Edge finite element approximation of Maxwell's equations in heterogeneous domains

Alexandre Ern and Jean-Luc Guermond

Université Paris-Est, CERMICS, ENPC and INRIA, Paris

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Outline

- Maxwell's equations
- Finite element setting
- Analysis tools
- Coercivity revisited
- $\boldsymbol{L}^2$-error estimate

Maxwell's equations

- Lipschitz polyhedron $D \subsetneq \mathbb{R}^3$ with connected boundary ∂D
- Model problem: Find $\boldsymbol{A} : D \to \mathbb{C}^3$ s.t.

$$\widetilde{\mu}\boldsymbol{A} + \nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f}, \qquad \boldsymbol{A}_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0}$$

  - homogeneous Dirichlet BC for simplicity
- Assumptions on $\widetilde{\mu}$ and κ
  - **boundedness:** $\widetilde{\mu}, \kappa \in L^\infty(D;\mathbb{C})$, set $\mu_\sharp = \|\widetilde{\mu}\|_{L^\infty}$, $\kappa_\sharp = \|\kappa\|_{L^\infty}$
  - **positivity:** there are real numbers θ, $\mu_\flat > 0$, $\kappa_\flat > 0$ s.t.

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re(\mathrm{e}^{i\theta}\widetilde{\mu}(\boldsymbol{x})) \geq \mu_\flat, \qquad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re(\mathrm{e}^{i\theta}\kappa(\boldsymbol{x})) \geq \kappa_\flat$$

  - **heterogeneous medium**: $\widetilde{\mu}$ and κ can have jumps, but are pcw. smooth ($W^{1,\infty}$) on a Lipschitz partition of D
  - define contrast factors $\mu_{\sharp/\flat} = \mu_\sharp/\mu_\flat$, $\kappa_{\sharp/\flat} = \kappa_\sharp/\kappa_\flat$
- Assumptions on source term: $\boldsymbol{f} \in \boldsymbol{L}^2(D)$ and $\nabla\cdot\boldsymbol{f} = 0 \Longrightarrow$

$$\nabla\cdot(\widetilde{\mu}\boldsymbol{A}) = 0$$

Two examples

- Time-harmonic regime (frequency ω)
- Source current $\boldsymbol{j}_s$
- **Helmholtz problem: $\boldsymbol{A} = \boldsymbol{E}$**

$$\widetilde{\mu} = -\omega^2\epsilon + i\omega\sigma, \quad \kappa = \mu^{-1}, \quad \boldsymbol{f} = -i\omega\boldsymbol{j}_s$$

ϵ: electric permittivity, μ: magnetic permeability, σ: electric conductivity

- **Eddy-current problem: $\boldsymbol{A} = \boldsymbol{H}$**

$$\widetilde{\mu} = i\omega\mu, \quad \kappa = \sigma^{-1}, \quad \boldsymbol{f} = \nabla\times(\sigma^{-1}\boldsymbol{j}_s)$$

Basic functional setting

- $\boldsymbol{V}_0 = \boldsymbol{H}_0(\mathrm{curl}; D) = \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(D),\ \boldsymbol{v}_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0}\}$
  - norm $\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{curl};D)} = \|\boldsymbol{v}\|^2_{L^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{v}\|^2_{L^2(D)}$
  - ℓ_D is a characteristic length of D (for dimensional coherence)

- Weak formulation: Find $\boldsymbol{A} \in \boldsymbol{V}_0$ s.t. $a(\boldsymbol{A}, \boldsymbol{b}) = \ell(\boldsymbol{b})$, $\forall \boldsymbol{b} \in \boldsymbol{V}_0$

$$a(\boldsymbol{A}, \boldsymbol{b}) = \int_D (\widetilde{\mu}\boldsymbol{A}\cdot\overline{\boldsymbol{b}} + \kappa\nabla\times\boldsymbol{A}\cdot\nabla\times\overline{\boldsymbol{b}})\,\mathrm{d}x, \quad \ell(\boldsymbol{b}) = \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}\,\mathrm{d}x$$

- $a(\cdot,\cdot)$ is bounded and coercive on $\boldsymbol{V}_0 \Longrightarrow$ the weak problem is well-posed (Lax–Milgram Lemma)

$$Re(\mathrm{e}^{i\theta}a(\boldsymbol{b}, \boldsymbol{b})) \geq \min(\mu_\flat, \ell_D^{-2}\kappa_\flat)\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}$$

- Coercivity parameter **not robust w.r.t.** $\mu_\flat$; this is relevant
  - in the low-frequency limit for the eddy-current problem
  - in the limit $\sigma \ll \omega\epsilon$ with $\kappa \in \mathbb{R}$ for the Helmholtz problem

Control on the divergence

- Since $\nabla\cdot\boldsymbol{f} = 0$, we have $\nabla\cdot(\widetilde{\mu}\boldsymbol{A}) = 0$, so that

$$\boldsymbol{A} \in \boldsymbol{X}_{0\widetilde{\mu}} = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid (\widetilde{\mu}\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in M_0\}, \quad M_0 = H^1_0(D)$$

- **Poincaré(–Steklov) inequality**

$$\exists \check{C}_{\mathrm{P},D} > 0 \ \text{ s.t. } \ \check{C}_{\mathrm{P},D}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \le \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\widetilde{\mu}}$$

  - $\check{C}_{\mathrm{P},D}$ depends on D and contrast factor $\mu_{\sharp/\flat}$
  - in the scalar-valued case, $\check{C}_{\mathrm{P},D}\ell_D^{-1}\|v\|_{L^2(D)} \le \|\nabla v\|_{\boldsymbol{L}^2(D)}$ for all $v \in H^1_0(D)$ [Poincaré 1894], [Steklov 1897]

- On $\boldsymbol{X}_{0\widetilde{\mu}}$, the coercivity of $a(\cdot,\cdot)$ is **robust w.r.t.** $\mu_\flat$

$$\begin{aligned}\Re(e^{i\theta}a(\boldsymbol{b},\boldsymbol{b})) &\ge \mu_\flat\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \kappa_\flat\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} \ge \kappa_\flat\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)}\\ &\ge \frac{1}{2}\kappa_\flat(\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \check{C}^2_{\mathrm{P},D}\ell_D^{-2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)})\\ &\ge \frac{1}{2}\kappa_\flat\ell_D^{-2}\min(1, \check{C}^2_{\mathrm{P},D})\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}\end{aligned}$$

Regularity pickup on $\boldsymbol{A}$

- Recall that ∂D is connected and $\widetilde{\mu}$ is pcw. smooth
- $\exists s > 0$ and $\check{C}_D$ (depending on D and contrast factor $\mu_{\sharp/\flat}$) s.t.

$$\check{C}_D \ell_D^{-1} \|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} \le \|\nabla \times \boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\widetilde{\mu}}$$

  with $\|\cdot\|_{\boldsymbol{H}^s} = (\|\cdot\|_{L^2}^2 + \ell_D^{2s} |\cdot|_{H^s}^2)^{1/2}$ and Sobolev–Slobodeckij seminorm

- $\Longrightarrow \boldsymbol{A} \in \boldsymbol{H}^s(D)$, $s > 0$, and typically $s < \frac{1}{2}$
- Proofs in [Jochmann 99] and [Bonito, Guermond, Luddens 13]
  - earlier results by [Birman, Solomyak 87; Costabel 90] for constant $\widetilde{\mu}$

$$\boldsymbol{X}_0 = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid \nabla \cdot \boldsymbol{b} = 0\} \hookrightarrow \boldsymbol{H}^s(D)$$

    with $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1]$ for a Lipschitz polyhedron [Amrouche, Bernardi, Dauge, Girault 98]

Regularity pickup on $\nabla\times\boldsymbol{A}$ (1)

- Assume that D is **simply connected**
- Recall that κ is pcw. smooth
- Let $\boldsymbol{V} = \boldsymbol{H}(\mathrm{curl}; D)$ and define

$$\boldsymbol{X}_{*\kappa^{-1}} = \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D) \mid (\kappa^{-1}\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in M_*\}$$

with $M_* := \{q \in H^1(D) \mid (q,1)_{L^2(D)} = 0\}$

- $\exists s' > 0$ and $\check{C}'_D$ (depending on D and contrast factor $\kappa_{\sharp/\flat}$) s.t.

$$\check{C}'_D \ell_D^{-1} \|\boldsymbol{b}\|_{\boldsymbol{H}^{s'}(D)} \leq \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{*\kappa^{-1}}$$

Regularity pickup on $\nabla\times\boldsymbol{A}$ (2)

- The field $\boldsymbol{R} = \kappa\nabla\times\boldsymbol{A}$ is in $\boldsymbol{X}_{*\kappa^{-1}}$, so that $\boldsymbol{R} \in \boldsymbol{H}^{s'}(D)$
- The material property κ being pcw. smooth, the following multiplier property holds: $\exists\tau > 0$, $C_{\kappa^{-1}} > 0$ s.t.

$$|\kappa^{-1}\boldsymbol{\xi}|_{\boldsymbol{H}^\tau(D)} \leq C_{\kappa^{-1}}|\boldsymbol{\xi}|_{\boldsymbol{H}^\tau(D)}, \quad \forall\boldsymbol{\xi} \in \boldsymbol{H}^\tau(D)$$

- Letting $s'' = \min(s', \tau) > 0$, we conclude that

$$\nabla\times\boldsymbol{A} \in \boldsymbol{H}^{s''}(D)$$

- **In summary, $\boldsymbol{A} \in \boldsymbol{H}^s(D)$ and $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^{s''}(D)$, $s, s'' > 0$,** and typically $s, s'' < \frac{1}{2}$

Finite element setting

- Shape-regular sequence of affine simplicial meshes $(\mathcal{T}_h)_{h>0}$
- De Rham sequence for canonical FE spaces

$$P^{\mathrm{g}}(\mathcal{T}_h) \xrightarrow{\nabla} \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \xrightarrow{\nabla\times} \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) \xrightarrow{\nabla\cdot} P^{\mathrm{b}}(\mathcal{T}_h)$$

  - Lagrange/Nédélec/Raviart–Thomas/dG FEM spaces
  - conforming in $H^1(D)/\boldsymbol{H}(\mathrm{curl}; D)/\boldsymbol{H}(\mathrm{div}; D)/L^2(D)$
  - degree $(k+1)/k/k/k$

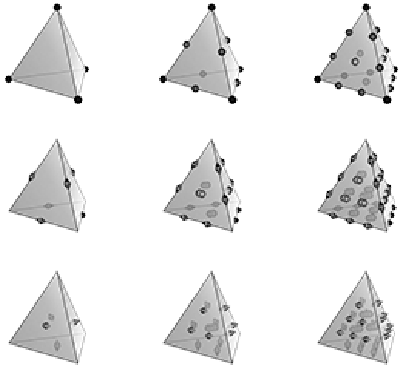
- Similar sequence with BCs

$$P_0^{\mathrm{g}}(\mathcal{T}_h) \xrightarrow{\nabla} \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) \xrightarrow{\nabla\times} \boldsymbol{P}_0^{\mathrm{d}}(\mathcal{T}_h) \xrightarrow{\nabla\cdot} P_0^{\mathrm{b}}(\mathcal{T}_h)$$

with $P_0^{\mathrm{g}}(\mathcal{T}_h) = P^{\mathrm{g}}(\mathcal{T}_h) \cap H_0^1(D)$, $\boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) = \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}; D)$, etc.

- Unified notation: $P(\mathcal{T}_h), P_0(\mathcal{T}_h)$ with $\mathbb{R}^q$-valued functions, $q \in \{1, 3\}$

Periodic table of finite elements [Arnold & Logg 14]



H^1

point evaluation	$(k \geq 1)$
edge integral	$(k \geq 2)$
face integral	$(k \geq 3)$
cell integral	$(k \geq 4)$

$\boldsymbol{H}(\mathrm{curl})$

edge integral	$(k \geq 0)$
face integral	$(k \geq 1)$
cell integral	$(k \geq 2)$

$\boldsymbol{H}(\mathrm{div})$

face integral	$(k \geq 0)$
cell integral	$(k \geq 1)$

Maxwell's equations

- Conforming edge FEM approximation in $\boldsymbol{V}_{h0} = \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) \subsetneq \boldsymbol{V}_0$
- Discrete problem: Find $\boldsymbol{A}_h \in \boldsymbol{V}_{h0}$ s.t. $a(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h)$, $\forall \boldsymbol{b}_h \in \boldsymbol{V}_{h0}$
- The discrete problem is well-posed (Lax–Milgram Lemma)
- **Main questions** to be addressed
  - $\mu_\flat$-robust coercivity in the discrete setting?
  - error estimates for $\boldsymbol{A} \in \boldsymbol{H}^s(D)$, $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^{s''}(D)$ when $s, s'' \in (0, \frac{1}{2})$?

Robust coercivity

- Since $\nabla P_0^{\mathrm{g}}(\mathcal{T}_h) \subsetneq \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h)$, we do have a discrete control on the divergence of $\boldsymbol{A}_h$

$$\boldsymbol{A}_h \in \boldsymbol{X}_{h0\widetilde{\mu}} = \{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid (\widetilde{\mu}\boldsymbol{b}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in P_0^{\mathrm{g}}(\mathcal{T}_h)\}$$

- The subtlety is that $\boldsymbol{X}_{h0\widetilde{\mu}}$ **is not a subspace** of $\boldsymbol{X}_{0\widetilde{\mu}}$ ...
- We need to establish a discrete PS inequality in $\boldsymbol{X}_{h0\widetilde{\mu}}$
  - usually, one invokes a **discrete compactness argument** [Kikuchi 89; Caorsi, Fernandes, Raffetto 00; Monk & Demkowicz 01]
  - this can be entirely bypassed by using some recently-devised commuting quasi-interpolation operators

Error estimates

- The canonical interpolation operators commute with differential operators ... but have **poor stability properties**
- For edge elements, stability only holds in $\boldsymbol{H}^s(D)$, $s > 1$ $(d = 3)$
  - using (subtle) trace theorems from [Amrouche et al. 98] shows stability in $\{\boldsymbol{v} \in \boldsymbol{H}^s(D), s > \frac{1}{2}, \nabla\times\boldsymbol{v} \in L^p(D), p > 2\}$ [Boffi, Gastaldi 06]
  - **regularity barrier** $s > \frac{1}{2}$ still remains ...
- To approximate fields in $\boldsymbol{H}^s$, $s > 0$, we invoke the recently-devised averaging quasi-interpolation operators from [AE, Guemond, 15-17]

FE analysis tools

- **Commuting quasi-interpolation**
  - [Schöberl 01; Christiansen & Winther 08; AE, Guermond, CMAM 16]
- **Averaging quasi-interpolation**
  - [AE, Guermond, arXiv 15, M2AN 17]

Commuting quasi-interpolation

There exist operators $\mathcal{J}_h : L^1(D;\mathbb{R}^q) \to P(\mathcal{T}_h)$ s.t.

- $\mathcal{J}_h$ leaves $P(\mathcal{T}_h)$ pointwise invariant ($\mathcal{J}_h \circ \mathcal{J}_h = \mathcal{J}_h$)
- $\mathcal{J}_h$ is L^p-stable for all $p \in [1,\infty]$
- $\mathcal{J}_h$ **commutes with the standard differential operators**

$$\begin{array}{ccccccc}
H^1(D) & \xrightarrow{\nabla} & \boldsymbol{H}(\mathrm{curl};D) & \xrightarrow{\nabla\times} & \boldsymbol{H}(\mathrm{div};D) & \xrightarrow{\nabla\cdot} & L^2(D) \\
\downarrow \mathcal{J}_h^{\mathrm{g}} & & \downarrow \mathcal{J}_h^{\mathrm{c}} & & \downarrow \mathcal{J}_h^{\mathrm{d}} & & \downarrow \mathcal{J}_h^{\mathrm{b}} \\
P^{\mathrm{g}}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P^{\mathrm{b}}(\mathcal{T}_h)
\end{array}$$

A similar construction is possible with **boundary prescription**

$$\mathcal{J}_{h0} : L^1(D;\mathbb{R}^q) \to P_0(\mathcal{T}_h)$$

Approximation

- **Stability and invariance imply approximation**
- For all $v \in L^p(D;\mathbb{R}^q)$,

$$\begin{aligned}\|v-\mathcal{J}_h(v)\|_{L^p(D;\mathbb{R}^q)} &= \inf_{v_h\in P(\mathcal{T}_h)} \|v-v_h-\mathcal{J}_h(v-v_h)\|_{L^p(D;\mathbb{R}^q)} \\ &\le \inf_{v_h\in P(\mathcal{T}_h)} (1+\|\mathcal{J}_h\|_{\mathcal{L}(L^p;L^p)})\|v-v_h\|_{L^p(D;\mathbb{R}^q)} \\ &\le c \inf_{v_h\in P(\mathcal{T}_h)} \|v-v_h\|_{L^p(D;\mathbb{R}^q)}\end{aligned}$$

- We will see how to obtain decay rates for the upper bound whenever $v \in W^{s,p}(D;\mathbb{R}^q)$ using the averaging quasi-interpolation operators
  - (relatively) complete results already exist for H^1-conforming FE $\rightarrow$ [Clément 75; Scott, Zhang 90]
  - literature was lacking on $\boldsymbol{H}$(curl)- and $\boldsymbol{H}(\mathrm{div})$-conforming FE $\rightarrow$ [AE, Guermond 15-17] aims at filling this gap

Main ideas of the construction

- Compose canonical interpolation $\widehat{\mathcal{I}}_h$ operator with some mollification operator $\mathcal{K}_\delta$, $\delta > 0$

$$L^1(D;\mathbb{R}^q) \xrightarrow{\mathcal{K}_\delta} C^\infty(\overline{D};\mathbb{R}^q) \xrightarrow{\widehat{\mathcal{I}}_h} P(\mathcal{T}_h)$$

- $\widehat{\mathcal{J}}_h := \widehat{\mathcal{I}}_h \circ \mathcal{K}_\delta$ achieves stability and commutation [Schöberl 01; Christiansen 07], ... but is not a projection
  - $\widehat{\mathcal{J}}_h$ is invertible on $P(\mathcal{T}_h)$ if $\delta \leq ch$, c small enough [Schöberl 05]
  - on shape-regular meshes, δ is a (smooth) space-dependent function [Christiansen & Winther 08]
  - $\mathcal{J}_h := (\widehat{\mathcal{J}}_h|_{P(\mathcal{T}_h)})^{-1} \circ \widehat{\mathcal{J}}_h$ **satisfies all the required properties**

- Boundary conditions can be prescribed
  - same mollification, just change the canonical interpolation operator

Shrinking-based mollification [AE, Guermond 16]

- Globally transversal field $\boldsymbol{j} \in \boldsymbol{C}^\infty(\mathbb{R}^d)$ to D [Hofmann, Mitrea, Taylor 07]
- Shrinking map $\varphi_\delta : \mathbb{R}^d \ni \boldsymbol{x} \mapsto \boldsymbol{x} - \delta \boldsymbol{j}(\boldsymbol{x}) \in \mathbb{R}^d$: There is $r > 0$ s.t.

$$\varphi_\delta(D) + B(\mathbf{0}, \delta r) \subset D, \qquad \forall \delta \in [0,1]$$

- **The shrinking technique avoids invoking extensions outside D**
- Shrinking-based mollification operators inspired from [Schöberl 01]

$$(\mathcal{K}_\delta^{\mathrm{g}} f)(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \rho(\boldsymbol{y}) f(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y$$

$$(\mathcal{K}_\delta^{\mathrm{c}} \boldsymbol{g})(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \rho(\boldsymbol{y}) \mathbb{J}_\delta^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{g}(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y, \quad \text{etc.}$$

with $\mathbb{J}_\delta(\boldsymbol{x})$ the Jacobian matrix of φ at $\boldsymbol{x} \in D$ and ρ is a smooth kernel supported in $B(\mathbf{0}, 1)$

Averaging quasi-interpolation

- Finite element generation
- Main result: no boundary prescription
- Main result with boundary prescription

Finite element generation

- Reference finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ of degree $k \geq 0$
  - $\mathbb{P}_{k,d}(\widehat{K}; \mathbb{R}^q) \subset \widehat{P} \subset W^{1,\infty}(\widehat{K}; \mathbb{R}^q)$
  - reference shape functions $\{\widehat{\theta}_i\}_{i\in\mathcal{N}}$ and dof's $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$
- For any mesh cell $K \in \mathcal{T}_h$, we consider
  - an affine geometric map $\boldsymbol{T}_K : \widehat{K} \to K$
  - a functional map $\psi_K : L^1(K; \mathbb{R}^q) \to L^1(\widehat{K}; \mathbb{R}^q)$ s.t.

$$\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K)$$

for some matrix $\mathbb{A}_K \in \mathbb{R}^{d\times d}$ (Piola transformations)

- FE generation in each mesh cell $K \in \mathcal{T}_h$

$$(K, P_K, \Sigma_K), \quad P_K = \psi_K^{-1} \circ \widehat{P}, \quad \Sigma_K = \widehat{\Sigma} \circ \psi_K$$

$\Longrightarrow$ local shape functions $\{\theta_{K,i}\}_{i\in\mathcal{N}}$ and dof's $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$

Finite element spaces

- Broken (or dG) FE space

$$P^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in L^\infty(D;\mathbb{R}^q) \mid v_{h|K} \in P_K,\ \forall K \in \mathcal{T}_h\}$$

leading to $P^{\mathrm{g,b}}(\mathcal{T}_h)$ for H^1-conf. FE, $\boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h)$ for $\boldsymbol{H}(\text{curl})$-conf. FE, etc.

- H^1-, $\boldsymbol{H}(\text{curl})$-, and $\boldsymbol{H}(\mathrm{div})$-conforming subspaces

$$P^{\mathrm{g}}(\mathcal{T}_h) = \{v_h \in P^{\mathrm{g,b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F = 0,\ \forall F \in \mathcal{F}_h^\circ\}$$
$$\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F{\times}\boldsymbol{n}_F = \mathbf{0},\ \forall F \in \mathcal{F}_h^\circ\}$$
$$\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F{\cdot}\boldsymbol{n}_F = 0,\ \forall F \in \mathcal{F}_h^\circ\}$$

where $\mathcal{F}_h^\circ$ collects the mesh interfaces

Fundamental property of face dof's

- Let $K \in \mathcal{T}_h$ be a mesh cell, let $F \in \mathcal{F}_K$ be a face of K
- Face unisolvence: $\exists$ nonempty subset $\mathcal{N}_{K,F} \subset \mathcal{N}$ s.t., for all $p \in P_K$,

$$[\,\sigma_{K,i}(p) = 0,\ \forall i \in \mathcal{N}_{K,F}\,] \iff [\,\gamma_{K,F}(p) = 0\,]$$

where $\gamma_{K,F}$ is one of the above trace operators from K to F

- This implies that for all $i \in \mathcal{N}_{K,F}$, there is a unique linear map $\sigma_{K,F,i} : P_{K,F} := \gamma_{K,F}(P_K) \to \mathbb{R}$ s.t. $\sigma_{K,i} = \sigma_{K,F,i} \circ \gamma_{K,F}$
- The fundamental property is that there is c, uniform, s.t.

$$|\sigma_{K,F,i}(q)| \le c\,\|\mathbb{A}_K\|_{\ell^2}\|q\|_{L^\infty(F;\mathbb{R}^t)} \qquad \forall q \in P_{K,F},\ \forall i \in \mathcal{N}_{K,F}$$

This assumption is satisfied by all FE elements from de Rham complex (all degree, all type, all kind)

Two-step construction procedure

$$\mathcal{I}_h : L^1(D;\mathbb{R}^q) \xrightarrow{\mathcal{I}_h^\sharp} P^{\mathrm{b}}(\mathcal{T}_h) \xrightarrow{\mathcal{I}_h^{\mathrm{av}}} P(\mathcal{T}_h)$$

- First apply the projection operator $\mathcal{I}_h^\sharp$ onto the broken FE space
  - L^2-orthogonal or oblique projection [Scott & Zhang 90]
  - $\mathcal{I}_h^\sharp$ enjoys **local stability and approximation** properties
- Then stitch the result by **averaging dof's** using $\mathcal{I}_h^{\mathrm{av}}$
  - the averaging step only handles discrete functions
- Literature
  - nodal-averaging for scalar FEM has a long history [Oswald 93; Brenner 93; Hoppe, Wohlmuth 96; Karakashian, Pascal 03; Burman, AE 07 ...]
  - see also [Peterseim 14], [Kornhuber & Yserentant 16] for recent two-step construction in scalar-valued case

Averaging operator (1)

- Recall the local shape functions $\theta_{K,i}$, $\forall (K,i) \in \mathcal{T}_h \times \mathcal{N}$
- Global shape functions φ_a, $\forall a \in \mathcal{A}_h$
  - connectivity array $\mathsf{a} : \mathcal{T}_h \times \mathcal{N} \to \mathcal{A}_h$ s.t. $\varphi_{\mathsf{a}(K,i)|K} = \theta_{K,i}$
  - connectivity set $\mathcal{C}_a := \{(K,i) \in \mathcal{T}_h \times \mathcal{N} \mid \mathsf{a}(K,i) = a\}$
- $\mathcal{I}_h^{\mathrm{av}} : P^{\mathrm{b}}(\mathcal{T}_h) \to P(\mathcal{T}_h)$ is defined by averaging dof's

$$\mathcal{I}_h^{\mathrm{av}}(v_h)(\boldsymbol{y}) = \sum_{a \in \mathcal{A}_h} \left(\frac{1}{\#(\mathcal{C}_a)} \sum_{(K,i) \in \mathcal{C}_a} \sigma_{K,i}(v_{h|K}) \right) \varphi_a(\boldsymbol{y})$$

Averaging operator (2)

- Bound on averaging error

$$|v_h - \mathcal{I}_h^{\mathrm{av}}(v_h)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{\frac{1}{p}-m} \sum_{F\in\mathcal{F}_K^\circ} \|[\![v_h]\!]_F\|_{L^p(F;\mathbb{R}^t)}$$

for all $m \in \{0{:}k+1\}$, all $p \in [1,\infty]$, all $v_h \in P^{\mathrm{b}}(\mathcal{T}_h)$

  - $\mathcal{F}_K^\circ$ is the collection of mesh interfaces sharing a dof with K

- A discrete trace inequality shows that $\mathcal{I}_h^{\mathrm{av}}$ is L^p-stable on $P^{\mathrm{b}}(\mathcal{T}_h)$

$$\|\mathcal{I}_h^{\mathrm{av}}(v_h)\|_{L^p(K;\mathbb{R}^q)} \le c\, \|v_h\|_{L^p(D_K;\mathbb{R}^q)}$$

  - D_K collects all the mesh cells sharing a dof with K

Theorem

$\mathcal{I}_h : L^1(D;\mathbb{R}^q) \to P(\mathcal{T}_h)$

- leaves $P(\mathcal{T}_h)$ pointwise invariant ($\mathcal{I}_h \circ \mathcal{I}_h = \mathcal{I}_h$)
- is L^p-stable for all $p \in [1,\infty]$
- has optimal local approximation properties

$$|v - \mathcal{I}_h(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{s-m} |v|_{W^{s,p}(D_K;\mathbb{R}^q)}$$

for all $s \in [0, k+1]$ and $m \in \{0{:}\lfloor s \rfloor\}$, all $p \in [1,\infty)$ ($p \in [1,\infty]$ if $s \in \mathbb{N}$), all $K \in \mathcal{T}_h$, all $v \in W^{s,p}(D_K;\mathbb{R}^q)$

In particular, we infer that

$$\inf_{w_h \in P_0(\mathcal{T}_h)} \|v - w_h\|_{L^p(D;\mathbb{R}^q)} \le c\, h^s |v|_{W^{s,p}(D;\mathbb{R}^q)}$$

Proof outline

- **Step 1.** Establish local $W^{m,p}$-stability

$$|\mathcal{I}_h(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\,|v|_{W^{m,p}(D_K;\mathbb{R}^q)}$$

for all $m \in \{0{:}k+1\}$, all $p \in [1,\infty]$, all $K \in \mathcal{T}_h$, all $v \in W^{m,p}(D_K;\mathbb{R}^q)$

- **Step 2.** Stability and polynomial-invariance imply

$$|v - \mathcal{I}_h(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c \inf_{p\in[\mathbb{P}_{k,d}]^q} |v-p|_{W^{m,p}(D_K;\mathbb{R}^q)}$$

- **Step 3.** Polynomial approximation in D_K for each component of v

$$\inf_{p\in\mathbb{P}_{k,d}} |v-p|_{W^{m,p}(D_K)} \leq c\, h_K^{s-m}|v|_{W^{s,p}(D_K)}$$

Details on Step 3

- Poincaré(–Steklov) in $W^{s,p}(D_K)$ ("pedestrian" proof)

$$\|v - \underline{v}_U\|_{L^p(U)} \leq h_U^s \left(\frac{h_U^d}{|U|}\right)^{\frac{1}{p}} |v|_{W^{s,p}(U)}$$

with $\underline{v}_U = \frac{1}{|U|}\int_U v\,\mathrm{d}x$

  - for $U = D_K$, use mesh regularity ($h_{D_K} \leq c|D_K|$)

- Poincaré(–Steklov) in $W^{1,p}(D_K)$

$$\|v - \underline{v}_{D_K}\|_{L^p(D_K)} \leq c\,h_K|v|_{W^{1,p}(D_K)}$$

  - D_K possibly nonconvex, cannot use the result from [Bebendorf 03]
  - break D_K into sub-simplices and combine PS in simplices with multiplicative trace inequality (see also [Veeser & Verfürth 12])

- Morrey's polynomial (1966) s.t. $\int_{D_K} \partial^\alpha(v - \pi(v))\,\mathrm{d}x = 0$, $\forall|\alpha| \leq k$

Main result with boundary prescription

- Two-step construction

$$\mathcal{I}_{h0} : L^1(D;\mathbb{R}^q) \xrightarrow{\mathcal{I}_h^\sharp} P^{\mathrm{b}}(\mathcal{T}_h) \xrightarrow{\mathcal{I}_{h0}^{\mathrm{av}}} P_0(\mathcal{T}_h)$$

  - BCs enforced at the second stage (on polynomials) by simply zeroing out the components of $\mathcal{I}_{h0}^{\mathrm{av}}(v_h)$ attached to boundary dof's

- **Theorem**
  - $\mathcal{I}_{h0}$ leaves $P_0(\mathcal{T}_h)$ pointwise invariant
  - $\|\mathcal{I}_{h0}\|_{\mathcal{L}(L^p;L^p)} \leq c$, $\forall p \in [1,\infty]$
  - best approximation: for all $s \in [0, k+1]$

$$\inf_{w_h \in P_0(\mathcal{T}_h)} \|v - w_h\|_{L^p} \leq \begin{cases} c\, h^s |v|_{W^{s,p}}, & \forall v \in W^{s,p}_{0,\gamma}(D;\mathbb{R}^q) \text{ if } sp > 1 \\ c\, h^s \ell_D^{-s} \|v\|_{W^{s,p}}, & \forall v \in W^{s,p}(D;\mathbb{R}^q) \text{ if } sp < 1 \end{cases}$$

  where $W^{s,p}_{0,\gamma}(D;\mathbb{R}^q) = \{v \in W^{s,p}(D;\mathbb{R}^q) \mid \gamma(v) = 0\}$
  - localized versions and bounds on higher-order norms available

Comments on the case $sp < 1$

$$\inf_{w_h \in P_0(\mathcal{T}_h)} \|v - w_h\|_{L^p} \le c\, h^s \ell_D^{-s} \|v\|_{W^{s,p}(D;\mathbb{R}^q)}$$

- v is not smooth enough to have a trace on ∂D ...
- yet, we achieve an h-optimal decay estimate of best approximation w.r.t. discrete functions **with boundary prescription**
- Main argument in the proof invokes a Hardy-type inequality
  - estimate cannot be localized close to ∂D
- This result seems to be new even in the H^1-conforming setting
  - see [Ciarlet Jr. 13] for Scott–Zhang operator and $sp > 1$

Robust coercivity for Maxwell's equations

- Recall $\boldsymbol{X}_{0\tilde{\mu}} = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid (\tilde{\mu}\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in H^1_0(D)\}$ and that

$$\check{C}_{\mathrm{P},D}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \le \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\tilde{\mu}}$$

- Recall $\boldsymbol{X}_{h0\tilde{\mu}} = \{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid (\tilde{\mu}\boldsymbol{b}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in P^{\mathrm{g}}_0(\mathcal{T}_h)\}$ and that $\boldsymbol{X}_{h0\tilde{\mu}}$ is not a subspace of $\boldsymbol{X}_{0\tilde{\mu}}$

- Our goal is to prove that there is $\check{C}_{\mathrm{P},\mathcal{T}_h} > 0$ s.t.

$$\check{C}_{\mathrm{P},\mathcal{T}_h}\ell_D^{-1}\|\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} \le \|\nabla\times\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b}_h \in \boldsymbol{X}_{h0\tilde{\mu}}$$

  - usually, one invokes a **discrete compactness argument** [Kikuchi 89; Caorsi, Fernandes, Raffetto 00; Monk & Demkowicz 01]
  - entirely bypassed by using the commuting quasi-interpolation operators $\mathcal{J}_{h0}$; this gives an explicit value for $\check{C}_{\mathrm{P},\mathcal{T}_h}$

$$\check{C}_{\mathrm{P},\mathcal{T}_h} = \mu_{\sharp/\flat}^{-1}\|\mathcal{J}^{\mathrm{c}}_{h0}\|^{-1}_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\check{C}_{\mathrm{P},D}$$

Sketch of proof

- Let $\boldsymbol{b}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$
- **Curl-preserving lifting operator** [Monk 92]
  - Let $\phi \in H^1_0(D)$ s.t. $(\widetilde{\mu}\nabla\phi, \nabla m)_{\boldsymbol{L}^2(D)} = (\widetilde{\mu}\boldsymbol{b}_h, \nabla m)_{\boldsymbol{L}^2(D)}$, $\forall m \in H^1_0(D)$
  - Then $\boldsymbol{b} := \boldsymbol{b}_h - \nabla\phi \in \boldsymbol{X}_{0\widetilde{\mu}}$ and $\nabla\times\boldsymbol{b} = \nabla\times\boldsymbol{b}_h$
- **Commuting quasi-interpolation operators**

$$\begin{aligned}\|\widetilde{\mu}^{\frac{1}{2}}\boldsymbol{b}_h\|^2_{\boldsymbol{L}^2} &= (\widetilde{\mu}\boldsymbol{b}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{b}_h))_{\boldsymbol{L}^2} = (\widetilde{\mu}\boldsymbol{b}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\nabla\phi))_{\boldsymbol{L}^2} + (\widetilde{\mu}\boldsymbol{b}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{b}))_{\boldsymbol{L}^2}\\ &= (\widetilde{\mu}\boldsymbol{b}_h, \nabla(\mathcal{J}^{\mathrm{g}}_{h0}(\phi)))_{\boldsymbol{L}^2} + (\widetilde{\mu}\boldsymbol{b}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{b}))_{\boldsymbol{L}^2} = (\widetilde{\mu}\boldsymbol{b}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{b}))_{\boldsymbol{L}^2}\end{aligned}$$

- Cauchy–Schwarz + continuous Poincaré–Steklov inequality

$$\|\widetilde{\mu}^{\frac{1}{2}}\boldsymbol{b}_h\|_{\boldsymbol{L}^2} \le \mu_\sharp^{\frac{1}{2}}\|\mathcal{J}^{\mathrm{c}}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2)}\|\boldsymbol{b}\|_{\boldsymbol{L}^2} \le \mu_\sharp^{\frac{1}{2}}\|\mathcal{J}^{\mathrm{c}}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2)}\check{C}_{\mathrm{P},D}\|\nabla\times\boldsymbol{b}_h\|_{\boldsymbol{L}^2}$$

- Proof inspired from FEEC analysis [Arnold, Falk & Winther 10]
  - see also [Bonelle & AE 15] for lowest-order polyhedral schemes

$\boldsymbol{H}$(curl)-error estimate (1)

- $\boldsymbol{A}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$ is s.t. $a(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h)$, $\forall \boldsymbol{b}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$
- Discrete PS inequality yields $\mu_\flat$-robust coercivity on $\boldsymbol{X}_{h0\widetilde{\mu}}$
- Standard error estimates [Xu, Zikatanov 03] lead to

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)} \leq \frac{2\max(\mu_\sharp, \ell_D^{-2}\kappa_\sharp)}{\kappa_\flat \ell_D^{-2}\min(1, \check{C}^2_{\text{P},\mathcal{T}_h})} \inf_{\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}} \|\boldsymbol{A} - \boldsymbol{x}_h\|_{\boldsymbol{H}(\text{curl};D)}$$

$\boldsymbol{H}$(curl)-error estimate (2)

- Using the commuting quasi-int. operators, one can show that $\boldsymbol{X}_{h0\widetilde{\mu}}$ has the **same approximation properties** as $\boldsymbol{V}_{h0}$ in $\boldsymbol{X}_{0\widetilde{\mu}}$

$$\inf_{\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}} \|\boldsymbol{A}-\boldsymbol{x}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \leq c\,\mu_{\sharp/\flat} \inf_{\boldsymbol{b}_h \in \boldsymbol{V}_{h0}} \|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}, \quad \forall \boldsymbol{A} \in \boldsymbol{X}_{0\widetilde{\mu}}$$

- In conclusion,

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \lesssim \inf_{\boldsymbol{b}_h \in \boldsymbol{V}_{h0}} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}$$

where hidden constant depends on the contrast factors $\mu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, and the magnetic Reynolds number $\gamma_{\mathrm{m}} = \mu_\sharp \ell_D^{-2} \kappa_\sharp^{-1}$

Convergence rates

- Convergence rates follow from

$$
\begin{aligned}
\inf_{\boldsymbol{b}_h\in \boldsymbol{V}_{h0}} \|\boldsymbol{A}-\boldsymbol{b}_h\|^2_{\boldsymbol{H}(\mathrm{curl};D)} &\le \|\boldsymbol{A}-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{H}(\mathrm{curl};D)} \\
&= \|\boldsymbol{A}-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{A}-\nabla\times\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} \\
&= \|\boldsymbol{A}-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{A}-\mathcal{J}^{\mathrm{d}}_{h0}(\nabla\times\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} \\
&\le c \inf_{\boldsymbol{b}_h\in \boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h)} \|\boldsymbol{A}-\boldsymbol{b}_h\|^2_{\boldsymbol{L}^2(D)} + c'\ell_D^2 \inf_{\boldsymbol{d}_h\in \boldsymbol{P}^{\mathrm{d}}_0(\mathcal{T}_h)} \|\nabla\times\boldsymbol{A}-\boldsymbol{d}_h\|^2_{\boldsymbol{L}^2(D)} \\
&\le c\|\boldsymbol{A}-\mathcal{I}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + c'\ell_D^2\|\nabla\times\boldsymbol{A}-\mathcal{I}^{\mathrm{d}}_{h0}(\nabla\times\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)}
\end{aligned}
$$

and we can now use the decay estimates from [AE, Guermond 15-17]

$\boldsymbol{L}^2$-error estimate

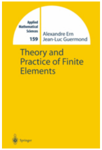
- We assume that D is simply connected (need both PS inequalities)
- Recall that $\boldsymbol{A} \in \boldsymbol{H}^s(D)$, $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^{s''}(D)$, let $\sigma = \min(s, s'') > 0$
- Main steps of the proof for $\boldsymbol{L}^2$-error estimate
  - duality argument + bound on curl-preserving lifting
  - main obstruction: dual solution is only in $\boldsymbol{H}^\sigma(D)$
  - see [Zhong, Shu, Wittum, Xu 09] with assumption $\sigma > \frac{1}{2}$
- **New result** [AE, Guermond 17]

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}} \left(\|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2} + h^\sigma \ell_D^{-\sigma} \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{curl})} \right)$$

where hidden constant depends on $\mu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, $\kappa_\sharp C_{\kappa^{-1}}$, γ_{m}

Summary

- New quasi-interpolation operators for FEM best-approximation
- Optimal $\boldsymbol{H}(\text{curl})$- and $\boldsymbol{L}^2$-estimates, Sobolev regularity $s \in (0, \frac{1}{2})$
- The material (and much more) can be found in **new book**
  - 10 chapters of 50 pages $\rightarrow$ 60 chapters of 14 pages with exercises
  - Spring 2018



$\rightarrow$

Thank you for your attention