The devising and analysis of Hybrid High-Order (HHO) methods

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Outline

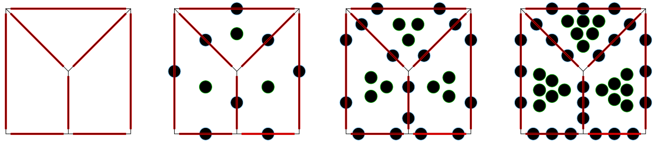
1. In a nutshell
2. Scalar elliptic PDEs
3. Computational Mechanics
4. Multiscale HHO

This talk is based on

- the two references where HHO methods were introduced
  - **[Di Pietro, AE, Lemaire, CMAM, 2014]** for diffusion
  - **[Di Pietro, AE, CMAME, 2015]** for elasticity
- plus a few more recent developments
  - [Cockburn, Di Pietro, AE, 2016] for building bridges
  - [Cicuttin, AE, Lemaire, 2016-] for multiscale HHO

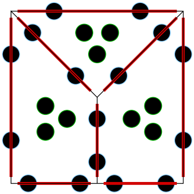
Discrete unknowns

- HHO methods attach discrete unknowns to **mesh faces**
  - one polynomial of order $k \geq 0$ on each mesh face
  - local variation of polynomial degree is possible
- HHO methods also use **cell unknowns**
  - elimination by static condensation (local Schur complement)
  - one simple choice is equal order (other choices are possible)



Devising HHO methods

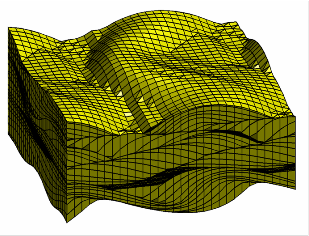
- Devising from **primal formulation** using two ideas
- Local reconstruction operator to build a higher-order field in each cell from cell and face unknowns
- Local stabilization operator to connect cell and face unknowns

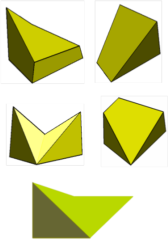


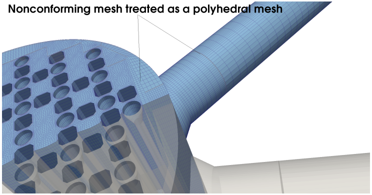
Main features

- Genericity
  - **arbitrary** polynomial order $k \geq 0$, energy-error of order $O(h^{k+1})$
  - construction **independent** of space dimension
  - **polytopal** meshes
  - library using **generic programming** [Cicuttin, Di Pietro, AE 17]
- Physical fidelity
  - local **conservation**
  - **robustness** (dominant advection, quasi-incompressible elasticity)
- Attractive costs
  - more compact stencil than, e.g., vertex-based methods
  - global system size $\sim k^2\#(\text{faces})$ (vs. $\sim (k+1)^3\#(\text{cells})$ for dG)
- **Industrial collaborations**: EDF R&D, BRGM, CEA, Saint Gobain

Some industrial motivations for POEMs (Courtesy IFPEN, EDF R&D)







Hybrid methods: a family portrait

- Lower-order methods
  - **Mimetic Finite Differences (MFD)** [Brezzi, Lipnikov, Shashkov 05]; recent textbook [Beirão da Veiga, Lipnikov, Manzini 14]
  - **Hybrid Finite Volumes (HFV)** [Droniou, Eymard 06; Eymard, Gallouet, Herbin 10]
  - **Crouzeix–Raviart FEM** (1973), see also [Di Pietro, Lemaire 15]
- Unifying settings
  - [Droniou, Eymard, Gallouet, Herbin 10, 13] for Hybrid Mimetic Mixed **(HMM)** methods and **Gradient Schemes**
  - [Bonelle, AE 14] for Compatible Discrete Operator **(CDO)** schemes
- Higher-order methods
  - **Hybridizable DG (HDG)** [Cockburn, Gopalakrishnan, Lazarov 09]
  - Weak Galerkin [Wang & Ye 13], equivalent to HDG [Cockburn 16]
  - **Higher-order MFD** [Lipnikov, Manzini 14]; **non-conforming VEM** [Ayuso, Lipnikov, Manzini 16]

Scalar elliptic PDEs

- Poisson model problem
- Advection-diffusion
- Building bridges

Poisson model problem

- Let $f \in L^2(D)$ and let $D \subset \mathbb{R}^d$ be a Lipschitz polyhedron
- Find $u \in V := H_0^1(D)$ s.t.

$$(\nabla u, \nabla w)_{\boldsymbol{L}^2(D)} = (f, w)_{L^2(D)} \quad \forall w \in V$$

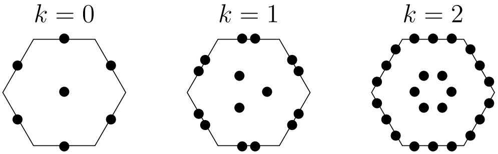
- Other BC's can be considered as well

Local viewpoint

- Consider a **mesh** $\mathcal{T} = \{T\}$ of D and a **polynomial degree** $k \geq 0$
  - broken polynomial space $\mathbb{P}^k(\partial T)$ (one poly. on each face of T)
- For all $T \in \mathcal{T}$, the discrete unknowns are

$$(v_T, v_{\partial T}) \in \mathbb{P}^k(T) \times \mathbb{P}^k(\partial T)$$

- Example in a hexagonal cell



Reconstruction operator

$$\mathrm{R}_T^{k+1} : \underbrace{\mathbb{P}^k(T) \times \mathbb{P}^k(\partial T)}_{\text{cell and face unknowns}} \longrightarrow \underbrace{\mathbb{P}^{k+1}(T)}_{\text{higher-order polynomial}}$$

- Let $(v_T, v_{\partial T}) \in \mathbb{P}^k(T) \times \mathbb{P}^k(\partial T)$
- Then $r := \mathrm{R}_T^{k+1}(v_T, v_{\partial T}) \in \mathbb{P}^{k+1}(T)$ solves, $\forall w \in \mathbb{P}^{k+1}(T)$,

$$\begin{aligned}(\nabla r, \nabla w)_{\boldsymbol{L}^2(T)} &= -(v_T, \Delta w)_{L^2(T)} + (v_{\partial T}, \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)} \\ &= (\nabla v_T, \nabla w)_{\boldsymbol{L}^2(T)} - (v_T - v_{\partial T}, \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)}\end{aligned}$$

  together with the mean-value condition $(r, 1)_{L^2(T)} = (v_T, 1)_{L^2(T)}$
  - well-posed local Neumann problem
  - Cholesky factorization of local stiffness matrix in $\mathbb{P}^{k+1}(T)$
  - **fully parallelizable**
- Note that $R_T^{k+1}(v_T, v_{T|\partial T}) = v_T$
  - no order pickup if trace and face values coincide

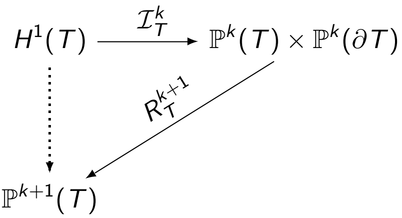
Reduction and approximation operators

- Reconstruction operator $\mathrm{R}_T^{k+1} : \mathbb{P}^k(T) \times \mathbb{P}^k(\partial T) \to \mathbb{P}^{k+1}(T)$
- Reduction operator $\mathcal{I}_T^k : H^1(T) \to \mathbb{P}^k(T) \times \mathbb{P}^k(\partial T)$ s.t.

$$\mathcal{I}_T^k(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v))$$

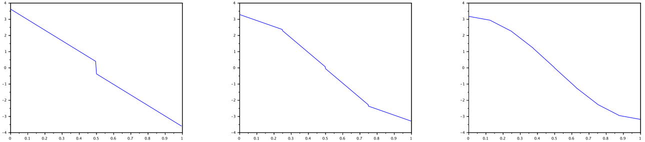
  with L^2-orthogonal projectors onto $\mathbb{P}^k(T)$ and $\mathbb{P}^k(\partial T)$ resp.

- $\mathrm{R}_T^{k+1} \circ \mathcal{I}_T^k : H^1(T) \to \mathbb{P}^{k+1}(T)$ acts as an approximation operator

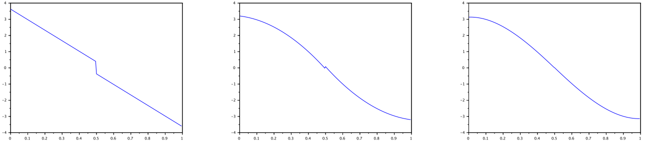


Numerical illustration

- h-approximation of $\cos(\pi x)$, $N = 2, 4, 8$, $k = 0$



- p-approximation of $\cos(\pi x)$, $N = 2$, $k = 0, 1, 2$



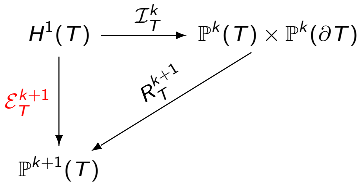
Elliptic projector

- Elliptic projector $\mathcal{E}_T^{k+1} : H^1(T) \to \mathbb{P}^{k+1}(T)$
  - $(\nabla(\mathcal{E}_T^{k+1}(v) - v), \nabla w)_{\boldsymbol{L}^2(T)} = 0,\ \forall w \in \mathbb{P}^{k+1}(T)$
  - $(\mathcal{E}_T^{k+1}(v) - v, 1)_{L^2(T)} = 0$

- The following holds true: $\mathrm{R}_T^{k+1} \circ \mathcal{I}_T^k = \mathcal{E}_T^{k+1}$; indeed $\forall w \in \mathbb{P}^{k+1}(T)$,

$$\begin{aligned}(\nabla R_T^{k+1}(\mathcal{I}_T^k(v)), \nabla w)_{\boldsymbol{L}^2(T)} &= (\nabla R_T^{k+1}(\Pi_T^k(v), \Pi_{\partial T}(v)), \nabla w)_{\boldsymbol{L}^2(T)} \\ &= -(\Pi_T^k(v), \Delta w)_{L^2(T)} + (\Pi_{\partial T}^k(v), \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)} \\ &= -(v, \Delta w)_{L^2(T)} + (v, \boldsymbol{n}_T \cdot \nabla w)_{L^2(\partial T)} = (\nabla v, \nabla w)_{\boldsymbol{L}^2(T)}\end{aligned}$$

- In summary, we have



Stabilization

- However, $\{\nabla \mathrm{R}_T^{k+1}(v_T, v_{\partial T}) = \mathbf{0}\} \not\Rightarrow \{v_T = v_{\partial T} = \mathrm{cst}\}$
- We "connect" cell and face unknowns by a Least-Squares penalty on

$$\mathrm{S}_{\partial T}^k(v_T, v_{\partial T}) = \Pi_{\partial T}^k\big(v_T - v_{\partial T} + (I - \Pi_T^k)(\mathrm{R}_T^{k+1}(v_T, v_{\partial T}))\big)$$

- The **higher-order term** is a **distinctive feature** of HHO methods
- Local mass matrices in $\mathbb{P}^k(T)$ and $\mathbb{P}^k(\partial T)$, **fully parallelizable**
- Note that $S_{\partial T}^k(v_T, v_{T|\partial T}) = 0$
  - hence, we have $S_{\partial T}^k(v_T, v_{\partial T}) = \tilde{S}_{\partial T}^k(v_T - v_{\partial T})$

$\mathbb{P}^{k+1}$-polynomial consistency

- Recall elliptic projector $\mathcal{E}_T^{k+1} : H^1(T) \to \mathbb{P}^{k+1}(T)$
- Recall reduction operator s.t. $\mathcal{I}_T^k(v) = (\Pi_T^k(v), \Pi_{\partial T}^k(v))$
- For all $v \in H^1(T)$, we have

$$\mathsf{S}_{\partial T}^k(\mathcal{I}_T^k(v)) = (\Pi_T^k - \Pi_{\partial T}^k)(v - \mathcal{E}_T^{k+1}(v))$$

Consequently, $\mathsf{S}_{\partial T}^k(\mathcal{I}_T^k(p)) = 0$, $\forall p \in \mathbb{P}^{k+1}(T)$

$$\begin{aligned}
\mathsf{S}_{\partial T}^k(\mathcal{I}_T^k(v)) &= \Pi_{\partial T}^k(\Pi_T^k(v) - \Pi_{\partial T}^k(v) + (I - \Pi_T^k)(\mathcal{E}_T^{k+1}(v))) \\
&= \Pi_{\partial T}^k(\Pi_T^k(v - \mathcal{E}_T^{k+1}(v)) - (\Pi_{\partial T}^k(v) - \mathcal{E}_T^{k+1}(v))) \\
&= \Pi_T^k(v - \mathcal{E}_T^{k+1}(v)) - \Pi_{\partial T}^k(v - \mathcal{E}_T^{k+1}(v))
\end{aligned}$$

since $\Pi_{\partial T}^k \Pi_T^k = \Pi_T^k$ and $\Pi_{\partial T}^k \Pi_{\partial T}^k = \Pi_{\partial T}^k$

- Without the higher-order term, $\mathsf{S}_{\partial T}^k(\mathcal{I}_T^k(p)) = 0$ only for $p \in \mathbb{P}^k(T)$

Local stability and boundedness

- Local bilinear form (with $\tau_{\partial T|F} \sim h_F^{-1}$ for all $F \subset \partial T$)

$$\hat{a}_T((v_T, v_{\partial T}), (w_T, w_{\partial T})) := \underbrace{(\nabla \mathrm{R}_T^{k+1}(v_T, v_{\partial T}), \nabla \mathrm{R}_T^{k+1}(w_T, w_{\partial T}))_{L^2(T)}}_{\text{Galerkin/reconstruction}}$$
$$+ \underbrace{(\tau_{\partial T} \mathrm{S}_{\partial T}^k(v_T, v_{\partial T}), \mathrm{S}_{\partial T}^k(w_T, w_{\partial T}))_{L^2(\partial T)}}_{\text{stabilization}}$$

- **Local stability and boundedness**:

$$\hat{a}_T((v_T, v_{\partial T}), (v_T, v_{\partial T})) \sim |(v_T, v_{\partial T})|^2_{\mathcal{H}^1(T)}$$

with the local seminorm

$$|(v_T, v_{\partial T})|^2_{\mathcal{H}^1(T)} = \|\nabla v_T\|^2_{L^2(T)} + \|\tau_{\partial T}^{\frac{1}{2}}(v_T - v_{\partial T})\|^2_{L^2(\partial T)}$$

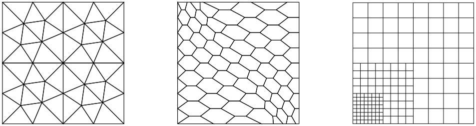
Note that $|(v_T, v_{\partial T})|_{\mathcal{H}^1(T)} = 0$ implies $v_T = v_{\partial T} = \text{cst}$

Interlude: variations on the cell unknowns

- Let $k \geq 0$ be the degree of the face unknowns
- Let $l \geq 0$ be the degree of the cell unknowns
- The equal-order case is $l = k$
- It is possible to choose $l = k - 1$ $(k \geq 1)$ while achieving the same stability and approximation properties
- It is possible to choose $l = k + 1$
  - no further gain in stability/approximation
  - simplified stabilization $S^k_{\partial T}(v_T, v_{\partial T}) = \Pi^k_{\partial T}(v_T - v_{\partial T})$, but more cell unknowns to eliminate
  - cf. [Lehrenfeld, Schöberl 10] stabilization for HDG

Assembling the discrete problem

- Mesh $\mathcal{M} = \{\mathcal{T}, \mathcal{F}\}$, cells collected in $\mathcal{T} = \{T\}$, faces in $\mathcal{F} = \{F\}$
  - polytopal cells and non-matching interfaces are possible



- The (global) discrete unknowns are in

$$(v_{\mathcal{T}}, v_{\mathcal{F}}) \in \mathcal{V}^k_{\mathcal{M}} := \mathbb{P}^k(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F})$$

  - one polynomial of order k per cell (or $l \in \{k-1, k, k+1\}$)
  - one polynomial of order k per face

- Let $(v_{\mathcal{T}}, v_{\mathcal{F}}) \in \mathcal{V}^k_{\mathcal{M}}$; the discrete unknowns attached to a cell $T \in \mathcal{T}$ and its faces $F \subset \partial T$ are denoted $(v_T, v_{\partial T})$

- To enforce homogeneous Dirichlet BCs, we restrict to $\mathcal{V}^k_{\mathcal{M},0}$ where global unknowns attached to boundary faces are set to zero

Discrete problem and local conservation

- The discrete problem is: Find $(u_{\mathcal{T}}, u_{\mathcal{F}}) \in \mathcal{V}^k_{\mathcal{M},0}$ s.t.

$$\sum_{T\in\mathcal{T}} \hat{a}_T((u_T, u_{\partial T}), (w_T, w_{\partial T})) = \sum_{T\in\mathcal{T}} (f, w_T)_{L^2(T)} \qquad (1)$$

  for all $(w_{\mathcal{T}}, w_{\mathcal{F}}) \in \mathcal{V}^k_{\mathcal{M},0}$

- Local conservation
  - for all $T \in \mathcal{T}$, there is a **numerical flux trace** $\phi_T \in \mathbb{P}^k(\partial T)$
  - testing (1) with $((p\delta_{T,T'})_{T'\in\mathcal{T}}, (0)_{F'\in\mathcal{F}})$ yields the **local balance**

$$(\nabla R_T^{k+1}(u_T, u_{\partial T}), \nabla p)_{L^2(T)} + (\phi_T, p)_{L^2(\partial T)} = (f, p)_{L^2(T)}, \quad \forall p \in \mathbb{P}^k(T)$$

  - testing (1) with $((0)_{T'\in\mathcal{T}}, (q\delta_{F,F'})_{F'\in\mathcal{F}})$, $\forall q \in \mathbb{P}^k(F)$, yields the **equilibration condition**

$$\phi_{T_1|F} + \phi_{T_2|F} = 0, \quad F = \partial T_1 \cap \partial T_2$$

  - we have $\phi_T = -\nabla R_T^{k+1}(u_T, u_{\partial T})\cdot \boldsymbol{n}_T + \alpha^{\mathrm{HHO}}_{\partial T}(u_T - u_{\partial T})$
  - see [Cockburn, Di Pietro, AE, 16]

Algebraic realization

- Ordering cell unknowns first and then face unknowns, we obtain the linear system

$$\begin{bmatrix} \mathbf{A}_{\mathcal{T}\mathcal{T}} & \mathbf{A}_{\mathcal{T}\mathcal{F}} \\ \mathbf{A}_{\mathcal{F}\mathcal{T}} & \mathbf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{\mathcal{T}} \\ \mathbf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\mathcal{T}} \\ \mathbf{0} \end{bmatrix}$$

- The system matrix is **SPD**
- Local elimination of cell unknowns
  - $\mathbf{A}_{\mathcal{T}\mathcal{T}}$ is block-diagonal $\rightarrow$ one can solve the Schur complement system in terms of face unknowns
  - size $\sim k^2 \#(\text{faces})$
  - compact stencil (two faces interact only if they belong to same cell)
  - can be interpreted as a **global transmission problem** [Cockburn 16]

Error analysis

- Stability and $\mathbb{P}^{k+1}$-consistency give (super)convergence
- $O(h^{k+1})$ energy-error estimate

$$\Big(\sum_{T\in\mathcal{T}} \|\nabla(u - \mathrm{R}_T^{k+1}(u_T, u_{\partial T}))\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \le c \Big(\sum_{T\in\mathcal{T}} h_T^{2(k+1)} |u|^2_{H^{k+2}(T)}\Big)^{\frac{1}{2}}$$

- Under (full) elliptic regularity, $O(h^{k+2})$ L^2-error estimate

$$\Big(\sum_{T\in\mathcal{T}} \|\Pi_T^k(u) - u_T\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \le c\, h \Big(\sum_{T\in\mathcal{T}} h_T^{2(k+1)} |u|^2_{H^{k+2}(T)}\Big)^{\frac{1}{2}}$$

Polytopal mesh regularity

- (Usual) assumption that each mesh cell is an agglomeration of **finitely many, shape-regular simplices**; we assume planar faces
- Polynomial approximation in polytopal cells in Sobolev norms
  - Poincaré–Steklov inequality:

$$\|v - \Pi_T^0(v)\|_{L^2(T)} \le C_{\mathrm{PS}} h_T \|\nabla v\|_{\boldsymbol{L}^2(T)}, \quad \forall v \in H^1(T)$$

    - $C_{\mathrm{PS}} = \pi^{-1}$ for convex T [Poincaré 1894; Steklov 1897; Payne, Weinberger 60 ($d = 2$), Bebendorf 03 ($d \ge 3$)]
    - on polytopal cells, combine PS on simplices with multiplicative trace inequality [Veeser, Verfürth 12; AE, Guermond 16]

$$\|v\|_{L^2(\partial T)} \le C_{\mathrm{MT}} \left(h_T^{-\frac{1}{2}} \|v\|_{L^2(T)} + \|v\|_{L^2(T)}^{\frac{1}{2}} \|\nabla v\|_{\boldsymbol{L}^2(T)}^{\frac{1}{2}} \right), \quad \forall v \in H^1(T)$$

  - higher-order polynomial approximation using Morrey's polynomial
  - this argument avoids a star-shapedness assumption on cells
  - both PS and MT inequalities allow for some **face degeneration**
- For further results on face degeneration, see [Cangiani, Georgoulis, Houston 14] and [Dong, PhD Thesis 2016]

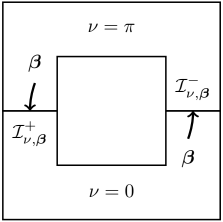
Péclet-robust advection-diffusion

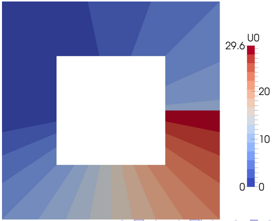
- Locally degenerate problem

$$\nabla\cdot(-\nu\nabla u + \beta u) + \mu u = f \quad \text{in } D$$

  with $\nu \geq 0$, $\beta = O(1)$ Lipschitz, $\mu > 0$

- Dirichlet BC on $\{\boldsymbol{x} \in \partial D \mid \nu > 0 \text{ or } \beta\cdot\boldsymbol{n} < 0\}$
- Exact solution **jumps** across diffusive/non-diffusive **interface** $I^-_{\nu,\beta}$ where β flows **from non-diffusive into diffusive region**
  - see [Gastaldi, Quarteroni 89; Di Pietro, AE, Guermond 08]





HHO discretization

- Main features of HHO method [Di Pietro, Droniou, AE 15]
  - arbitrary polynomial degree $k \geq 0$
  - local reconstruction in $\mathbb{P}^{k+1}(T)$ and local stabilization
  - local advective derivative reconstruction in $\mathbb{P}^k(T)$
  - local upwind stabilization between face and cell unknowns
  - weak enforcement of BC's à la Nitsche
  - no need to duplicate face unknowns on $I^-_{\nu,\beta}$
- Inf-sup stability norm

$$\|(v_T, v_{\partial T})\|_{\nu\beta,T} = |(v_T, v_{\partial T})|_{\nu,T} + |(v_T, v_{\partial T})|_{\beta,T} + h_T \beta_T^{-1} \|G^k_{\beta,T}(v_T, v_{\partial T})\|_{L^2(T)}$$

  - $|\cdot|_{\nu,T}$: ν-scaled diffusive norm
  - $|\cdot|_{\beta,T} = \|v_T\|_{L^2(T)} + \||\beta{\cdot}\boldsymbol{n}|^{\frac{1}{2}}(v_T - v_{\partial T})\|_{L^2(\partial T)}$

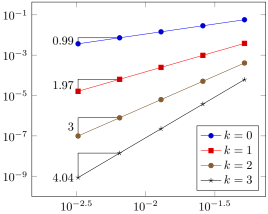
Error estimate and convergence

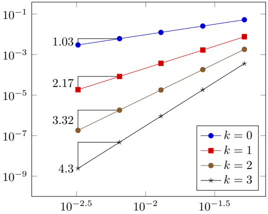
- Error estimate captures full range of Péclet numbers $\mathrm{Pe}_T \in [0,\infty]$

$$\left(\sum_{T\in\mathcal{T}} \|\mathcal{I}_T^k(u) - (u_T, u_{\partial T})\|^2_{\nu\beta,T}\right)^{\frac{1}{2}} \leq$$

$$c\left(\sum_{T\in\mathcal{T}} \nu_T h_T^{2(k+1)} |u|^2_{H^{k+2}(T)} + \beta_T \min(1,\mathrm{Pe}_T) h_T^{2(k+\frac{1}{2})} |u|^2_{H^{k+1}(T)}\right)^{\frac{1}{2}}$$

- Numerical decay rates for energy (left) and L^2 (right) errors





Building bridges

- Following [Cockburn, Di Pietro, AE 16], we bridge the viewpoints of **HHO, HDG & ncVEM**, focusing on Poisson's model problem
- **Usual presentation of HDG**
  - approximate the triple $(\boldsymbol{\sigma}, u, \lambda)$, with $\boldsymbol{\sigma} = -\nabla u$, $\lambda = u_{|\mathcal{F}}$
  - $(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}, \lambda_{\mathcal{F}}) \in \boldsymbol{S}_{\mathcal{T}} \times V_{\mathcal{T}} \times V_{\mathcal{F}}$ with local spaces $\boldsymbol{S}_T$, V_T, V_F
  - discrete HDG problem: $\forall(\boldsymbol{\tau}_T, w_T, \mu_F) \in \boldsymbol{S}_T \times V_T \times V_F$,

$$
\begin{aligned}
&(\boldsymbol{\sigma}_T, \boldsymbol{\tau}_T)_{\boldsymbol{L}^2(T)} - (u_T, \nabla\cdot\boldsymbol{\tau}_T)_{L^2(T)} + (\lambda_{\partial T}, \boldsymbol{\tau}_T\cdot\boldsymbol{n}_T)_{L^2(\partial T)} = 0 \\
&-(\boldsymbol{\sigma}_T, \nabla w_T)_{\boldsymbol{L}^2(T)} + (\phi_T, w_T)_{L^2(\partial T)} = (f, w_T)_{L^2(T)} \\
&(\phi_{T_1} + \phi_{T_2}, \mu_F)_{L^2(F)} = 0, \quad F = \partial T_1 \cap \partial T_2
\end{aligned}
$$

with the numerical flux trace

$$
\phi_T = \boldsymbol{\sigma}_T\cdot\boldsymbol{n}_T + \alpha_{\partial T}^{\mathrm{HDG}}(u_T - \lambda_{\partial T})
$$

HHO meets HDG

- HDG method specified through $\boldsymbol{S}_T$, V_T, V_F and $\alpha_{\partial T}^{\mathrm{HDG}}$
  - $\boldsymbol{S}_T = \mathbb{P}^k(T;\mathbb{R}^d)$, $V_T = \mathbb{P}^k(T)$, $V_F = \mathbb{P}^k(F)$, $\alpha_{\partial T}^{\mathrm{HDG}}$ acts pointwise

- HHO as HDG method
  - $\boldsymbol{S}_T = \nabla\mathbb{P}^{k+1}(T)$, V_T, V_F as above, $\alpha_{\partial T}^{\mathrm{HHO}} = \tilde{\mathrm{S}}_{\partial T}^{k*}(\tau_{\partial T}\tilde{\mathrm{S}}_{\partial T}^{k})$
    - $(\tilde{\mathrm{S}}_{\partial T}^{k*}(\lambda), \mu)_{L^2(\partial T)} = (\lambda, \tilde{\mathrm{S}}_{\partial T}^{k}(\mu))_{L^2(\partial T)}$
  - 1st HDG eq: $\boldsymbol{\sigma}_T = -\nabla\mathrm{R}_T^{k+1}(u_T, \lambda_{\partial T})$
  - 2nd HDG eq: HHO tested with $(w_T, 0)$
  - 3rd HDG eq: HHO tested with $(0, \mu_F)$

- Comments
  - HHO uses a **smaller flux space** (avoids curl-free functions)
  - HHO uses a **nonlocal stabilization design** for polytopal super-CV
  - alternative route to super-CV for HDG based on building space triplets by ***M*-decompositions** [Cockburn, Fu, Sayas 16]

HHO meets ncVEM (1)

- Consider the (finite-dimensional) subspace

$$V^{k+1}(T) = \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^k(T),\ \boldsymbol{n}_T \cdot \nabla v \in \mathbb{P}^k(\partial T)\}$$

- $\mathbb{P}^{k+1}(T) \subsetneq V^{k+1}(T)$; other functions are not explicitly known
- Recall reduction operator $\mathcal{I}_T^k(v) = (\Pi_T^k(v), \Pi_{\partial T}^k(v))$; then

$$\mathcal{I}_T^k : V^{k+1}(T) \longleftrightarrow \mathbb{P}^k(T) \times \mathbb{P}^k(\partial T) \text{ is an isomorphism}$$

- Let $\varphi \in V^{k+1}(T)$
  - $\mathcal{E}_T^{k+1}(\varphi) = R_T^{k+1}(\mathcal{I}_T^k(\varphi))$ is computable from the dof's $\mathcal{I}_T^k(\varphi)$ of φ
  - same for $\check{S}_{\partial T}^k(\varphi) = S_{\partial T}^k(\mathcal{I}_T^k(\varphi))$

HHO meets ncVEM (2)

- Consider the following bilinear form on $V^{k+1}(T) \times V^{k+1}(T)$:

$$\check{a}_T(\varphi,\psi) = (\nabla\mathcal{E}_T^{k+1}(\varphi), \nabla\mathcal{E}_T^{k+1}(\psi))_{\boldsymbol{L}^2(T)} + (\tau_{\partial T}\check{S}_{\partial T}^k(\varphi), \check{S}_{\partial T}^k(\psi))_{L^2(\partial T)}$$

where $\check{S}_{\partial T}^k(\varphi) = (\Pi_T^k - \Pi_{\partial T}^k)(\varphi - \mathcal{E}_T^{k+1}(\varphi))$

- We have $\check{a}_T(\varphi,\psi) = \hat{a}_T(\mathcal{I}_T^k(\varphi), \mathcal{I}_T^k(\psi))$
- Equivalent HHO stabilization with LS penalty on cells and on faces

$$(\mathcal{S}_T^k(\cdot), \mathcal{S}_T^k(\cdot))_{L^2(T)} + (\mathcal{S}_{\partial T}^k(\cdot), \mathcal{S}_{\partial T}^k(\cdot))_{L^2(\partial T)}$$

  - $\mathcal{S}_T^k(v_T, v_{\partial T}) = h_T^{-1}\Pi_T^k(v_T - R_T^{k+1}(v_T, v_{\partial T}))$
  - $\mathcal{S}_{\partial T}^k(v_T, v_{\partial T}) = h_{\partial T}^{-1/2}\Pi_{\partial T}^k(v_{\partial T} - R_T^{k+1}(v_T, v_{\partial T}))$
  - both operators can be rewritten as operators on $v_{T|\partial T} - v_{\partial T}$ ...
- Similar for mixed-order HHO (cell unknowns of order $k-1$)

$$V^{k+1}(T) = \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^{k-1}(T),\ \boldsymbol{n}_T\cdot\nabla v \in \mathbb{P}^k(\partial T)\}$$

$$\mathcal{I}_T^k(v) = (\Pi_T^{k-1}(v), \Pi_{\partial T}^k(v))$$

Computational mechanics

- Let us focus on linear elasticity
- Model problem $\nabla{\cdot}\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0}$ in D, $\boldsymbol{u} = \mathbf{0}$ on ∂D (for simplicity)

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\varepsilon(\boldsymbol{u}) + \lambda\, tr(\varepsilon(\boldsymbol{u}))\mathbb{I} \qquad \varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathsf{T}})$$

  with Lamé coefficients $\mu > 0$ and $\lambda + \frac{2}{3}\mu > 0$

- Incompressible limit $\lambda \gg \mu$
  - need to represent accurately non-trivial divergence-free fields
  - locking phenomenon for (lower-order) conforming FEM
  - IPDG [Hansbo, Larson 03]
  - HDG [Kabaria, Lew, Cockburn 15; Fu, Cockburn, Stolarski 16]
  - VEM [Beirão da Veiga, Brezzi, Marini 13; Gain, Talischi, Paulino 14; Beirão da Veiga, Lovadina, Mora 15]

HHO discretization

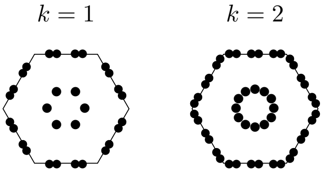
- Arbitrary polynomial degree $k \geq 1$
- Dimension-independent construction
- Displacement-based (primal) formulation, SPD linear system
- Error estimates: $O(h^{k+1})$ (energy-norm) and $O(h^{k+2})$ (L^2-norm, elliptic regularity) on polytopal meshes
- Robust in the incompressible limit
- Local traction balance in each mesh cell

Devising the HHO method

- Let $k \geq 1$, let T be a mesh cell; the discrete unknowns are

$$(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in \boldsymbol{\mathcal{V}}_T^k := \mathbb{P}^k(T;\mathbb{R}^d) \times \mathbb{P}^k(\partial T;\mathbb{R}^d)$$

- Example in a hexagonal cell



- **Lowest-order version**
  - 2D: 4#(faces) dofs $\rightarrow \sim$ 6#(cells) on triangles
  - 3D: 9#(faces) dofs $\rightarrow \sim$ 18#(cells) on tetrahedra
  - $\sim$ 25#(cells) for recent 3D DPG in [Carstensen, Hellwig 17]

Local displacement reconstruction

- Reconstruction of polynomial displacement in $\mathbb{P}^{k+1}(T;\mathbb{R}^d)$

$$\mathrm{R}_T^{k+1} : \underbrace{\mathcal{V}_T^k}_{\text{cell and face unknowns}} \longrightarrow \underbrace{\mathbb{P}^{k+1}(T;\mathbb{R}^d)}_{\text{higher-order polynomial}}$$

- Let $(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in \mathcal{V}_T^k$
- $\boldsymbol{r} := \mathrm{R}_T^{k+1}(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$ solves, $\forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T;\mathbb{R}^d)$,

$$(\varepsilon(\boldsymbol{r}), \varepsilon(\boldsymbol{w}))_{\boldsymbol{L}^2(T)} = (\varepsilon(\boldsymbol{v}_T), \varepsilon(\boldsymbol{w}))_{\boldsymbol{L}^2(T)} - (\boldsymbol{v} - \boldsymbol{v}_{\partial T}, \varepsilon(\boldsymbol{w})\boldsymbol{n}_T)_{\boldsymbol{L}^2(\partial T)}$$

  and the rigid-body motions of $\boldsymbol{r}$ are prescribed from $\boldsymbol{v}_T$ (translation) and $\boldsymbol{v}_{\partial T}$ (rotation)
  - symmetric strain-based reconstruction
  - well-posed local Neumann problem
  - Cholesky factorization of local stiffness matrices
  - **fully parallelizable**

Local divergence reconstruction

- $D_T^k : \mathcal{V}_T^k \to \mathbb{P}^k(T)$
- Let $(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in \mathcal{V}_T^k$
- $d := D_T^k(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in \mathbb{P}^k(T)$ solves, $\forall q \in \mathbb{P}^k(T)$,

$$(d, q)_{L^2(T)} = (\nabla{\cdot}\boldsymbol{v}_T, q)_{L^2(T)} - (\boldsymbol{v}_T - \boldsymbol{v}_{\partial T}, q\boldsymbol{n}_T)_{\boldsymbol{L}^2(\partial T)}$$

- **Commuting diagram property** (key for incompressible limit)

$$\begin{array}{ccc} \boldsymbol{H}^1(T) & \xrightarrow{\nabla\cdot} & L^2(T) \\ \downarrow \mathcal{I}_T^k & & \downarrow \Pi_T^k \\ \mathcal{V}_T^k & \xrightarrow{D_T^k} & \mathbb{P}^k(T) \end{array}$$

with reduction operator s.t. $\mathcal{I}_T^k(\boldsymbol{v}) := (\Pi_T^k(\boldsymbol{v}), \Pi_{\partial T}^k(\boldsymbol{v}))$

Discrete problem

- Exact bilinear form composed locally of

$$a_{\mu,T}(\boldsymbol{v},\boldsymbol{w}) = 2\mu(\varepsilon(\boldsymbol{v}),\varepsilon(\boldsymbol{w}))_{\boldsymbol{L}^2(T)}$$
$$a_{\lambda,T}(\boldsymbol{v},\boldsymbol{w}) = \lambda(\nabla\cdot\boldsymbol{v},\nabla\cdot\boldsymbol{w})_{L^2(T)}$$

- Discrete bilinear form composed locally of

$$\hat{a}_{\mu,T}((\boldsymbol{v}_T,\boldsymbol{v}_{\partial T}),(\boldsymbol{w}_T,\boldsymbol{w}_{\partial T})) = 2\mu(\varepsilon(\mathrm{R}_T^{k+1}(\boldsymbol{v}_T,\boldsymbol{v}_{\partial T})),\varepsilon(\mathrm{R}_T^{k+1}(\boldsymbol{w}_T,\boldsymbol{w}_{\partial T}))) + \mathrm{stab}$$
$$\hat{a}_{\lambda,T}((\boldsymbol{v}_T,\boldsymbol{v}_{\partial T}),(\boldsymbol{w}_T,\boldsymbol{w}_{\partial T})) = \lambda(D_T^k(\boldsymbol{v}_T,\boldsymbol{v}_{\partial T}),D_T^k(\boldsymbol{w}_T,\boldsymbol{w}_{\partial T}))_{L^2(T)}$$

with stab devised component-wise as in the scalar diffusive case

- Discrete problem assembled cell-wise
  - cell displacement unknowns eliminated by static condensation
  - global SPD matrix of size $\sim k^2\#(\text{faces})$ (3D)

Error analysis

- Stability and $\mathbb{P}^{k+1}$-consistency yield (super)convergence; the commuting diagram property yields λ-robustness
- $O(h^{k+1})$ energy-error estimate

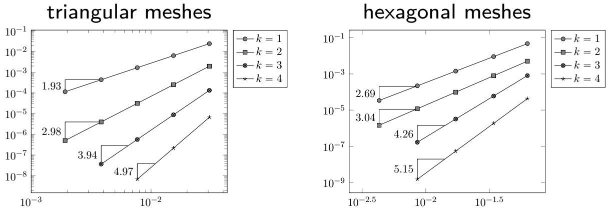
$$\Big(\sum_{T\in\mathcal{T}} 2\mu\|\varepsilon(\boldsymbol{u}-\mathrm{R}_T^{k+1}(\boldsymbol{u}_T,\boldsymbol{u}_{\partial T}))\|^2_{\boldsymbol{L}^2(T)}\Big)^{\frac12} \le$$

$$c\Big(\sum_{T\in\mathcal{T}} 2\mu h_T^{2(k+1)}|\boldsymbol{u}|^2_{\boldsymbol{H}^{k+2}(T)} + \lambda h_T^{2(k+1)}|\nabla\cdot\boldsymbol{u}|^2_{H^{k+1}(T)}\Big)^{\frac12}$$

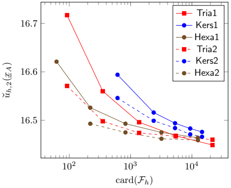
- $O(h^{k+2})$ $\boldsymbol{L}^2$-error estimate under (full) elliptic regularity

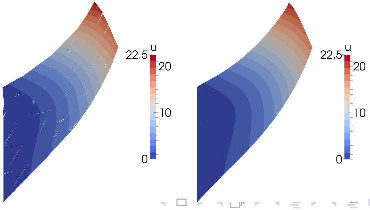
Numerical results

- **Convergence rates** with analytical solution, $\frac{\lambda}{\mu} = 10^3$



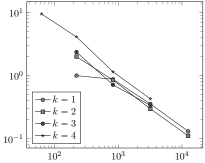
- **Cook's membrane test case**, $\mu = 0.375$ and $\lambda = 7.5\times10^6$
  - deformed configuration (22 vs. 4192 cells)





(Brief) performance assessment

- Assembly time τ_{ass} / solution time τ_{sol}, hexagonal meshes, $k \in \{1, 2, 3, 4\}$



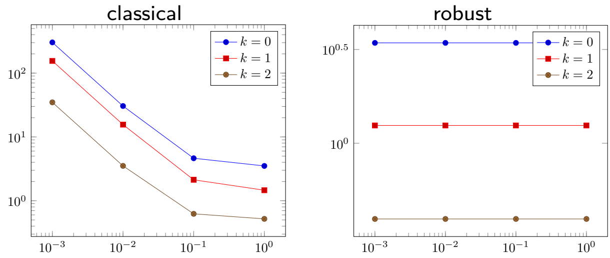
- $\tau_{\text{ass}}/\tau_{\text{sol}}$ decreases as $\sim (\#\text{dofs})^{-1}$
- rate and value fairly insensitive to k
- no parallelism exploited in assembly

Incompressible Stokes flows

- Model problem $-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$, $\nabla\cdot\boldsymbol{u} = 0$ in D
- Main features of HHO method [DP, AE, Linke, Schieweck 16]
  - arbitrary polynomial order $k \geq 0$
  - local velocity in $\mathbb{P}^k(T;\mathbb{R}^d) \times \mathbb{P}^k(\partial T;\mathbb{R}^d)$ and pressure in $\mathbb{P}^k(T)$
  - after static condensation, global saddle-point system of size $k^2\#(\text{faces}) + 1\#(\text{cells})$ (3D)
  - energy-velocity and L^2-pressure $O(h^{k+1})$ error estimates
  - $\boldsymbol{L}^2$-velocity $O(h^{k+2})$ error estimates under full elliptic regularity
  - local momentum and mass balance in each mesh cell
- Some recent literature on hybrid methods for Stokes flows
  - hybrid FE [Jeon, Park, Sheen 14]
  - HDG [Egger, Waluga 13; Cockburn, Sayas 14; Cockburn, Shi 14; Lehrenfeld, Schöberl 15], WG [Mu, Wang, Ye 15]

Large irrotational body forces

- Examples: Coriolis, centrifugal, electrokinetics [Linke 14]
- Pointwise divergence-free velocity reconstruction to test momentum balance; here, Raviart–Thomas reconstruction on tet meshes
- Velocity error for 3D Green–Taylor vortex flow vs. viscosity



A multiscale HHO method

- Find $u_\varepsilon \in V = H_0^1(D)$ s.t. $(\mathbb{A}_\varepsilon \nabla u_\varepsilon, \nabla w)_{\boldsymbol{L}^2(D)} = (f, w)_{L^2(D)}, \forall w \in V$
  - $f \in L^2(D)$ and Lipschitz polyhedron $D \subset \mathbb{R}^d$
  - $\mathbb{A}_\varepsilon$ unif. > 0, **oscillatory with length scale** $\varepsilon \ll \ell_D$
  - periodic case $\mathbb{A}_\varepsilon(\boldsymbol{x}) = \mathbb{A}(\boldsymbol{x}/\varepsilon)$, where $\mathbb{A}(\cdot)$ is $\mathbb{Z}^d$-periodic in $\mathbb{R}^d$
- Theoretical results (periodic setting) [Allaire 02]
  - $(\mathbb{A}_\varepsilon)_{\varepsilon>0}$ G-converges to $\mathbb{A}_0$ s.t. $[\mathbb{A}_0]_{ij} = \int_Q \mathbb{A}\,(\boldsymbol{e}_j + \nabla\mu_j)\cdot(\boldsymbol{e}_i + \nabla\mu_i)$
  - **corrector** $\mu_i \in H^1_{\mathrm{per}}(Q)$ s.t. $\nabla\cdot(\mathbb{A}(\nabla\mu_i + \boldsymbol{e}_i)) = 0$ on $Q = (0,1)^d$
  - **homogenized sol.** $u_0 \in V$, $(\mathbb{A}_0 \nabla u_0, \nabla w)_{\boldsymbol{L}^2(D)} = (f, w)_{L^2(D)}, \forall w \in V$
  - **first-order two-scale expansion**

$$\mathcal{L}^1_\varepsilon(u_0) = u_0 + \varepsilon \sum_{l\in\{1:d\}} \mu_l(\cdot/\varepsilon)\partial_l u_0$$

  - if $u_0 \in H^2(D) \cap W^{1,\infty}(D)$, $\|\mathbb{A}_\varepsilon^{\frac{1}{2}} \nabla(u_\varepsilon - \mathcal{L}^1_\varepsilon(u_0))\|_{\boldsymbol{L}^2(D)} = O(\varepsilon^{\frac{1}{2}})$ [Jikov, Kozlov, Oleinik 94]

msFEM

- The goal is to approximate the **oscillatory** solution u_ε on a **coarse** mesh $\mathcal{T}_H$ with $\varepsilon \le H \le \ell_D$
  - if $H \le \varepsilon$, any mono-scale method can be used
- The main idea is to replace the usual FEM basis functions by **oscillatory basis functions** pre-computed offline [Hou, Wu 97]
- The error analysis assumes a periodic setting
  - in lowest-order case, $\|\mathbb{A}_\varepsilon^{\frac{1}{2}}\nabla(u_\varepsilon - u_{\varepsilon,H})\|_{L^2(D)} = O(\varepsilon^{\frac{1}{2}} + H + (\varepsilon/H)^{\frac{1}{2}})$ where the last term is the **resonance error**, visible when $H \sim \varepsilon$
  - for higher-order msFEM [Allaire, Brizzi 05], the upper bound is $O(\varepsilon^{\frac{1}{2}} + H^k + (\varepsilon/H)^{\frac{1}{2}})$; increasing k pays off: the upper bound is minimal for higher H and takes a smaller value
  - the resonance error can be tamed by **oversampling** [Efendiev, Hou, Wu 00] and essentially eliminated by local decompositions with **larger supports** [Malqvist, Peterseim 14; Kornhuber, Yserentant 16]

msHHO

- Let $\mathcal{T}_H$ be a coarse (polytopal) mesh with $\varepsilon \leq H \leq \ell_D$
- The main ideas of the msHHO method are
  - **oscillatory cell and face basis functions** precomputed offline
  - discrete unknowns are polynomials of order $k \geq 0$ on mesh faces and of order $l \geq 0$ on mesh cells **(as in mono-scale case)**
  - equal-order ($l = k$) and mixed-order ($l = k - 1$) variants
  - reconstruction operator based on **oscillatory basis functions**
- For $k = 0$, the method is analyzed in [Le Bris, Legoll, Lozinski 13] under the name msFEM à la Crouzeix–Raviart
  - error estimate $O(\varepsilon^{\frac{1}{2}} + H + (\varepsilon/H)^{\frac{1}{2}})$ on simplicial meshes
- The msHHO method provides an extension to arbitrary order with an analysis on polytopal meshes
- See [Cicuttin, AE, Lemaire 16] and Lemaire's talk this Friday

Oscillatory basis functions

- Cell basis functions (for $k \geq 1$)

$$\varphi_{\varepsilon,T}^{k+1,i} = \underset{\substack{\varphi\in H^1(T)\\ \Pi_F^k(\varphi)=0,\ \forall F\subset\partial T}}{\arg\min} \int_T \left[\frac{1}{2}\mathbb{A}_\varepsilon\nabla\varphi\cdot\nabla\varphi - \Phi_T^{k-1,i}\varphi\right]$$

where $(\Phi_T^{k-1,i})_{1\leq i\leq N_d^{k-1}}$ is a basis of $\mathbb{P}^{k-1}(T)$

- Face basis functions (for $k \geq 0$)

$$\varphi_{\varepsilon,T,F}^{k+1,j} = \underset{\substack{\varphi\in H^1(T)\\ \Pi_F^k(\varphi)=\Phi_F^{k,j}\\ \Pi_\sigma^k(\varphi)=0,\ \forall\sigma\subset\partial T\setminus\{F\}}}{\arg\min} \int_T \left[\frac{1}{2}\mathbb{A}_\varepsilon\nabla\varphi\cdot\nabla\varphi\right]$$

where $(\Phi_F^{k,j})_{1\leq j\leq N_{d-1}^k}$ is a basis of $\mathbb{P}^k(F)$

- The oscillatory functions form a basis of

$$V_\varepsilon^{k+1}(T) = \{v \in H^1(T) \mid \nabla\cdot(\mathbb{A}_\varepsilon\nabla v) \in \mathbb{P}^{k-1}(T),\ \boldsymbol{n}_T\cdot\mathbb{A}_\varepsilon\nabla v \in \mathbb{P}^k(\partial T)\}$$

Multiscale reconstruction operator

$$R^{k+1}_{\varepsilon,T} : \underbrace{\mathbb{P}^l(T) \times \mathbb{P}^k(\partial T)}_{\text{cell and face unknowns}} \longrightarrow \underbrace{V^{k+1}_{\varepsilon}(T)}_{\text{oscillatory function}}$$

- Let $(v_T, v_{\partial T}) \in \mathbb{P}^l(T) \times \mathbb{P}^k(\partial T)$
- Then $r := R^{k+1}_{\varepsilon,T}(v_T, v_{\partial T}) \in V^{k+1}_{\varepsilon}(T)$ solves, $\forall w \in V^{k+1}_{\varepsilon}(T)$,

  $$(\mathbb{A}_\varepsilon \nabla r, \nabla w)_{\boldsymbol{L}^2(T)} = -(v_T, \nabla\cdot(\mathbb{A}_\varepsilon \nabla w))_{L^2(T)} + (v_{\partial T}, \boldsymbol{n}_T \cdot \mathbb{A}_\varepsilon \nabla w)_{L^2(\partial T)}$$

  together with the mean-value condition $(r, 1)_{L^2(T)} = (v_T, 1)_{L^2(T)}$
- In practice, the oscillatory basis functions are precomputed offline by meshing T (of size $H > \varepsilon$) with subcells of size $h < \varepsilon$, and using a mono-scale method to approximate the minimizers

Mixed-order msHHO

- Let us take $l = k - 1$
- The local msHHO bilinear form is

$$\hat{a}_{\varepsilon,T}(\cdot,\cdot) = (\mathbb{A}_\varepsilon \nabla R_{\varepsilon,T}^{k+1}(\cdot), \nabla R_{\varepsilon,T}^{k+1}(\cdot))_{\boldsymbol{L}^2(T)}$$

- **No need for stabilization**: we explored the whole space $V_\varepsilon^{k+1}(T)$ to resolve the oscillatory nature of the problem
- Error estimate: if $u_0 \in H^{k+2}(D) \cap W^{1,\infty}(D)$ and $\mu_l \in W^{1,\infty}(\mathbb{R}^d)$,

$$\left(\sum_{T \in \mathcal{T}_H} \|\mathbb{A}_\varepsilon^{\frac{1}{2}} \nabla (u_\varepsilon - R_{\varepsilon,T}^{k+1}(u_T, u_{\partial T}))\|_{\boldsymbol{L}^2(T)}^2 \right)^{\frac{1}{2}} \leq c\left(\varepsilon^{\frac{1}{2}} + H^{k+1} + (\varepsilon/H)^{\frac{1}{2}}\right)$$

Key approximation lemma

- $V_\varepsilon^{k+1}(T)$ no longer contains high-order polynomials ...
- Recall homogenized solution u_0 and first-order expansion $\mathcal{L}_\varepsilon^1(u_0)$
- Key idea: build a function $\mathcal{J}_{\varepsilon,T}^{k+1}(u_0) \in V_\varepsilon^{k+1}(T)$ from $\Pi_T^{k+1}(u_0)$
  - $\nabla\cdot(\mathbb{A}_\varepsilon \nabla \mathcal{J}_{\varepsilon,T}^{k+1}(u_0)) = \nabla\cdot(\mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0)) \in \mathbb{P}^{k-1}(T)$
  - $\boldsymbol{n}_T\cdot\mathbb{A}_\varepsilon \nabla \mathcal{J}_{\varepsilon,T}^{k+1}(u_0) = \boldsymbol{n}_T\cdot\mathbb{A}_0 \nabla \Pi_T^{k+1}(u_0) \in \mathbb{P}^k(\partial T)$
- Assume $u_0 \in H^{k+2}(T) \cap W^{1,\infty}(T)$ and $\mu_l \in W^{1,\infty}(\mathbb{R}^d)$; then

$$\|\mathbb{A}_\varepsilon^{\frac{1}{2}} \nabla(\mathcal{L}_\varepsilon^1(u_0) - \mathcal{J}_{\varepsilon,T}^{k+1}(u_0))\|_{\boldsymbol{L}^2(T)} \le c\left(\varepsilon |u_0|_{H^2} + H_T^{k+1}|u_0|_{H^{k+2}} + (\varepsilon|\partial T|)^{\frac{1}{2}}|u_0|_{W^{1,\infty}}\right)$$

proof combines HHO analysis tools with [Jikov, Kozlov, Oleinik 94]

Equal-order msHHO

- Let us take $l = k$
- We still reconstruct in $V_\varepsilon^{k+1}(T)$ using $R_{\varepsilon,T}^{k+1}$
- The local msHHO bilinear form is

$$\hat{a}_{\varepsilon,T}(\cdot,\cdot) = (\mathbb{A}_\varepsilon \nabla R_{\varepsilon,T}^{k+1}(\cdot), \nabla R_{\varepsilon,T}^{k+1}(\cdot))_{\boldsymbol{L}^2(T)} + (\tau_{\partial T} S_{\varepsilon,\partial T}^{k}(\cdot), S_{\varepsilon,\partial T}^{k}(\cdot))_{L^2(\partial T)}$$

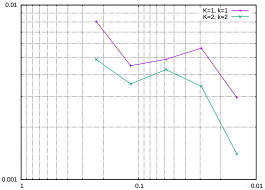
$$S_{\varepsilon,\partial T}^{k}(v_T, v_{\partial T}) = v_T - \Pi_T^k(R_{\varepsilon,T}^{k+1}(v_T, v_{\partial T}))$$

  - stabilization needed because we reconstruct in $V_{\varepsilon,T}^{k+1}$ and not in $\tilde{V}_{\varepsilon,T}^{k+1} = \{v \in H^1(T) \mid \nabla{\cdot}(\mathbb{A}_\varepsilon \nabla v) \in \mathbb{P}^k(T),\ \boldsymbol{n}_T{\cdot}\mathbb{A}_\varepsilon \nabla v \in \mathbb{P}^k(\partial T)\}$
  - stabilization can be avoided by computing additional oscillatory basis functions to span $\tilde{V}_{\varepsilon,T}^{k+1}$; see [Le Bris, Legoll, Lozinski 14] for $k = 0$ (one additional basis function)

- **Same error estimate** as in mixed-order case

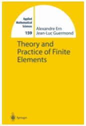
Numerical illustration

- Periodic setting with $\mathbb{A}_\varepsilon(x,y) = a(x/\varepsilon, y/\varepsilon)\mathbb{I}_2$,, $\varepsilon = \pi/150 \approx 0.021$
$$a(x,y) = 1 + 100\cos^2(\pi x)\sin^2(\pi y)$$
- Hierarchical triangular meshes of size $H_l = 0.23 \times 2^{-l}$, $l \in \{0{:}11\}$
  - resonance expected for $H_4 < \varepsilon < H_3$
- Mixed-order msHHO with $k \in \{1,2\}$
- Oscillatory basis functions computed with (equal-order) mono-scale HHO of degree $k' = k$, refining the coarse mesh cells six times
- Energy-error as a function of H_l for $k \in \{1,2\}$



Conclusions

- **Outlook**
  - further HHO developments: *hp*-analysis, Leray–Lions, Cahn–Hilliard, Navier–Stokes, fractured porous media ($\rightarrow$ talk by Di Pietro)
  - HHO for nonlinear mechanics ($\rightarrow$ talk by M. Botti, PhD of Pignet)
  - nonplanar faces ($\rightarrow$ talk by L. Botti)
- New Finite Element book with J.-L. Guermond (Spring 2018)
  - 10 chap. of 50 pages $\rightarrow$ 60 chap. of 12/14 pages (incl. exercises)



$\rightarrow$

Thank you for your attention