\frac12} \sup_{(\boldsymbol{b}, \boldsymbol{e}) \in \boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}} \frac{\big|a_h\big((\boldsymbol{b}_h', \boldsymbol{e}_h'), (\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{b}), \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{e}))\big)\big|}{\|\boldsymbol{b}\|_{\boldsymbol{X}_0^{\mathrm{c}}} + \|\boldsymbol{e}\|_{\boldsymbol{X}^{\mathrm{c}}}}.\end{aligned} \tag{5.11}$$

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5.4. Consistency and boundedness. To establish consistency/boundedness, we proceed in two steps. We first introduce $\tilde{T}_h : L^{\mathrm{c}} \to L_h^{\mathrm{c}} \subset L^{\mathrm{c}}$ so that, for all $(\boldsymbol{f},\boldsymbol{g}) \in L^{\mathrm{c}}$, $\tilde{T}_h(\boldsymbol{f},\boldsymbol{g}) := (\tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{E}}_h)$ is the unique pair in L_h^{c} s.t., for all $(\boldsymbol{b}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}$,

$$a_h\big((\tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{E}}_h), (\boldsymbol{b}_h, \boldsymbol{e}_h)\big) = ((\boldsymbol{I} - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{f}), \boldsymbol{b}_h)_{\boldsymbol{L}^2(D)} + ((\boldsymbol{I} - \boldsymbol{\Pi}^{\mathrm{c}})(\boldsymbol{g}), \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}. \tag{5.12}$$

To bound the consistency error induced by $\tilde{T}_h$, we augment the norm $\|\cdot\|_{\flat,h}$ by defining the following mesh-dependent norm on $\boldsymbol{H}^s(D)\times\boldsymbol{H}^s(D) + L_h^{\mathrm{c}}$ for all $s \in (\frac{1}{2}, 1]$:

$$\begin{aligned}\|(\boldsymbol{b},\boldsymbol{e})\|_{\sharp,h} &:= \|(\boldsymbol{b},\boldsymbol{e})\|_{\flat,h} + \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} + \|\tilde{h}^{-\frac{1}{2}}\boldsymbol{e}\|_{\boldsymbol{L}^2(D)} \\ &\quad + \left\{\sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_K} \|\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{b})\|_{\boldsymbol{L}^2(F)}^2 + \|\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{e})\|_{\boldsymbol{L}^2(F)}^2\right\}^{\frac{1}{2}}. \end{aligned} \tag{5.13}$$

LEMMA 5.7 (consistency/boundedness). *For all $(\boldsymbol{f},\boldsymbol{g}) \in L^{\mathrm{c}}$, let $(\boldsymbol{B},\boldsymbol{E}) := T(\boldsymbol{f},\boldsymbol{g})$ and $(\tilde{\boldsymbol{B}}_h, \tilde{\boldsymbol{E}}_h) := \tilde{T}_h(\boldsymbol{f},\boldsymbol{g})$. Define $\tilde{\boldsymbol{\theta}}_h^{\boldsymbol{b}} := \tilde{\boldsymbol{B}}_h - \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{B})$, $\boldsymbol{\xi}^{\boldsymbol{b}} := \boldsymbol{B} - \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{B})$ and $\tilde{\boldsymbol{\theta}}_h^{\boldsymbol{e}} = \tilde{\boldsymbol{E}}_h - \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{E})$, $\boldsymbol{\xi}^{\boldsymbol{e}} := \boldsymbol{E} - \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{E})$. The following holds for all $(\boldsymbol{b}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}$:*

$$\big|a_h\big((\tilde{\boldsymbol{\theta}}_h^{\boldsymbol{b}}, \tilde{\boldsymbol{\theta}}_h^{\boldsymbol{e}}), (\boldsymbol{b}_h, \boldsymbol{e}_h)\big)\big| \lesssim \|(\boldsymbol{\xi}^{\boldsymbol{b}}, \boldsymbol{\xi}^{\boldsymbol{e}})\|_{\sharp,h} \|(\boldsymbol{b}_h, \boldsymbol{e}_h)\|_{\flat,h}. \tag{5.14}$$

The second step in the consistency error analysis consists of estimating $T_h - \tilde{T}_h$.

LEMMA 5.8 (bound on $(\tilde{T}_h - T_h)$). *We have* $\lim_{\mathcal{H}\ni h\to 0} \|\tilde{T}_h - T_h\|_{\mathcal{L}(L^{\mathrm{c}};L^{\mathrm{c}})} = 0$.

Remark 5.9 (T_h vs. $\tilde{T}_h$). The fact that $\lim_{\mathcal{H}\ni h\to 0} \|T - \tilde{T}_h\|_{\mathcal{L}(L^{\mathrm{c}},L^{\mathrm{c}})} = 0$ is not sufficient to prove the spectral correctness of the dG approximation using the sesquilinear form $\hat{a}_h$ defined in (5.4). As for the grad-div problem (see Remark 4.10), one needs to prove that $\lim_{\mathcal{H}\ni h\to 0} \|T - T_h\|_{\mathcal{L}(L^{\mathrm{c}},L^{\mathrm{c}})} = 0$.

5.5. Conclusion. We are now ready to state the main result of this section.

THEOREM 5.10 (convergence). *We have* $\lim_{\mathcal{H}\ni h\to 0} \|T - T_h\|_{\mathcal{L}(L^{\mathrm{c}};L^{\mathrm{c}})} = 0$.

Appendix A. Helmholtz decompositions. We recall results characterizing the kernel and the image of the gradient, curl, and divergence operators. These results are mostly drawn from Amrouche et al. [2], Dautray and Lions [14], and Girault and Raviart [19]; see also Fernandes and Gilardi [18] for the case of mixed boundary conditions.

A.1. Topology of $\boldsymbol{D}$. Recall that we assumed that D is an open, bounded, Lipschitz polyhedron of $\mathbb{R}^d$, $d \in \{2,3\}$. We denote Γ_0 the boundary of the only unbounded connected component of $\mathbb{R}^d\backslash\overline{D}$. If ∂D is not connected, i.e., $\partial D \neq \Gamma_0$, we denote $\{\Gamma_i\}_{i\in\{1:I\}}$ the connected components of ∂D that are different from Γ_0 (see, e.g., [19, p. 37], [2, p. 835], [14, p. 217]). If D is not simply connected, we assume that there exist J cuts ($(d-1)$-dimensional smooth manifolds) $\{\Sigma_j\}_{j\in\{1:J\}}$ that make the open set $D^\Sigma := D\backslash\bigcup_{j\in\{1:J\}}\Sigma_j$ simply connected. Additional regularity assumptions on these cuts as stated in [2, Hyp. 3.3, p. 836] are assumed to hold true. For all $q \in L^2(D)$ such that $q|_{D^\Sigma} \in H^1(D^\Sigma)$, we denote by $\nabla_\Sigma q$ the broken gradient of q such that $(\nabla_\Sigma q)(\boldsymbol{x}) = (\nabla q|_{D^\Sigma})(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in D$.

For all $i \in \mathbb{N}$, let c_i denote any real number. We define

$$H_\Gamma^1(D) := \{q \in H^1(D) \mid q|_{\Gamma_0} = 0, q|_{\Gamma_i} = c_i \ \ \forall i \in \{1{:}I\}\}, \tag{A.1a}$$

$$H_\Sigma^1(D) := \{q \in L^2(D) \mid q|_{D^\Sigma} \in H^1(D^\Sigma),\ [\![q]\!]|_{\Sigma_j} = c_j \ \ \forall j \in \{1{:}J\}\}. \tag{A.1b}$$

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We also consider the subspaces

$$\boldsymbol{H}^{\Gamma}(\mathrm{div}=0;D) := \left\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div}=0;D) \mid \int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s=0\ \ \forall i\in\{1{:}I\}\right\}, \tag{A.2a}$$

$$\boldsymbol{H}_0^{\Sigma}(\mathrm{div}=0;D) := \left\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div}=0;D) \mid \int_{\Sigma_j}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s=0\ \ \forall j\in\{1{:}J\}\right\}. \tag{A.2b}$$

The following spaces characterize the topology of D:

$$\boldsymbol{K}_{\mathrm{T}}(D) := \boldsymbol{H}_0(\mathrm{div}=0;D)\cap\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D), \tag{A.3a}$$

$$\boldsymbol{K}_{\mathrm{N}}(D) := \boldsymbol{H}(\mathrm{div}=0;D)\cap\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D). \tag{A.3b}$$

We have $\dim(\boldsymbol{K}_{\mathrm{N}}(D))=I$ [2, Prop. 3.18] and $\dim(\boldsymbol{K}_{\mathrm{T}}(D))=J$ [2, Prop. 3.14].

THEOREM A.1 (orthogonal decompositions, [14, p. 314]). *The following decompositions hold true and are orthogonal in* $\boldsymbol{L}^2(D)$*:*

$$\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) = \nabla H^1(D) \overset{\perp}{\oplus} \boldsymbol{K}_{\mathrm{T}}(D), \tag{A.4a}$$

$$\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) = \nabla_0 H_0^1(D) \overset{\perp}{\oplus} \boldsymbol{K}_{\mathrm{N}}(D), \tag{A.4b}$$

$$\boldsymbol{H}(\mathrm{div}=0;D) = \boldsymbol{H}^{\Gamma}(\mathrm{div}=0;D) \overset{\perp}{\oplus} \boldsymbol{K}_{\mathrm{N}}(D), \tag{A.4c}$$

$$\boldsymbol{H}_0(\mathrm{div}=0;D) = \boldsymbol{H}_0^{\Sigma}(\mathrm{div}=0;D) \overset{\perp}{\oplus} \boldsymbol{K}_{\mathrm{T}}(D), \tag{A.4d}$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) \overset{\perp}{\oplus} \boldsymbol{H}^{\Gamma}(\mathrm{div}=0;D), \tag{A.4e}$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) \overset{\perp}{\oplus} \boldsymbol{H}_0^{\Sigma}(\mathrm{div}=0;D). \tag{A.4f}$$

A.2. Helmholtz decompositions.

THEOREM A.2 (decompositions for grad-div problem). *The following decompositions hold true and are orthogonal in* $L^2(D)$ *and* $\boldsymbol{L}^2(D)$*, respectively:*

$$L^2(D) = \{0\} \overset{\perp}{\oplus} \nabla\cdot\boldsymbol{H}(\mathrm{div};D), \tag{A.5a}$$

$$L^2(D) = \mathbb{P}_0 \overset{\perp}{\oplus} \nabla_0\cdot\boldsymbol{H}_0(\mathrm{div};D), \tag{A.5b}$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}_0(\mathrm{div}=0;D) \overset{\perp}{\oplus} \nabla H^1(D), \tag{A.5c}$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}(\mathrm{div}=0;D) \overset{\perp}{\oplus} \nabla_0 H_0^1(D). \tag{A.5d}$$

THEOREM A.3 (decompositions for curl-curl problem). *The following decompositions hold true and are orthogonal in* $\boldsymbol{L}^2(D)$*:*

$$\boldsymbol{L}^2(D) = \nabla H_{\Gamma}^1(D) \overset{\perp}{\oplus} \nabla\times\boldsymbol{H}(\mathbf{curl};D), \tag{A.6a}$$

$$\boldsymbol{L}^2(D) = \nabla_{\Sigma} H_{\Sigma}^1(D) \overset{\perp}{\oplus} \nabla_0\times\boldsymbol{H}_0(\mathbf{curl};D). \tag{A.6b}$$

Remark A.4 (uniqueness of decomposition). The potentials in the Helmholtz decompositions from Theorems A.2 and A.3 can be made unique:

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$$L^2(D) = \{0\} \overset{\perp}{\oplus} \nabla\cdot(\boldsymbol{H}(\mathrm{div}; D) \cap \boldsymbol{H}(\mathrm{div} = 0; D)^{\perp}),$$

$$L^2(D) = \mathbb{P}_0 \overset{\perp}{\oplus} \nabla_0\cdot(\boldsymbol{H}_0(\mathrm{div}; D) \cap \boldsymbol{H}_0(\mathrm{div} = 0; D)^{\perp}),$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}_0(\mathrm{div} = 0; D) \overset{\perp}{\oplus} \nabla(H^1(D) \cap \mathbb{P}_0^{\perp}),$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}(\mathrm{div} = 0; D) \overset{\perp}{\oplus} \nabla_0 H_0^1(D),$$

$$\boldsymbol{L}^2(D) = \nabla(H_\Gamma^1(D) \cap \mathbb{P}_0^{\perp}) \overset{\perp}{\oplus} \nabla\times(\boldsymbol{H}(\mathbf{curl}; D) \cap \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^{\perp}),$$

$$\boldsymbol{L}^2(D) = \nabla_\Sigma(H_\Sigma^1(D) \cap \mathbb{P}_0^{\perp}) \overset{\perp}{\oplus} \nabla_0\times(\boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}).$$

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