MATHEMATICS OF COMPUTATION
Volume 95, Number 360, July 2026, Pages 1573–1602
https://doi.org/10.1090/mcom/4091
Article electronically published on May 28, 2025

A PRIORI AND A POSTERIORI ANALYSIS OF THE DISCONTINUOUS GALERKIN APPROXIMATION OF THE TIME-HARMONIC MAXWELL'S EQUATIONS UNDER MINIMAL REGULARITY ASSUMPTIONS

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Abstract. We derive a priori and a posteriori error estimates for the discontinuous Galerkin (dG) approximation of the time-harmonic Maxwell's equations. Specifically, we consider an interior penalty dG method, and establish error estimates that are valid under minimal regularity assumptions and involving constants that do not depend on the frequency for sufficiently fine meshes. The key result of our a priori error analysis is that the dG solution is asymptotically optimal in an augmented energy norm that contains the dG stabilization. Specifically, up to a constant that tends to one as the mesh is refined, the dG solution is as accurate as the best approximation in the same norm. The main insight is that the quantities controlling the smallness of the mesh size are essentially those already appearing in the conforming setting. We also show that for fine meshes, the inf-sup stability constant is as good as the continuous one up to a factor two. Concerning the a posteriori analysis, we consider a residual-based error estimator under the assumption of piecewise constant material properties on a fixed partition to which the mesh is conforming. We derive a global upper bound and local lower bounds on the error with constants that (i) only depend on the shape-regularity of the mesh if it is sufficiently refined and (ii) are independent of the stabilization bilinear form.

1. Introduction

Let $D \subset \mathbb{R}^d$, $d = 3$, be an open, bounded, Lipschitz polyhedron with boundary ∂D and outward unit normal $\boldsymbol{n}_D$. We do not make any simplifying assumption on the topology of D. We use boldface fonts for vectors, vector fields, and functional spaces composed of such fields. More details on the notation are given in Section 2.

Given a positive real number $\omega > 0$ representing a frequency and a source term $\boldsymbol{J} : D \to \mathbb{R}^3$, and focusing for simplicity on homogeneous Dirichlet boundary conditions (a.k.a. perfect electric conductor boundary conditions), the model problem consists in finding $\boldsymbol{E} : D \to \mathbb{R}^3$ such that

$$-\omega^2 \boldsymbol{\epsilon} \boldsymbol{E} + \nabla\times(\boldsymbol{\mu}^{-1}\nabla\times\boldsymbol{E}) = \boldsymbol{J} \quad \text{in } D, \tag{1.1a}$$

$$\boldsymbol{E}\times\boldsymbol{n}_D = \boldsymbol{0} \quad \text{on } \partial D, \tag{1.1b}$$

where $\boldsymbol{\epsilon}$ represents the electric permittivity of the materials contained in D and $\boldsymbol{\mu}$ their magnetic permeability. Both material properties can vary in D and take

Received by the editor May 28, 2024, and, in revised form, December 16, 2024, and March 18, 2025.

2020 *Mathematics Subject Classification.* Primary 65N30, 78M10, 65N15.

Key words and phrases. Time-harmonic Maxwell's equations, discontinuous Galerkin, interior penalty, duality argument, asymptotic optimality, a posteriori error analysis.

symmetric positive-definite values with eigenvalues uniformly bounded from above and from below away from zero. We assume that ω is not a resonant frequency, so that (1.1) is uniquely solvable in $\boldsymbol{H}_0(\mathbf{curl}; D)$ for every $\boldsymbol{J}$ in the topological dual space $\boldsymbol{H}_0(\mathbf{curl}; D)'$. The time-harmonic Maxwell's equations (1.1a) and (1.1b) are one of the central models of electrodynamics. Therefore, efficient discretizations are a cornerstone for the computational modelling of electromagnetic wave propagation [24, 35]. In this work, we focus on the discontinuous Galerkin (dG) method.

DG methods employ approximation spaces composed of nonconforming (discontinuous, broken) polynomials on the mesh. They are attractive since they easily allow for more flexibility in the mesh and for local variations of the polynomial degree. DG methods exist in many flavors. One popular approach for the Poisson model problem is the interior penalty dG method, which hinges on a consistency term involving the mean-value of the normal flux at the mesh faces, possibly a symmetry term, and a stabilization term penalizing the jumps across the mesh interfaces and the values at the mesh boundary faces (see, e.g., [2, 18] and the references therein). The expression of the consistency term involving the mean-value of the normal flux is convenient for efficient implementation, but for the analysis, it is useful to consider an (equivalent) reformulation involving jump liftings. The approach was first considered in [4] and analyzed in [6]. One important outcome is the notion of discrete gradient obtained by adding the jump liftings to the broken (piecewise) gradient. Indeed, the discrete gradient enjoys a compactness property that plays a central role in various nonlinear problems [7, 9, 17]. Another attractive feature is that the discrete gradient admits a bounded extension to H^1, whereas the standard consistency term can only be extended to H^{1+s} with $s > \frac{1}{2}$. It is also possible to define bounded extensions of the consistency term to H^{1+s} for $s > 0$ arbitrarily small by proceeding as in [22].

In the context of the time-harmonic Maxwell's equations, interior penalty dG methods were devised and analyzed in [26, 37], and the notion of discrete curl, obtained by adding the liftings of the tangential jumps to the broken curl, has been considered in the method formulation and analysis. However, the combined use of discrete curls (allowing for minimal regularity requirements) with Schatz's duality argument seems to be lacking in the literature, contrary to the case of the Helmholtz equation where such a result has been recently achieved in [10]. Our first main contribution is to fill this gap. Indeed, we show that an *asymptotically optimal* error estimate holds true with an augmented energy norm including a nonconformity measure. Asymptotic optimality means that the ratio between the approximation error and the best-approximation error tends to one as the mesh size h is sent to zero. An important aspect in our analysis is that the (frequency dependent) quantities controlling the smallness of the mesh size are essentially those already appearing in the conforming setting. Our second main contribution is to establish asymptotic optimality under minimal regularity assumptions. Specifically, we assume that the source term sits in $\boldsymbol{L}^2(D)$ (rather than in the dual space $\boldsymbol{H}_0(\mathbf{curl}; D)'$) with no further assumption on $\nabla\cdot\boldsymbol{J}$ and that the material coefficients are bounded from above and from below away from zero, with no further regularity assumption on the exact solution. The third main contribution of the paper is an a posteriori error analysis in the present indefinite setting and using, for the first time, a duality argument. We establish global upper bounds and local lower bounds on the error, where the constants are independent of the frequency, again in the limit as the mesh is refined. An important insight is, once again, that the behavior of the constants is the same as for a conforming approximation. The a posteriori error analysis is of residual-type and requires to tighten slightly the assumption on the source term so that $\nabla\cdot\boldsymbol{J} \in L^2(D)$, and we assume piecewise constant material properties on a fixed partition to which the mesh is conforming.

Let us put our results in perspective with the literature. Concerning the a priori error analysis, a quasi-optimal (but not asymptotically optimal) error estimate under minimal regularity is derived in [31], but for a different interior penalty dG method, where a Lagrange multiplier related to the divergence constraint is introduced together with the corresponding stabilization term. A similar result is achieved in [13] using a hydridizable dG (HDG) method for $\boldsymbol{E}$ and a continuous finite element approximation for the Lagrange multiplier. Moreover, asymptotically optimal error estimates for the time-harmonic Maxwell's equations approximated using conforming edge elements have been derived quite recently in [33] and in [11]. The analysis in [33] considers impedance boundary conditions (allowing for an explicit frequency analysis), but requires the domain boundary to be smooth and connected. Instead, the analysis in [11] considers Dirichlet boundary conditions and allows for general domains, material coefficients, and right-hand side. The present analysis leverages the ideas developed in [11], but needs to address two additional, nontrivial difficulties: (i) the lack of strong consistency under minimal regularity, leading to the appearance of new terms in the analysis related to the consistency defect; (ii) the nonconforming nature of the approximation, which calls for a careful handling of the stabilization. In particular, we notice that we allow for a rather general stabilization bilinear form and provide explicit design assumptions for the analysis to hold true.

Concerning the a posteriori error analysis, we leverage the ideas proposed in [12] for the conforming edge finite element approximation of the time-harmonic Maxwell's equations. Here, the novelty is twofold. First, we additionally deal with the consistency defect and the presence of stabilization in the discontinuous Galerkin setting, by extending ideas introduced in [10] for the Helmholtz equation. Moreover, we tighten some arguments from [12] in the proof of the error upper bound so that the involved constants only depend on the shape-regularity parameter of the mesh, whereas in [12] some constants are frequency-dependent in the low-frequency regime. Specifically, instead of invoking the regular decomposition results from [25, Theorem 2.1] as in [12], we make use of Galerkin orthogonality on conforming test functions to invoke the regular decomposition results from [39, Theorem 1]. Finally, we observe that the general form of the error indicators is the same as the one derived in [27] in the positive definite setting.

The paper is organized as follows. In Section 2, we briefly present the continuous setting, and in Section 3, we do the same for the discrete setting. In particular, we introduce various (nondimensional) approximation and divergence-conformity factors to be used in the analysis. These factors are important to support our claim that the smallness condition on the mesh size made in the error analysis essentially behaves (in terms of frequency) as the corresponding condition for the conforming approximation. In Section 4, we introduce the dG approximation in a rather general setting and show that the setting covers, in particular, the well-known interior penalty approach. Moreover, we establish in Lemma 4.4 a key estimate on the weak consistency of the dG approximation. In Section 5, we deal with the a priori error analysis and inf-sup stability. The main results in this section are Theorem 5.5 and Theorem 5.7. In Section 6, we perform the a posteriori residual-based error analysis. The main results in this section are Theorem 6.4 and Theorem 6.5. Finally, in Section 7, we establish bounds on the approximation and divergence conformity factors. These bounds prove that these factors tend to zero with the mesh size.

2. Continuous setting

In this section, we briefly recall the functional setting for the time-harmonic Maxwell's equations and formulate the model problem.

2.1. **Functional spaces.** We use standard notation for Lebesgue and Sobolev spaces. To alleviate the notation, the inner product and associated norm in the spaces $L^2(D)$ and $\boldsymbol{L}^2(D)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The material properties $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu} := \boldsymbol{\mu}^{-1}$ are measurable functions that take symmetric positive-definite values in D with eigenvalues uniformly bounded from above and from below away from zero. It is convenient to introduce the inner product and associated norm weighted by either $\boldsymbol{\epsilon}$ or $\boldsymbol{\nu}$, leading to the notation $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$, $\|\cdot\|_{\boldsymbol{\epsilon}}$, $(\cdot,\cdot)_{\boldsymbol{\nu}}$ and $\|\cdot\|_{\boldsymbol{\nu}}$. Whenever no confusion can arise, we use the symbol $^\perp$ to denote orthogonality with respect to the inner product $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$. Moreover, all the projection operators denoted using the symbol $\boldsymbol{\Pi}$ are meant to be orthogonal with respect to this inner product; we say that the projections are $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal.

We consider the Hilbert Sobolev spaces

$$\boldsymbol{H}(\mathbf{curl};D) := \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla{\times}\boldsymbol{v} \in \boldsymbol{L}^2(D)\}, \tag{2.1a}$$

$$\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};D) \mid \nabla{\times}\boldsymbol{v} = \mathbf{0}\}, \tag{2.1b}$$

$$\boldsymbol{H}_0(\mathbf{curl};D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};D) \mid \gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{v}) = \mathbf{0}\}, \tag{2.1c}$$

$$\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};D) \mid \nabla_0{\times}\boldsymbol{v} = \mathbf{0}\}, \tag{2.1d}$$

where the tangential trace operator $\gamma^{\mathrm{c}}_{\partial D} : \boldsymbol{H}(\mathbf{curl};D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ is the extension by density of the tangent trace operator such that $\gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{v}) = \boldsymbol{v}|_{\partial D}{\times}\boldsymbol{n}_D$ for smooth fields. The subscript $_0$ indicates the curl operator acting on fields respecting homogeneous Dirichlet conditions. Notice that $\nabla\times$ and $\nabla_0\times$ are adjoint to each other, i.e., $(\nabla_0{\times}\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v},\nabla{\times}\boldsymbol{w})$ for all $(\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{H}_0(\mathbf{curl};D) \times \boldsymbol{H}(\mathbf{curl};D)$. We equip the space $\boldsymbol{H}(\mathbf{curl};D)$ and its subspaces defined in (2.1) with the following (dimensionally consistent) energy norm:

$$|\!|\!|\boldsymbol{v}|\!|\!|^2 := \omega^2\|\boldsymbol{v}\|^2_{\boldsymbol{\epsilon}} + \|\nabla{\times}\boldsymbol{v}\|^2_{\boldsymbol{\nu}}. \tag{2.2}$$

We consider the subspace

$$\boldsymbol{X}^{\mathrm{c}}_0 := \boldsymbol{H}_0(\mathbf{curl};D) \cap \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp, \tag{2.3}$$

and we introduce the $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal projection

$$\boldsymbol{\Pi}^{\mathrm{c}}_0 : \boldsymbol{L}^2(D) \to \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D). \tag{2.4}$$

Since $\nabla H^1_0(D) \subset \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$, any field $\boldsymbol{\xi} \in \boldsymbol{X}^{\mathrm{c}}_0$ is such that $\nabla{\cdot}(\boldsymbol{\epsilon}\boldsymbol{\xi}) = 0$ in D. Hence, $\boldsymbol{X}^{\mathrm{c}}_0$ compactly embeds into $\boldsymbol{L}^2(D)$ [40].

Remark 2.1 (Topology of D). We have $\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp \subset \{\boldsymbol{v} \in \boldsymbol{L}^2(D),\ \nabla{\cdot}(\boldsymbol{\epsilon}\boldsymbol{v}) = 0\}$ with equality if only if ∂D is connected (see, e.g., [1]).

2.2. **Model problem.** Given a positive real number $\omega > 0$ and a source term $\boldsymbol{J} \in (\boldsymbol{H}_0(\mathbf{curl}; D))'$ (the topological dual space of $\boldsymbol{H}_0(\mathbf{curl}; D)$), the model problem amounts to finding $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ such that

$$b(\boldsymbol{E}, \boldsymbol{w}) = \langle \boldsymbol{J}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D), \tag{2.5}$$

with the bilinear form defined on $\boldsymbol{H}_0(\mathbf{curl}; D) \times \boldsymbol{H}_0(\mathbf{curl}; D)$ such that

$$b(\boldsymbol{v}, \boldsymbol{w}) := -\omega^2 (\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{\epsilon}} + (\nabla_0 \times \boldsymbol{v}, \nabla_0 \times \boldsymbol{w})_{\boldsymbol{\nu}}, \tag{2.6}$$

and where the brackets on the right-hand side of (2.5) denote the duality product between $(\boldsymbol{H}_0(\mathbf{curl}; D))'$ and $\boldsymbol{H}_0(\mathbf{curl}; D)$. In what follows, we assume that ω^2 is not an eigenvalue of the $\boldsymbol{\epsilon}^{-1} \nabla \times (\boldsymbol{\nu} \nabla_0 \times \cdot)$ operator in D. As a result, the model problem (2.5) is well-posed. We observe that the bilinear form b satisfies $|b(\boldsymbol{v}, \boldsymbol{w})| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|$. The following inf-sup stability result is established in [11, Lemma 2].

Lemma 2.2 (Inf-sup stability)**.** *The following holds:*

$$\frac{1}{1 + 2\beta_{\mathrm{st}}} \leq \inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D) \\ \|\boldsymbol{v}\| = 1}} \sup_{\substack{\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D) \\ \|\boldsymbol{w}\| = 1}} |b(\boldsymbol{v}, \boldsymbol{w})| \leq \frac{1}{\beta_{\mathrm{st}}}, \tag{2.7}$$

with the (nondimensional) stability constant

$$\beta_{\mathrm{st}} := \sup_{\substack{\boldsymbol{g} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp} \\ \|\boldsymbol{g}\|_{\boldsymbol{\epsilon}} = 1}} \omega \|\boldsymbol{v}_{\boldsymbol{g}}\|. \tag{2.8}$$

Here, for all $\boldsymbol{g} \in \boldsymbol{L}^2(D)$, $\boldsymbol{v}_{\boldsymbol{g}} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ denotes the unique solution to (2.5) with right-hand side $(\boldsymbol{g}, \boldsymbol{w})_{\boldsymbol{\epsilon}}$, i.e., $b(\boldsymbol{v}_{\boldsymbol{g}}, \boldsymbol{w}) = (\boldsymbol{g}, \boldsymbol{w})_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D)$.

3. Discrete setting

In this section, we introduce the discrete setting and various important tools, such as the discrete curl operator, two approximation norms and a nonconformity measure, and some important approximation and divergence conformity factors. All these tools are of broader interest than the dG method presented in the next section; they can indeed be applied to analyze other nonconforming approximation methods.

3.1. **Mesh and polynomial spaces.** Let $\mathcal{T}_h$ be an affine simplicial mesh covering D exactly. A generic mesh cell is denoted K, its diameter h_K and its outward unit normal $\boldsymbol{n}_K$. We define the piecewise constant functions $\tilde{h}$ and $\tilde{\nu}$ such that

$$\tilde{h}|_K := h_K, \qquad \tilde{\nu}|_K := \min_{\substack{\boldsymbol{u} \in \mathbb{R}^d \\ |\boldsymbol{u}| = 1}} \boldsymbol{\nu}|_K \boldsymbol{u} \cdot \boldsymbol{u}, \qquad \forall K \in \mathcal{T}_h. \tag{3.1}$$

We write $\mathcal{F}_h$ for the set of mesh faces, $\mathcal{F}_h^{\circ}$ for the subset of mesh interfaces (shared by two distinct mesh cells, K_l, K_r), and $\mathcal{F}_h^{\partial}$ for the subset of mesh boundary faces (shared by one mesh cell, K_l, and the boundary, ∂D). Every mesh interface $F \in \mathcal{F}_h^{\circ}$ is oriented by the unit normal, $\boldsymbol{n}_F$, pointing from K_l to K_r (the orientation is arbitrary, but fixed). Every boundary face $F \in \mathcal{F}_h^{\partial}$ is oriented by the unit normal $\boldsymbol{n}_F := \boldsymbol{n}_D|_F$. For all $K \in \mathcal{T}_h$, $\mathcal{F}_K$ is the collection of the mesh faces composing ∂K.

Let $k \geq 1$ be the polynomial degree. Let $\mathbb{P}_{k,d}$ be the space composed of d-variate polynomials of total degree at most k and set $\mathbf{P}_{k,d} := [\mathbb{P}_{k,d}]^d$. The dG approximation hinges on the following broken polynomial space:

$$\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^2(D) \mid \boldsymbol{v}_h|_K \in \mathbf{P}_{k,d}, \, \forall K \in \mathcal{T}_h\}. \tag{3.2}$$

Moreover, the error analysis makes use of the following subspaces:

$$\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}; D), \tag{3.3a}$$

$$\boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}; D), \tag{3.3b}$$

$$\boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D). \tag{3.3c}$$

The superscript $^{\mathrm{c}}$ in the above subspaces is meant to remind us that all these subspaces are $\boldsymbol{H}(\mathbf{curl}; D)$-conforming. The $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal projection

$$\boldsymbol{\Pi}_{h0}^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \tag{3.4}$$

plays a key role in what follows. In particular, we introduce the subspace

$$\boldsymbol{X}_h^{\mathrm{b}} := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^{\perp}, \tag{3.5}$$

which is composed of fields $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ such that $\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h) = \mathbf{0}$.

3.2. **Jumps and discrete curl operator.** For all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_K$, and all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we define the local trace operators such that $\gamma_{K,F}^{\mathrm{g}}(\boldsymbol{v}_h)(\boldsymbol{x}) := \boldsymbol{v}_h|_K(\boldsymbol{x})$, $\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{v}_h)(\boldsymbol{x}) := \boldsymbol{v}_h|_K(\boldsymbol{x}) \times \boldsymbol{n}_F$, for a.e. $\boldsymbol{x} \in F$. Then, for all $F \in \mathcal{F}_h^{\circ}$ and $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}\}$, we define the jump and average operators such that

$$[\![\boldsymbol{v}_h]\!]_F^{\mathrm{x}} := \gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{v}_h) - \gamma_{K_r,F}^{\mathrm{x}}(\boldsymbol{v}_h), \quad \{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{x}} := \frac{1}{2}\big(\gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{v}_h) + \gamma_{K_r,F}^{\mathrm{x}}(\boldsymbol{v}_h)\big). \tag{3.6}$$

We also set $[\![\boldsymbol{v}_h]\!]_F^{\mathrm{x}} := \{\!\{\boldsymbol{v}_h\}\!\}_F^{\mathrm{x}} := \gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{v}_h)$ for all $F \in \mathcal{F}_h^{\partial}$.

For every field $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, $\nabla_h \times \boldsymbol{v}_h$ denotes the broken curl of $\boldsymbol{v}_h$ (evaluated cellwise). Let $\ell \geq k-1 \geq 0$. We define the discrete curl operator $\boldsymbol{C}_{h,0}^{k,\ell} : \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{P}_\ell^{\mathrm{b}}(\mathcal{T}_h)$ such that, for all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h) := \nabla_h \times \boldsymbol{v}_h + \boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{v}_h), \tag{3.7}$$

where the jump lifting operator $\boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{v}_h) \in \boldsymbol{P}_\ell^{\mathrm{b}}(\mathcal{T}_h)$ is defined by requiring that

$$(\boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{v}_h), \boldsymbol{\phi}_h) := \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{\phi}_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)} \tag{3.8}$$

for all $\boldsymbol{\phi}_h \in \boldsymbol{P}_\ell^{\mathrm{b}}(\mathcal{T}_h)$. Taking the polynomial degree ℓ larger than $k-1$ is useful to improve the consistency property of the discrete curl operator; see Lemma 4.4.

It is convenient to introduce the infinite-dimensional space

$$\boldsymbol{V}_\sharp := \boldsymbol{H}_0(\mathbf{curl}; D) + \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h), \tag{3.9}$$

where the error $(\boldsymbol{E} - \boldsymbol{E}_h)$ lives. Although the sum in (3.9) is not direct, any field $\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ satisfies $[\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} = \mathbf{0}$ for all $F \in \mathcal{F}_h$, as well as $\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h) = \nabla_0 \times \boldsymbol{v}_h$. It is therefore legitimate to extend the curl and jump operators to $\boldsymbol{V}_\sharp$ by setting, for all $\boldsymbol{v} = \tilde{\boldsymbol{v}} + \boldsymbol{v}_h \in \boldsymbol{V}_\sharp$ with $\tilde{\boldsymbol{v}} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ and $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}) := \nabla_0 \times \tilde{\boldsymbol{v}} + \nabla_h \times \boldsymbol{v}_h + \boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{v}_h), \qquad [\![\boldsymbol{v}]\!]_F^{\mathrm{c}} := [\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}. \tag{3.10}$$

3.3. **Approximation norms and nonconformity measure.** We consider the following two norms:

$$\|\!|\boldsymbol{v}_h|\!\|_{\mathrm{ap}}^2 := \|\tilde{\nu}^{\frac{1}{2}}\tilde{h}^{-1}\boldsymbol{v}_h\|^2 + \|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h)\|_{\boldsymbol{\nu}}^2 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h), \tag{3.11a}$$

$$\|\!|\boldsymbol{v}|\!\|_{\mathrm{ap}*}^2 := \|\boldsymbol{v}\|_{\boldsymbol{\nu}^{-1}}^2 + \|\tilde{\nu}^{-\frac{1}{2}}\tilde{h}\nabla\times\boldsymbol{v}\|^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; D). \tag{3.11b}$$

(Recall that $\tilde{h}$ and $\tilde{\nu}$ are defined in (3.1).) We introduce the following nonconformity measure:

$$|\boldsymbol{v}|_{\mathrm{nc}} := \min_{\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)} \|\!|\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}}|\!\|_{\mathrm{ap}} \qquad \forall \boldsymbol{v} := \tilde{\boldsymbol{v}} + \boldsymbol{v}_h \in \boldsymbol{V}_\sharp. \tag{3.12}$$

Notice that the definition (3.12) is independent of the decomposition $\boldsymbol{v} := \tilde{\boldsymbol{v}} + \boldsymbol{v}_h$. The $\|\!|\cdot|\!\|_{\mathrm{ap}}$-norm is used to measure the nonconformity in (3.12), whereas the $\|\!|\cdot|\!\|_{\mathrm{ap}*}$-norm is used in the next section to estimate the approximability properties of some dual solution.

3.4. **Approximation and divergence conformity factors.** Here, we introduce three factors to be used in the error analysis. We prove in Section 7 that these factors tend to zero (possibly with a certain rate) as the mesh size tends to zero.

For all $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$, we consider the adjoint problem consisting of finding $\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ such that

$$b(\boldsymbol{w}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = (\boldsymbol{w}, \boldsymbol{\theta})_{\boldsymbol{\epsilon}} \qquad \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D). \tag{3.13}$$

Taking any test function $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D) \subset \boldsymbol{H}_0(\mathbf{curl}; D)$ shows that

$$\omega^2(\boldsymbol{w}, \boldsymbol{\zeta}_{\boldsymbol{\theta}})_{\boldsymbol{\epsilon}} = b(\boldsymbol{w}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = (\boldsymbol{w}, \boldsymbol{\theta})_{\boldsymbol{\epsilon}} = 0, \tag{3.14}$$

where the first equality follows from $\nabla_0\times\boldsymbol{w} = \mathbf{0}$, the second from the definition of the adjoint solution, and the third from the assumption $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$. Since $\boldsymbol{w}$ is arbitrary in $\boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)$, this proves that $\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$. Thus, $\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{X}_0^{\mathrm{c}}$. We introduce the (nondimensional) approximation factors

$$\gamma_{\mathrm{ap}} := \sup_{\substack{\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^\perp \\ \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}} = 1}} \min_{\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)} \omega \|\!|\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{v}_h^{\mathrm{c}}|\!\|, \tag{3.15a}$$

$$\gamma_{\mathrm{ap}*} := \sup_{\substack{\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)^\perp \\ \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}} = 1}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}} \in \boldsymbol{P}_{\ell}^{\mathrm{c}}(\mathcal{T}_h)} \omega \|\!|\boldsymbol{\nu}\nabla_0\times\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{\Phi}_h^{\mathrm{c}}|\!\|_{\mathrm{ap}*}. \tag{3.15b}$$

The approximation factor γ_{ap} uses the triple norm $\|\!|\cdot|\!\|$ defined in (2.2), whereas $\gamma_{\mathrm{ap}*}$ uses the norm $\|\!|\cdot|\!\|_{\mathrm{ap}*}$ defined in (3.11b). Finally, we introduce the (nondimensional) divergence conformity factor

$$\gamma_{\mathrm{dv}} := \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{X}_h^{\mathrm{b}} \\ \|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h)\|_{\boldsymbol{\nu}}^2 + |\boldsymbol{v}_h|_{\mathrm{nc}}^2 = 1}} \omega \|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h)\|_{\boldsymbol{\epsilon}}. \tag{3.16}$$

Loosely speaking, γ_{dv} measures how much discretely divergence-free fields depart from being exactly divergence-free.

4. Discontinuous Galerkin approximation

In this section, we formulate the dG approximation of the model problem (2.5) in a rather general setting and show that the interior penalty dG method fits the proposed framework. We then examine the Galerkin orthogonality and weak consistency property of the proposed dG method. We assume from now on that $\boldsymbol{J} \in \boldsymbol{L}^2(D)$; notice though that we do not make any further assumption on $\nabla{\cdot}\boldsymbol{J}$ for the a priori error analysis. Moreover, the sole assumption on the material properties is uniform boundedness from above and from below away from zero. The main novel result in this section is Lemma 4.4 which leads to a weak consistency property of the dG method without requiring any additional regularity property on the exact solution.

4.1. **Stabilization and extended bilinear form.**

We consider a stabilization bilinear form $s_\sharp$ defined on $\boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp$ for which we make the following assumptions:

$$\text{(i) } s_\sharp \text{ is symmetric positive semidefinite,} \tag{4.1a}$$

$$\text{(ii) } s_\sharp(\boldsymbol{v},\cdot) = s_\sharp(\cdot,\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D). \tag{4.1b}$$

(Notice that the first equality in (4.1b) follows from (4.1a)). We also assume that

$$\exists \rho > 0 \quad \text{s.t.} \quad \rho |\boldsymbol{v}_h|_{\mathrm{nc}} \leq s_\sharp(\boldsymbol{v}_h,\boldsymbol{v}_h)^{\frac{1}{2}} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.2}$$

where the constant ρ is independent of the mesh size and the frequency. The value of ρ can depend on the mesh shape-regularity and the polynomial degree. This assumption is needed only for the a priori error analysis, but not for the a posteriori analysis. Furthermore, we notice that although the converse bound $\tilde{\rho}|\boldsymbol{v}_h|_{\mathrm{s}} \leq |\boldsymbol{v}_h|_{\mathrm{nc}}$ for some $\tilde{\rho} > 0$ is not required anywhere in the analysis, it is reasonable to assume it to avoid ill-conditioned linear systems. Finally, we notice that, as usual in dG methods, the stabilization bilinear form $s_\sharp$ is not bounded in the $\boldsymbol{H}(\mathbf{curl}; D)$-norm uniformly with respect to the mesh size. Its role is to deal with the nonconformity of the discretization when handling the curl operator by enforcing some penalty on the tangential jumps of discrete fields.

We define the bilinear forms $b_\sharp$ and $b_{\sharp\mathrm{s}}$ on $\boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp$ such that

$$b_\sharp(\boldsymbol{v},\boldsymbol{w}) := -\omega^2(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{\epsilon}} + (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}), \boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{w}))_{\boldsymbol{\nu}}, \tag{4.3a}$$

$$b_{\sharp\mathrm{s}}(\boldsymbol{v},\boldsymbol{w}) := b_\sharp(\boldsymbol{v},\boldsymbol{w}) + s_\sharp(\boldsymbol{v},\boldsymbol{w}). \tag{4.3b}$$

The bilinear form $b_{\sharp\mathrm{s}}$ is used to define the discrete problem and perform the a priori error analysis; the bilinear form $b_\sharp$ is useful in the a posteriori error analysis. Owing to (3.10) and (4.1b), we have the following (minimal) consistency property:

$$b_{\sharp\mathrm{s}}(\boldsymbol{v},\boldsymbol{w}) = b_\sharp(\boldsymbol{v},\boldsymbol{w}) = b(\boldsymbol{v},\boldsymbol{w}) \qquad \forall \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D). \tag{4.4}$$

We extend the $\|\cdot\|$-norm defined in (2.2) to $\boldsymbol{V}_\sharp$ by setting, for all $\boldsymbol{v} \in \boldsymbol{V}_\sharp$,

$$\|\boldsymbol{v}\|_\sharp^2 := \omega^2 \|\boldsymbol{v}\|_{\boldsymbol{\epsilon}}^2 + \|\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v})\|_{\boldsymbol{\nu}}^2, \tag{4.5a}$$

$$\|\boldsymbol{v}\|_{\sharp\mathrm{s}}^2 := \|\boldsymbol{v}\|_\sharp^2 + |\boldsymbol{v}|_{\mathrm{s}}^2, \qquad |\boldsymbol{v}|_{\mathrm{s}}^2 := s_\sharp(\boldsymbol{v},\boldsymbol{v}). \tag{4.5b}$$

(The definition of $|\boldsymbol{v}|_{\mathrm{s}}$ is legitimate owing to (4.1a).) This leads to the following boundedness properties on the bilinear form $b_{\sharp\mathrm{s}}$:

$$|b_{\sharp\mathrm{s}}(\boldsymbol{v},\boldsymbol{w})| \leq \|\boldsymbol{v}\|_{\sharp\mathrm{s}} \, \|\boldsymbol{w}\|_{\sharp\mathrm{s}} \qquad \forall (\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp, \tag{4.6a}$$

$$|b_{\sharp\mathrm{s}}(\boldsymbol{v},\boldsymbol{w})| \leq \|\boldsymbol{v}\|_\sharp \, \|\boldsymbol{w}\| \qquad \forall (\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{V}_\sharp \times \boldsymbol{H}_0(\mathbf{curl}; D). \tag{4.6b}$$

4.2. **Discrete problem.** The discrete problem reads as follows: Find $\boldsymbol{E}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ such that

$$b_{\sharp\mathrm{s}}(\boldsymbol{E}_h, \boldsymbol{w}_h) = (\boldsymbol{J}, \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} \qquad \forall \boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{4.7}$$

Notice that we use here the assumption that $\boldsymbol{J} \in \boldsymbol{L}^2(D)$.

One simple, yet important, observation is that Galerkin orthogonality holds true whenever the discrete test functions are required to be $\boldsymbol{H}_0(\mathbf{curl}; D)$-conforming.

Lemma 4.1 (Galerkin orthogonality on conforming test functions)**.** *If $\boldsymbol{E}_h$ solves* (4.7)*, the following holds true:*

$$b_{\sharp\mathrm{s}}(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{v}_h^{\mathrm{c}}) = 0 \qquad \forall \boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h). \tag{4.8}$$

In particular, we have

$$\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{E} - \boldsymbol{E}_h) = \boldsymbol{0}. \tag{4.9}$$

Proof. The property (4.8) follows from the definition of $b_{\sharp\mathrm{s}}$ which implies that $b_{\sharp\mathrm{s}}(\boldsymbol{E}, \boldsymbol{v}_h^{\mathrm{c}}) = b(\boldsymbol{E}, \boldsymbol{v}_h^{\mathrm{c}}) = (\boldsymbol{J}, \boldsymbol{v}_h^{\mathrm{c}})$ for all $\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$. Moreover, since $\boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h) \subset \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$, (4.8) implies that $(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} = 0$ for all $\boldsymbol{w}_h \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h)$, which proves (4.9). □

4.3. **Example: Interior penalty discontinuous Galerkin method.** The classical interior penalty dG formulation for the model problem (2.5) is based upon the following discrete bilinear form [37]: For all $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$\begin{aligned} b_h(\boldsymbol{v}_h, \boldsymbol{w}_h) := &-\omega^2(\boldsymbol{v}_h, \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} + (\nabla_h \times \boldsymbol{v}_h, \nabla_h \times \boldsymbol{w}_h)_{\boldsymbol{\nu}} + \eta_* s_h(\boldsymbol{v}_h, \boldsymbol{w}_h) \\ &+ \sum_{F \in \mathcal{F}_h} \left\{ (\{\!\{\boldsymbol{\nu} \nabla_h \times \boldsymbol{v}_h\}\!\}_F^{\mathrm{g}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} + ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{\nu} \nabla_h \times \boldsymbol{w}_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)} \right\}, \end{aligned} \tag{4.10}$$

with the stabilization bilinear form

$$s_h(\boldsymbol{v}_h, \boldsymbol{w}_h) := \sum_{F \in \mathcal{F}_h} \frac{\tilde{\nu}_F}{h_F} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}, \tag{4.11}$$

and the user-dependent parameter $\eta_* > 0$ is to be taken large enough. In (4.11), h_F denotes the diameter of $F \in \mathcal{F}_h$ and $\tilde{\nu}_F := \max_{K \in \mathcal{T}_F} \tilde{\nu}_K$ with $\mathcal{T}_F := \{K \in \mathcal{T}_h \,|\, F \in \mathcal{F}_K\}$. Notice that s_h is readily extended to $\boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp$ using (3.10).

The discrete bilinear form b_h defined in (4.10) can be extended to $\boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp$ using the bilinear form $b_{\sharp\mathrm{s}}$ defined in (4.3b) provided the polynomial degree for the jump lifting satisfies $\ell \geq k - 1 \geq 0$ and provided the material property $\boldsymbol{\nu}$ is piecewise constant on the mesh. In this situation, $b_{\sharp\mathrm{s}}$ is indeed an extension of b_h, i.e., $b_{\sharp\mathrm{s}}|_{\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)} = b_h$, provided the stabilization bilinear form $s_\sharp$ is defined as follows:

$$s_\sharp(\boldsymbol{v}, \boldsymbol{w}) := \eta_* s_h(\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{v}), \boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{w}))_{\boldsymbol{\nu}} \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp. \tag{4.12}$$

(Notice that $s_\sharp$ is not an extension of s_h.) The bilinear form $s_\sharp$ defined in (4.12) trivially satisfies (4.1b) and is symmetric. It is positive semidefinite, i.e., (4.1a) also holds true, if the factor η_* is chosen large enough [36]. The minimal threshold classically depends on the mesh shape-regularity and the polynomial degree ℓ. Finally, it is possible to choose the parameter $\eta_* > 0$ large enough so that (4.2) holds true. To this purpose, one can, for instance, bound $|\boldsymbol{v}_h|_{\mathrm{nc}}$ by taking $\boldsymbol{v}_h^{\mathrm{c}} := \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v}_h)$ defined using the $\boldsymbol{H}_0(\mathbf{curl}; D)$-conforming averaging operator analyzed in [20]. We observe in passing that the value of the parameter η_* is independent of the frequency, as it is only related to the nonconformity in the discretization of the curl operator.

Remark 4.2 (Weighted averages). Whenever the jumps of (the eigenvalues of) $\boldsymbol{\nu}$ are large across the mesh interfaces, it can be useful to consider weighted $\boldsymbol{\nu}$-dependent averages to evaluate the last two terms on the right-hand side of (4.10). We refer the reader, e.g., to [23] for an example in the context of scalar diffusion problems. Such weighted averages can be handled in our framework by modifying the definition of the lifting operator accordingly.

4.4. **Weak consistency.** We now consider the consistency error produced by the discrete curl operator when tested against general fields. For all $\boldsymbol{\Psi} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ such that $\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} \in \boldsymbol{H}(\mathbf{curl}; D)$ and for all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we define the weak consistency error on the discrete curl as

$$\delta_{\mathrm{wkc}}(\boldsymbol{v}_h, \boldsymbol{\Psi}) := (\boldsymbol{v}_h, \nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi})) - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h), \nabla_0\times\boldsymbol{\Psi})_{\boldsymbol{\nu}}. \tag{4.13}$$

The weak consistency error δ_{wkc} allows us to measure the consistency defect of the discrete primal problem (4.7) and the adjoint problem (3.13). (Compare with Lemma 4.1 for the consistency of the discrete primal problem restricted to conforming test functions.)

Lemma 4.3 (Weak consistency of primal and dual problems)**.** *If $\boldsymbol{E}_h$ solves the discrete primal problem* (4.7)*, the following holds true:*

$$b_{\sharp\mathrm{s}}(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{w}_h) = -\delta_{\mathrm{wkc}}(\boldsymbol{w}_h, \boldsymbol{E}) \qquad \forall \boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{4.14a}$$

If $\boldsymbol{\zeta_\theta} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ solve the adjoint problem (3.13) *with data $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$, the following holds true:*

$$b_{\sharp\mathrm{s}}(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta}) - (\boldsymbol{w}_h, \boldsymbol{\theta})_{\boldsymbol{\epsilon}} = -\delta_{\mathrm{wkc}}(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta}) \qquad \forall \boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{4.14b}$$

Proof. Recall the definition (4.3b) of $b_{\sharp\mathrm{s}}$ and the assumption (4.1b) on $s_\sharp$. To prove (4.14a), we observe that

$$\begin{aligned} b_{\sharp\mathrm{s}}(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{w}_h) &= b_{\sharp\mathrm{s}}(\boldsymbol{E}, \boldsymbol{w}_h) - (\boldsymbol{J}, \boldsymbol{w}_h) \\ &= b_\sharp(\boldsymbol{E}, \boldsymbol{w}_h) - (\boldsymbol{J}, \boldsymbol{w}_h) \\ &= (\nabla_0\times\boldsymbol{E}, \boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{w}_h))_{\boldsymbol{\nu}} - (\nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}), \boldsymbol{w}_h) \\ &= -\delta_{\mathrm{wkc}}(\boldsymbol{w}_h, \boldsymbol{E}). \end{aligned}$$

To prove (4.14b), we observe that

$$\begin{aligned} b_{\sharp\mathrm{s}}(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta}) = b_\sharp(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta}) &= -\omega^2(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta})_{\boldsymbol{\epsilon}} + (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{w}_h), \nabla_0\times\boldsymbol{\zeta_\theta})_{\boldsymbol{\nu}} \\ &= (\boldsymbol{w}_h, -\omega^2\boldsymbol{\epsilon}\boldsymbol{\zeta_\theta} + \nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\zeta_\theta})) - \delta_{\mathrm{wkc}}(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta}) \\ &= (\boldsymbol{w}_h, \boldsymbol{\theta})_{\boldsymbol{\epsilon}} - \delta_{\mathrm{wkc}}(\boldsymbol{w}_h, \boldsymbol{\zeta_\theta}). \end{aligned}$$

This completes the proof. $\square$

The above result motivates the need to bound the weak consistency error on the discrete curl.

Lemma 4.4 (Weak consistency)**.** *For all $\boldsymbol{\Psi} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ such that $\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} \in \boldsymbol{H}(\mathbf{curl}; D)$ and for all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, the following holds true:*

$$|\delta_{\mathrm{wkc}}(\boldsymbol{v}_h, \boldsymbol{\Psi})| \le |\boldsymbol{v}_h|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}} \in \boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*}, \tag{4.15}$$

with $\boldsymbol{P}^{\mathrm{c}}_{\ell}(\mathcal{T}_h) := \boldsymbol{P}^{\mathrm{b}}_{\ell}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}; D)$.

Proof. We follow the idea of [10] for the Helmholtz problem. For any field $\boldsymbol{\Phi}^{\mathrm{c}}_h \in \boldsymbol{P}^{\mathrm{c}}_{\ell}(\mathcal{T}_h)$, integration by parts gives

$$(\boldsymbol{v}_h, \nabla\times\boldsymbol{\Phi}^{\mathrm{c}}_h) = (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}_h), \boldsymbol{\Phi}^{\mathrm{c}}_h).$$

We infer that

$$\delta_{\mathrm{wkc}}(\boldsymbol{v}_h, \boldsymbol{\Psi}) = (\boldsymbol{v}_h, \nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}^{\mathrm{c}}_h)) - (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}_h), \boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}^{\mathrm{c}}_h).$$

Let $\boldsymbol{v}^{\mathrm{c}}_h \in \boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)$ and observe that $\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}^{\mathrm{c}}_h) = \nabla_0\times\boldsymbol{v}^{\mathrm{c}}_h$. Integration by parts using $\boldsymbol{\zeta} := \boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}^{\mathrm{c}}_h \in \boldsymbol{H}(\mathbf{curl}; D)$ gives

$$(\boldsymbol{v}^{\mathrm{c}}_h, \nabla\times\boldsymbol{\zeta}) = (\nabla_0\times\boldsymbol{v}^{\mathrm{c}}_h, \boldsymbol{\zeta}) = (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}^{\mathrm{c}}_h), \boldsymbol{\zeta}).$$

Putting everything together yields

$$\begin{aligned}\delta_{\mathrm{wkc}}(\boldsymbol{v}_h, \boldsymbol{\Psi}) &= (\boldsymbol{v}_h - \boldsymbol{v}^{\mathrm{c}}_h, \nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}^{\mathrm{c}}_h)) - (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}_h - \boldsymbol{v}^{\mathrm{c}}_h), \boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}^{\mathrm{c}}_h) \\ &\leq \|\boldsymbol{v}_h - \boldsymbol{v}^{\mathrm{c}}_h\|_{\mathrm{ap}} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{\Psi} - \boldsymbol{\Phi}^{\mathrm{c}}_h\|_{\mathrm{ap}*},\end{aligned}$$

where we used the Cauchy–Schwarz inequality and the definitions (3.11a) and (3.11b) of the norms $\|\cdot\|_{\mathrm{ap}}$ and $\|\cdot\|_{\mathrm{ap}*}$, respectively. Taking the infimum over $\boldsymbol{v}^{\mathrm{c}}_h \in \boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)$ and over $\boldsymbol{\Phi}^{\mathrm{c}}_h \in \boldsymbol{P}^{\mathrm{c}}_{\ell}(\mathcal{T}_h)$ completes the proof. ☐

5. A priori error analysis and inf-sup stability

This section is devoted to the error analysis of the dG approximation. As usual with Schatz-like arguments, we first establish an error estimate by assuming that the discrete solution $\boldsymbol{E}_h$ exists and then we prove that the discrete problem (4.7) is indeed well-posed if h is small enough.

5.1. **Error decomposition and best approximation.** We define the approximation error $\boldsymbol{e} := \boldsymbol{E} - \boldsymbol{E}_h$ and consider the error decomposition

$$\boldsymbol{e} = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_\Pi, \tag{5.1}$$

with

$$\boldsymbol{\theta}_0 := (I - \boldsymbol{\Pi}^{\mathrm{c}}_0)(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \qquad \boldsymbol{\theta}_\Pi := \boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D). \tag{5.2}$$

Let us define the bilinear forms $b^+_\sharp$ and $b^+_{\sharp\mathrm{S}}$ on $\boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp$ such that

$$b^+_\sharp(\boldsymbol{v}, \boldsymbol{w}) := \omega^2(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{\epsilon}} + (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{v}), \boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{w}))_{\boldsymbol{\nu}}, \tag{5.3a}$$

$$b^+_{\sharp\mathrm{S}}(\boldsymbol{v}, \boldsymbol{w}) := b^+_\sharp(\boldsymbol{v}, \boldsymbol{w}) + s_\sharp(\boldsymbol{v}, \boldsymbol{w}). \tag{5.3b}$$

The difference with $b_\sharp$ and $b_{\sharp\mathrm{S}}$ lies in the sign of the zero-order term. We define the best-approximation operator $\mathcal{B}^{\mathrm{b}}_h : \boldsymbol{V}_\sharp \to \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$ as follows: For all $\boldsymbol{v} \in \boldsymbol{V}_\sharp$, $\mathcal{B}^{\mathrm{b}}_h(\boldsymbol{v}) \in \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$ is such that

$$b^+_{\sharp\mathrm{S}}(\boldsymbol{v} - \mathcal{B}^{\mathrm{b}}_h(\boldsymbol{v}), \boldsymbol{w}_h) = 0 \quad \forall \boldsymbol{w}_h \in \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h). \tag{5.4}$$

The best-approximation error is defined to be

$$\boldsymbol{\eta} := \boldsymbol{E} - \mathcal{B}^{\mathrm{b}}_h(\boldsymbol{E}). \tag{5.5}$$

Lemma 5.1 (Properties of $\mathcal{B}_h^{\mathrm{b}}$)**.** *The best-approximation operator $\mathcal{B}_h^{\mathrm{b}}$ defined in* (5.4) *enjoys the following two properties:*

$$\|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v})\|_{\sharp\mathrm{s}} \leq \|\boldsymbol{v}\|_{\sharp\mathrm{s}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_\sharp, \tag{5.6a}$$

$$\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v}) \in \boldsymbol{X}_h^{\mathrm{b}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp. \tag{5.6b}$$

In particular, the error $\boldsymbol{e} = \boldsymbol{E} - \boldsymbol{E}_h$ satisfies

$$\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}) \in \boldsymbol{X}_h^{\mathrm{b}}. \tag{5.6c}$$

Proof. (5.6a) follows from the fact that the bilinear form $b_{\sharp\mathrm{s}}^+$ is the inner product associated with the $\|\cdot\|_{\sharp\mathrm{s}}$-norm. To prove (5.6b), consider any $\boldsymbol{v} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$. Take any $\boldsymbol{w}_h \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ in (5.4) and observe that $\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{w}_h) = \mathbf{0}$ and $s_\sharp(\cdot, \boldsymbol{w}_h) = 0$. Since

$$\omega^2(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v}), \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} = \omega^2(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v}) - \boldsymbol{v}, \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} = b_{\sharp\mathrm{s}}^+(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v}) - \boldsymbol{v}, \boldsymbol{w}_h) = 0,$$

we infer that $\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v}) \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$. Moreover, $\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{v}) \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ by construction. This proves (5.6b). Finally, (5.6c) follows from (4.9) and (5.6b). □

5.2. **Preliminary bounds.**

Lemma 5.2 (Bound on $\boldsymbol{\theta}_0$)**.** *We have*

$$\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}} \leq \gamma_{\mathrm{ap}} \|\boldsymbol{e}\|_\sharp + \gamma_{\mathrm{ap}*} |\boldsymbol{e}|_{\mathrm{nc}}, \tag{5.7}$$

with the approximation factors γ_{ap} and $\gamma_{\mathrm{ap}*}$ defined in (3.15a) *and* (3.15b)*, respectively.*

Proof. Let $\boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ solve the adjoint problem (3.13) with data $\boldsymbol{\theta} := \boldsymbol{\theta}_0$. Since $\boldsymbol{E}, \boldsymbol{\zeta}_{\boldsymbol{\theta}} \in \boldsymbol{H}_0(\mathbf{curl}; D)$, we infer from (4.4) and the definition of the adjoint solution that

$$b_{\sharp\mathrm{s}}(\boldsymbol{E}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = b(\boldsymbol{E}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = (\boldsymbol{E}, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}}.$$

Owing to (4.14b), we infer that

$$b_{\sharp\mathrm{s}}(\boldsymbol{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = (\boldsymbol{E}_h, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}} - \delta_{\mathrm{wkc}}(\boldsymbol{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}}).$$

Combining the above two identities and using $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_\Pi)_{\boldsymbol{\epsilon}} = 0$ gives

$$\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2 = \omega(\boldsymbol{e}, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}} = \omega b_{\sharp\mathrm{s}}(\boldsymbol{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) - \omega\delta_{\mathrm{wkc}}(\boldsymbol{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}}). \tag{5.8}$$

Owing to Galerkin orthogonality on conforming test functions (see (4.8)), we infer that, for all $\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$,

$$\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2 = \omega b_{\sharp\mathrm{s}}(\boldsymbol{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{v}_h^{\mathrm{c}}) - \omega\delta_{\mathrm{wkc}}(\boldsymbol{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}}). \tag{5.9}$$

Invoking the boundedness property (4.6b) on $b_{\sharp\mathrm{s}}$, and using the definition of the approximation factor γ_{ap} gives

$$\omega|b_{\sharp\mathrm{s}}(\boldsymbol{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{v}_h^{\mathrm{c}})| \leq \|\boldsymbol{e}\|_\sharp\, \omega\|\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \boldsymbol{v}_h^{\mathrm{c}}\| \leq \|\boldsymbol{e}\|_\sharp\, \gamma_{\mathrm{ap}}\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}.$$

Moreover, invoking Lemma 4.4 to bound the weak consistency error and recalling the definition of the approximation factor $\gamma_{\mathrm{ap}*}$ gives

$$\omega|\delta_{\mathrm{wkc}}(\boldsymbol{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}})| \leq |\boldsymbol{E}_h|_{\mathrm{nc}}\, \gamma_{\mathrm{ap}*}\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}} = |\boldsymbol{e}|_{\mathrm{nc}}\, \gamma_{\mathrm{ap}*}\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}.$$

Putting the above two bounds together proves (5.7). □

Lemma 5.3 (Bound on $\boldsymbol{\theta}_\Pi$)**.** *We have*

$$\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \leq \omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{\epsilon}} + \gamma_{\mathrm{dv}}\left\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\right\}^{\frac{1}{2}}. \tag{5.10}$$

Proof. We observe that

$$\boldsymbol{\theta}_\Pi + \boldsymbol{\theta}_0 = \boldsymbol{e} = \mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}) + (I - \mathcal{B}_h^{\mathrm{b}})(\boldsymbol{e}) = \mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}) + \boldsymbol{\eta},$$

since $(I - \mathcal{B}_h^{\mathrm{b}})(\boldsymbol{E}_h) = \mathbf{0}$. This gives $\boldsymbol{\theta}_\Pi = \mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}) + \boldsymbol{\eta} - \boldsymbol{\theta}_0$. Since $(\boldsymbol{\theta}_\Pi, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}} = 0$, we infer that

$$\begin{aligned}\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}^2 &= (\boldsymbol{\theta}_\Pi, \mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))_{\boldsymbol{\epsilon}} + (\boldsymbol{\theta}_\Pi, \boldsymbol{\eta})_{\boldsymbol{\epsilon}} \\ &= (\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}_0^{\mathrm{c}}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})))_{\boldsymbol{\epsilon}} + (\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}))_{\boldsymbol{\epsilon}} =: \Theta_1 + \Theta_2,\end{aligned}$$

where we used that $\boldsymbol{\theta}_\Pi = \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\theta}_\Pi)$ and that $\boldsymbol{\Pi}_0^{\mathrm{c}}$ is self-adjoint for the inner product $(\cdot,\cdot)_{\boldsymbol{\epsilon}}$. We bound Θ_1 as follows:

$$\begin{aligned}|\Theta_1| &\le \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \, \|\boldsymbol{\Pi}_0^{\mathrm{c}}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\epsilon}} \\ &\le \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \, \gamma_{\mathrm{dv}} \omega^{-1} \big\{ \|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2 \big\}^{\frac{1}{2}},\end{aligned}$$

where we used the divergence conformity factor defined in (3.16) (this is legitimate since $\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}) \in \boldsymbol{X}_h^{\mathrm{b}}$ owing to (5.6c)). Moreover, the Cauchy–Schwarz inequality gives

$$|\Theta_2| \le \|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \, \|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{\epsilon}}.$$

Putting the above two bounds together proves the assertion. ☐

Lemma 5.4 (Bound on $\|\boldsymbol{\theta}_0\|_{\sharp\mathrm{S}}$). *We have*

$$\begin{aligned}\|\boldsymbol{\theta}_0\|_{\sharp\mathrm{S}}^2 \le{}& \|(I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\|_{\sharp\mathrm{S}}^2 + 2\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \, \gamma_{\mathrm{dv}} \big\{ \|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2 \big\}^{\frac{1}{2}} \\ &+ 2|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}} \in \boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0 \times \boldsymbol{E} - \boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*} + 4\omega^2 \|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2. \qquad (5.11)\end{aligned}$$

Proof. Since $\boldsymbol{e} = \boldsymbol{\eta} + \mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})$ as shown in the above proof, we have $\boldsymbol{\theta}_0 = (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{e}) = (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta}) + (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))$. This gives

$$\begin{aligned}b_{\sharp\mathrm{S}}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) &= b_{\sharp\mathrm{S}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})) + b_{\sharp\mathrm{S}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))) \\ &= b_{\sharp\mathrm{S}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})) + b_{\sharp\mathrm{S}}(\boldsymbol{e}, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))),\end{aligned}$$

where the second equality follows from $b_{\sharp\mathrm{S}}(\boldsymbol{\theta}_\Pi, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\cdot)) = 0$. The first term on the right-hand side is bounded by invoking the continuity property (4.6a), giving

$$|b_{\sharp\mathrm{S}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta}))| \le \|\boldsymbol{\theta}_0\|_{\sharp\mathrm{S}} \, \|(I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\|_{\sharp\mathrm{S}}.$$

The second term is decomposed as $b_{\sharp\mathrm{S}}(\boldsymbol{e}, (I - \boldsymbol{\Pi}_0^{\mathrm{c}})(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))) = \beta_1 + \beta_2$ with

$$\beta_1 := b_{\sharp\mathrm{S}}(\boldsymbol{e}, \mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})), \qquad \beta_2 := -b_{\sharp\mathrm{S}}(\boldsymbol{e}, \boldsymbol{\Pi}_0^{\mathrm{c}}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))) = \omega^2(\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}_0^{\mathrm{c}}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})))_{\boldsymbol{\epsilon}}.$$

Recalling (4.14a), i.e., the weak consistency of the discrete primal problem for all test functions in $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we have $\beta_1 = -\delta_{\mathrm{wkc}}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}), \boldsymbol{E})$. Hence, invoking Lemma 4.4 gives

$$|\beta_1| \le |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}} \in \boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0 \times \boldsymbol{E} - \boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*}.$$

Moreover, using the Cauchy–Schwarz inequality and since $\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}) \in \boldsymbol{X}_h^{\mathrm{b}}$ (see (5.6c)), we have

$$|\beta_2| \le \omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}} \, \gamma_{\mathrm{dv}} \big\{ \|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2 \big\}^{\frac{1}{2}}.$$

Altogether, this gives

$$b_{\sharp\mathrm{s}}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) \le \|\boldsymbol{\theta}_0\|_{\sharp\mathrm{s}}\, \|(I-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\|_{\sharp\mathrm{s}} + \omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\,\gamma_{\mathrm{dv}}\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\}^{\frac12} + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*}.$$

Since $\|\boldsymbol{\theta}_0\|_{\sharp\mathrm{s}}^2 = b_{\sharp\mathrm{s}}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0)+2\omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2$, we infer that

$$\|\boldsymbol{\theta}_0\|_{\sharp\mathrm{s}}^2 \le \|\boldsymbol{\theta}_0\|_{\sharp\mathrm{s}}\, \|(I-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\|_{\sharp\mathrm{s}} + \omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}\,\gamma_{\mathrm{dv}}\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\}^{\frac12} + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*} + 2\omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2.$$

Dealing with the first term on the right-hand side by Young's inequality gives (5.11). □

5.3. A priori error estimate.

We are now ready to establish our main result on the a priori error analysis which establishes asymptotic optimality. Importantly, the (frequency dependent) constants controlling the smallness of the mesh size are essentially those appearing in a conforming approximation.

Theorem 5.5 (Asymptotically optimal error estimate and discrete well-posedness)**.** *Assume* (4.2). *The following holds:*

$$(5.12)\quad (1-c_\gamma)\|\boldsymbol{e}\|_{\sharp\mathrm{s}}^2 \le (1+4\gamma_{\mathrm{dv}})\|\boldsymbol{\eta}\|_{\sharp\mathrm{s}}^2 + 2\rho^{-1}\|\boldsymbol{e}\|_{\sharp\mathrm{s}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*},$$

with $c_\gamma := 8\max(\gamma_{\mathrm{ap}}^2,\rho^{-2}\gamma_{\mathrm{ap}*}^2)+\max(1,\rho^{-2})(\gamma_{\mathrm{dv}}+3\gamma_{\mathrm{dv}}^2)$. *Consequently, if the mesh size is small enough so that* $c_\gamma<1$, *the discrete problem* (4.7) *is well-posed.*

Proof. We use (5.10) in (5.11) to infer that

$$\|\boldsymbol{\theta}_0\|_{\sharp\mathrm{s}}^2 \le \|(I-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\|_{\sharp\mathrm{s}}^2 + 2\omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{\epsilon}}\,\gamma_{\mathrm{dv}}\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\}^{\frac12} + 2\gamma_{\mathrm{dv}}^2\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\} + 2|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*} + 4\omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2.$$

We now square (5.10) and add the result to the above estimate. Since

$$\|\boldsymbol{e}\|_{\sharp\mathrm{s}}^2 = \|\boldsymbol{\theta}_0\|_{\sharp\mathrm{s}}^2+\omega^2\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}^2, \qquad \|\boldsymbol{\eta}\|_{\sharp\mathrm{s}}^2 = \|(I-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{\eta})\|_{\sharp\mathrm{s}}^2+\omega^2\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{\epsilon}}^2,$$

we obtain

$$\|\boldsymbol{e}\|_{\sharp\mathrm{s}}^2 \le \|\boldsymbol{\eta}\|_{\sharp\mathrm{s}}^2 + 4\omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{\epsilon}}\,\gamma_{\mathrm{dv}}\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\}^{\frac12} + 3\gamma_{\mathrm{dv}}^2\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\} + 2|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*} + 4\omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2.$$

We deal with the second term on the right-hand side by Young's inequality. Since $\omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{\epsilon}} \le \omega\|\boldsymbol{\eta}\|_{\boldsymbol{\epsilon}} \le \|\boldsymbol{\eta}\|_{\sharp\mathrm{s}}$, this gives

$$\|\boldsymbol{e}\|_{\sharp\mathrm{s}}^2 \le (1+4\gamma_{\mathrm{dv}})\|\boldsymbol{\eta}\|_{\sharp\mathrm{s}}^2 + (\gamma_{\mathrm{dv}}+3\gamma_{\mathrm{dv}}^2)\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\} + 2|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*} + 4\omega^2\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2.$$

We invoke (5.7) to bound the last term on the right-hand side. This yields

$$\|\!|\boldsymbol{e}|\!\|_{\sharp\mathrm{s}}^2 \le (1+4\gamma_{\mathrm{dv}})\|\!|\boldsymbol{\eta}|\!\|_{\sharp\mathrm{s}}^2 + (\gamma_{\mathrm{dv}}+3\gamma_{\mathrm{dv}}^2)\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2\big\}$$
$$+ 2|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*} + 8\big(\gamma_{\mathrm{ap}}^2\|\!|\boldsymbol{e}|\!\|_\sharp^2 + \gamma_{\mathrm{ap}*}^2|\boldsymbol{e}|_{\mathrm{nc}}^2\big).$$

We observe that

$$\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}}^2 \le \max(1,\rho^{-2})\big\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e}))\|_{\boldsymbol{\nu}}^2 + |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{s}}^2\big\}$$
$$\le \max(1,\rho^{-2})\|\!|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|\!\|_{\sharp\mathrm{s}}^2 \le \max(1,\rho^{-2})\|\!|\boldsymbol{e}|\!\|_{\sharp\mathrm{s}}^2,$$

where the last bound follows from (5.6a). Moreover, we have

$$\gamma_{\mathrm{ap}}^2\|\!|\boldsymbol{e}|\!\|_\sharp^2 + \gamma_{\mathrm{ap}*}^2|\boldsymbol{e}|_{\mathrm{nc}}^2 \le \max(\gamma_{\mathrm{ap}}^2,\rho^{-2}\gamma_{\mathrm{ap}*}^2)(\|\!|\boldsymbol{e}|\!\|_\sharp^2 + |\boldsymbol{e}|_{\mathrm{s}}^2) = \max(\gamma_{\mathrm{ap}}^2,\rho^{-2}\gamma_{\mathrm{ap}*}^2)\|\!|\boldsymbol{e}|\!\|_{\sharp\mathrm{s}}^2.$$

Combining the above bounds shows that

$$(1-c_\gamma)\|\!|\boldsymbol{e}|\!\|_{\sharp\mathrm{s}}^2 \le (1+4\gamma_{\mathrm{dv}})\|\!|\boldsymbol{\eta}|\!\|_{\sharp\mathrm{s}}^2 + 2|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}}\in\boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}-\boldsymbol{\Phi}_h^{\mathrm{c}}\|_{\mathrm{ap}*}.$$

Since $\rho|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{nc}} \le |\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|_{\mathrm{s}} \le \|\!|\mathcal{B}_h^{\mathrm{b}}(\boldsymbol{e})|\!\|_{\sharp\mathrm{s}} \le \|\!|\boldsymbol{e}|\!\|_{\sharp\mathrm{s}}$, this readily gives (5.12). □

Remark 5.6 (Error estimate (5.12)). The last term on the right-hand side of (5.12) stems from the weak consistency of the discrete formulation and somewhat pollutes the asymptotic optimality of the a priori error estimate. We notice that this term can be made superconvergent already with the choice $\ell = k$ (provided $\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}$ is smooth enough). The (slight) price to pay is to choose the stabilization factor η_* large enough so that $s_\sharp$ is indeed positive semidefinite for $\ell = k$ (see (4.12)).

5.4. **Inf-sup stability.** Here, we establish the discrete inf-sup stability of the bilinear form $b_{\sharp\mathrm{s}}$ on $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\times\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$. As for the error estimate from Theorem 5.5, the main insight is that the (frequency dependent) constants controlling the smallness of the mesh size are essentially those appearing in a conforming approximation. The inf-sup stability constant of the discrete problem also depends on the frequency through the stability constant β_{st} of the exact problem; again, this is the same situation as for a conforming approximation.

Theorem 5.7 (Inf-sup stability)**.** *Under assumption* (4.2)*, we have*

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\\ \|\!|\boldsymbol{v}_h|\!\|_{\sharp\mathrm{s}}=1}} \max_{\substack{\boldsymbol{w}_h\in\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\\ \|\!|\boldsymbol{w}_h|\!\|_{\sharp\mathrm{s}}=1}} |b_{\sharp\mathrm{s}}(\boldsymbol{v}_h,\boldsymbol{w}_h)| \ge \frac{1-c'_\gamma}{1+2\beta_{\mathrm{st}}}, \tag{5.13}$$

with $c'_\gamma := 2\big(\gamma_{\mathrm{ap}} + \frac{1}{2}\rho^{-1}\gamma_{\mathrm{ap}*} + \max(1,\rho^{-2})\gamma_{\mathrm{dv}}^2\big)$.

Proof. Let $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$. We build a suitable $\boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ so that $\|\!|\boldsymbol{w}_h|\!\|_{\sharp\mathrm{s}} \le (1+2\beta_{\mathrm{st}})\|\!|\boldsymbol{v}_h|\!\|_{\sharp\mathrm{s}}$ and $b_{\sharp\mathrm{s}}(\boldsymbol{v}_h,\boldsymbol{w}_h) \ge (1-c'_\gamma)\|\!|\boldsymbol{v}_h|\!\|_{\sharp\mathrm{s}}^2$.

(1) Set $\boldsymbol{v}_{h0} := (I-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\boldsymbol{v}_h) \in \boldsymbol{X}_h^{\mathrm{b}}$ and $\boldsymbol{v}_{h\Pi} := \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h) \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)$, so that $\boldsymbol{v}_h = \boldsymbol{v}_{h0} + \boldsymbol{v}_{h\Pi}$. We further decompose $\boldsymbol{v}_{h0}$ as $\boldsymbol{v}_{h0} = \boldsymbol{\phi}_0 + \boldsymbol{\phi}_\Pi$ with $\boldsymbol{\phi}_0 := (I-\boldsymbol{\Pi}_0^{\mathrm{c}})(\boldsymbol{v}_{h0})$ and $\boldsymbol{\phi}_\Pi := \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_{h0})$. Let $\boldsymbol{\xi}_0 \in \boldsymbol{H}_0(\mathbf{curl};D)$ be the unique adjoint solution such that $b(\boldsymbol{w},\boldsymbol{\xi}_0) = \omega^2(\boldsymbol{w},\boldsymbol{\phi}_0)_{\boldsymbol{\epsilon}}$ for all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};D)$. We set $\boldsymbol{\xi}_{h0} := \mathcal{B}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}_0)$, where $\mathcal{B}_{h0}^{\mathrm{c}} : \boldsymbol{H}_0(\mathbf{curl};D) \to \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$ is uniquely defined by requiring that $b_\sharp^+(\boldsymbol{v}-\mathcal{B}_{h0}^{\mathrm{c}}(\boldsymbol{v}),\boldsymbol{w}_h) = 0$, for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};D)$ and all $\boldsymbol{w}_h \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$. Finally, we set

$$\boldsymbol{w}_h := \boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \boldsymbol{v}_{h\Pi} \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h).$$

(2) Upper bound on $\|\boldsymbol{w}_h\|_{\sharp\mathrm{s}}$. The same argument as in the proof of Lemma 5.1 shows that $\|\boldsymbol{\xi}_{h0}\| \leq \|\boldsymbol{\xi}_0\|$. Moreover, since $\boldsymbol{\phi}_0 \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$, we have

$$\beta_{\mathrm{st}}^{-1}\|\boldsymbol{\xi}_0\| \leq \omega\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}} \leq \omega\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}} \leq \omega\|\boldsymbol{v}_h\|_{\boldsymbol{\epsilon}} \leq \|\boldsymbol{v}_h\|_\sharp \leq \|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}.$$

This gives

$$\begin{aligned}\|\boldsymbol{w}_h\|_{\sharp\mathrm{s}}^2 = \|\boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0}\|_{\sharp\mathrm{s}}^2 + \omega^2\|\boldsymbol{v}_{h\Pi}\|_{\boldsymbol{\epsilon}}^2 &\leq \left(\|\boldsymbol{v}_{h0}\|_{\sharp\mathrm{s}} + 2\|\boldsymbol{\xi}_{h0}\|\right)^2 + \omega^2\|\boldsymbol{v}_{h\Pi}\|_{\boldsymbol{\epsilon}}^2 \\ &\leq (1+2\beta_{\mathrm{st}})^2\|\boldsymbol{v}_{h0}\|_{\sharp\mathrm{s}}^2 + \omega^2\|\boldsymbol{v}_{h\Pi}\|_{\boldsymbol{\epsilon}}^2 \leq (1+2\beta_{\mathrm{st}})^2\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2.\end{aligned}$$

(3) Lower bound on $b_{\sharp\mathrm{s}}(\boldsymbol{v}_h, \boldsymbol{w}_h)$. We first observe that

$$\begin{aligned}b_{\sharp\mathrm{s}}(\boldsymbol{v}_h, \boldsymbol{\xi}_{h0}) &= b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0}) + b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h\Pi}, \boldsymbol{\xi}_{h0}) \\ &= b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0}) = b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) + b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0).\end{aligned}$$

Owing to (4.14b), we have

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) = \omega^2(\boldsymbol{v}_{h0}, \boldsymbol{\phi}_0)_{\boldsymbol{\epsilon}} - \delta_{\mathrm{wkc}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) = \omega^2\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}^2 - \delta_{\mathrm{wkc}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0).$$

Invoking Lemma 4.4 and the definition (3.15b) of the approximation factor $\gamma_{\mathrm{ap}*}$ gives

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) \geq \omega^2\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}^2 - |\boldsymbol{v}_{h0}|_{\mathrm{nc}}\gamma_{\mathrm{ap}*}\omega\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}.$$

Using the above bound on $\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}$ together with $|\boldsymbol{v}_{h0}|_{\mathrm{nc}} = |\boldsymbol{v}_h|_{\mathrm{nc}}$ and assumption (4.2), we infer that

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_0) \geq \omega^2\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}^2 - \rho^{-1}\gamma_{\mathrm{ap}*}|\boldsymbol{v}_h|_{\mathrm{s}}\|\boldsymbol{v}_h\|_\sharp \geq \omega^2\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}^2 - \tfrac{1}{2}\rho^{-1}\gamma_{\mathrm{ap}*}\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2, \tag{5.14}$$

where the last bound follows from Young's inequality. Furthermore, using the definition (3.15a) of the approximation factor γ_{ap} together with the boundedness property (4.6b) gives

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0) \geq -\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}\gamma_{\mathrm{ap}}\omega\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}} \geq -\gamma_{\mathrm{ap}}\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2.$$

Combining this lower bound with (5.14), we infer that

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0}) \geq \omega^2\|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}^2 - \left(\gamma_{\mathrm{ap}} + \tfrac{1}{2}\rho^{-1}\gamma_{\mathrm{ap}*}\right)\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2. \tag{5.15}$$

Furthermore, using the divergence conformity factor γ_{dv} yields

$$\begin{aligned}\omega^2\|\boldsymbol{\phi}_\Pi\|_{\boldsymbol{\epsilon}}^2 = \omega^2\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_{h0})\|_{\boldsymbol{\epsilon}}^2 &\leq \gamma_{\mathrm{dv}}^2\{\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_{h0})\|_{\boldsymbol{\nu}}^2 + |\boldsymbol{v}_{h0}|_{\mathrm{nc}}^2\} \\ &\leq \max(1,\rho^{-2})\gamma_{\mathrm{dv}}^2\|\boldsymbol{v}_{h0}\|_{\sharp\mathrm{s}}^2 \leq \max(1,\rho^{-2})\gamma_{\mathrm{dv}}^2\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2.\end{aligned}$$

Since $\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}}^2 = \|\boldsymbol{\phi}_0\|_{\boldsymbol{\epsilon}}^2 + \|\boldsymbol{\phi}_\Pi\|_{\boldsymbol{\epsilon}}^2$, combining this bound with (5.15) gives

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_{h0}, \boldsymbol{\xi}_{h0}) \geq \omega^2\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}}^2 - \tfrac{1}{2}c_\gamma'\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2.$$

Finally, since $b_{\sharp\mathrm{s}}(\boldsymbol{v}_h, \boldsymbol{v}_{h0} - \boldsymbol{v}_{h\Pi}) = \|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2 - 2\omega^2\|\boldsymbol{v}_{h0}\|_{\boldsymbol{\epsilon}}^2$, we infer that

$$b_{\sharp\mathrm{s}}(\boldsymbol{v}_h, \boldsymbol{w}_h) = b_{\sharp\mathrm{s}}(\boldsymbol{v}_h, \boldsymbol{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \boldsymbol{v}_{h\Pi}) \geq (1 - c_\gamma')\|\boldsymbol{v}_h\|_{\sharp\mathrm{s}}^2.$$

This completes the proof. □

Remark 5.8 (Discrete inf-sup constant). The discrete inf-sup constant appearing on the left-hand side of (5.13) tends to $(1+2\beta_{\mathrm{st}})^{-1}$ as the mesh is refined, thus approaching, by up to a factor of two at most, the inf-sup constant from the continuous setting.

6. A posteriori residual-based error analysis

In this section, we estimate the error $\boldsymbol{e} := \boldsymbol{E} - \boldsymbol{E}_h$ by means of local residual-based quantities called error indicators. We derive both a global upper error bound (reliability) and local lower error bounds (local efficiency). The only property required for the discrete object $\boldsymbol{E}_h$ in the a posteriori error analysis is to satisfy the Galerkin orthogonality (4.8) on conforming test functions, i.e., $b_{\sharp\mathrm{s}}(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{v}_h^{\mathrm{c}}) = b_{\sharp}(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{v}_h^{\mathrm{c}}) = 0$ for all $\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$. Lemma 4.1 shows that the dG solution solving (4.7) satisfies this property. For simplicity, we keep the notation $\boldsymbol{E}_h$ in this section. For the a posteriori error analysis, we assume that $\nabla\cdot\boldsymbol{J} \in L^2(D)$ and that the material properties are piecewise constant on the mesh on a fixed partition to which the mesh is conforming.

6.1. Notation and interpolation operators.

For all $K \in \mathcal{T}_h$, the element patch K^{v} (resp., K^{e}, K^{f}) denotes the domain covered by all the cells $K' \in \mathcal{T}_h$ sharing at least one vertex (resp., edge, face) with K. Similarly, the extended patch K^{vv} (resp., K^{vvv}) is the domain covered by all the cells $K'' \in \mathcal{T}_h$ sharing at least one vertex with a cell $K' \subset K^{\mathrm{v}}$ (resp., $K' \subset K^{\mathrm{vv}}$). For a face $F \in \mathcal{F}_h$, $\widetilde{F}$ is the domain covered by the one or two cells sharing F. Whenever no confusion can arise, we also employ the symbols K^{v}, K^{e}, K^{vv}, K^{vvv}, $\widetilde{F}$ for the set of cells covering the domains. We employ the symbol $\kappa_{\mathcal{T}_h}$ for the shape-regularity parameter of the mesh $\mathcal{T}_h$, and $C(\kappa_{\mathcal{T}_h})$ denotes any generic constant solely depending on $\kappa_{\mathcal{T}_h}$ and whose value can change at each occurrence. For any subset $\mathcal{T} \subset \mathcal{T}_h$, we introduce the notation

$$
\begin{aligned}
\varepsilon_{\max,\mathcal{T}} &:= \max_{K\in\mathcal{T}} \max_{\boldsymbol{x}\in K} \max_{\substack{\boldsymbol{u}\in\mathbb{R}^d\\|\boldsymbol{u}|=1}} \max_{\substack{\boldsymbol{v}\in\mathbb{R}^d\\|\boldsymbol{v}|=1}} \boldsymbol{\epsilon}(\boldsymbol{x})\boldsymbol{u}\cdot\boldsymbol{v}, \\
\varepsilon_{\min,\mathcal{T}} &:= \min_{T\in\mathcal{T}} \min_{\boldsymbol{x}\in K} \min_{\substack{\boldsymbol{u}\in\mathbb{R}^d\\|\boldsymbol{u}|=1}} \boldsymbol{\epsilon}(\boldsymbol{x})\boldsymbol{u}\cdot\boldsymbol{u},
\end{aligned} \tag{6.1}
$$

and define $\nu_{\max,\mathcal{T}}$ and $\nu_{\min,\mathcal{T}}$ similarly. Then, $\vartheta_{\mathcal{T}} := (\nu_{\min,\mathcal{T}}/\varepsilon_{\max,\mathcal{T}})^{\frac{1}{2}}$ stands for the minimum velocity in the subdomain covered by the cells in $\mathcal{T}$. We write $\|v\|_{\mathcal{T}}^2 := \sum_{K\in\mathcal{T}} \|v\|_{L^2(K)}^2$ and employ a similar notation if v is vector-valued. We also write $\|v\|_{\mathcal{F}}^2 := \sum_{F\in\mathcal{F}} \|v\|_{L^2(F)}^2$ for every subset $\mathcal{F} \subset \mathcal{F}_h$. For simplicity, we assume that $\ell \in \{k-1, k\}$ in the discrete curl operator and do not track the dependency on ℓ of the constants.

We employ the quasi-interpolation operators from [30] (see also [32] and see [19, Corollary 2.5] for using the seminorm in the extended patch K^{vv}). Specifically, there exists an operator $\mathcal{I}_{h0}^{\mathrm{g}} : H_0^1(D) \to \mathcal{P}_{k+1}^{\mathrm{b}}(\mathcal{T}_h) \cap H_0^1(D)$ such that, for all $q \in H_0^1(D)$ and all $K \in \mathcal{T}_h$,

$$
\frac{k^2}{h_K^2}\|q - \mathcal{I}_{h0}^{\mathrm{g}}(q)\|_K^2 + \frac{k}{h_K}\|q - \mathcal{I}_{h0}^{\mathrm{g}}(q)\|_{\partial K}^2 \le C(\kappa_{\mathcal{T}_h})\|\nabla q\|_{K^{\mathrm{vv}}}^2. \tag{6.2}
$$

Similarly, there exists an operator $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{c}} : \boldsymbol{H}_0^1(D) \to \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$ such that, for all $\boldsymbol{w} \in \boldsymbol{H}_0^1(D)$ and all $K \in \mathcal{T}_h$,

$$
\frac{k^2}{h_K^2}\|\boldsymbol{w} - \boldsymbol{\mathcal{I}}_{h0}^{\mathrm{c}}(\boldsymbol{w})\|_K^2 + \frac{k}{h_K}\|(\boldsymbol{w} - \boldsymbol{\mathcal{I}}_{h0}^{\mathrm{c}}(\boldsymbol{w}))\times\boldsymbol{n}_K\|_{\partial K}^2 \le C(\kappa_{\mathcal{T}_h})\|\nabla\boldsymbol{w}\|_{K^{\mathrm{vv}}}^2. \tag{6.3}
$$

We will also need the quasi-interpolation averaging operator $\mathcal{I}_{h0}^{\mathrm{c,av}} : \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$ from [20] which is such that there is $c_k^{\mathrm{av}} \ge 1$ so that, for all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$

and all $K \in \mathcal{T}_h$,

$$\text{(6.4)}\quad \frac{k}{h_K}\|\boldsymbol{v}_h - \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v}_h)\|_K + \|\nabla\times(\boldsymbol{v}_h - \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v}_h))\|_K \leq C(\kappa_{\mathcal{T}_h}) c_k^{\mathrm{av}} \left(\frac{k^2}{h_K}\right)^{\frac{1}{2}} \left\{ \sum_{K' \in K^{\mathrm{e}}} \|[\![\boldsymbol{v}_h]\!]_{\partial K'}^{\mathrm{c}}\|_{\partial K'}^2 \right\}^{\frac{1}{2}}.$$

The dependency of c_k^{av} on k has been explored in some specific cases for $d = 2$ [8,28]. We keep this factor here as the analysis of the behavior of c_k^{av} in k goes beyond the present scope. Invoking a discrete trace inequality (see, e.g., [21, Lem. 12.10]) yields

$$\text{(6.5)}\quad \|\nabla\times\boldsymbol{v}_h - \boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h)\|_K = \|\boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{v}_h)\|_K \leq C(\kappa_{\mathcal{T}_h})\left(\frac{k^2}{h_K}\right)^{\frac{1}{2}} \|[\![\boldsymbol{v}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}.$$

Combining (6.4) and (6.5) gives

$$\text{(6.6)}\quad \frac{k}{h_K}\|\boldsymbol{v}_h - \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v}_h)\|_K + \|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h - \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v}_h))\|_K \leq C(\kappa_{\mathcal{T}_h}) c_k^{\mathrm{av}} \left(\frac{k^2}{h_K}\right)^{\frac{1}{2}} \left\{ \sum_{K' \in K^{\mathrm{e}}} \|[\![\boldsymbol{v}_h]\!]_{\partial K'}^{\mathrm{c}}\|_{\partial K'}^2 \right\}^{\frac{1}{2}},$$

and

$$\text{(6.7)}\quad \|\!|\boldsymbol{v}_h - \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v}_h)|\!\|_{\sharp} \leq C(\kappa_{\mathcal{T}_h}) c_k^{\mathrm{av}} \left(1 + \max_{K\in\mathcal{T}_h} \frac{\omega h_K}{k\vartheta_{\min,K^{\mathrm{e}}}}\right) \left\{ \sum_{K\in\mathcal{T}_h} \nu_{\max,K^{\mathrm{e}}} \frac{k^2}{h_K} \|[\![\boldsymbol{v}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2 \right\}^{\frac{1}{2}}.$$

6.2. **Estimator and error measure.** The a posteriori error estimator is written as the sum over the mesh cells of local error indicators η_K for all $K \in \mathcal{T}_h$. The local error indicator consists of three pieces. The first two respectively measure the residuals of the divergence constraint and of Maxwell's equations:

$$\text{(6.8a)}\quad \eta_{K,\mathrm{div}}^2 := \varepsilon_{\min,K^{\mathrm{vvv}}}^{-1} \left\{ \frac{h_K^2}{\omega^2 k^2} \|\nabla\cdot(\boldsymbol{J} + \omega^2 \boldsymbol{\epsilon}\boldsymbol{E}_h)\|_K^2 + \frac{\omega^2 h_K}{k} \|[\![\boldsymbol{\epsilon}\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{d}}\|_{\partial K\setminus\partial\Omega}^2 \right\},$$

and

$$\text{(6.8b)}\quad \eta_{K,\mathrm{curl}}^2 := \nu_{\min,K^{\mathrm{vvv}}}^{-1} \left\{ \frac{h_K^2}{k^2} \|\boldsymbol{J} + \omega^2 \boldsymbol{\epsilon}\boldsymbol{E}_h - \nabla\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_K^2 + \frac{h_K}{k} \|[\![\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h)]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K\setminus\partial\Omega}^2 \right\},$$

where $[\![\boldsymbol{\epsilon}\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{d}}|_F := [\![\boldsymbol{\epsilon}\boldsymbol{E}_h]\!]_F^{\mathrm{g}}\cdot\boldsymbol{n}_F$ and $[\![\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k}(\boldsymbol{E}_h)]\!]_{\partial K}^{\mathrm{c}}|_F := [\![\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k}(\boldsymbol{E}_h)]\!]_F^{\mathrm{g}}\times\boldsymbol{n}_F$ for all $F \in \mathcal{F}_K$. The last part of the estimator controls the nonconformity of the discrete field $\boldsymbol{E}_h$ as follows:

$$\text{(6.8c)}\quad \eta_{K,\mathrm{nc}}^2 := c_k^{\mathrm{av}} \frac{\nu_{\max,K^{\mathrm{e}}} k^2}{h_K} \|[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2,$$

where $[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}|_F := [\![\boldsymbol{E}_h]\!]_F^{\mathrm{g}}\times\boldsymbol{n}_F$ for all $F \in \mathcal{F}_K$. For shortness, we also introduce the following notation:

$$\text{(6.9)}\quad \eta_K^2 := \eta_{K,\mathrm{div}}^2 + \eta_{K,\mathrm{curl}}^2 + \eta_{K,\mathrm{nc}}^2, \qquad \eta_{\bullet}^2 := \sum_{K\in\mathcal{T}_h} \eta_{\bullet,K}^2, \qquad \eta^2 := \sum_{K\in\mathcal{T}_h} \eta_K^2,$$

with $\bullet \in \{\mathrm{div}, \mathrm{curl}, \mathrm{nc}\}$.

For all $\mathcal{T} \subset \mathcal{T}_h$, we define the error measure

$$\|\!|\boldsymbol{e}|\!\|^2_{\dagger,\mathcal{T}} := \sum_{K\in\mathcal{T}} \left\{ \omega^2 \|\boldsymbol{e}\|^2_{\boldsymbol{\epsilon},K} + \|\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{e})\|^2_{\boldsymbol{\nu},K} + c^{\mathrm{av}}_k \frac{\nu_{\max,K^{\mathrm{e}}} k^2}{h_K} \|[\![\boldsymbol{e}]\!]^{\mathrm{c}}_{\partial K}\|^2_{\partial K} \right\}, \tag{6.10}$$

and we omit the subscript $\mathcal{T}$ whenever $\mathcal{T} = \mathcal{T}_h$. A crucial observation is that the last term in the norm measuring the nonconformity can be chosen independently of the stabilization in the dG scheme. In particular, it does not have to be large enough.

Remark 6.1 (Broken curl). In view of (6.5), we can freely replace the discrete curl $\boldsymbol{C}^{k,\ell}_{h,0}$ by the broken curl in the definition of the estimator η and the error measure $\|\!|\cdot|\!\|_\dagger$.

6.3. **Error upper bound (reliability).** We start by controlling the PDE residual in Lemma 6.2. Lemma 6.2 is similar to [12, Lemma 3.2], but the result proposed here is sharper. In particular, the constant only depends on the shape-regularity of the mesh. Notice also that we consider here only conforming test functions so that we can work with the bilinear form $b_\sharp$ rather than $b_{\sharp\mathrm{S}}$.

Lemma 6.2 (Residual)**.** *For all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D)$, we have*

$$|b_\sharp(\boldsymbol{e}, \boldsymbol{v})| \le C(\kappa_{\mathcal{T}_h}) \eta_{\mathrm{dc}} \|\!|\boldsymbol{v}|\!\|, \tag{6.11}$$

with $\eta^2_{\mathrm{dc}} := \eta^2_{\mathrm{div}} + \eta^2_{\mathrm{curl}}$.

Proof. Here, we invoke [39, Theorem 1], which states that, given any $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D)$, there exists $\boldsymbol{S}^{\mathrm{c}}_{h0}(\boldsymbol{w}) \in \boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)$, such that

$$\boldsymbol{w} - \boldsymbol{S}^{\mathrm{c}}_{h0}(\boldsymbol{w}) = \nabla q + \boldsymbol{\phi}, \tag{6.12}$$

with $q \in H^1_0(D)$, $\boldsymbol{\phi} \in \boldsymbol{H}^1_0(D)$ such that, for all $K \in \mathcal{T}_h$,

$$\begin{aligned} h_K^{-1} \|q\|_K + \|\nabla q\|_K &\le C(\kappa_{\mathcal{T}_h}) \|\boldsymbol{w}\|_{K^{\mathrm{v}}}, \\ h_K^{-1} \|\boldsymbol{\phi}\|_K + \|\nabla \boldsymbol{\phi}\|_K &\le C(\kappa_{\mathcal{T}_h}) \|\nabla_0 \times \boldsymbol{w}\|_{K^{\mathrm{v}}}. \end{aligned} \tag{6.13}$$

We now pick an arbitrary test function $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D)$. We have

$$b_\sharp(\boldsymbol{e}, \boldsymbol{v}) = b_\sharp(\boldsymbol{e}, \boldsymbol{v} - \boldsymbol{S}^{\mathrm{c}}_{h0}(\boldsymbol{v})) = b_\sharp(\boldsymbol{e}, \nabla q + \boldsymbol{\phi}),$$

where q and $\boldsymbol{\phi}$ are the components of the decomposition in (6.12). We then estimate separately the two parts of the residual associated with the decomposition.

For the gradient part, we write

$$\begin{aligned}
b_\sharp(\boldsymbol{e}, \nabla q) &= b_\sharp(\boldsymbol{e}, \nabla(q - \mathcal{I}^{\mathrm{g}}_{h0}(q))) \\
&= -\omega^2(\boldsymbol{\epsilon}\boldsymbol{e}, \nabla(q - \mathcal{I}^{\mathrm{g}}_{h0}(q))) \\
&= \sum_{K\in\mathcal{T}_h} \omega^2(\nabla\cdot(\boldsymbol{\epsilon}\boldsymbol{e}), q - \mathcal{I}^{\mathrm{g}}_{h0}(q))_K - \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} \omega^2([\![\boldsymbol{\epsilon}(\boldsymbol{e})]\!]^{\mathrm{d}}_F, q - \mathcal{I}^{\mathrm{g}}_{h0}(q))_F \\
&= \sum_{K\in\mathcal{T}_h} -(\nabla\cdot(\boldsymbol{J} + \omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h), q - \mathcal{I}^{\mathrm{g}}_{h0}(q))_K + \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} \omega^2([\![\boldsymbol{\epsilon}\boldsymbol{E}_h]\!]^{\mathrm{d}}_F, q - \mathcal{I}^{\mathrm{g}}_{h0}(q))_F \\
&\le \sum_{K\in\mathcal{T}_h} \big\{\|\nabla\cdot(\boldsymbol{J} + \omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h)\|_K \|q - \mathcal{I}^{\mathrm{g}}_{h0}(q)\|_K \\
&\qquad\qquad + \omega^2\|[\![\boldsymbol{\epsilon}\boldsymbol{E}_h]\!]^{\mathrm{d}}_{\partial K}\|_{\partial K\setminus\partial\Omega}\|q - \mathcal{I}^{\mathrm{g}}_{h0}(q)\|_{\partial K\setminus\partial\Omega}\big\} \\
&\le \sum_{K\in\mathcal{T}_h} \eta_{K,\mathrm{div}}\varepsilon^{\frac12}_{\min,K^{\mathrm{vvv}}}\omega \left\{\frac{k}{h_K}\|q - \mathcal{I}^{\mathrm{g}}_{h0}(q)\|_K + \left(\frac{k}{h_K}\right)^{\frac12}\|q - \mathcal{I}^{\mathrm{g}}_{h0}(q)\|_{\partial K}\right\}.
\end{aligned}$$

For all $K \in \mathcal{T}_h$, invoking (6.2) and (6.13), we have

$$\begin{aligned}
&\frac{k}{h_K}\|q - \mathcal{I}^{\mathrm{g}}_{h0}(q)\|_K + \sqrt{\frac{k}{h_K}}\|q - \mathcal{I}^{\mathrm{g}}_{h0}(q)\|_{\partial K} \\
&\quad \le C(\kappa_{\mathcal{T}_h})\|\nabla q\|_{K^{\mathrm{vv}}} \\
&\quad \le C(\kappa_{\mathcal{T}_h})\|\boldsymbol{v}\|_{K^{\mathrm{vvv}}} \le C(\kappa_{\mathcal{T}_h})\varepsilon^{-\frac12}_{\min,K^{\mathrm{vvv}}}\|\boldsymbol{v}\|_{\boldsymbol{\epsilon},K^{\mathrm{vvv}}}.
\end{aligned}$$

Summing over $K \in \mathcal{T}_h$ and since the number of overlaps is uniformly controlled by $\kappa_{\mathcal{T}_h}$, we obtain

$$|b_\sharp(\boldsymbol{e}, \nabla q)| \le C(\kappa_{\mathcal{T}_h})\eta_{\mathrm{div}}\omega\|\boldsymbol{v}\|_{\boldsymbol{\epsilon}}. \tag{6.14}$$

For the $\boldsymbol{H}^1_0(D)$-part, proceeding similarly gives

$$\begin{aligned}
b_\sharp(\boldsymbol{e}, \boldsymbol{\phi}) &= b_\sharp(\boldsymbol{e}, \boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi})) \\
&= (\boldsymbol{J} + \omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h, \boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi})) - (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}_h), \nabla_0\times(\boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi})))_{\boldsymbol{\nu}} \\
&= \sum_{K\in\mathcal{T}_h} (\boldsymbol{J} + \omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h - \nabla\times(\boldsymbol{\nu}\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}_h)), \boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi}))_K \\
&\quad - \sum_{F\in\mathcal{F}_h^{\mathrm{int}}} ([\![\boldsymbol{\nu}\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}_h)]\!]^{\mathrm{c}}_F, \boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi}))_F \\
&\le \sum_{K\in\mathcal{T}_h} \eta_{K,\mathrm{curl}}\nu^{\frac12}_{\min,K^{\mathrm{vvv}}}\left\{\frac{k}{h_K}\|\boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi})\|_K \right. \\
&\qquad\qquad \left. + \left(\frac{k}{h_K}\right)^{\frac12}\|(\boldsymbol{\phi} - \boldsymbol{\mathcal{I}}^{\mathrm{c}}_{h0}(\boldsymbol{\phi}))\times\boldsymbol{n}\|_{\partial K\setminus\partial\Omega}\right\} \\
&\le C(\kappa_{\mathcal{T}_h}) \sum_{K\in\mathcal{T}_h} \eta_{K,\mathrm{curl}}\nu^{\frac12}_{\min,K^{\mathrm{vvv}}}\|\nabla\boldsymbol{\phi}\|_{K^{\mathrm{vv}}} \\
&\le C(\kappa_{\mathcal{T}_h}) \sum_{K\in\mathcal{T}_h} \eta_{K,\mathrm{curl}}\nu^{\frac12}_{\min,K^{\mathrm{vvv}}}\|\nabla_0\times\boldsymbol{v}\|_{K^{\mathrm{vvv}}} \\
&\le C(\kappa_{\mathcal{T}_h}) \sum_{K\in\mathcal{T}_h} \eta_{K,\mathrm{curl}}\|\nabla_0\times\boldsymbol{v}\|_{\boldsymbol{\nu},K^{\mathrm{vvv}}},
\end{aligned}$$

so that

$$|b_\sharp(\boldsymbol{e},\boldsymbol{\phi})| \le C(\kappa_{\mathcal{T}_h})\eta_{\rm curl}\|\nabla_0\times\boldsymbol{v}\|_{\boldsymbol{\nu}}. \tag{6.15}$$

Combining (6.14) and (6.15) concludes the proof. □

The next step is an Aubin–Nitsche-type duality argument to estimate the $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-norm of the error. Here, the weak consistency estimate from Lemma 4.4 is crucial to treat the nonconformity of the dG solution.

Lemma 6.3 ($\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-norm reliability estimate)**.** *We have*

$$\omega\|\boldsymbol{e}\|_{\boldsymbol{\epsilon}} \le C(\kappa_{\mathcal{T}_h})(1+\gamma_{\rm ap}+\gamma_{\rm ap*})\eta. \tag{6.16}$$

Proof. Recall the $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-orthogonal decomposition $\boldsymbol{e}=\boldsymbol{\theta}_0+\boldsymbol{\theta}_\Pi$ with $\boldsymbol{\theta}_0\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp$ and $\boldsymbol{\theta}_\Pi\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$ (see (5.1)).

For the first component, recalling (5.8) and using (4.14b), we have

$$\omega\|\boldsymbol{\theta}_0\|^2_{\boldsymbol{\epsilon}} = \omega(\boldsymbol{e},\boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}} = \omega b_\sharp(\boldsymbol{e},\boldsymbol{\zeta}_{\boldsymbol{\theta}}) - \omega\delta_{\rm wkc}(\boldsymbol{E}_h,\boldsymbol{\zeta}_{\boldsymbol{\theta}}). \tag{6.17}$$

Since $\boldsymbol{E}_h$ satisfies the Galerkin orthogonality for conforming test functions and invoking the bound (6.11) established in Lemma 6.2, we have, for all $\boldsymbol{v}_h^{\rm c}\in\boldsymbol{P}^{\rm c}_{k,0}(\mathcal{T}_h)$,

$$\omega b_\sharp(\boldsymbol{e},\boldsymbol{\zeta}_{\boldsymbol{\theta}}) = \omega b_\sharp(\boldsymbol{e},\boldsymbol{\zeta}_{\boldsymbol{\theta}}-\boldsymbol{v}_h^{\rm c}) \le C(\kappa_{\mathcal{T}_h})\eta_{\rm dc}\omega\|\boldsymbol{\zeta}_{\boldsymbol{\theta}}-\boldsymbol{v}_h^{\rm c}\| \le C(\kappa_{\mathcal{T}_h})\gamma_{\rm ap}\eta_{\rm dc}\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}, \tag{6.18}$$

where we used the definition (3.15a) of $\gamma_{\rm ap}$ in the last inequality (since $\boldsymbol{v}_h^{\rm c}$ is arbitrary in $\boldsymbol{P}^{\rm c}_{k,0}(\mathcal{T}_h)$). Moreover, owing to the estimate (4.15) from Lemma 4.4, we have

$$\omega|\delta_{\rm wkc}(\boldsymbol{E}_h,\boldsymbol{\zeta}_{\boldsymbol{\theta}})| \le |\boldsymbol{E}_h|_{\rm nc}\gamma_{\rm ap*}\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}.$$

Recalling the definition (3.12) of the $|{\cdot}|_{\rm nc}$-seminorm and using (6.6), we infer that

$$|\boldsymbol{E}_h|_{\rm nc} \le \|\boldsymbol{E}_h-\mathcal{I}^{\rm c,av}_{h0}(\boldsymbol{E}_h)\|_{\rm ap} \le C(\kappa_{\mathcal{T}_h})\eta_{\rm nc}.$$

This gives

$$\omega|\delta_{\rm wkc}(\boldsymbol{E}_h,\boldsymbol{\zeta}_{\boldsymbol{\theta}})| \le C(\kappa_{\mathcal{T}_h})\eta_{\rm nc}\gamma_{\rm ap*}\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}. \tag{6.19}$$

Combining (6.17), (6.18) and (6.19), we arrive at

$$\omega\|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}} \le C(\kappa_{\mathcal{T}_h})(\gamma_{\rm ap}+\gamma_{\rm ap*})\eta. \tag{6.20}$$

For the other part of the error, since $\boldsymbol{\theta}_\Pi\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$, we can use (6.11) to write

$$\omega^2\|\boldsymbol{\theta}_\Pi\|^2_{\boldsymbol{\epsilon}} = -b_\sharp(\boldsymbol{e},\boldsymbol{\theta}_\Pi) \le C(\kappa_{\mathcal{T}_h})\eta_{\rm dc}\|\boldsymbol{\theta}_\Pi\| = C(\kappa_{\mathcal{T}_h})\eta_{\rm dc}\omega\|\boldsymbol{\theta}_\Pi\|_{\boldsymbol{\epsilon}}. \tag{6.21}$$

Combining (6.20) and (6.21) proves (6.16). □

We are now ready to establish a reliability estimate with an argument similar to the one used in [10] for the scalar Helmholtz problem.

Theorem 6.4 (Reliability)**.** *We have*

$$\|\boldsymbol{e}\|_\dagger \le C(\kappa_{\mathcal{T}_h})\left(1+\max_{K\in\mathcal{T}_h}\frac{\omega h_K}{k\vartheta_{\min,K^{\rm e}}}+\gamma_{\rm ap}+\gamma_{\rm ap*}\right)\eta. \tag{6.22}$$

Proof. Since $\|\boldsymbol{e}\|^2_\dagger = \|\boldsymbol{e}\|^2_\sharp+\eta^2_{\rm nc}$, we only need to estimate $\|\boldsymbol{e}\|^2_\sharp$. To this purpose, recall that the bilinear form $b^+_\sharp$ defined in (5.3b) is the inner product associated with the $\|{\cdot}\|_\sharp$-norm. We introduce the $\boldsymbol{H}_0(\mathbf{curl};D)$-conforming projection $\mathcal{B}^{\rm c}_0:\boldsymbol{V}_\sharp\to$

$\boldsymbol{H}_0(\mathbf{curl}; D)$ such that $b_\sharp^+(\boldsymbol{v} - \mathcal{B}_0^{\rm c}(\boldsymbol{v}), \boldsymbol{w}) =$ for all $\boldsymbol{v} \in \boldsymbol{V}_\sharp$ and all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D)$. Reasoning as in the proof of Lemma 5.1 proves the following Pythagorean identity:

$$\|\boldsymbol{e}\|_\sharp^2 = \|\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|_\sharp^2 + \|\boldsymbol{E}_h - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|_\sharp^2. \tag{6.23}$$

We estimate separately the two terms on the right-hand side.

For the first term, we observe that $\mathcal{B}_0^{\rm c}(\boldsymbol{E}) = \boldsymbol{E}$ and $\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h) \in \boldsymbol{H}_0(\mathbf{curl}; D)$, and write that

$$\begin{aligned}\|\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|_\sharp^2 &= b_\sharp^+(\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h), \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)) \\ &= b_\sharp^+(\mathcal{B}_0^{\rm c}(\boldsymbol{e}), \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)) \\ &= b_\sharp^+(\boldsymbol{e}, \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)) \\ &= b_\sharp(\boldsymbol{e}, \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)) + 2\omega^2(\boldsymbol{e}, \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h))_{\boldsymbol{\epsilon}}.\end{aligned}$$

Since the second argument in the first term on the right-hand side is conforming, this term can be estimated by means of the estimate (6.11) from Lemma 6.2. This gives

$$|b_\sharp(\boldsymbol{e}, \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h))| \leq C(\kappa_{\mathcal{T}_h})\eta_{\rm dc}\|\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|.$$

We apply the Cauchy–Schwarz inequality to bound the second term, leading to

$$\omega^2|(\boldsymbol{e}, \boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h))_{\boldsymbol{\epsilon}}| \leq \omega\|\boldsymbol{e}\|_{\boldsymbol{\epsilon}}\omega\|\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|_{\boldsymbol{\epsilon}} \leq \omega\|\boldsymbol{e}\|_{\boldsymbol{\epsilon}}\|\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|.$$

This yields the following estimate:

$$\|\boldsymbol{E} - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|_\sharp \leq C(\kappa_{\mathcal{T}_h})\eta_{\rm dc} + \omega\|\boldsymbol{e}\|_{\boldsymbol{\epsilon}} \leq C(\kappa_{\mathcal{T}_h})(1 + \gamma_{\rm ap} + \gamma_{\rm ap*})\eta,$$

where we employed the $\boldsymbol{L}^2_{\boldsymbol{\epsilon}}$-estimate (6.16) from Lemma 6.3.

For the second term on the right-hand side of (6.23), invoking (6.7) gives

$$\|\boldsymbol{E}_h - \mathcal{B}_0^{\rm c}(\boldsymbol{E}_h)\|_\sharp \leq \|\boldsymbol{E}_h - \mathcal{I}_{h0}^{\rm c,av}(\boldsymbol{E}_h)\|_\sharp \leq C(\kappa_{\mathcal{T}_h})\left(1 + \max_{K\in\mathcal{T}_h}\frac{\omega h_K}{k\vartheta_{\min,K^{\rm e}}}\right)\eta_{\rm nc}.$$

Putting everything together yields the assertion. □

6.4. **Local error lower bound (local efficiency).** We now derive efficiency estimates. To do so, we will need bubble functions (see [34] and [12, Section 2.7]). Specifically, for all $K \in \mathcal{T}_h$, there exists a function $b_K \in H_0^1(K)$ with $b_K \leq 1$ such that, for all $\boldsymbol{w}_K \in \mathbb{P}_{k,d}$ (recall that $k \geq 1$ by assumption),

$$\|\boldsymbol{w}_K\|_K \leq C(\kappa_{\mathcal{T}_h})k\|b_K^{\frac{1}{2}}\boldsymbol{w}_K\|_K, \tag{6.24}$$

and

$$\|\nabla(b_K\boldsymbol{w}_K)\|_K \leq C(\kappa_{\mathcal{T}_h})\frac{k}{h_K}\|b_K^{\frac{1}{2}}\boldsymbol{w}_K\|_K. \tag{6.25}$$

Similarly, for all $F \in \mathcal{F}_h$, there exists a function $b_F \in H_0^1(F)$ such that, for all $\boldsymbol{w}_F \in \mathbb{P}_{k,d-1}$,

$$\|\boldsymbol{w}_F\|_F \leq C(\kappa_{\mathcal{T}_h})k\|b_F^{\frac{1}{2}}\boldsymbol{w}_F\|_F, \tag{6.26}$$

and an extension operator $\mathcal{E}_F : \mathbb{P}_{k,d-1} \to \boldsymbol{H}_0^1(\widetilde{F})$ such that, for all $\boldsymbol{w}_F \in \mathbb{P}_{k,d-1}$, $\mathcal{E}_F(b_F\boldsymbol{w}_F)|_F = b_F\boldsymbol{w}_F$ and

$$kh_F^{-\frac{1}{2}}\|\mathcal{E}_F(b_F\boldsymbol{w}_F)\|_{\widetilde{F}} + k^{-1}h_F^{\frac{1}{2}}\|\nabla\mathcal{E}_F(b_F\boldsymbol{w}_F)\|_{\widetilde{F}} \leq C(\kappa_{\mathcal{T}_h})\|b_F^{\frac{1}{2}}\boldsymbol{w}_F\|_F. \tag{6.27}$$

Theorem 6.5 (Local efficiency)**.** *For all $K \in \mathcal{T}_h$, we have*

$$\eta_K \le C(\kappa_{\mathcal{T}_h})\mathcal{K}_K^{\frac{1}{2}}k^{\frac{3}{2}}\left\{\left(1+\frac{\omega h_K}{k\vartheta_{\min,K^{\rm f}}}\right)\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\dagger,K^{\rm f}} + \operatorname{osc}_{K^{\rm f}}\right\}, \tag{6.28}$$

with the data oscillation term

$$\operatorname{osc}^2_{K^{\rm f}} := \frac{1}{\omega^2}\varepsilon^{-1}_{\min,K^{\rm f}} \times \sum_{K'\in K^{\rm f}} \min_{\boldsymbol{J}_h\in\boldsymbol{P}^{\rm b}_k(\mathcal{T}_h)}\left\{\frac{\omega^2 h^2_{K'}}{k^2\vartheta^2_{\min,K^{\rm f}}}\|\boldsymbol{J}-\boldsymbol{J}_h\|^2_{K'} + \frac{h^2_{K'}}{k^2}\|\nabla\cdot(\boldsymbol{J}-\boldsymbol{J}_h)\|^2_{K'}\right\}, \tag{6.29}$$

and the contrast coefficient $\mathcal{K}_K := \max\left\{\frac{\varepsilon_{\max,K^{\rm f}}}{\varepsilon_{\min,K^{\rm vvv}}}, \frac{\nu_{\max,K^{\rm f}}}{\nu_{\min,K^{\rm vvv}}}\right\}$.

Proof. Fix $K \in \mathcal{T}_h$. The proof contains three parts, respectively dedicated to providing upper bounds for $\eta_{K,\rm div}$, $\eta_{K,\rm curl}$ and $\eta_{K,\rm nc}$.

(i) The proof that $\eta_{K,\rm div} \le C(\kappa_{\mathcal{T}_h})k^{\frac{3}{2}}\left(\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},K^{\rm v}} + \operatorname{osc}_{K^{\rm v}}\right)$ can be found in [12, Lemma 3.5]. This proof is established for conforming edge finite elements, but it holds verbatim in the discontinuous Galerkin setting.

(ii) For $\eta_{K,\rm curl}$, we need to slightly adapt the proof from [12, Lemma 3.6]. The volumic residual and jump term in $\eta_{K,\rm curl}$ are estimated separately.

(iia) For the volume term, we introduce $\boldsymbol{r}_K := \boldsymbol{J}_K + \omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h - \nabla\times(\boldsymbol{\nu}\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}_h))|_K$ and $\boldsymbol{v}_K = b_K\boldsymbol{r}_K$, with $\boldsymbol{J}_K$ arbitrary in $\mathbb{P}_{k,d}$. We observe that $\boldsymbol{v}_K$ vanishes on ∂K, so that, letting $\boldsymbol{v}_h$ be the zero-extension of $\boldsymbol{v}_K$ to D, we infer that

$$\begin{aligned}\|b_K^{\frac{1}{2}}\boldsymbol{r}_K\|^2_K &= (\boldsymbol{r}_K,\boldsymbol{v}_K)_K \\ &= (\boldsymbol{J}_K,\boldsymbol{v}_K)_K - b_{\sharp\rm s}(\boldsymbol{E}_h,\boldsymbol{v}_h) = b_{\sharp\rm s}(\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{v}_h) - (\boldsymbol{J}-\boldsymbol{J}_K,\boldsymbol{v}_K)_K.\end{aligned}$$

For the first term, we employ $b_K \le 1$ and (6.25) to show that

$$\begin{aligned}&|b_{\sharp\rm s}(\boldsymbol{E}-\boldsymbol{E}_h,\boldsymbol{v}_h)| \\ &\le \omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},K}\omega\epsilon^{\frac{1}{2}}_{\max,K}\|\boldsymbol{v}_K\|_K + \|\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},K}\nu^{\frac{1}{2}}_{\max,K}\|\nabla\times\boldsymbol{v}_K\|_K \\ &\le C(\kappa_{\mathcal{T}_h})\left((\omega\epsilon^{\frac{1}{2}}_{\max,K})\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},K} + \nu^{\frac{1}{2}}_{\max,K}\frac{k}{h_K}\|\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},K}\right)\|b_K^{\frac{1}{2}}\boldsymbol{r}_K\|_K \\ &\le C(\kappa_{\mathcal{T}_h})\nu^{\frac{1}{2}}_{\max,K}\frac{k}{h_K}\left(\frac{\omega h_K}{k\vartheta_K}\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},K} + \|\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},K}\right)\|b_K^{\frac{1}{2}}\boldsymbol{r}_K\|_K.\end{aligned}$$

The second term is simply estimated as follows

$$\begin{aligned}|(\boldsymbol{J}-\boldsymbol{J}_K,\boldsymbol{v}_K)_K| &\le \|\boldsymbol{J}-\boldsymbol{J}_K\|_K\|\boldsymbol{v}_K\|_K \\ &\le C(\kappa_{\mathcal{T}_h})\left(\nu^{\frac{1}{2}}_{\max,K}\frac{k}{h_K}\right)\nu^{-\frac{1}{2}}_{\max,K}\frac{h_K}{k}\|\boldsymbol{J}-\boldsymbol{J}_K\|_K\|b_K^{\frac{1}{2}}\boldsymbol{r}_K\|_K.\end{aligned}$$

Combining these two bounds leads to

$$\begin{aligned}&\nu^{-\frac{1}{2}}_{\max,K}\frac{h_K}{k}\|b_K^{\frac{1}{2}}\boldsymbol{r}_K\|_K \\ &\qquad\le C(\kappa_{\mathcal{T}_h})\left\{\left(1+\frac{\omega h_K}{k\vartheta_K}\right)\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\sharp,K} + \nu^{-\frac{1}{2}}_{\max,K}\frac{h_K}{k}\|\boldsymbol{J}-\boldsymbol{J}_K\|_K\right\},\end{aligned}$$

and therefore

$$\begin{aligned}
&\nu_{\max,K}^{-\frac{1}{2}}\frac{h_K}{k}\|\boldsymbol{r}_K\|_K \\
&\quad\le C(\kappa_{\mathcal{T}_h})k\nu_{\max,K}^{-\frac{1}{2}}\frac{h_K}{k}\|b_K^{\frac{1}{2}}\boldsymbol{r}_K\|_K \\
&\quad\le C(\kappa_{\mathcal{T}_h})k\left\{\left(1+\frac{\omega h_K}{k\vartheta_K}\right)\|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!\|_{\sharp,K}+\nu_{\max,K}^{-\frac{1}{2}}\frac{h_K}{k}\|\boldsymbol{J}-\boldsymbol{J}_K\|_K\right\}.
\end{aligned}$$

Invoking the triangle inequality leads to

$$\begin{aligned}
(6.30)\quad &\nu_{\max,K}^{-\frac{1}{2}}\frac{h_K}{k}\|\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_K \\
&\quad\le C(\kappa_{\mathcal{T}_h})k\left\{\left(1+\frac{\omega h_K}{k\vartheta_K}\right)\|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!\|_{\sharp,K}+\nu_{\max,K}^{-\frac{1}{2}}\frac{h_K}{k}\|\boldsymbol{J}-\boldsymbol{J}_K\|_K\right\}.
\end{aligned}$$

(iib) For the jump term, we introduce $\boldsymbol{r}_F := [\![\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h)]\!]_F^{\mathrm{c}}$ and $\boldsymbol{v}_F := \mathcal{E}_F(b_F\boldsymbol{r}_F) \in \boldsymbol{H}_0^1(\widetilde{F})$. Since $\boldsymbol{v}_F \in \boldsymbol{H}_0^1(\widetilde{F})$ and $\boldsymbol{v}_F|_F = b_F\boldsymbol{r}_F$, we have

$$\begin{aligned}
(6.31)\quad \|b_F^{\frac{1}{2}}\boldsymbol{r}_F\|_F^2 &= (\boldsymbol{r}_F,\boldsymbol{v}_F)_F \\
&= (\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h)),\boldsymbol{v}_F)_{\widetilde{F}}-(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h),\nabla_h\times\boldsymbol{v}_F)_{\widetilde{F}} \\
&= (\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h),\nabla\times\boldsymbol{v}_F)_{\widetilde{F}}-(\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)),\boldsymbol{v}_F)_{\widetilde{F}}.
\end{aligned}$$

For the first term, we can immediately write that

$$\begin{aligned}
|(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h),\nabla\times\boldsymbol{v}_F)_{\widetilde{F}}| &\le \nu_{\max,\widetilde{F}}^{\frac{1}{2}}\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},\widetilde{F}}\|\nabla\times\boldsymbol{v}_F\|_{\widetilde{F}} \\
&\le C(\kappa_{\mathcal{T}_h})kh_F^{-\frac{1}{2}}\nu_{\max,\widetilde{F}}^{\frac{1}{2}}\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},\widetilde{F}}\|b_F^{\frac{1}{2}}\boldsymbol{r}_F\|_F,
\end{aligned}$$

where we employed (6.27). We infer that

$$\begin{aligned}
(6.32)\quad &\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\left(\frac{h_F}{k}\right)^{\frac{1}{2}}|(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h),\nabla\times\boldsymbol{v}_F)_{\widetilde{F}}| \\
&\qquad\le C(\kappa_{\mathcal{T}_h})k^{\frac{1}{2}}\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},\widetilde{F}}\|b_F^{\frac{1}{2}}\boldsymbol{r}_F\|_F.
\end{aligned}$$

For the second term, we first observe that

$$\begin{aligned}
\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)) &= \boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h)) \\
&= \omega^2\boldsymbol{\epsilon}(\boldsymbol{E}-\boldsymbol{E}_h)+\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h)),
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\|\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\widetilde{F}} \\
&\quad\le \omega\epsilon_{\max,\widetilde{F}}^{\frac{1}{2}}\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},\widetilde{F}}+\|\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_{\widetilde{F}} \\
&\quad\le \nu_{\max,\widetilde{F}}^{\frac{1}{2}}\frac{k}{h_F}\left(\frac{\omega h_F}{k\vartheta_{\widetilde{F}}}\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},\widetilde{F}}\right. \\
&\qquad\qquad\left.+\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\frac{h_F}{k}\|\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_{\widetilde{F}}\right).
\end{aligned}$$

Combining this last estimate with (6.27) and invoking the Cauchy–Schwarz inequality gives

(6.33)

$$\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\left(\frac{h_F}{k}\right)^{\frac{1}{2}}|(\nabla\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)),\boldsymbol{v}_F)_{\widetilde{F}}| \leq C(\kappa_{\mathcal{T}_h})\times$$
$$\left(\frac{\omega h_F}{k\vartheta_{\widetilde{F}}}\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},\widetilde{F}}+\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\frac{h_F}{k}\|\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_{\widetilde{F}}\right)k^{\frac{1}{2}}\|b_F^{\frac{1}{2}}\boldsymbol{r}_F\|_F.$$

We can now plug (6.32) and (6.33) in (6.31), leading to

$$\begin{aligned}
&\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\left(\frac{h_F}{k}\right)^{\frac{1}{2}}\|b_F^{\frac{1}{2}}\boldsymbol{r}_F\|_F \\
&\qquad\leq C(\kappa_{\mathcal{T}_h})k^{\frac{1}{2}}\left(\frac{\omega h_F}{k\vartheta_{\widetilde{F}}}\omega\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{\epsilon},\widetilde{F}}\right.\\
&\qquad\qquad\qquad\left.+\,\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\frac{h_F}{k}\|\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_{\widetilde{F}}\right)\\
&\qquad\quad+C(\kappa_{\mathcal{T}_h})k^{\frac{1}{2}}\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{\nu},\widetilde{F}}\\
&\qquad\leq C(\kappa_{\mathcal{T}_h})k^{\frac{1}{2}}\left(1+\frac{\omega h_F}{k\vartheta_{\widetilde{F}}}\right)|\!|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!|\!|_{\sharp,\widetilde{F}}\\
&\qquad\quad+C(\kappa_{\mathcal{T}_h})k^{\frac{1}{2}}\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\frac{h_F}{k}\|\boldsymbol{J}+\omega^2\boldsymbol{\epsilon}\boldsymbol{E}_h-\nabla_h\times(\boldsymbol{\nu}\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h))\|_{\widetilde{F}}\\
&\qquad\leq C(\kappa_{\mathcal{T}_h})k^{\frac{1}{2}}\left(\left(1+\frac{\omega h_F}{k\vartheta_{\widetilde{F}}}\right)|\!|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!|\!|_{\sharp,\widetilde{F}}+\nu_{\max,\widetilde{F}}^{-\frac{1}{2}}\frac{h_F}{k}\|\boldsymbol{J}-\boldsymbol{J}_h\|_{\widetilde{F}}\right),
\end{aligned}$$

owing to (6.30) and the shape-regularity of the mesh. Recalling the definition of K^{f}, it follows from (6.26) that

$$\eta_{K,\mathrm{curl}} \leq C(\kappa_{\mathcal{T}_h})k^{\frac{3}{2}}\left(\left(1+\frac{\omega h_K}{k\vartheta_{K^{\mathrm{f}}}}\right)|\!|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!|\!|_{\sharp,K^{\mathrm{e}}}+\nu_{\max,K^{\mathrm{f}}}^{-\frac{1}{2}}\frac{h_K}{k}\|\boldsymbol{J}-\boldsymbol{J}_h\|_{K^{\mathrm{f}}}\right).$$

(iii) For the last part of the estimator, we simply use that $\eta_{K,\mathrm{nc}}^2 \leq |\!|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!|\!|_{\sharp,K}^2 + \eta_{K,\mathrm{nc}}^2 = |\!|\!|\boldsymbol{E}-\boldsymbol{E}_h|\!|\!|_{\dagger,K}^2$.

□

7. Bound on approximation and divergence conformity factors

In this section, we show that the factors introduced in Section 3.4 tend to zero as the mesh is refined. For positive real numbers A and B, we abbreviate as $A \lesssim B$ the inequality $A \leq CB$ with a generic (nondimensional) constant C whose value can change at each occurrence as long as it is independent of the mesh size, the frequency parameter ω, and, whenever relevant, any function involved in the bound. The constant C can depend on the shape-regularity of the mesh, the polynomial degree k, the (global) contrast in the coefficients (i.e. $\epsilon_{\max}/\epsilon_{\min}$ and $\nu_{\max}/\nu_{\min}$), and the shape of the domain D (but not on its size).

For simplicity, we focus on the case where the parameters $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ are piecewise constant on a fixed polyhedral partition of D, and refer the reader to Remark 7.4 for the more general case where the material coefficients are just bounded from above and from below away from zero. Under the above assumption of piecewise constant coefficients (see [5, 16, 29]), there exists $s \in (0, \frac{1}{2})$ such that, for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ with $\boldsymbol{\epsilon v} \in \boldsymbol{H}(\mathrm{div} = 0; D)$, and for all $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}; D)$ with $\boldsymbol{\nu}^{-1}\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div} = 0; D)$, we have $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^s(D)$ with

$$|\boldsymbol{v}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \nu_{\min}^{-\frac{1}{2}} \|\nabla_0 \times \boldsymbol{v}\|_{\boldsymbol{\nu}}, \qquad |\boldsymbol{w}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \epsilon_{\max}^{\frac{1}{2}} \|\nabla \times \boldsymbol{w}\|_{\boldsymbol{\epsilon}^{-1}}. \tag{7.1}$$

The length scale ℓ_D is equal to the diameter of D and is introduced for dimensional consistency. We will also need commuting (quasi-)interpolation operators, $\mathcal{J}_h^{\mathrm{c}} : \boldsymbol{H}(\mathbf{curl}; D) \to \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ and $\mathcal{J}_h^{\mathrm{d}} : \boldsymbol{H}(\mathrm{div}; D) \to \boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}; D)$ such that $\nabla \times (\mathcal{J}_h^{\mathrm{c}}(\boldsymbol{v})) = \mathcal{J}_h^{\mathrm{d}}(\nabla \times \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; D)$ and

$$\|\boldsymbol{v} - \mathcal{J}_h^{\mathrm{c}}(\boldsymbol{v})\| \lesssim h^s |\boldsymbol{v}|_{\boldsymbol{H}^s(D)}, \qquad \|\boldsymbol{w} - \mathcal{J}_h^{\mathrm{d}}(\boldsymbol{w})\| \lesssim \|\boldsymbol{w}\|, \tag{7.2}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; D) \cap \boldsymbol{H}^s(D)$ and all $\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div}; D)$. Since we are working on simplicial meshes, we can invoke the operators devised in [21, Section 20] (see also [3, 14, 15, 38]) using edge (Nédélec) and Raviart–Thomas finite elements.

Proposition 7.1 (Primal approximation factor). *Let* γ_{ap} *be defined in* (3.15a). *We have*

$$\gamma_{\mathrm{ap}} \lesssim (1 + \beta_{\mathrm{st}}) \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^s,$$

where β_{st} *is the stability constant introduced in* (2.8).

Proof. See [11, Lemma 5.1]. □

Lemma 7.2 (Dual approximation factor). *Let* $\gamma_{\mathrm{ap}*}$ *be defined in* (3.15b). *We have*

$$\gamma_{\mathrm{ap}*} \lesssim (1 + \beta_{\mathrm{st}}) \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^s. \tag{7.3}$$

Proof. Consider a right-hand side $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$ and the associated adjoint solution $\boldsymbol{\zeta_\theta} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ defined in (3.13). Set $\boldsymbol{\phi_\theta} := \boldsymbol{\nu} \nabla_0 \times \boldsymbol{\zeta_\theta}$. The strong form of Maxwell's equations ensures that

$$\nabla \times \boldsymbol{\phi_\theta} = \boldsymbol{\epsilon \theta} + \omega^2 \boldsymbol{\epsilon \zeta_\theta},$$

so that $\boldsymbol{\phi_\theta} \in \boldsymbol{H}(\mathbf{curl}; D)$ with

$$\|\nabla \times \boldsymbol{\phi_\theta}\|_{\boldsymbol{\epsilon}^{-1}} \leq \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}} + \omega^2 \|\boldsymbol{\zeta_\theta}\|_{\boldsymbol{\epsilon}} \leq (1 + \beta_{\mathrm{st}}) \|\boldsymbol{\theta}\|_{\boldsymbol{\epsilon}}. \tag{7.4}$$

Besides, since $\boldsymbol{\nu}^{-1} \boldsymbol{\phi_\theta} \in \boldsymbol{H}_0(\mathrm{div} = 0; D)$, we infer that $\boldsymbol{\phi_\theta} \in \boldsymbol{H}^s(D)$ with

$$|\boldsymbol{\phi_\theta}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \epsilon_{\max}^{\frac{1}{2}} \|\nabla \times \boldsymbol{\phi_\theta}\|_{\boldsymbol{\epsilon}^{-1}}. \tag{7.5}$$

We are now ready to bound $\gamma_{\mathrm{ap}*}$. We notice that

$$\begin{aligned}\gamma_{\mathrm{ap}*}^2 &\leq \omega^2 \|\!|\boldsymbol{\phi_\theta} - \mathcal{J}_h^{\mathrm{c}}(\boldsymbol{\phi_\theta})\|\!|_{\mathrm{ap}*}^2 \\ &= \omega^2 \|\boldsymbol{\phi_\theta} - \mathcal{J}_h^{\mathrm{c}}(\boldsymbol{\phi_\theta})\|_{\boldsymbol{\nu}^{-1}}^2 + \omega^2 \|\tilde{\nu}^{-\frac{1}{2}} \tilde{h} (\nabla \times \boldsymbol{\phi_\theta} - \mathcal{J}_h^{\mathrm{d}}(\nabla \times \boldsymbol{\phi_\theta}))\|^2,\end{aligned}$$

where we employed the commuting property satisfied by $\mathcal{J}_h^{\mathrm{c}}$ and $\mathcal{J}_h^{\mathrm{d}}$. We bound the two terms on the right-hand side. For the first term, invoking (7.2) and (7.5), we have

$$\begin{aligned}\omega \|\boldsymbol{\phi_\theta} - \mathcal{J}_h^{\mathrm{c}}(\boldsymbol{\phi_\theta})\|_{\boldsymbol{\nu}^{-1}} &\lesssim \omega \nu_{\min}^{-\frac{1}{2}} h^s |\boldsymbol{\phi_\theta}|_{\boldsymbol{H}^s(D)} \\ &\lesssim \omega \ell_D^{1-s} \frac{h^s}{\vartheta_{\min}} \|\nabla \times \boldsymbol{\phi_\theta}\|_{\boldsymbol{\epsilon}^{-1}} = \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^s \|\nabla \times \boldsymbol{\phi_\theta}\|_{\boldsymbol{\epsilon}^{-1}}.\end{aligned} \tag{7.6}$$

For the second term, using (7.2), we can write

$$\omega\|\tilde{\nu}^{-\frac{1}{2}}\tilde{h}(\nabla\times\boldsymbol{\phi}_{\boldsymbol{\theta}} - \mathcal{J}_h^{\mathrm{d}}(\nabla\times\boldsymbol{\phi}_{\boldsymbol{\theta}}))\| \le \omega h\nu_{\min}^{-\frac{1}{2}}\|\nabla\times\boldsymbol{\phi}_{\boldsymbol{\theta}} - \mathcal{J}_h^{\mathrm{d}}(\nabla\times\boldsymbol{\phi}_{\boldsymbol{\theta}})\|$$

$$\lesssim \omega h\nu_{\min}^{-\frac{1}{2}}\|\nabla\times\boldsymbol{\phi}_{\boldsymbol{\theta}}\| \lesssim \frac{\omega h}{\vartheta_{\min}}\|\nabla\times\boldsymbol{\phi}_{\boldsymbol{\theta}}\|_{\boldsymbol{\epsilon}^{-1}}. \tag{7.7}$$

Combining (7.4), (7.6) and (7.7) and observing that $h \le \ell_D$ proves the assertion. □

Lemma 7.3 (Divergence conformity factor). *Let γ_{dv} be defined in* (3.16)*. We have*

$$\gamma_{\mathrm{dv}} \lesssim \left(\frac{\omega\ell_D}{\vartheta_{\min}}\right)^{1-s}\left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}. \tag{7.8}$$

Proof. Let $\boldsymbol{v}_h \in \boldsymbol{X}_h^{\mathrm{b}}$ and consider an arbitrary $\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$. Since $\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h) = \boldsymbol{0}$ by assumption, we have

$$\begin{aligned}\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h) &= \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}}) + \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}})\\ &= \boldsymbol{\Pi}_0^{\mathrm{c}}((I - \boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}})) + \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}})).\end{aligned}$$

Multiplying by ω, and invoking the triangle inequality and the $\boldsymbol{L}_{\boldsymbol{\epsilon}}^2$-stability of the projection operators, we infer that

$$\omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h)\|_{\boldsymbol{\epsilon}} \le \omega\|\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}}\|_{\boldsymbol{\epsilon}} + \omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}}))\|_{\boldsymbol{\epsilon}}. \tag{7.9}$$

For the second term on the right-hand side of (7.9), we invoke [11, Lemma 5.2] which gives

$$\begin{aligned}\omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}}))\|_{\boldsymbol{\epsilon}} &\lesssim \left(\frac{\omega\ell_D}{\vartheta_{\min}}\right)^{1-s}\left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}\|\nabla_0\times(\boldsymbol{v}_h^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}}))\|_{\boldsymbol{\nu}}\\ &\lesssim \left(\frac{\omega\ell_D}{\vartheta_{\min}}\right)^{1-s}\left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}(\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h)\|_{\boldsymbol{\nu}} + \|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}})\|_{\boldsymbol{\nu}}),\end{aligned}$$

where we used that $\nabla_0\times\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{v}_h^{\mathrm{c}}) = \boldsymbol{0}$ and the triangle inequality. For the first term on the right-hand side of (7.9), we observe that

$$\omega\|\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}}\|_{\boldsymbol{\epsilon}} \le \frac{\omega h}{\vartheta_{\min}}\|\tilde{\nu}^{\frac{1}{2}}\tilde{h}^{-1}(\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}})\| \le \frac{\omega h}{\vartheta_{\min}}|\!|\!|\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}}|\!|\!|_{\mathrm{ap}}.$$

Combining this bound with the above two bounds and since $h \le \ell_D$, this gives

$$\omega\|\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{v}_h)\|_{\boldsymbol{\epsilon}} \lesssim \left(\frac{\omega\ell_D}{\vartheta_{\min}}\right)^{1-s}\left(\frac{\omega h}{\vartheta_{\min}}\right)^{s}\left(\|\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v}_h)\|_{\boldsymbol{\nu}}^2 + |\!|\!|\boldsymbol{v}_h - \boldsymbol{v}_h^{\mathrm{c}}|\!|\!|_{\mathrm{ap}}^2\right)^{\frac{1}{2}}.$$

The bound (7.8) follows by taking the minimum over $\boldsymbol{v}_h^{\mathrm{c}} \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)$ and recalling the definition (3.12) of the $|\cdot|_{\mathrm{nc}}$-seminorm. □

Remark 7.4 (Rough coefficients). It is possible to show that the above factors tend to zero (without a specific algebraic rate) when the material coefficients are just bounded from above and from below away from zero. The proof hinges on a compactness result from [40]. We refer the reader to the discussions in [11, Section 5.2] which can be readily extended to the present nonconforming setting. Details are skipped for brevity.

Remark 7.5 (Improved decay rates). One can show improved convergence rates for the approximation factors by assuming extra regularity on the material coefficients and the domain. We refer the reader to the discussion in [11, Remark 5.5] for more details.

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