Homogenization of some particle systems with two-body interactions and of the dislocation dynamics

Nicolas Forcadel[1], Cyril Imbert[2], Régis Monneau[1]

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Abstract. This paper is concerned with the homogenization of some particle systems with two-body interactions in dimension one and of dislocation dynamics in higher dimensions.

The dynamics of our particle systems are described by some ODEs. We prove that the rescaled "cumulative distribution function" of the particles converges towards the solution of a Hamilton-Jacobi equation. In the case when the interactions between particles have a slow decay at infinity as $1/x$, we show that this Hamilton-Jacobi equation contains an extra diffusion term which is a half Laplacian. We get the same result in the particular case where the repulsive interactions are exactly $1/x$, which creates some additional difficulties at short distances.

We also study a higher dimensional generalisation of these particle systems which is particularly meaningful to describe the dynamics of dislocations lines. One main result of this paper is the discovery of a satisfactory mathematical formulation of this dynamics, namely a Slepčev formulation. We show in particular that the system of ODEs for particle systems can be naturally imbedded in this Slepčev formulation. Finally, with this formulation in hand, we get homogenization results which contain the particular case of particle systems.

Keywords: periodic homogenization, Hamilton-Jacobi equations, moving fronts, two-body interactions, integro-differential operators, Lévy operator, dislocation dynamics, Slepčev formulation, particle systems.
Mathematics Subject Classification: 35B27, 35F20, 45K05, 47G20, 49L25, 35B10.

1 Introduction

1.1 Homogenization of particle systems

In this work, we study a system of ODEs describing the dynamics of particles with two-body interactions. The system we consider has the following form:

$$\dot{y}_i = F - V_0'(y_i) - \sum_{j\in\{1,\dots,N_\varepsilon\}\setminus\{i\}} V'(y_i - y_j) \quad \text{for } i = 1,\dots, N_\varepsilon \tag{1}$$

where F is a constant given force, V_0 is a 1-periodic potential and V is a potential taking into account two-body interactions. The typical example we have in mind is the dynamics of dislocation straight lines. In this case, the potential $V(y)$ is given by $-\ln|y|$ and y_i is the "position" of dislocation straight lines. More generally, the main assumption on V is that $V(y) = v(|y|)$ is convex on $(0,+\infty)$ and that $V''(y)$ behaves like $\frac{1}{y^2}$ at infinity, *i.e.*,

$$\text{there exists } R_0 > 0 \text{ and a constant } g_0 \text{ such that } V''(y)y^2 = g_0 \text{ for } |y| \geq R_0. \tag{2}$$

Our aim is to get homogenization results for this system. More precisely, we want to describe the collective motion of particles as the number of particles goes to infinity. To do so, we introduce the

[1]CERMICS, Université Paris Est - ENPC, 6 & 8 avenue Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
[2]CEREMADE, Université Paris-Dauphine, Pl. de Lattre de Tassigny, 75775 Paris cedex 16, France

rescaled "cumulative distribution function" ρ^ε of particles defined by

$$\rho^\varepsilon(t,x) = \varepsilon\left(-\frac{1}{2} + \sum_{i=1}^{N_\varepsilon} H(x - \varepsilon y_i(t/\varepsilon))\right)$$

(where H is the Heavyside function — see below for a precise definition).

We will prove that the limit ρ^0 of ρ^ε as $\varepsilon \to 0$ exists and is the (unique) solution of a homogenized (or effective) equation. The function ρ^0 is understood as a "cumulative distribution function" associated with particles and its gradient represents the particle density. The effective equation is given by

$$\partial_t \rho^0 = \overline{H}^0\left(\mathcal{I}_1[\rho^0(t,\cdot)], \nabla\rho^0\right) \quad \text{in} \quad (0,+\infty)\times\mathbb{R} \tag{3}$$

where $\overline{H}^0$ is a continuous function and $\mathcal{I}_1$ is a Lévy operator of order 1 associated with the function g_0 appearing in (2). It is defined for any function $U \in C_b^2(\mathbb{R})$ for $r > 0$ by

$$\mathcal{I}_1[U](x) = \int_{z\in\mathbb{R}} (U(x+z) - U(x) - z\cdot\nabla_x U(x)\ 1_{\{|z|\le r\}})\ \frac{g_0}{|z|^2} dz \tag{4}$$

(notice that the latter expression is independent of r).

1.2 Homogenization of dislocation dynamics

The key fact to obtain our homogenization result is that, under proper assumptions on V_0 and V, for some c and J well-chosen, for a suitable initial data and for $N = 1$, the function ρ^ε is a solution of the following non-local equation:

$$\begin{cases} \partial_t u^\varepsilon = \left(c\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) + M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x)\right)|\nabla u^\varepsilon| & \text{in} \quad (0,+\infty)\times\mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{5}$$

where M^ε is a 0 order non-local operator defined by

$$M^\varepsilon[U](x) = \int_{\mathbb{R}^N} dz\, J(z) E\left(U(x+\varepsilon z) - U(x)\right) \tag{6}$$

where E is a modification of the integer part, defined by

$$E(\alpha) = k + \frac{1}{2} \quad \text{if} \quad k \le \alpha < k+1. \tag{7}$$

Let us mention that equation (5) (for $\varepsilon = 1$) describes the motion of several dislocation lines moving in interactions (see for instance Alvarez *et al.* [1] for the model corresponding to a single dislocation line). See Section 3 for a physical explanation of this model.

The interactions are completely characterized by the kernel J. We assume that $J \in W^{1,1}(\mathbb{R}^N)$ is an *even nonnegative* function with the following behaviour at infinity

$$\exists\ R_0 > 0 \text{ and } \exists\ g \in C^0(\mathbf{S}^{N-1}),\ g \ge 0 \text{ such that } J(z) = \frac{1}{|z|^{N+1}}\, g\left(\frac{z}{|z|}\right) \text{ for } |z| \ge R_0. \tag{8}$$

Let us mention that such an assumption is natural for dislocations and can be (slightly) generalized. In the special case where J has a bounded support (choose $g = 0$), we also assume

$$\inf_{e\in[0,1)^N} \int_{\mathbb{R}^N} dz\ \min(J(z), J(z+e)) > 0 \qquad \text{if } N \ge 2. \tag{9}$$

As far as the forcing term c and the initial datum are concerned, we assume

$$\begin{cases} c(\tau, y) \text{ is Lipschitz continuous and } \mathbb{Z}^{N+1}\text{-periodic w.r.t. } (\tau, y); \\ u_0 \in W^{2,\infty}(\mathbb{R}^N). \end{cases} \tag{10}$$

The model we study here has some analogies with [19], but also represents a major improvement. Indeed for the model (5) studied in the present paper, our main result is the discovery of a satisfactory mathematical formulation, namely a formulation "à la Slepčev" (*i.e.* a notion of viscosity solutions for non-local equations introduced by Slepčev [25]). This formulation consists in considering the simultaneous evolution of all the level sets of the function u^ε (see Definition 4.1). Such a definition ensures the stability of solutions, a key property in the viscosity solution approach. The main point is that this formulation is able to include exactly the case of particle systems in interactions. On the contrary, the model studied in [19] was only an approximation for particle systems. Analogously, as far as in higher dimension is concerned, the model (5) is of sharp interface type, *i.e.* it permits to describe the geometric motion of dislocations lines (see Theorem 4.7 in Subsection 4.5), while the model considered in [19] was of phasefield type.

The aim of this work is to pass from a discrete and microscopic model involving the evolution of a finite number of dislocation lines to a continuous and macroscopic one describing the evolution of a dislocation density. To do so, we prove that u^ε converges as $\varepsilon \to 0$ to some u^0. The function u^0 is a solution of the following effective equation

$$\begin{cases} \partial_t u^0 = \overline{H}^0 \left(\mathcal{I}_1[u^0(t,\cdot)], \nabla u^0\right) & \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \\ u^0(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{11}$$

where $\overline{H}^0$ is a continuous function and $\mathcal{I}_1$ is an anisotropic Lévy operator of order 1 associated with the function g appearing in (8). It is defined for any function $U \in C_b^2(\mathbb{R}^N)$ for $r > 0$ by

$$\mathcal{I}_1[U](x) = \int_{\mathbb{R}^N} (U(x+z) - U(x) - z \cdot \nabla_x U(x) 1_{B(0,r)}) \, \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \tag{12}$$

(notice that the latter expression is independent of r since J is even). This Lévy operator $\mathcal{I}_1$ only keeps the memory of the long range interactions between dislocations, as the *effective Hamiltonian* $\overline{H}^0$ will keep the memory of the short range interactions (see the proof of Lemma 6.1 below).

We would like to conclude this introduction by mentioning that our work is focused on a particular equation with a particular scaling in ε, directly inspired from the dislocation dynamics. It would be interesting to consider several extensions of this model. A first extension could take into account the presence of Franck-Read sources or of mean curvature motion terms. The study of different scalings in ε and different decays at infinity for the kernel J is another interesting question. We also want to point out that getting some error estimates, both for the homogenization process and for the numerical computation of the effective Hamiltonian, would be also very interesting. To finish with, let us mention that we will study in a future work the homogenization of classical Frenkel-Kontorova models [21] (see also [16]) which are systems of ODEs. The main idea is to redefine the nonlocal operator M^ε in (6) by truncating the modified integer part E. Precisely, one can consider a Slepčev formulation of (5)-(6) where E is replaced with $T_{\frac{3}{2}}(E)(r) = \max(-\frac{3}{2}, \min(\frac{3}{2}, E(r)))$.

Organization of the article. The paper is organized as follows. In Section 2, we present our main results. In Section 3, we give a physical derivation of equation (5). In Section 4, we recall the definition of viscosity solutions for equations like (5), we give a stability result, a comparison principle and existence results. The construction of the effective Hamiltonian $\overline{H}^0$ and the proof of the ergodicity of the problem (Theorem 2.1) is presented in Section 5. Section 6 is devoted to the proof of the convergence of u^ε to u^0 (Theorem 2.5). In Section 7, we establish the qualitative properties of the effective Hamiltonian described in Theorem 2.6. Finally, in Section 8, we apply our approach to the case of particle systems (Theorem 2.11). We deal in particular with the singular potential $V(x) = -\ln|x|$.

Notation. The open ball of radius r centered at x is classically denoted $B_r(x)$. When x is the origin, $B_r(0)$ is simply denoted B_r and the unit ball B_1 is denoted B. The cylinder $(t-\tau, t+\tau)\times B_r(x)$ is denoted $Q_{\tau,r}(t,x)$. The indicator function of a subset $A \subset C$ is denoted by 1_A: it equals 1 on A and 0 on $C \setminus A$.

The quantity $\lfloor x \rfloor$ denotes the floor integer parts of a real number x. Let $H(x)$ denote the Heaviside function:

$$H(r) = \begin{cases} 1 & \text{if } r \geq 0, \\ 0 & \text{if } r < 0. \end{cases}$$

It is convenient to introduce the unbounded measure on $\mathbb{R}^N$ defined on $\mathbb{R}^N \setminus \{0\}$ by:

$$\mu(dz) = \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \tag{13}$$

and such that $\mu(\{0\}) = 0$. For the reader's convenience, we recall here the five integro-differential operators appearing in this work:

$$\begin{aligned}
M^\varepsilon[U](x) &= \int_{\mathbb{R}^N} E(U(x+\varepsilon z) - U(x)) J(z)\, dz \\
M^\alpha_p[U](x) &= \int_{\mathbb{R}^N} \{E(U(x+z) - U(x) + p\cdot z + \alpha) - p\cdot z\} J(z)\, dz, \\
\text{for } \alpha = 0, \quad M_p[U](x) &= \int_{\mathbb{R}^N} \{E(U(x+z) - U(x) + p\cdot z) - p\cdot z\} J(z)\, dz, \\
\text{for } \alpha = 0 \text{ and } p = 0, \quad M[U](x) &= \int_{\mathbb{R}^N} \{E(U(x+z) - U(x))\} J(z)\, dz, \\
\mathcal{I}_1[U](x) &= \int_{\mathbb{R}^N} (U(x+z) - U(x) - 1_{B_r}(z)\nabla_x U(x)\cdot z)\ \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz.
\end{aligned}$$

To each operator M, we associate $\tilde{M}$ which is defined in the same way but where E is replaced with E_* defined as follows

$$E_*(\alpha) = k + \frac{1}{2} \quad \text{if} \quad k < \alpha \leq k+1.$$

See Section 4 for further details.

2 Main results

2.1 General homogenization results

We explained in the introduction that our first aim is to get homogenization results for (5). In other words, we want to say what happens to the solution u^ε of (5) as $\varepsilon \to 0$. We classically try to prove that u^ε converges to the solution u^0 of an effective equation. In order to both determine the effective equation and prove the convergence, it is also classical to perform a formal expansion, that is to write u^ε as $u^0 + \varepsilon v$. One must next find an equation (E) solved by v. The function v is classically called a corrector and the equation it satisfies is referred to as the cell equation or cell problem. In our case, such a problem is associated with any constant $L \in \mathbb{R}$ and any $p \in \mathbb{R}^N$:

$$\lambda + \partial_\tau v = \Big(c(\tau,y) + L + M_p[v(\tau,\cdot)](y)\Big)|p + \nabla v| \text{ in } \quad (0,+\infty)\times\mathbb{R}^N \tag{14}$$

where

$$M_p[U](y) = \int dz\, J(z) \left\{E\left(U(y+z) - U(y) + p\cdot z\right) - p\cdot z\right\}.$$

The construction of correctors v satisfying (14) is one of the important problem we have to solve. It is done by considering the solution w of

$$\begin{cases} \partial_\tau w = \Big(c(\tau, y) + L + M_p[w(\tau,\cdot)](y)\Big)|p + \nabla w| & \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \\ w(0,y) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{15}$$

and by looking for some $\lambda \in \mathbb{R}$ such that $w - \lambda\tau$ is bounded. Here is the precise result.

Theorem 2.1 (Ergodicity)**.** *Under the assumptions* (8)-(9)-(10)*, for any* $L \in \mathbb{R}$ *and* $p \in \mathbb{R}^N$*, there exists a unique* $\lambda \in \mathbb{R}$ *such that the continuous viscosity solution of* (15) *(in the sense of Definition 4.1) satisfies:* $\frac{w(\tau,y)}{\tau}$ *converges towards* λ *as* $\tau \to +\infty$*, locally uniformly in* y*. The real number* λ *is denoted by* $\overline{H}^0(L,p)$*. Moreover, the function* $\overline{H}^0$ *satisfies*

$$\overline{H}^0 \text{ is continuous in } (L,p) \text{ and nondecreasing in } L. \tag{16}$$

Remark 2.2. Condition (9) is related to the periodicity assumption on the velocity. Indeed, assumption (9) is crucial in our analysis and we do not know if ergodicity holds or not if this assumption is not fullfilled in dimension $N \geq 2$.

Remark 2.3. Condition (16) ensures the existence of solutions for the homogenized equation (11) (see Theorem 4.6).

Remark 2.4. A superscript 0 appears in the effective Hamiltonian. The reason is that we will have to study the ergodicity of a *family* of Hamiltonians in order to prove the convergence. With the notation of Section 5, we have $\overline{H}^0(L,p) = \overline{H}(L,p,0)$.

With correctors in hand, we can now prove the convergence of the sequence u^ε. The second main result of this paper is the following convergence result.

Theorem 2.5 (Convergence)**.** *Under the assumptions* (8)-(9)-(10)*, the bounded continuous viscosity solution* u^ε *of* (5) *(in the sense of Definition 4.1) with initial data* $u_0 \in W^{2,\infty}(\mathbb{R}^N)$*, converges as* $\varepsilon \to 0$ *locally uniformly in* (t,x) *towards the unique bounded viscosity solution* u^0 *of* (11)*.*

Recall that the first homogenization problem we are trying to solve comes from dislocation theory. We thus would like to be able to get an interpretation of the homogenization result we obtain in terms of dislocation theory. This is the reason why we look for qualitative properties of $\overline{H}^0$. Considering the one dimensional special case and a driving force independent of time, we obtain the following

Theorem 2.6 (Qualitative properties of $\overline{H}^0$)**.** *Under the assumptions* $N = 1$*,* $c = c(y)$ *and* $\int_{(0,1)} c = 0$*, the function* $\overline{H}^0(L,p)$ *is continuous and satisfies the following properties:*

1. *If* $c \equiv 0$ *then* $\overline{H}^0(L,p) = L|p|$*.*
2. **(Bound)** *We have*
$$\left| \frac{\overline{H}^0(L,p)}{|p|} - L \right| \leq \|c\|_\infty \quad \textit{for} \quad (L,p) \in \mathbb{R} \times \mathbb{R}.$$
3. **(Sign of the Hamiltonian)**
$$\overline{H}^0(L,p)L \geq 0 \quad \textit{for} \quad (L,p) \in \mathbb{R} \times \mathbb{R}.$$
4. **(Monotonicity in** L**)** *The function* $\overline{H}^0(L,p)$ *satisfies for* $C = \|c\|_\infty + (|p| + \frac{1}{2})\|J\|_{L^1}$*:*
$$\frac{|\overline{H}^0|}{(|L| - C)^+} \geq \frac{\partial \overline{H}^0}{\partial L} \geq \frac{|\overline{H}^0|}{|L| + C}.$$

5. **(Modulus of continuity in** L**)** *There exists a constant* C_1 *only depending on* $\|\nabla c\|_\infty$ *such that*

$$0 \le \overline{H}^0(L+L',p) - \overline{H}^0(L,p) \le \frac{C_1|p|}{|\ln L'|} \quad \text{for } 0 < L' \le \frac{1}{2}.$$

6. **(Antisymmetry in** L**)** *If there exists* $a \in \mathbb{R}$ *such that* $-c(y) = c(y+a)$*, then:*

$$\overline{H}^0(-L,-p) = -\overline{H}^0(L,p).$$

7. **(Symmetry in** p**)** *If there exists* $a \in \mathbb{R}$ *such that* $c(-y) = c(y+a)$*, then:*

$$\overline{H}^0(L,-p) = \overline{H}^0(L,p).$$

8. **(0-plateau property)** *If* $c \not\equiv 0$*, then there exists* $r_0 > 0$ *(only depending on* $\|c\|_\infty$ *and* $J_{|\mathbb{R}\setminus[-1,1]}$*) such that:*

$$\overline{H}^0(L,p) = 0 \quad \textit{for} \quad (L,p) \in B_{r_0}(0) \subset \mathbb{R}^2.$$

9. **(Non-zero Hamiltonian for large** p**)** *Let us assume that* $c \in W^{2,\infty}(\mathbb{R})$*,* $J \in W^{1,\infty}(\mathbb{R})$ *and:*

$$h_0, h_1 \in L^1(\mathbb{R})$$

where

$$h_0(z) = \sup_{a\in[-1/2,1/2]} |J(z+a)|, \quad h_1(z) = \sup_{a\in[-1/2,1/2]} |J'(z+a)|.$$

Then there exists a constant $C > 0$ *(depending on* $\|c\|_{W^{2,\infty}}$*,* $\|h_0\|_{L^1}$*,* $\|h_1\|_{L^1}$*) such that:*

$$L\overline{H}^0(L,p) > 0 \quad \textit{for} \quad |L| > C/|p| \quad \textit{and} \quad |p| > C.$$

Remark 2.7. Notice that assuming $\int c = 0$ is not a restriction at all.

Remark 2.8. The qualitative properties 8 and 9 of the homogenized Hamiltonian shows that there is a cooperative collective behaviour. More precisely, increasing the dislocation density allows to move the dislocations that were locked for small densities (and small enough L).
The 0-plateau property for small p is related to the work of Aubry [3] on the breaking of analyticity of the hull function. In particular, from De La Llave [8], it is possible to see that for any Diophantine number p and for any c small enough (depending on p), with c and V analytic, we have

$$L\overline{H}^0(L,p) > 0 \quad \text{for } L \neq 0.$$

The threshold $\frac{C}{p}$ for large p is related to the well-known pile-up effect for dislocations in front of an obstacle. Indeed, it is know that for an applied stress L_a and a pile-up of p dislocations stuck on the obstacle, the internal stress field created by the the obstacle is $F = pL_a$ (see [15] page 766 for further details). Therefore to make the dislocations to move, we need to apply a stress of the order F/p, which is exactly the result we get.

Remark 2.9. In the case $c \equiv 0$, self-similar solutions of (11) were obtained by Head (see for instance [9, 14]).

The typical profile of $\overline{H}^0$ is represented in Figure 1. We also refer to Ghorbel [12] and Ghorbel, Hoch, Monneau [13] for simulations.

Remark 2.10. The boundary of the set $\{\overline{H}^0(L,p) = 0\}$ is given by two graphs $h^-(p) \le L \le h^+(p)$, but it is not known if h^+ and h^- are continuous.

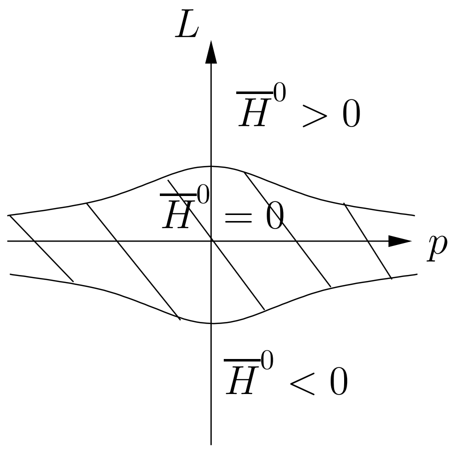

Figure 1: Schematic representation of the effective Hamiltonian

2.2 Application to the homogenization of particle systems with two-body interactions

As explained in the Introduction, we are able to apply the homogenization results of (5) to the system of ODEs (1) because under appropriate assumptions the function ρ of (τ, y) defined by

$$\rho(\tau, y) = -\frac{1}{2} + \sum_{i=1}^{N_\varepsilon} H(y - y_i(\tau)) \tag{17}$$

(where H is the Heaviside function — see the Introduction for a definition) is a solution of (5) with $\varepsilon = 1$ and c independent on time where

$$c(y) = V_0'(y) - F \qquad \text{and} \qquad J = V'' \text{ on } \mathbb{R}\backslash\{0\}. \tag{18}$$

See Theorem 8.1 for a precise statement.

Before presenting the results for (1), let us make precise the assumptions on F, V_0 and V and make some comments on them. We recall that F is a constant given force and V_0 is a 1-periodic potential. As far as V is concerned, we assume

Assumption (H)

$(H0)$ $V \in W^{1,\infty}_{\mathrm{Loc}}(\mathbb{R})$ and $V'' \in W^{1,1}(\mathbb{R}\backslash\{0\})$,

$(H1)$ V is symmetric, *i.e.* $V(-y) = V(y)$,

$(H2)$ V is nonincreasing and convex on $(0, +\infty)$,

$(H3)$ $V'(y) \to 0$ as $|y| \to +\infty$,

$(H4)$ there exists $R_0 > 0$ and a constant g_0 such that $V''(y)y^2 = g_0$ for $|y| \geq R_0$.

In $(H4)$, V'' will play the role of the function J appearing in (6) and condition $(H4)$ is equivalent to (8). It is possible to slightly generalize Assumption $(H4)$ by only assuming an asymptotic behaviour of V'' instead of assuming that it coincides with g_0/y^2 outside a fixed ball. The system of ODEs (1) has some similarities with the overdamped Frenkel-Kontorova model [21], except that in the classical Frenkel-Kontorova model only interactions between nearest neighboors are considered (see Hu, Qin, Zheng [16]; see also Aubry [3], Aubry, Le Daeron [4] as far as stationnary solutions are concerned). We plan to study the homogenization of the classical Frenkel-Kontorova model in a future work.

Then we have the following homogenization result for our particle system.

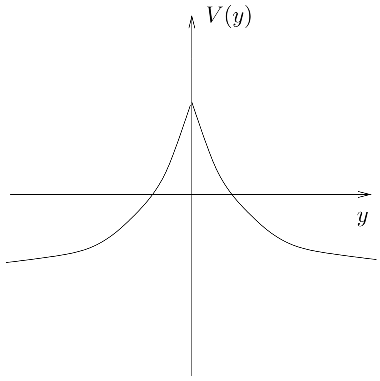

Figure 2: Typical profile for a potential V satisfying assumptions (H).

Theorem 2.11 (Homogenization of the particle system)**.** *Assume that V_0 is 1-periodic, V_0' is Lipschitz continuous and V satisfies (H). Assume that $y_1(0) < ... < y_{N_\varepsilon}(0)$ are given by the discontinuities of the function $\rho_0^\varepsilon(x) = \varepsilon E\left(\frac{u_0(x)}{\varepsilon}\right)$ with E defined by* (7)*, for some given nondecreasing function $u_0 \in W^{2,\infty}(\mathbb{R})$. Define ρ as in* (17) *and consider*

$$\rho^\varepsilon(t,x) = \varepsilon\rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Then ρ^ε converges towards the solution $u^0(t,x)$ of (11) *where the operator $\mathcal{I}_1$ is defined by* (12) *with $g\left(z/|z|\right) = g_0$ and $\overline{H}^0$ is given in Theorem 2.1 with c and J defined in* (18)*.*

Remark 2.12. In the case of short range interactions, *i.e.*, $g_0 = 0$ in $(H4)$, the homogenized equation (11) is a local Hamilton-Jacobi equation and the dislocation density $\frac{\partial u^0}{\partial x}$ satisfies formally a hyperbolic equation.

Remark 2.13. Theorem 2.11 is still true (with other constants depending possibly on these quantities) with a potential $V_0(t,x)$ periodic in x and t (see Theorem 2.5). Moreover, the regularity of the initial data u_0 can be considerably weakened if necessary.

Since our first goal was to study dislocation dynamics, the following generalization is of special interest.

Theorem 2.14 (The dislocation case)**.** *Theorems 2.11 and 2.6 (excepted point 4. and point 9.) are still true with $V(x) = -\ln|x|$ and c and J defined by* (18)*.*

Remark 2.15. Here the annihilation of particles is not included in (1), but our approach with equation (5) could be developed in this case.

3 Physical derivation of the model for dislocation dynamics

Dislocations are line defects in crystals. Their typical length is of the order of $10^{-6}m$ and their thickness of the order of $10^{-9}m$. When the material is submitted to shear stress, these lines can move in the crystallographic planes and their complicated dynamics is one of the main explanation of the plastic behaviour of metals.

In the present paper we are interested in describing the effective dynamics for the collective motion of dislocation lines with the same Burgers's vector and all contained in a *single* slip plane $\{x_3 = 0\}$ of coordinates $x = (x_1, x_2)$, and moving in a periodic medium. At the end of this derivation, we will see that the dynamics of dislocations is described by equation (5) in dimension $N = 2$.

Several obstacles to the motion of dislocation lines can exist in real life: precipitates, inclusions, other pinned dislocations or other moving dislocations, *etc.* We will describe all these obstacles by a given field

$$c(t,x) \tag{19}$$

that we assume to be periodic in space and time. Another natural force exists: this is the Peach-Koehler force acting on a dislocation j. This force is the sum of the interactions with the other dislocations k for $k \neq j$, and of the self-force created by the dislocation j itself.

The level set approach for describing dislocation dynamics at this scale consists in considering a function v such that the dislocation $k \in \mathbb{Z}$ is basically described by the level set $\{v = k\}$. Let us first assume that v is smooth.

As explained in [1], the Peach-Koehler force at the point x created by a dislocation j is well-described by the expression

$$c_0 \star 1_{\{v \geq j\}}$$

where $1_{\{v\geq j\}}$ is the characteristic function of the set $\{v \geq j\}$ which is equal to 1 or 0. In a general setting, the kernel c_0 can change sign. In the special case where the dislocations have the same Burgers vector and move in the same slip plane, a monotone formulation (see Alvarez et al. [1], Da Lio et al. [7]) is physically acceptable. Indeed, the kernel can be chosen as

$$c_0 = J - \delta_0$$

where J is nonnegative and δ_0 denotes the Dirac mass. The negative part of the kernel is somehow concentrated at the origin. Moreover, we assume that J satisfies the symmetry condition: $J(-z) = J(z)$ and $\int_{\mathbb{R}^2} J = 1$ so that we have (at least formally) $\int_{\mathbb{R}^2} c_0 = 0$. The kernel J can be computed from physical quantities (like the elastic coefficients of the crystal, the Burgers vector of the dislocation line, the slip plane of the dislocation, the Peierls-Nabarro parameter, *etc.*). We set formally

$$\left(\delta_0 \star 1_{\{v\geq j\}}\right)(x) := \begin{cases} 1 & \text{if} \quad v(x) > j \\ \frac{1}{2} & \text{if} \quad v(x) = j \\ 0 & \text{if} \quad v(x) < j \end{cases}$$

We remark in particular that the Peach-Koehler force is discontinuous on the dislocation line in this modeling (see Figure 3).

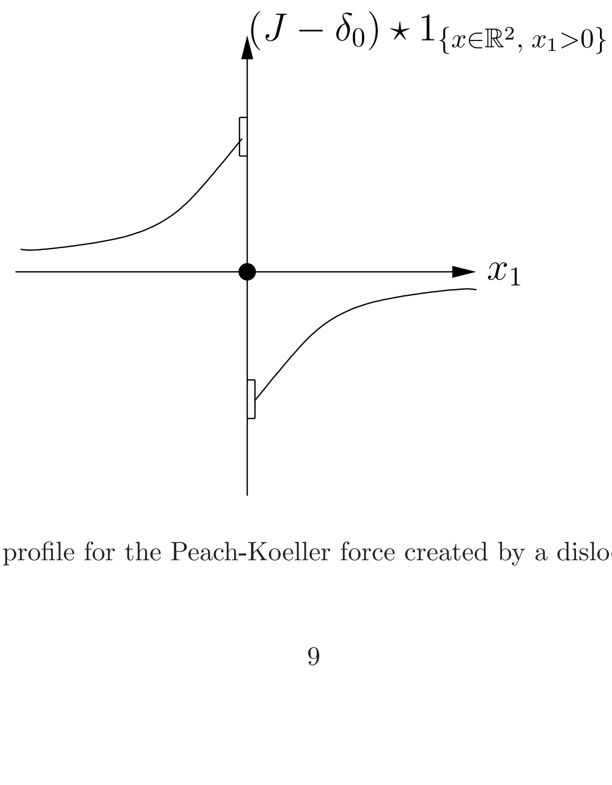

Figure 3: Typical profile for the Peach-Koeller force created by a dislocation straight line.

Let us now assume that for integers $N_1, N_2 \geq 0$, we have $-N_1 - 1/2 < v < N_2 + 1/2$. Then the Peach-Koehler force at the point x on the dislocation j (*i.e.* $v(x) = j$) created by dislocations for $k = -N_1, ..., N_2$ is given by the sum

$$\left((J - \delta_0) \star \sum_{k=-N_1}^{N_2} 1_{\{v \geq k\}} \right)(x) = (J \star E(v - v(x)))(x) \tag{20}$$

with E defined in (7). Defining the normal velocity to dislocation lines as the sum of the periodic field (19) and the Peach-Koehler force (20), we see that the dislocation line $\{v = j\}$ for integer j, is formally a solution of the following level set (or eikonal) equation:

$$v_t = (c + J \star E(v - v(x))) |\nabla v| \tag{21}$$

which is exactly (5) with $\varepsilon = 1$. More generally, if u solves (5) with $\varepsilon = 1$, then for any fixed $\alpha \in [0, 1)$, we can recover the dislocations Γ_k^α at time t by

$$\left\{ x \in \mathbb{R}^N : u(x, t) = k + \alpha \right\}.$$

We refer in particular to [19] for a mechanical interpretation of the homogenized equation and the references therein for other studies of models with dislocation densities. See in particular [24] for the homogenization of one-dimensional models giving some rate-independent plasticity macroscopic models.

4 Viscosity solutions for non-local equations (5) and (15)

In this paper, we have to deal with Hamilton-Jacobi equations involving integro-differential operators. For Equations (5) and (15), we will use a definition of viscosity solutions first introduced by Slepčev [25]. As far as Equation (11) is concerned, the reader is referred to [19] for a definition for viscosity solution and for the proof of a comparison principle in the class of bounded functions.

Let us first recall the definition of relaxed lower semi-continuous (lsc for short) and upper semi-continuous (usc for short) limits of a family of functions u^ε which is locally bounded uniformly w.r.t. ε:

$$\limsup{}^* u^\varepsilon(t, x) = \limsup_{\varepsilon \to 0, s \to t, y \to x} u^\varepsilon(s, y) \quad \text{and} \quad \liminf{}_* u^\varepsilon(t, x) = \liminf_{\varepsilon \to 0, s \to t, y \to x} u^\varepsilon(s, y).$$

If the family contains only one element, we recognize the usc envelope and the lsc envelope of a locally bounded function u:

$$u^*(t, x) = \limsup_{s \to t, y \to x} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{s \to t, y \to x} u(s, y).$$

4.1 Definition of viscosity solutions

In this subsection, we will give the definition of viscosity solution for the following problem

$$\begin{cases} \partial_t u = \big(c(t, x) + M_p^\alpha[u(t, \cdot)](x)\big) |p + \nabla u| & \text{in} \quad (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{22}$$

where M_p^α is defined by

$$M_p^\alpha[U](x) = \int_{\mathbb{R}^N} dz\; J(z) \left\{ E(U(x + z) - U(x) + p \cdot z + \alpha) - p \cdot z \right\}.$$

It will be convenient to define the following associated operator

$$\tilde{M}_p^\alpha[U](x) = \int_{\mathbb{R}^N} dz\; J(z) \left\{ E_*(U(x + z) - U(x) + p \cdot z + \alpha) - p \cdot z \right\}.$$

where we recall that

$$E_*(\alpha) = k + \frac{1}{2} \quad \text{if} \quad k < \alpha \le k+1.$$

We now recall the definition of viscosity solutions introduced by Slepčev in [25]:

Definition 4.1 (Viscosity solutions for (22))**.** *A upper semi-continuous (resp. lower semi-continuous) function $u : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of* (22) *if $u(0,x) \le u_0^*(x)$ in $\mathbb{R}^N$ (resp. $u(0,x) \ge (u_0)_*(x)$) and for any $(t,x) \in (0,\infty) \times \mathbb{R}^N$ and any test function $\phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^N)$ such that $u - \phi$ attains a maximum (resp. a minimum) at the point $(t,x) \in (0,+\infty) \times \mathbb{R}^N$, then we have*

$$\partial_t \phi(t,x) \le \big(c(t,x) + M_p^\alpha[u(t,\cdot)](x)\big)\, |p + \nabla\phi|$$

$$\left(\text{resp.} \quad \partial_t \phi(t,x) \ge \left(c(t,x) + \tilde{M}_p^\alpha[u(t,\cdot)](x)\right) |p + \nabla\phi|\right).$$

A function u is a viscosity solution of (22) *if u^* is a viscosity subsolution and u_* is a viscosity supersolution.*

4.2 Stability results for (22)

In this subsection, we will prove a general stability result for the non-local term. The following proposition permits to show all the classical stability results for viscosity solutions we need.

Proposition 4.2 (Stability of the solutions of (22))**.** *Let $(u_n)_n$ be a sequence of uniformly bounded usc functions (resp. lsc functions) and let $\overline{u}$ denote $\limsup^* u^n$ (resp. $\underline{u} = \liminf^* u^n$). Let $(t_n, x_n, p_n, \alpha_n) \to (t_0, x_0, p, \alpha)$ in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ be such that $u_n(t_n,x_n) \to \overline{u}(t_0,x_0)$ (resp. $u_n(t_n,x_n) \to \underline{u}(t_0,x_0)$). Then*

$$\limsup_{n\to\infty} M_{p_n}^{\alpha_n}[u_n(t_n,\cdot)](x_n) \le M_p^\alpha[\overline{u}(t_0,\cdot)](x_0) \tag{23}$$

$$\left(\textit{resp.} \ \liminf_{n\to\infty} \tilde{M}_{p_n}^{\alpha_n}[u_n(t_n,\cdot)](x_n) \ge \tilde{M}_p^\alpha[\underline{u}(t_0,\cdot)](x_0)\right).$$

This result is a consequence of the stability of the Slepčev definition and in particular of the following lemma (whose proof is given in [25]):

Lemma 4.3. *Let $(f_n)_n$ be a sequence of measurable functions on $\mathbb{R}^N$, and consider*

$$\overline{f} = \limsup{}^* f_n$$

and

$$\underline{f} = \liminf{}^* f_n.$$

Let $(a_n)_n$ be a sequence of $\mathbb{R}$ converging to zero. Then

$$\mathcal{L}(\{f_n \ge a_n\} \backslash \{\overline{f} \ge 0\}) \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \backslash \{f_n > a_n\}) \to 0 \quad \text{as} \quad n \to \infty$$

where $\mathcal{L}(A)$ denotes the Lebesgue measure of measurable set A.

Proof of Proposition 4.2. We just prove the result for $\overline{u}$. Let $\varepsilon > 0$. Using the strong decay at infinity of J and the fact that $|E(r) - r| \le \frac{1}{2}$, we know that there exists R such that for any $n \in \mathbb{N}$

$$\left| \int_{|z|\ge R} J(z)\{E(u_n(t_n, x_n+z) - u_n(t_n,x_n) + p_n\cdot z + \alpha_n) - p_n \cdot z\} \right| \le \frac{\varepsilon}{4},$$

$$\left| \int_{|z|\ge R} J(z)\{E(\overline{u}(t_0, x_0+z) - \overline{u}(t_0,x_0) + p\cdot z + \alpha) - p \cdot z\} \right| \le \frac{\varepsilon}{4}. \tag{24}$$

Moreover, using the uniform bound on the sequence $(u_n)_n$, we deduce that for $|z| \leq R$, there exists $N_0 \in \mathbb{N}$, $N_0 \geq 1$, such that

$$|u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z + \alpha_n| \leq N_0$$

and

$$|\overline{u}(t_0, x_0 + z) - \overline{u}(t_0, x_0) + p \cdot z + \alpha| \leq N_0.$$

We notice that

$$E(\beta) = \sum_{k \geq 1} 1_{\{\beta \geq k\}} - \sum_{k \leq 0} 1_{\{\beta < k\}} + \frac{1}{2}. \tag{25}$$

We then get that

$$\begin{aligned}
&\int_{|z|\leq R} J(z)\{E(u_n(t_n, x_n + z) - u_n(t_n, x_n) + p_n \cdot z + \alpha_n) - p_n \cdot z\} \\
&\quad - \int_{|z|\leq R} J(z)\{E(\overline{u}(t_0, x_0 + z) - \overline{u}(t_0, x_0) + p \cdot z + \alpha) - p \cdot z\} \\
&\leq \int_{|z|\leq R} J(z)\Bigg\{ \sum_{k=1}^{N_0} \big(1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n \geq k\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha\geq k\}}\big) \\
&\qquad - \sum_{k=-N_0}^{0} \big(1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n < k\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha< k\}}\big) \Bigg\}
\end{aligned} \tag{26}$$

Since the two sums are finite, we get, using Lemma 4.3, that for n big enough

$$\sum_{k=1}^{N_0} \int_{|z|\leq R} dz\ J(z) \big(1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n \geq k\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha\geq k\}}\big) \leq \frac{\varepsilon}{4}$$

$$\sum_{k=-N_0}^{0} \int_{|z|\leq R} dz\ J(z) \big(1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p\cdot z+\alpha< k\}} - 1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n\cdot z+\alpha_n < k\}}\big) \leq \frac{\varepsilon}{4}. \tag{27}$$

Using (24), (26) and (27) we deduce that

$$M_p^\alpha[u_n(t_n, \cdot)](x_n) \leq M_p^\alpha[\overline{u}(t_0, \cdot)](x_0) + \varepsilon$$

for n big enough. This implies (23). □

4.3 Comparison principles

In this subsection, we will prove a comparison principle for (22).

Theorem 4.4 (Comparison Principle)**.** *Let $T > 0$ and assume that $J \in W^{1,1}(\mathbb{R}^N)$. Consider an initial datum $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ and a forcing term $c \in W^{1,\infty}([0, +\infty) \times \mathbb{R}^N)$. Let u be a bounded upper semi-continuous subsolution of (22) and v be a bounded lower semi-continuous supersolution. Then $u(t, x) \leq v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^N$.*

Proof. Suppose by contradiction that $M_0 = \sup_{(0,T)\times\mathbb{R}^N}(u(t, x) - v(t, x)) > 0$. For all $0 < \gamma < 1$, $\eta > 0$, $\beta > 0$, we define

$$\Phi^\eta_{\gamma,\beta}(t, x, y) = u(t, x) - v(t, y) - \eta t + p \cdot (x - y) - e^{K_0 t}\left(\frac{|x - y|^2}{2\gamma} + \beta(|x|^2 + |y|^2)\right)$$

where K_0 is a constant which will be chosen latter. We observe that $\limsup_{|x|,|y|\to\infty} \Phi^\eta_{\gamma,\beta}(t,x,y) = -\infty$ so $\Phi^\eta_{\gamma,\beta}$ reaches its maximum at a point $(\bar{t},\bar{x},\bar{y}) \in [0,T]\times\mathbb{R}^N\times\mathbb{R}^N$. Moreover, the constant K_0 being fixed, we have $\overline{M}_0 = \sup \Phi^\eta_{\gamma,\beta} \geq \frac{M_0}{2}$ for η and β small enough. Standard arguments show that

$$|\bar{x}-\bar{y}|^2 \leq C_0\gamma, \quad \beta(|\bar{x}|^2+|\bar{y}|^2) \leq C_0 \tag{28}$$

with C_0 depending on $\|u\|_\infty$, $\|v\|_\infty$, p and K_0.

We claim that there exists $0<\gamma<1$ such that for all β small enough, we have $\bar{t}>0$. Indeed, if for all $0<\gamma<1$ there exists $\beta>0$ (small) such that $\bar{t}=0$, then the following estimates holds:

$$\begin{aligned}\frac{M_0}{2} \leq \overline{M}_0 &\leq u(0,\bar{x}) - v(0,\bar{y}) + |p||\bar{x}-\bar{y}| \\ &\leq (\|Du_0\|_\infty + |p|)|\bar{x}-\bar{y}|.\end{aligned}$$

Using (28), we get a contradiction if γ is small enough and we prove the claim. This implies using Ishii's Lemma (see Crandall, Ishii, Lions [6, Lemma 8.3]), that there are $a,b\in\mathbb{R}$ and $\bar{p},\bar{q}_x,\bar{q}_y\in\mathbb{R}^N$ such that

$$\begin{aligned}a-b &= \eta + K_0 e^{K_0\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2+|\bar{y}|^2)\right), \\ \bar{p} &= e^{K_0\bar{t}}\frac{\bar{x}-\bar{y}}{\gamma}, \quad \bar{q}_x = 2\beta e^{K_0\bar{t}}\bar{x}, \quad \bar{q}_y = 2\beta e^{K_0\bar{t}}\bar{y}, \\ &a - \left(c(\bar{t},\bar{x}) + M^\alpha_p[u(\bar{t},\cdot)](\bar{x})\right)|\bar{p}+\bar{q}_x| \leq 0\end{aligned}$$

and

$$b - \left(c(\bar{t},\bar{y}) + \tilde{M}^\alpha_p[v(\bar{t},\cdot)](\bar{y})\right)|\bar{p}-\bar{q}_y| \geq 0.$$

Subtracting the two last inequalities, we get

$$\begin{aligned}&\eta + K_0 e^{K_0\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2+|\bar{y}|^2)\right) \\ &\leq \left(c(\bar{t},\bar{x}) + M^\alpha_p[u(\bar{t},\cdot)](\bar{x})\right)|\bar{p}+\bar{q}_x| - \left(c(\bar{t},\bar{y}) + \tilde{M}^\alpha_p[v(\bar{t},\cdot)](\bar{y})\right)|\bar{p}-\bar{q}_y|.\end{aligned} \tag{29}$$

We define

$$\mathcal{A} = \{z : E(u(\bar{t},z) - u(\bar{t},\bar{x}) + p\cdot(z-\bar{x}) + \alpha) \leq E_*(v(\bar{t},z) - v(\bar{t},\bar{y}) + p\cdot(z-\bar{y}) + \alpha)\}. \tag{30}$$

The inequality $\Phi^\eta_{\gamma,\beta}(\bar{t},\bar{x},\bar{y}) \geq \Phi^\eta_{\gamma,\beta}(\bar{t},x,x)$ yields

$$u(\bar{t},z)-u(\bar{t},\bar{x})+p\cdot(z-\bar{x})+\alpha \leq v(\bar{t},z)-v(\bar{t},\bar{y})+p\cdot(z-\bar{y})+\alpha - e^{K_0\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2+|\bar{y}|^2) - 2\beta|z|^2\right).$$

This implies that

$$\mathcal{A}^c \subset \{|z| \geq R_{\gamma,\beta}\},$$

where

$$(R_{\gamma,\beta})^2 = \frac{1}{2\beta}\left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma} + \beta(|\bar{x}|^2+|\bar{y}|^2)\right).$$

We now distinguish two cases.

Case 1. There exists a constant $\tilde{C}_\gamma > 0$ such that for any β small enough we have

$$\frac{|\bar{x}-\bar{y}|}{2\gamma} \geq \tilde{C}_\gamma.$$

In this case, we have

$$\{|z-\bar{x}| \geq R_{\gamma,\beta}\} \subset \{|z| \geq \tilde{R}_{\gamma,\beta}\} \tag{31}$$

where $\tilde{R}_{\gamma,\beta} = -|\bar{x}| + R_{\gamma,\beta} \to +\infty$ as $\beta \to 0$ (see Da Lio *et al.* [7, Lemma 2.5]). This implies that

$$\begin{aligned} M_p^\alpha[u(\bar{t},\cdot)](\bar{x}) &= \int_{\mathbb{R}^N} dz\ J(\bar{x}-z)\left\{E(u(\bar{t},z)-u(\bar{t},\bar{x})+p\cdot(z-\bar{x})+\alpha)-p\cdot(z-\bar{x})\right\} \\ &\leq \int_{\mathbb{R}^N} dz\ J(\bar{x}-z)\left\{E_*(v(\bar{t},z)-v(\bar{t},\bar{y})+p\cdot(z-\bar{y})+\alpha)-p\cdot(z-\bar{x})\right\}+o_\beta(1). \end{aligned}$$

Using (29) we then get

$$\begin{aligned} &\eta + K_0 e^{K_0\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{2\gamma}+\beta(|\bar{x}|^2+|\bar{y}|^2)\right) \\ \leq &\,(c(\bar{t},\bar{x})-c(\bar{t},\bar{y}))\,|\bar{p}+\bar{q}_x| + c(\bar{t},\bar{y})\,(|\bar{p}+\bar{q}_x|-|\bar{p}-\bar{q}_y|) \\ &+ M_p^\alpha[u(\bar{t},\cdot)](\bar{x})\,(|\bar{p}+\bar{q}_x|-|\bar{p}-\bar{q}_y|) \\ &+ \left(M_p^\alpha[u(\bar{t},\cdot)](\bar{x}) - \tilde{M}_p^\alpha[v(\bar{t},\cdot)](\bar{y})\right)|\bar{p}-\bar{q}_y| \\ \leq &\|\nabla c\|_\infty|\bar{x}-\bar{y}||\bar{p}+\bar{q}_x| + \|c\|_\infty|\bar{q}_x+\bar{q}_y| + \left(2\|u\|_\infty+\alpha+\frac{1}{2}\right)\|J\|_{L^1(\mathbb{R}^N)}|\bar{q}_x+\bar{q}_y| \\ &+ \left(\int_{\mathbb{R}^N} dz\ J(\bar{x}-z)\left\{E_*(v(\bar{t},z)-v(\bar{t},\bar{y})+p\cdot(z-\bar{y})+\alpha)-p\cdot(z-\bar{y})\right\} + \int_{\mathbb{R}^N} dz\ J(\bar{x}-z)(p\cdot(\bar{x}-\bar{y}))\right. \\ &\left. -\int_{\mathbb{R}^N} dz\ J(\bar{y}-z)\left\{E_*(v(\bar{t},z)-v(\bar{t},\bar{y})+p\cdot(z-\bar{y})+\alpha)-p\cdot(z-\bar{y})\right\}\right)|\bar{p}-\bar{q}_y| \\ &+ o_\beta(1)|\bar{p}-\bar{q}_y| \\ \leq &e^{K_0\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{2\gamma}\left(2\|\nabla c\|_\infty + 2\|\nabla J\|_{L^1(\mathbb{R}^N)}\left(2\|v\|_\infty+\frac{1}{2}+\alpha\right)+2\|J\|_{L^1(\mathbb{R}^N)}\right)+o_\beta(1) \end{aligned}$$

where we have used the definition of $\bar{p}$ and that $|\bar{q}_x|,\ |\bar{q}_y| = o_\beta(1)$. Taking

$$K_0 = 2\|Dc\|_\infty + 2\|DJ\|_{L^1(\mathbb{R}^N)}(2\|v\|_\infty+\frac{1}{2}+\alpha)+2\|J\|_{L^1(\mathbb{R}^N)},$$

we get a contradiction for β small enough.

Case 2. there exists a subsequence β_n, such that

$$\frac{|\bar{x}-\bar{y}|}{2\gamma} \to 0 \quad \text{as} \quad n \to +\infty.$$

In this case, we have $|\bar{p}+\bar{q}_x| \to 0$ and $|\bar{p}-\bar{q}_y| \to 0$ as $n \to +\infty$. Sending $n \to +\infty$ in (29), we get a contradiction.

This ends the proof of the theorem. □

4.4 Existence results

Theorem 4.5. *Consider $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, $c \in W^{1,\infty}([0,+\infty)\times\mathbb{R}^N)$ and $J \in W^{1,1}(\mathbb{R}^N)$. For $\varepsilon > 0$, there exists a (unique) bounded continous viscosity solution u^ε of* (5)*. Moreover, there exists a constant C independent on $\varepsilon > 0$ such that,*

$$|u^\varepsilon(t,x) - u_0(x)| \leq Ct. \tag{32}$$

Proof. As it is explained in [19, Proof of Theorem 6] (see also Alvarez, Tourin [2] or Imbert [17, Theorem 3]), to apply the Perron's method for non-local equations, it suffices to prove that there exists a constant

$C > 0$ (independent of ε) such that $u_0 \pm Ct$ are respectively a super and a subsolution. The only difficulty is to bound, for every C, the term $\left|M^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right]\right|_\infty$ by a constant C_1 independent of C and ε. To do this, it suffices to remark that

$$\left|M^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right]\right|_\infty \leq \frac{1}{2}\|J\|_{L^1} + \int_{\mathbb{R}^N} dz\ J(z)\left|\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon} - \nabla u_0(x)\cdot\frac{z}{\varepsilon}\mathbf{1}_B(z)\right|$$

and to use [19, Proof of Theorem 6] to get a constant $\tilde{C}_1 = \tilde{C}_1(R_0, N, ||u_0||_{W^{2,\infty}})$ such that

$$\int_{\mathbb{R}^N} dz\ J(z)\left|\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon} - \nabla u_0(x)\cdot\frac{z}{\varepsilon}\mathbf{1}_B(z)\right| \leq \tilde{C}_1$$

(with R_0 appearing in (8)). Then taking $C_1 = \frac{1}{2}\|J\|_{L^1} + \tilde{C}_1$ and $C = (\|c\|_\infty + C_1)\|\nabla u_0\|$, we get that $u_0 \pm Ct$ are respectively a super and a subsolution. This achieves the proof of the theorem. □

We recall the existence and uniqueness result for (11).

Theorem 4.6 ([19, Proposition 3]). *Assume that $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, $g \geq 0$, $g \in C^0(\mathbf{S}^{N-1})$ and $\overline{H}^0$ is continuous in (L,p) and nondecreasing in L, then the homogenized equation (11) has a unique bounded continuous viscosity solution u^0.*

4.5 Consistency of the definition of the geometric motion

As explained in Sections 1 and 3, Eq. (5) is the rescaled level set equation corresponding to the motion of N fronts submitted to monotone two-body interactions. The classical level set approach is well adapted for describing the motion of fronts since it can be proved (at least for local equations) that if (5) is solved for two initial data u_0 and v_0 that have the same 0-level set, then so have the two corresponding solutions. It turns out that the classical proof of [5] can be adapted to our framework. Before explaining it, let us state precisely the result.

Theorem 4.7. *Consider two bounded uniformly continuous functions u_0, v_0 and two corresponding solutions u and v of (5) with $\varepsilon = 1$. Fix any $\alpha \in [0,1)$, and assume that u_0 and v_0 satisfy for any $k \in \mathbb{Z}$*

$$\{u_0 < k+\alpha\} = \{v_0 < k+\alpha\} \qquad \& \qquad \{u_0 > k+\alpha\} = \{v_0 > k+\alpha\}.$$

Then the solutions u and v satisfy

$$\{u(t,\cdot) < k+\alpha\} = \{v(t,\cdot) < k+\alpha\} \qquad \& \qquad \{u(t,\cdot) > k+\alpha\} = \{v(t,\cdot) > k+\alpha\}.$$

The proof of this theorem relies on the invariance of the set of sub/super solutions of a level set equation under the action of monotone semicontinuous functions. Such a result is classical in the level set approach literature.

Proposition 4.8. *Assume that $\theta : \mathbb{R} \to \mathbb{R}$ is nondecreasing and upper semicontinuous (resp. lower semicontinuous). Assume also that*

$$\theta(v) - v \text{ is } 1-\text{periodic in } v. \tag{33}$$

Assume that $\varepsilon = 1$ in (5). Consider also a subsolution (resp. supersolution) u of (5). Then $\theta(u)$ is also a subsolution (resp. supersolution) of (5).

The proof of this proposition is postoned and we now explain how to use it to prove Theorem 4.7.

Proof of Theorem 4.7. We only do the proof for bounded fronts since the general case imply further technicalities we want to avoid, see for instance [20]. This is the reason why we assume that for any $k \in \mathbb{Z}$ (case $\alpha = 0$)

$$\{u_0 = k\} = \{v_0 = k\} \text{ is bounded.}$$

We now follow the lines of the original proof of [5]. Hence, we introduce two nondecreasing functions ϕ and ψ

$$\phi(r) = \begin{cases} \inf\{v_0(y) : u_0(y) \geq r\} & \text{if } r \leq M \\ \phi(M) & \text{if } r > M \end{cases} \qquad \psi(r) = \begin{cases} \sup\{v_0(y) : u_0(y) < r\} & \text{if } r \geq m \\ \psi(m) & \text{if } r < m \end{cases}$$

where $M = \sup u_0$ and $m = \inf u_0$. It is clear that ϕ is upper semicontinuous and ψ is lower semicontinuous. We now consider increasing extension $\tilde{\phi}$, $\tilde{\psi}$ of ϕ and ψ that satisfy $\tilde{\psi}(k) = k = \tilde{\phi}(k)$ for $k \in \mathbb{Z}$ and define

$$\begin{cases} \bar{\phi}(v) = \inf_{k \in \mathbb{Z}} \{\tilde{\phi}(v+k) - k\} \\ \bar{\psi}(v) = \sup_{k \in \mathbb{Z}} \{\tilde{\psi}(v+k) - k\} \end{cases}$$

(in fact the infimum or supremum are only on finite values of k because u_0 and v_0 are bounded). By noticing that $\bar{\phi}(u_0) \leq v_0 \leq \bar{\psi}(u_0)$ (because $v_0 \leq \Psi(u_0 + \varepsilon)$ for any $\varepsilon > 0$) and using Proposition 4.8, we conclude that $\bar{\phi}(u) \leq v \leq \bar{\psi}(u)$. It is now easy to conclude. □

It remains to prove Proposition 4.8.

Proof of Proposition 4.8. We just need to check that the non-local term can be handled in the classical proof. We only treat the case of subsolutions. Consider first $\theta \in C^1(\mathbb{R})$ such that $\theta' > 0$. Consider φ a test function from above satisfying $\theta(u) \leq \varphi$ with equality at (t_0, x_0). Then $u \leq \theta^{-1}(\varphi)$ and

$$\partial_t(\theta^{-1}(\varphi)) \leq c[u]|\nabla_x \theta^{-1}(\varphi)|$$

with

$$c[u] = c(t_0, x_0) + M[u(t_0, \cdot)](x_0).$$

Because from (25)

$$E(u(t_0, x_0 + z) - u(t_0, x_0)) \leq E(\theta(u(t_0, x_0 + z)) - \theta(u(t_0, x_0)))$$

we deduce that

$$\partial_t \varphi \leq c[\theta(u)]|\nabla_x \varphi|,$$

i.e. $\theta(u)$ is a subsolution in the sense of Definition 4.1.

In the general case, use the following lemma whose proof is left to the reader.

Lemma 4.9. *For a usc nondecreasing function θ, there exists $\theta^\varepsilon \in C^1$ such that $(\theta^\varepsilon)' > 0$, $\theta^\varepsilon \geq \theta$ and $\limsup^* \theta^\varepsilon = \theta$.*

On one hand, one can prove that such an approximation satisfies $\limsup^* \theta^\varepsilon(u) = \theta(u)$. On the other hand, θ^ε still satisfies (33). Hence $\theta^\varepsilon(u)$ is a subsolution of (5) by the previous case and we conclude that so is $\theta(u)$ by the stability result. □

5 Ergodicity

As explained in the Introduction, we will need in the proof of convergence to add a parameter $\alpha > 0$ in the cell problem. For the solution w of

$$\begin{cases} \partial_\tau w = \big(c(\tau, y) + L + M_p^\alpha[w(\tau, \cdot)](y)\big)\,|p + \nabla_y w| & \text{in} \quad (0, +\infty) \times \mathbb{R}^N \\ w(0, y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{34}$$

we prove a result that is stronger than Theorem 2.1.

Theorem 5.1 (Estimates for the initial value problem)**.** *There exists a unique* $\lambda = \lambda(L, p, \alpha)$ *such that the (unique) bounded continuous viscosity solution* $w \in C([0,T] \times \mathbb{R}^N)$ *of* (34) *satisfies:*

$$|w(\tau, y) - \lambda\tau| \le C_3, \tag{35}$$
$$|w(\tau, y) - w(\tau, z)| \le C_1, \quad \text{for all } y,\ z \in \mathbb{R}^N \tag{36}$$
$$|\lambda - (L + \alpha\|J\|_{L^1})|p|| \le \|c\|_\infty |p| =: C_2 \tag{37}$$

where

$$C_1 = \frac{(2\|c\|_\infty + \|J\|_{L^1})}{c_0}, \quad C_3 = 5C_1 + 2C_2, \quad c_0 = \inf_{\delta \in [0,1/2)^N} \int_{\mathbb{R}^N} dz\ \min\left(J(z - \delta), J(z + \delta)\right) > 0. \tag{38}$$

In the case $N = 1$*, we can choose:*

$$C_1 = |p|. \tag{39}$$

Theorem 2.1 is a consequence of (35). The existence of bounded solutions $v(\tau, y) = w(\tau, y) - \lambda\tau$ of

$$\lambda + \partial_\tau v = \big(c(\tau, y) + L + M_p^\alpha[v(\tau, \cdot)](y)\big)\, |p + \nabla_y v| \quad \text{on } (0, +\infty) \times \mathbb{R}^N \tag{40}$$

is a straighforward consequence of the previous theorem:

Corollary 5.2 (Existence of bounded correctors)**.** *There exists a solution* v *of* (40) *that satisfies:*

$$|v(\tau, y)| \le C_3,$$
$$|v(\tau, y) - v(\tau, z)| \le C_1.$$

Remark 5.3. To construct periodic sub and supersolution of (14) we can also classically consider

$$\delta v^\delta + v_\tau^\delta = \big(c_1(\tau, y) + L + M_p[v^\delta(\tau, \cdot)](y)\big)\, |p + \nabla_y v^\delta|$$

and take the limit $\delta \to 0$.

In order to solve the homogenized equation and to prove the convergence theorem, further properties of the number λ given by Theorem 5.1 are needed.

Corollary 5.4 (Properties of the effective Hamiltonian)**.** *The real number* λ *defines a continuous function* $\overline{H} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *that satisfies:*

$$\overline{H}(L, p, \alpha) \to \pm\infty \quad \text{as} \quad L \to \pm\infty, \tag{41}$$
$$\overline{H}(L, p, \alpha) \to \pm\infty \quad \text{as} \quad \alpha \to \pm\infty. \tag{42}$$

Moreover, $\overline{H}(L, p, \alpha)$ *is nonincreasing in* L *and* α*.*

In a first subsection, we successively prove Theorem 5.1 in dimension $N \ge 2$ and Corollary 5.4. Theorem 5.1 in the case $N = 1$ will be proved in a second subsection.

5.1 Proof of Theorem 5.1 and Corollary 5.4

Proof of Theorem 5.1 in the case $N \ge 2$. We proceed in several steps.

STEP 1: BARRRIERS AND EXISTENCE OF A SOLUTION. We proceed as in the proof of Theorem 4.5. We remark that $w^\pm(\tau, y) = C^\pm \tau$ with $C^\pm = (L + \alpha\|J\|_{L^1})|p| \pm C_2$ (with C_2 defined in (37)) are respectively a super- and a subsolution of (34) (use that $M_p[0] \equiv 0$). Hence, there exists a unique bounded continuous viscosity solution of (34) that satisfies:

$$|w(\tau, y) - (L + \alpha\|J\|_{L^1})|p|\tau| \le \|c\|_\infty |p|\tau. \tag{43}$$

Remark that by uniqueness, w is $\mathbb{Z}^N$-periodic with respect to y.

STEP 2: CONTROL OF THE OSCILLATIONS W.R.T. SPACE, UNIFORMLY IN TIME. We proceed as in [19] by considering the functions $M(\tau)$, $m(\tau)$ and $q(\tau)$ defined by:

$$M(\tau) = \sup_{y\in\mathbb{R}^N} w(\tau, y), \quad m(\tau) = \inf_{y\in\mathbb{R}^N} w(\tau, y) \quad \text{and} \quad q(\tau) = M(\tau) - m(\tau) \geq 0.$$

The supremum and infimum are attained since w is 1-periodic with respect to y. In particular, we can assume that:

$$M(\tau) = w(\tau, Y_\tau) \quad \text{and} \quad m(\tau) = w(\tau, y_\tau) \quad \text{and} \quad Y_\tau - y_\tau \in [0, 1)^N.$$

Now m, M and q satisfy in the viscosity sense:

$$\begin{aligned}
\frac{dM}{d\tau}(\tau) \leq & \Big(c(\tau, Y_\tau) + L + M_p^\alpha[w(\tau,\cdot)](Y_\tau)\Big)|p| \\
\leq & (\|c\|_\infty + \frac{1}{2}\|J\|_{L^1} + L + \alpha\|J\|_{L^1})|p| + \int dz\, J(z)(w(\tau, Y_\tau + z) - w(\tau, Y_\tau))|p|, \\
\frac{dm}{d\tau}(\tau) \geq & \Big(c(\tau, y_\tau) + L + M_p^\alpha[w(\tau,\cdot)](y_\tau)\Big)|p| \\
\geq & (-\|c\|_\infty - \frac{1}{2}\|J\|_{L^1} + L + \alpha\|J\|_{L^1})|p| + \int dz\, J(z)(w(\tau, y_\tau + z) - w(\tau, y_\tau))|p|, \\
\frac{dq}{d\tau}(\tau) \leq & (2\|c\|_\infty + \|J\|_{L^1})|p| + \mathcal{L}(\tau)|p|
\end{aligned}$$

where

$$\mathcal{L}(\tau) = \int dz\, J(z)(w(\tau, Y_\tau + z) - w(\tau, Y_\tau)) - \int dz\, J(z)(w(\tau, y_\tau + z) - w(\tau, y_\tau)).$$

Let us estimate $\mathcal{L}(\tau)$ from above by using the definition of y_τ and Y_τ. To do so, let us introduce $\delta_\tau = \frac{Y_\tau - y_\tau}{2} \in [0, \frac{1}{2})^N$ and $c_\tau = \frac{Y_\tau + y_\tau}{2}$ and write:

$$\begin{aligned}
\mathcal{L}(\tau) = & \int dz\, J(z - \delta_\tau)(w(\tau, c_\tau + z) - w(\tau, Y_\tau)) - \int dz\, J(z + \delta_\tau)(w(\tau, c_\tau + z) - w(\tau, y_\tau)) \\
& \leq \min_{\delta\in[0,\frac{1}{2})^N} \int dz\, \min\{J(z - \delta), J(z + \delta)\}(w(\tau, y_\tau) - w(\tau, Y_\tau)) = -c_0 q(\tau).
\end{aligned}$$

We conclude that q satisfies in the viscosity sense:

$$\frac{dq}{d\tau}(\tau) \leq (2\|c\|_\infty + \|J\|_{L^1})|p| - c_0|p|q(\tau)$$

with $q(0) = 0$ from which we obtain $q(\tau) \leq C_1$ for any $\tau \geq 0$ which can be rewritten under the following form:

$$|w(\tau, y) - w(\tau, z)| \leq C_1 \quad \text{for any } \tau \geq 0, \quad y, z \in \mathbb{R}^N. \tag{44}$$

STEP 3: CONTROL OF THE OSCILLATIONS W.R.T. TIME. We keep following the construction of the correctors of [19] by introducing, in order to estimate oscillations w.r.t. time, the two quantities:

$$\lambda^+(T) = \sup_{\tau\geq 0} \frac{w(\tau + T, 0) - w(\tau, 0)}{T} \quad \text{and} \quad \lambda^-(T) = \inf_{\tau\geq 0} \frac{w(\tau + T, 0) - w(\tau, 0)}{T}$$

and proving that they have a common limit as $T \to +\infty$. In order to do so, we first estimate λ^+ from above. This is a consequence of the comparison principle for (34) on the time interval $[\tau, \tau + \tau_0]$ for every $\tau_0 > 0$: since $w(\tau, 0) + C_1 + w^+(t)$ is a supersolution, we get for $t \in [0, \tau_0]$:

$$w(\tau + t, y) \leq w(\tau, 0) + C_1 + C^+ t \tag{45}$$

where $C^{\pm} = (L + \alpha \|J\|_{L^1})|p| \pm C_2$. Similarly, we get

$$w(\tau, 0) - C_1 + C^- t \leq w(\tau + t, y). \tag{46}$$

We then obtain for $\tau_0 = t = T$:

$$(L + \alpha \|J\|_{L^1})|p| - C_2 - \frac{C_1}{T} \leq \lambda^-(T) \leq \lambda^+(T) \leq (L + \alpha \|J\|_{L^1})|p| + C_2 + \frac{C_1}{T}$$

By definition of $\lambda^{\pm}(T)$, for any $\delta > 0$, there exists $\tau^{\pm} \geq 0$ such that

$$\left| \lambda^{\pm}(T) - \frac{w(\tau^{\pm} + T, 0) - w(\tau^{\pm}, 0)}{T} \right| \leq \delta.$$

Let us consider $\beta \in [0, 1)$ such that $\tau^+ - \tau^- - \beta = k$ is an integer. Next, consider $\Delta = w(\tau^+, 0) - w(\tau^- + \beta, 0)$. From (44), we get:

$$w(\tau^+, y) \leq w(\tau^- + \beta, y) + 2C_1 + \Delta = w(\tau^+ - k, y) + 2C_1 + \Delta.$$

The comparison principle for (34) on the time interval $[\tau^+, \tau^+ + T]$ (using the fact that $c(\tau, y)$ is $\mathbb{Z}$-periodic in τ) therefore implies that:

$$\begin{aligned} w(\tau^+ + T, y) &\leq w(\tau^+ - k + T, y) + 2C_1 + \Delta \\ &= w(\tau^- + \beta + T, y) + 2C_1 + w(\tau^+, 0) - w(\tau^- + \beta, 0). \end{aligned}$$

Choosing $y = 0$ in the previous inequality yields:

$$w(\tau^+ + T, 0) - w(\tau^+, 0) \leq w(\tau^- + \beta + T, 0) - w(\tau^- + \beta, 0) + 2C_1$$

and setting $t = \beta \leq 1$ and $\tau = \tau^- + T$ in (45) and $\tau = \tau^-$ in (46) finally yields:

$$T\lambda^+(T) \leq T\lambda^-(T) + 2\delta + 2(C_1 + C_2) + 2C_1.$$

Since this is true for any $\delta > 0$, we conclude that:

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{4C_1 + 2C_2}{T}.$$

Now arguing as in [18, 19], we conclude that $\lim_{T \to +\infty} \lambda^{\pm}(T)$ exist and are equal to λ and:

$$|\lambda^{\pm}(T) - \lambda| \leq \frac{4C_1 + 2C_2}{T}. \tag{47}$$

Step 4: conclusion. Estimate (43) implies (37). ¿From (47), we conclude that:

$$\left| \frac{w(\tau + T, 0) - w(\tau, 0)}{T} - \lambda \right| \leq \frac{4C_1 + 2C_2}{T}$$

which implies that:

$$|w(T, 0) - \lambda T| \leq 4C_1 + 2C_2$$

and finally (35) derives from this inequality and (44). The uniqueness of λ follows from (35) for instance. □

Proof of Corollary 5.4. The only point to be proved is the continuity of $\overline{H}$ since (37) implies the other properties. Let us consider a sequence $(L_n, p_n, \alpha_n) \to (L_0, p_0, \alpha_0)$ and set $\lambda_n = \lambda(L_n, p_n, \alpha_n)$. We remark that by (35), we have for any $\tau > 0$

$$\left| \lambda_n - \frac{w_n(\tau, 0)}{\tau} \right| \leq \frac{C_3}{\tau}$$

for some constant C_3 that we can choose independent on n. Stability of viscosity solutions for (34) implies that $w_n \to w_0$ locally uniformly w.r.t. (τ, y). This implies that $\limsup_{n \to +\infty} |\lambda_n - \lambda_0| \leq \frac{2C_3}{\tau}$ for any $\tau > 0$. Hence, we conclude that $\lim_{n \to +\infty} \lambda_n = \lambda_0$.

Finally the monotonicity in L and α of $\overline{H}(L, p, \alpha)$ comes from the comparison priniciple.

□

5.2 Proof of Theorem 5.1 in the case $N = 1$

Before proving Theorem 5.1 in the one dimensional case, we need the following lemma:

Lemma 5.5. *Let w be the solution of* (34) *in dimension $N = 1$. Then the function $y \mapsto p(p \cdot y + w(\tau, y))$ is nondecreasing for any $\tau \geq 0$, i.e.,*

$$p(p + w_y(\tau, y)) \geq 0.$$

Proof. We only do the proof in the case $p > 0$, since the case $p < 0$ is similar and the case $p = 0$ is trivial. We want to prove that

$$M_0 = \inf_{\Omega_T} \{w(\tau, x) - w(\tau, y) + p \cdot (x - y)\} \geq 0$$

where $\Omega_T = \{(\tau, x, y),\ 0 \leq \tau \leq T,\ y \leq x\}$. By contradiction, assume that $M_0 \leq -\delta < 0$. For $\eta > 0$, we consider

$$\Phi_\eta(\tau, x, y) = w(\tau, x) - w(\tau, y) + p \cdot (x - y) + \frac{\eta}{T - \tau}$$

and

$$M_\eta = \inf_{\Omega_T} \Phi_\eta(\tau, x, y) \leq -\frac{\delta}{2} \tag{48}$$

for η small enough.

By the space periodicity of w, we remark that

$$\begin{cases} \Phi_\eta(\tau, x + 1, y + 1) = \Phi_\eta(\tau, x, y) \\ \Phi_\eta(\tau, x - 1, y) = \Phi_\eta(\tau, x, y) - p \quad \text{if } x - 1 \geq y \end{cases}$$

so the minimum is reached at a point $(\bar{\tau}, \bar{x}, \bar{y})$ with $0 \leq \bar{x} - \bar{y} < 1$ and $\bar{\tau} < T$ (because w is bounded by the barrier functions). Moreover $\bar{x} > \bar{y}$ and $\bar{t} > 0$; indeed, otherwise we can check easily that the minimum would be nonnegative. Using Ishii's Lemma (see Crandall, Ishii, Lions [6, Lemma 8.3]), we then get that there exist $(a, -p) \in \overline{D}^+ w(\bar{\tau}, \bar{y})$ and $(b, -p) \in \overline{D}^- w(\bar{\tau}, \bar{x})$ with

$$a - b = \frac{\eta}{(T - \bar{\tau})^2}$$

such that (see equation (34))

$$a \leq 0 \quad \text{and} \quad b \geq 0.$$

Subtracting the two above inequalities yields a contradiction. □

Proof of Theorem 5.1 in the case $N = 1$. Let us assume that $p > 0$ since the case $p < 0$ is proved similarly and the case $p = 0$ is trivial.

Consider the solution w of (34). Lemma 5.5 ensures that $u(\tau, y) = w(\tau, y) + p \cdot y$ is nondecreasing:

$$\nabla_y u \geq 0.$$

To simplify the notation, let us drop the time dependence. We also know that w is 1-periodic in y, so for all $0 \leq y \leq z \leq 1$, we have

$$py + w(y) \leq pz + w(z) \leq p(y + 1) + w(y).$$

This implies that

$$|w(y) - w(z)| \leq p.$$

The rest of the proof in the same as in the case $N \geq 2$. □

6 The proof of convergence

This Section is devoted to the proof of Theorem 2.5.

Proof of Theorem 2.5. We consider the upper semicontinuous function $\overline{u} = \limsup{}^* u^\varepsilon$. By Theorem 4.5, it is bounded for bounded times and $\overline{u}(0,x) = u_0(x)$. As usual, we are going to prove that it is a subsolution of (11). Similarly, we can prove that $\underline{u} = \liminf{}_* u^\varepsilon$ is a bounded supersolution of (11) such that $\underline{u}(0,x) = u_0(x)$. Theorem 4.6 thus yield the result.

Let us prove that $\overline{u}$ is a subsolution of (11). We argue (classically) by contradiction by assuming that there exists a point (t_0,x_0), $t_0>0$, and a test function $\phi \in C^2$ such that $\overline{u}-\phi$ attains a global zero strict maximum at (t_0,x_0) and:

$$\partial_t \phi(t_0,x_0) = \overline{H}^0(L_0,p) + \theta = \overline{H}(L_0,p,0) + \theta$$

with

$$L_0 = \int_{|z|\le 2} \{\phi(t_0,x_0+z) - \phi(t_0,x_0) - \nabla\phi(t_0,x_0)\cdot z\}\mu(dz) + \int_{|z|\ge 2}\{\overline{u}(t_0,x_0+z) - \overline{u}(t_0,x_0)\}\mu(dz) \quad (49)$$

and $p = \nabla\phi(t_0,x_0)$ and $\theta > 0$. By Corollary 5.4, there exists $\alpha > 0$ and $\beta > 0$ such that:

$$\partial_t\phi(t_0,x_0) = \overline{H}(L_0+\beta,p,\alpha) + \frac{\theta}{2}. \quad (50)$$

In the following, λ denotes $\overline{H}(L_0+\beta,p,\alpha)$.

We now construct a supersolution ϕ^ε of (5) on a small ball centered at (t_0,x_0) by using the perturbed test function method (see [10, 11]). Precisely, we consider:

$$\phi^\varepsilon(t,x) = \begin{cases} \phi(t,x) + \varepsilon\, v\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) - \eta_r & \text{if } (t,x)\in(t_0/2,2t_0)\times B_1(x_0),\\ u^\varepsilon(t,x) & \text{if not}\end{cases}$$

where η_r is chosen later and the corrector v is a bounded solution of (40) associated with $(L,p,\alpha) = (L_0+\beta,p,\alpha)$ given by Corollary 5.2. We will prove that ϕ^ε is a supersolution of (5) on $B_r(t_0,x_0)$ (for r and ε small enough – this is made precise later) and that $\phi^\varepsilon \ge u^\varepsilon$ outside. In particular, r is chosen small enough so that $B_r(t_0,x_0)\subset(t_0/2,2t_0)\times B_1(x_0)$.

Let us first focus on "boundary conditions". For ε small enough (*i.e.* $0<\varepsilon\le\varepsilon_0(r)<r$), since $\overline{u}-\phi$ attains a strict maximum at (t_0,x_0), we can ensure that:

$$u^\varepsilon(t,x) \le \phi(t,x) + \varepsilon v\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) - \eta_r \quad \text{for } (t,x)\in(t_0/3,3t_0)\times B_3(x_0)\setminus B_r(t_0,x_0) \quad (51)$$

for some $\eta_r = o_r(1) > 0$. Hence, we conclude that $\phi^\varepsilon \ge u^\varepsilon$ outside of $B_r(t_0,x_0)$.

We now turn to the equation. Consider a test function ψ such that $\phi^\varepsilon - \psi$ attains a local minimum at $(\overline{t},\overline{x})\in B_r(t_0,x_0)$. This implies that $v-\Gamma$ attains a local minimum at $(\overline{\tau},\overline{y})$ where $\overline{\tau} = \frac{\overline{t}}{\varepsilon}$, $\overline{y} = \frac{\overline{x}}{\varepsilon}$ and

$$\Gamma(\tau,y) = \frac{1}{\varepsilon}(\psi-\phi)(\varepsilon\tau,\varepsilon y).$$

Since v is a viscosity solution of (40), we conclude that:

$$\lambda + \partial_\tau\Gamma(\overline{\tau},\overline{y}) \ge \left(c(\overline{\tau},\overline{y}) + L + \tilde{M}_p^\alpha[v(\overline{\tau},\cdot)](\overline{y})\right)|p + \nabla\Gamma(\overline{\tau},\overline{y})|$$

from which we deduce:

$$\begin{aligned}&\left(\partial_t\phi(t_0,x_0) - \frac{\theta}{2}\right) + \partial_t\psi(\overline{t},\overline{x}) - \partial_t\phi(\overline{t},\overline{x})\\ \ge&\left(c(\overline{\tau},\overline{y}) + L + \tilde{M}_p^\alpha[v(\overline{\tau},\cdot)](\overline{y})\right)\left|\nabla\psi(\overline{t},\overline{x}) - (\nabla\phi(\overline{t},\overline{x}) - \nabla\phi(t_0,x_0))\right|.\end{aligned}$$

Hence, we get:

$$\partial_t\psi(\bar{t},\bar{x}) \geq \left(c\left(\frac{\bar{t}}{\varepsilon},\frac{\bar{x}}{\varepsilon}\right) + L_0 + \beta + \tilde{M}_p^\alpha\left[v\left(\frac{\bar{t}}{\varepsilon},\cdot\right)\right]\left(\frac{\bar{x}}{\varepsilon}\right)\right)|\nabla\psi(\bar{t},\bar{x}) + o_r(1)| + \frac{\theta}{2} + o_r(1)$$

where $o_r(1)$ only depends on local bounds of ϕ and its derivatives. Since c is bounded and:

$$|\tilde{M}_p^\alpha[v(\overline{\tau},\cdot)](\overline{y})| \leq (\text{osc}\, v(\tau,\cdot) + \alpha\|J\|_{L^1}) + \frac{1}{2}\|J\|_{L^1} \leq \left(C_1 + \alpha + \frac{1}{2}\right)\|J\|_{L^1},$$

where C_1 is given in Theorem 5.1, we conclude that:

$$\partial_t\psi(\bar{t},\bar{x}) \geq \left(c\left(\frac{\bar{t}}{\varepsilon},\frac{\bar{x}}{\varepsilon}\right) + L_0 + \beta + \tilde{M}_p^\alpha\left[v\left(\frac{\bar{t}}{\varepsilon},\cdot\right)\right]\left(\frac{\bar{x}}{\varepsilon}\right)\right)|\nabla\psi(\bar{t},\bar{x})|. \tag{52}$$

Now recall that L_0 is defined by (49) and use the following lemma:

Lemma 6.1. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, $r \leq r_0$ and $(t,x) \in B_r(t_0,x_0)$:*

$$\tilde{M}^\varepsilon\left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right](x) \leq L_0 + \beta + \tilde{M}_p^\alpha\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{\varepsilon}\right). \tag{53}$$

The proof of this lemma is postponed. Combining (52) and (53), we conclude that for any $\varepsilon \leq \varepsilon_0$ and $r \leq r_0$:

$$\partial_t\psi(\bar{t},\bar{x}) \geq \left(c\left(\frac{\bar{t}}{\varepsilon},\frac{\bar{x}}{\varepsilon}\right) + \tilde{M}^\varepsilon\left[\frac{\phi^\varepsilon(t,\cdot)(x)}{\varepsilon}\right]\right)|\nabla\psi(\bar{t},\bar{x})|.$$

We conclude that ϕ^ε is a supersolution of (5) on $B_r(t_0,x_0)$ and $\phi^\varepsilon \geq u^\varepsilon$ outside. Using the comparison principle, this implies $\phi^\varepsilon(t,x) \geq u^\varepsilon(t,x)$. Passing to the supremum limit at the point (t_0,x_0), we obtain: $\phi(t_0,x_0) \geq \overline{u}(t_0,x_0) + \eta_r$ which is a contradiction. The proof of Theorem 2.5 is now complete. □

It remains to prove Lemma 6.1.

Proof of Lemma 6.1. It is convenient to use the notation: $\tau = t/\varepsilon$ and $y = x/\varepsilon$. We simply divide the domain of integration in two parts: short range interaction and long range interaction. Precisely:

$$\begin{aligned}
\tilde{M}^\varepsilon\left[\frac{\phi^\varepsilon(t,\cdot)(x)}{\varepsilon}\right] &= \int dz\, J(z)\, E_*\left(\frac{\phi^\varepsilon(t,x+\varepsilon z) - \phi^\varepsilon(t,x)}{\varepsilon}\right) \\
&= \int_{|z|\leq r_\varepsilon} dz\,\{\dots\} + \int_{\varepsilon|z|\geq \varepsilon r_\varepsilon} dz\,\{\dots\} = T_1 + T_2
\end{aligned}$$

and we choose r_ε such that $r_\varepsilon \to +\infty$ and $\varepsilon r_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us estimate from above each term. For $(t,x) \in B_r(t_0,x_0)$ and $|z| \leq r_\varepsilon$, we are sure that $(t, x+\varepsilon z) \in (t_0/2, 2t_0) \times B_1(x_0)$ for ε small enough, and:

$$\begin{aligned}
T_1 &= \int_{|z|\leq r_\varepsilon} dz\, J(z)\, E_*\left(\frac{\phi^\varepsilon(t,x+\varepsilon z) - \phi^\varepsilon(t,x)}{\varepsilon}\right) \\
&= \int_{|z|\leq r_\varepsilon} dz\, J(z)\, E_*\left(\frac{\phi(t,x+\varepsilon z) - \phi(t,x)}{\varepsilon} + v(\tau,y+z) - v(\tau,y)\right) \\
&= \int_{|z|\leq r_\varepsilon} dz\, J(z)\Bigg\{E_*\Bigg(\nabla\phi(t,x)\cdot z + v(\tau,y+z) - v(\tau,y) \\
&\qquad\qquad + \frac{\phi(t,x+\varepsilon z) - \phi(t,x) - \varepsilon\nabla\phi(t,x)\cdot z}{\varepsilon}\Bigg) - \nabla\phi(t,x)\cdot z\Bigg\}.
\end{aligned}$$

To get the last line of the previous inequality, we used that J is even. Choose next ε small enough and r_ε big enough so that

$$\frac{\phi(t,x+\varepsilon z)-\phi(t,x)-\varepsilon\nabla\phi(t,x)\cdot z}{\varepsilon}\leq C\varepsilon(r^\varepsilon)^2\leq\frac{\alpha}{2}$$

$$\int_{|z|\geq r_\varepsilon}dz\,J(z)\left\{E_*\left(v\left(\cdot,\frac{x}{\varepsilon}+z\right)-v\left(\cdot,\frac{x}{\varepsilon}\right)+\frac{\alpha}{2}+z\cdot\nabla\phi(t,x)\right)-z\cdot\nabla\phi(t,x)\right\}\leq\frac{\beta}{4}.$$

Hence we obtain

$$T_1\leq\tilde{M}^{\alpha/2}_{\nabla\phi(t,x)}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{\varepsilon}\right)+\frac{\beta}{4}. \tag{54}$$

We now claim that:

$$\tilde{M}^{\alpha/2}_{\nabla\phi(t,x)}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{t}\right)\leq\tilde{M}^{\alpha}_{\nabla\phi(t_0,x_0)}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{t}\right)+\beta/4 \tag{55}$$

for r small enough. To see this, consider $R_\beta>0$ such that:

$$\int_{|z|\geq R_\beta}dz\,J(z)\{E_*(\nabla\phi(t,x)\cdot z+v(\tau,y+z)-v(\tau,y))-\nabla\phi(t,x)\cdot z\}\leq\beta/8,$$

$$\int_{|z|\geq R_\beta}dz\,J(z)\{E_*(\nabla\phi(t_0,x_0)\cdot z+v(\tau,y+z)-v(\tau,y))-\nabla\phi(t_0,x_0)\cdot z\}\leq\beta/8.$$

Now for $|z|\leq R_\beta$ and r small enough:

$$|(\nabla\phi(t,x)\cdot z-\nabla\phi(t_0,x_0))\cdot z|\leq\alpha/2$$

and we get (55). Combining this inequality with (54), we obtain:

$$T_1\leq\tilde{M}^{\alpha}_{p}\left[v\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{t}\right)+\beta/2. \tag{56}$$

We now turn to T_2. We can choose $r_\varepsilon\geq R_0$ where R_0 appears in (8) so that $J(z)=g(z/|z|)|z|^{-N-1}$ for $|z|\ \geq r_\varepsilon$. Hence

$$T_2=\int_{\varepsilon|z|\geq\varepsilon r_\varepsilon}dz\,J(z)\,E_*\left(\frac{\phi^\varepsilon(t,x+\varepsilon z)-\phi^\varepsilon(t,x)}{\varepsilon}\right)=\int_{|q|\geq\varepsilon r_\varepsilon}\mu(dq)\,E^\varepsilon_*(\phi^\varepsilon(t,x+q)-\phi^\varepsilon(t,x))$$

with μ defined by (13) and where $E^\varepsilon_*(\alpha)=\varepsilon E_*\left(\frac{\alpha}{\varepsilon}\right)$. Remark that $|E^\varepsilon_*(\alpha)-\alpha|\leq\frac{\varepsilon}{2}$ and use (51) to get:

$$\begin{aligned}T_2&\leq\int_{|q|\geq\varepsilon r_\varepsilon}\mu(dq)\,(\phi^\varepsilon(t,x+q)-\phi^\varepsilon(t,x))+\frac{\varepsilon}{2}\mu(\mathbb{R}^N\setminus B_{\varepsilon r_\varepsilon})\\&\leq\int_{\varepsilon r_\varepsilon\leq|q|\leq 2}\mu(dq)\,\{\phi(t,x+q)-\phi(t,x)+\varepsilon\operatorname{osc}v(\tau,\cdot)\}\\&\quad+\int_{|q|\geq 2}\mu(dq)\,\{u^\varepsilon(t,x+q)-\phi(t,x)\}+\frac{C}{r_\varepsilon}+C'\eta_r.\end{aligned}$$

Now use that μ is even and get

$$\begin{aligned}T_2&\leq\int_{\varepsilon r_\varepsilon\leq|q|\leq 2}\mu(dq)\,\{\phi(t,x+q)-\phi(t,x)-\nabla\phi(t,x)\cdot q\}\\&\quad+\int_{|q|\geq 2}\mu(dq)\,\{u^\varepsilon(t,x+q)-\phi(t,x)\}+C'\left(\frac{1}{r_\varepsilon}+\varepsilon+\eta_r\right).\end{aligned}$$

Now remark that:

$$\int_{\varepsilon r_\varepsilon \le |q| \le 2} \mu(dq) \left\{ \phi(t, x+q) - \phi(t,x) - \nabla\phi(t,x) \cdot q \right\}$$
$$\le \int_{|q|\le 2} \mu(dq) \left\{ \phi(t_0, x_0+q) - \phi(t_0,x_0) - \nabla\phi(t_0,x_0) \cdot q \right\} + C\varepsilon r_\varepsilon + o_r(1)$$

and keeping in mind that $\phi(t_0, x_0) = \overline{u}(t_0, x_0)$, we also have

$$\int_{2\le|q|} \mu(dq) \left\{ u^\varepsilon(t, x+q) - \phi(t,x) \right\} \le \int_{2\le |q|} \mu(dq) \left\{ \overline{u}(t_0, x_0+q) - \overline{u}(t_0,x_0) \right\} + o_r(1).$$

Indeed, it is equivalent to

$$\limsup_{y\to x_0, \varepsilon\to 0} \int_{2\le|q|} \mu(dq) \left\{ u^\varepsilon(t, y+q) - \phi(t,y) \right\} \le \int_{2\le |q|} \mu(dq) \left\{ \overline{u}(t_0, x_0+q) - \overline{u}(t_0,x_0) \right\}$$

and such an inequality is a consequence of Fatou's lemma.

Combining all the estimates yields:

$$\begin{aligned} T_2 \quad &\le \quad \int_{|q|\le 2} \mu(dq) \left\{ \phi(t_0, x_0+q) - \phi(t_0,x_0) - \nabla\phi(t_0,x_0) \cdot q \right\} \\ &\quad + \int_{2\le|q|} \mu(dq) \left\{ \overline{u}(t_0, x_0+q) - \overline{u}(t_0,x_0) \right\} + C\varepsilon r_\varepsilon + C' \left(\frac{1}{r_\varepsilon} + \varepsilon \right) + o_r(1) + \beta/4 \\ &\le \quad L_0 + \beta/2. \end{aligned} \tag{57}$$

Combining (56) and (57) yields (53). □

7 Qualitative properties of the effective Hamiltonian

In this section, we consider the special case of the one-dimensional space and of a driving force independent of time: $c(\tau, y) = c(y)$. Before proving Theorem 2.6 in Subsection 7.3, we establish gradient estimates in Subsection 7.1 and then construct sub/super/correctors independent on time in Subsection 7.2.

7.1 Gradient estimates

We recall that w denotes the solution of the following Cauchy problem

$$\begin{cases} \partial_t w = \Big(c(x) + L + M_p[w(t,\cdot)](x) \Big) |p + \nabla w| & \text{in} \quad (0, +\infty) \times \mathbb{R}^N, \\ w(0,x) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{58}$$

Lemma 7.1 (Lipschitz estimates on the solution)**.** *The solution w of* (58) *is Lipschitz continuous w.r.t. x and satisfies:*

$$\|p + \nabla_x w(t,\cdot)\|_\infty \le |p| e^{t\|\nabla c\|_\infty}. \tag{59}$$

Proof. The function $u(t,x) = p\cdot x + w(t,x)$ is a solution of

$$\partial_t u = (c(x) + L + M[u(t,\cdot)](x)) \, |\nabla u| \quad \text{on } (0,+\infty)\times\mathbb{R}. \tag{60}$$

Consider the sup-convolution of u:

$$u^\beta(t,x) = \sup_{y\in\mathbb{R}^N} \left\{ u(t,y) - e^{Kt} \frac{|x-y|^2}{2\beta} \right\} = u(t, x_\beta) - e^{Kt} \frac{|x-x_\beta|^2}{2\beta}. \tag{61}$$

We claim that u^β is a subsolution of the equation (60) for K large enough. Indeed, for any $(\eta, q) \in D^{1,+}u^\beta(t,x)$, it is classical that $\left(\eta + Ke^{Kt}\ \frac{|x-x_\beta|^2}{2\beta}, q\right) \in D^{1,+}u(t,x_\beta)$, $q = -e^{Kt}\frac{x-x_\beta}{\beta}$ and $|x - x_\beta| \ \leq C\sqrt{\beta}$ where C depends on $\|w\|_\infty$, so that:

$$\begin{aligned} \eta + Ke^{Kt}\frac{|x-x_\beta|^2}{2\beta} &\leq (c(x_\beta) + M_p[w(t,\cdot)](x_\beta))\,|q| \\ &\leq \big(c(x) + M_p[w^\beta(t,\cdot)](x)\big)\,|q| + \|\nabla c\|_\infty e^{Kt}\frac{|x-x_\beta|^2}{\beta} \end{aligned}$$

where we used that

$$w(t, x_\beta + z) - w(t, x_\beta) \leq w^\beta(t, x+z) - w^\beta(t,x) \quad \text{with} \quad w^\varepsilon(t,x) = u(t,x) - p\cdot x$$

(this comes from (61) and the fact that $u(t, x_\beta + z) - e^{Kt}\frac{|x-x_\beta|^2}{2\beta} \leq u^\beta(t, x+z)$). Choosing now $K = 2\|\nabla c\|_\infty$ permits to get that u^β is a subsolution. Next, remark that:

$$u^\beta(0,x) \leq u(0,x) + \sup_{r>0}\left\{\|\ \nabla u_0\|_\infty r - \frac{r^2}{2\beta}\right\} \leq u(0,x) + \beta\frac{\|\nabla u_0\|^2}{2}.$$

Hence, the comparison principle (see Theorem 4.4 applied to $u^\beta(t,x) - p\cdot x$ and $w(t,x)$) implies that

$$u^\beta \leq u + \beta\frac{\|\nabla u_0\|^2}{2}.$$

Rewrite this inequality as follows

$$u(t,y) \leq u(t,x) + \beta\frac{\|\nabla u_0\|^2}{2} + e^{Kt}\ \frac{|x-y|^2}{2\beta}.$$

Optimizing with respect to β permits to conclude.

□

7.2 Sub- and supercorrectors

We give an alternative characterization of the ergodicity of (58) that complements Theorem 2.1 in the special case $c(\tau, y) = c(y)$. We will use it repeatedly in the proof of Theorem 2.6. More precisely, we are interested in the following stationnary equation:

$$\lambda = (c(y) + L + M_p[v](y))\,|p + \nabla_y v| \quad \text{on } \mathbb{R}^N. \tag{62}$$

Lemma 7.2. *The function* $\overline{H}^0$ *satisfies:*

$$\begin{aligned} \overline{H}^0(p,L) &= \max\{\lambda : \text{there exists a 1-periodic subsolution of (62)}\} \\ &= \min\{\lambda : \text{there exists a 1-periodic supersolution of (62)}\}. \end{aligned} \tag{63}$$

Remark 7.3. Such a characterization is classical in the context of homogenization of Hamilton-Jacobi equation. See for instance [22] for such a characterization for local Hamilton-Jacobi equations.

Proof. We first remark that we can construct λ^- and λ^+ such that $v = 0$ is respectively a sub- and a supersolution of (62) so that the sets at stake in (63) are not empty. Let $\lambda^{\min}$ and $\lambda^{\max}$ denote respectively the maximum and the minimum defined in (63). The fact that the infimum and the supremum defining $\lambda^{\min}$ and $\lambda^{\max}$ are attained is a consequence of the stability of viscosity solutions and L^∞ a priori bounds (that are easy to obtain).

Let w be the solution of (58) and consider the upper relaxed limit $\overline{v}_\infty$ as n goes to infinity (resp. the lower relaxed limit $\underline{v}_\infty$) of $v_n(\tau, y) = v(\tau + n, y)$ with $v(\tau, y) = w(\tau, y) - \overline{H}^0(p, L)\tau$. Then consider the suppremum in time of $\overline{v}_\infty$ (resp. the infimum in time of $\underline{v}_\infty$). This allows us to construct a subsolution (resp. a supersolution) of (62). This implies that $\overline{H}^0(p, L) \le \lambda^{\min}$ (resp. $\lambda^{\max} \le \overline{H}^0(p, L)$).

Now let v^- be a periodic subsolution of (62) for $\lambda^{\min}$. Since (62) does not see the constants, we can assume that $v^- \le 0$. We have that $v^- + \lambda^{\min}\tau$ is a subsolution of (58). By comparison principle, we deduce that

$$w \ge v^- + \lambda^{\min}\tau$$

where w is the solution of (58). Dividing by τ and sending $\tau \to \infty$, we get that $\overline{H}^0(p, L) \ge \lambda^{\min}$. The proof that $\lambda^{\max} \ge \overline{H}^0(p, L)$ is similar. This ends the proof of the lemma. □

The second technical lemma we need in the proof of Theorem 2.6 is the construction of sub- and supercorrectors of (62) with some monotonicity properties and with precise estimates on their oscillations.

Lemma 7.4. (Existence of sub and supercorrectors) *For any $p \in \mathbb{R}$ and $L \in \mathbb{R}$, there exists $\lambda \in \mathbb{R}$, a subcorrector $\underline{v}(y)$ and a supercorrector $\overline{v}(y)$ which are 1-periodic in y and satisfy*

$$\begin{aligned} &\lambda \le (c + L + M_p[\underline{v}])\, |p + \nabla_y \underline{v}|, \quad \text{with} \quad p(p + \nabla_y \underline{v}) \ge 0 \quad \text{on} \quad \mathbb{R}, \\ &\lambda \ge \left(c + L + \tilde{M}_p[\overline{v}]\right) |p + \nabla_y \overline{v}|, \quad \text{with} \quad p(p + \nabla_y \overline{v}) \ge 0 \quad \text{on} \quad \mathbb{R} \end{aligned}$$

such that

$$\max \underline{v} - \min \underline{v} \le |p| \quad \text{and} \quad \max \overline{v} - \min \overline{v} \le |p|. \tag{64}$$

There exists a discontinuous corrector v which satisfies

$$\lambda = (c + L + M_p[v])\, |p + \nabla_y v| \quad \text{on} \quad \mathbb{R}, \tag{65}$$

$$\max v - \min v \le 2|p| \quad \text{and} \quad |v|_\infty \le |p|.$$

The unique solution w of (58) satisfies for all $\tau \ge 0$

$$|w(\tau, y) - \lambda\tau|_\infty \le 2|p|. \tag{66}$$

Proof. When $p = 0$, we observe that (62) is satisfied with $\lambda = 0$ and $v = 0$. Hence (64) is clearly satisfied. Let us now assume that $p > 0$ since the case $p < 0$ is similar.

Step 1. Consider the solution w of (58), *i.e.*, the solution w of (34) with $\alpha = 0$. The proof of Theorem 5.1 in the case $N = 1$ implies that $\tilde{v}(\tau, y) = w(\tau, y) - \lambda\tau$ satisfies

$$|\tilde{v}(\tau, y) - \tilde{v}(\tau, z)| \le |p| \quad \text{and } p(p + \nabla_y \tilde{v}) \ge 0$$

and so

$$\max_y \tilde{v} - \min_y \tilde{v} \le |p|.$$

Step 2. Consider the upper relaxed limit $\overline{v}_\infty$ as n goes to infinity (resp. the lower relaxed limit $\underline{v}_\infty$) of $v_n(\tau, y) = \tilde{v}(\tau + n, y)$ Then consider the supremum in time of $\overline{v}_\infty$ (resp. the infimum in time of $\underline{v}_\infty$). This allows us to build a subsolution $\overline{v}$ (resp. a supersolution $\underline{v}$) of (65) that satisfies the expected properties.

Step 3. Finally, we have $\underline{v} + \operatorname{osc} v \ge \overline{v}$ and $\underline{v} + \operatorname{osc} v$ is still a supersolution of (65). Then the Perron's method allows us to build a discontinuous periodic corrector v which satisfy $\operatorname{osc} v \le 2p$. Moreover, since the equation does not see constants, we have that $v(y) - K$ is solution of (65) and so we can assume that $|v|_\infty \le |p|$ for a good choice of the constant K.

To prove that the solution w of (58) satisfies $|w - \lambda\tau|_\infty \le 2|p|$, it suffices to remark that $\underline{v} + \operatorname{osc} \underline{v} + \lambda\tau$ (resp. $\overline{v} + \operatorname{osc} \overline{v} + \lambda\tau$) is supersolution (resp. subsolution) of (58) and to use the comparison principle for equation (58). This ends the proof of the lemma. □

7.3 Proof of Theorem 2.6

We now turn to the proof itself.

Proof of Theorem 2.6.
1. When $c \equiv 0$, notice that $\lambda = L|p|$ and $v = 0$ do satisfy (62).

2. This is a consequence of (37).

3. Using the monotonicity of $\overline{H}^0(L,p)$ in L (see Corollary 5.4) we only need to prove that $\overline{H}^0(0,p) = 0$. Thanks to Lemma 7.2, it is enough to construct a sub- and a supersolution of:

$$0 = c + M_p[v].$$

Let us consider the solution v of:

$$\begin{cases} \partial_\tau v = c(y) + M_{p,\delta}[v(\tau,\cdot)](y) & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ v(0,y) = 0 & \text{in } \mathbb{R} \end{cases}$$

with

$$M_{p,\delta}[v](y) = \int_{\mathbb{R}} dz\ J(z) \left\{ E^\delta\left(v(y+z) - v(y) + p\cdot z\right) - p\cdot z \right\}$$

where E^δ is a smooth approximation of E, such that $E^\delta(z) - z$ is 1-periodic, $E^\delta(-z) = -E^\delta(z)$ and E^δ is increasing. From the 1-periodicity of c_1, we deduce the 1-periodicity of v. It is also easy to prove (by adaptating the proof of Lemma 7.1) that v is Lipschitz continuous in space and time for all finite time. Let us consider $p = P/Q$ with $P \in \mathbb{Z}$, $Q \in \mathbb{N}\setminus\{0\}$ and dropping for a while the dependence on τ, we set $u(y) = v(y) + p\cdot y$. Assuming temporarily that J decays to zero at infinity sufficiently quickly, recalling that J is even and using the fact that $v(y)$ is 1-periodic in y, we compute:

$$\begin{aligned} &\int_{(-Q/2,Q/2)} dy \int_{\mathbb{R}} dz\ J(z) \left\{ E^\delta\left(v(y+z) - v(y) + p\cdot z\right) - p\cdot z \right\} \\ &= \int_{(-Q/2,Q/2)} dy \sum_{k\in\mathbb{Z}} \int_{((k-1/2)Q,(k+1/2)Q)} dz\ J(z) \left\{ E^\delta\left(u(y+z) - u(y)\right) \right\} \\ &= K + \int_{(-Q/2,Q/2)} dy \sum_{k\in\mathbb{Z}} \int_{(-Q/2,Q/2)} dz\ J(z+kQ) \left\{ E^\delta\left(u(y+z) - u(y)\right) \right\} \end{aligned}$$

where

$$\begin{aligned} K &= \int_{(-Q/2,Q/2)} dy \int_{(-Q/2,Q/2)} dz \sum_{k\in\mathbb{Z}} kQJ(z+kQ) \\ &= P\left(\int_{(-Q/2,Q/2)} dz \sum_{k\in\mathbb{Z}} (z+kQ)J(z+kQ) - \int_{(-Q/2,Q/2)} dz J_Q(z) \right) \\ &= P \int_{\mathbb{R}} d\bar{z}\ \bar{z}J(\bar{z}) \\ &= 0. \end{aligned}$$

Hence, with the notation

$$J_Q(z) = \sum_{k\in\mathbb{Z}} J(z+kQ) = J_Q(-z),$$

we get

$$\begin{aligned}
&\int_{(-Q/2,Q/2)} dy \int_{\mathbb{R}} dz\ J(z) \left\{E^\delta \left(v(y+z)-v(y)+p\cdot z\right)-p\cdot z\right\}\\
&= \int_{(-Q/2,Q/2)} dy \int_{(-Q/2,Q/2)} dz\ J_Q(z) \left\{E^\delta \left(u(y+z)-u(y)\right)\right\}\\
&= \int_{(-Q/2,Q/2)} dz\ J_Q(z) \int_{(z-Q/2,z+Q/2)} dx\ \left\{E^\delta \left(u(x)-u(x-z)\right)\right\}\\
&= \int_{(-Q/2,Q/2)} dz\ J_Q(z) \int_{(-z-Q/2,-z+Q/2)} dx\ \left\{E^\delta \left(u(x)-u(x+z)\right)\right\}\\
&= -\int_{(-Q/2,Q/2)} dz\ J_Q(z) \int_{(-z-Q/2,-z+Q/2)} dx\ \left\{E^\delta \left(u(x+z)-u(x)\right)\right\}\\
&= -\int_{(-Q/2,Q/2)} dz\ J_Q(z) \int_{(-Q/2,Q/2)} dx\ \left\{E^\delta \left(u(x+z)-u(x)\right)\right\}
\end{aligned}$$

(the last line is obtained by using the 1-periodicity of $u(x+z)-u(x)$ in x). Finally comparing the second line and the last one of the previous equality, we deduce that:

$$\int_{(-Q/2,Q/2)} dy \int_{\mathbb{R}} dz\ J(z) \left\{E^\delta \left(v(y+z)-v(y)+p\cdot z\right)-p\cdot z\right\} = 0.$$

By taking a limit, we deduce that this is still true without assuming that J has strong decay at infinity.

Using now the fact that the equation is satisfied almost everywhere (because the solution is Lipschitz continuous) we get by integration of the equation and the fact that $\int_{(0,1)} c = 0$:

$$\partial_\tau \left(\int_{(-Q/2,Q/2)} dy\ v(\tau,y)\right) = 0.$$

We deduce (by periodicity of v) that

$$\int_{(0,1)} dy\ v(\tau,y) = 0 \quad \text{for any} \quad \tau \geq 0$$

for any rational p. We conclude that this is true for any $p \in \mathbb{R}$ (just by taking a limit). On the other hand, we deduce as in the proof of Theorem 5.1 in the case $N = 1$ that

$$\max_y v(\tau,y) - \min_y v(\tau,y) \leq |p|.$$

and similarly that

$$|v(\tau,y)| \leq C(p).$$

Considering the semi-relaxed limits of $v(\tau,y)$ with the supremum (resp. the infimum) in time, we build a subsolution $\underline{v}$ (resp. a supersolution $\overline{v}$) of the following equation

$$0 = c + M_{p,\delta}[v].$$

Taking now the limit as δ goes to zero, we build a subsolution $\underline{v}$ (resp. a supersolution $\overline{v}$) of the limit equation

$$0 = c + M_p[v]$$

and the proof of $\overline{H}^0(0,p) = 0$ is complete.

4. The monotonicity of $\overline{H}^0(L,p)$ in L is a straightforward consequence of the comparison principle.

We next consider $L_2 > L_1 > 0$ with $\lambda_i = \overline{H}^0(L_i, p)$ for $i = 1, 2$ and $\lambda_1 > 0$ and $p \neq 0$. The other cases can be treated similarly. ¿From Lemma 7.4, we get a subcorrector $\underline{v}_1$ satisfying

$$0 < \lambda_1 \leq |p + \nabla_y \underline{v}_1| \left(c + L_1 + M_p[\underline{v}_1]\right)$$

such that (64) holds true. Therefore

$$M_p[\underline{v}_1] \leq \int_{\mathbb{R}} dz \; J(z)[E(|p| + p \cdot z)) - p \cdot z] \leq \left(|p| + \frac{1}{2}\right) \|J\|_{L^1}$$

and we get:

$$0 \leq c + L_1 + M_p[\underline{v}_1] \quad \text{and} \quad 0 < \delta \leq |p + \nabla_y \underline{v}_1|$$

for some constant $\delta \geq \lambda_1/(L_1 + C_p) > 0$ with $C_p = \|c\|_\infty + \left(|p| + \frac{1}{2}\right) \|J\|_{L^1}$. And then

$$\lambda_1 + \delta(L_2 - L_1) \leq |p + \nabla_y \underline{v}_1| \left(c + L_2 + M_p[\underline{v}_1]\right)$$

which implies by Lemma 7.2 that $\lambda_2 \geq \lambda_1 + \delta(L_2 - L_1)$, *i.e.*

$$\frac{\lambda_2 - \lambda_1}{L_2 - L_1} \geq \frac{\lambda_1}{L_1 + C_p}.$$

More generally, we have for $L_2 L_1 > 0$, $|L_2| > |L_1|$ and for some universal constant $C > 0$

$$\frac{\overline{H}^0(L_2, p) - \overline{H}^0(L_1, p)}{L_2 - L_1} \geq \frac{|\overline{H}^0(L_1, p)|}{|L_1| + C_p}.$$

Similarly, let us consider a supercorrector $\overline{v}_1$ given by Lemma 7.4 which satisfies:

$$\lambda_1 \geq |p + \nabla_y \overline{v}_1| \left(c + L_1 + \tilde{M}_p[\overline{v}_1]\right). \tag{67}$$

As soon as $L_1 > C_p$, we get again that

$$c + L_1 + \tilde{M}_p[\overline{v}_1] > 0$$

from which we deduce

$$\frac{\lambda_1}{L_1 - C_p} \geq |p + \nabla_y \overline{v}_1|. \tag{68}$$

Next, using (67) and (68), write:

$$\begin{aligned} \lambda_1 + (L_2 - L_1)\frac{\lambda_1}{L_1 - C_p} &\geq \lambda_1 + (L_2 - L_1)|p + \nabla_y \overline{v}_1| \\ &\geq |p + \nabla_y \overline{v}_1| \left(c + L_2 + M_p[\overline{v}_1]\right) \end{aligned}$$

and deduce from Lemma 7.2 that:

$$\lambda_2 \leq \lambda_1 + (L_2 - L_1)\frac{\lambda_1}{L_1 - C_p}.$$

This implies the result.

5. Let $T > 0$. The function w denotes the solution of (15) and w' denotes the solution of

$$\begin{cases} \partial_\tau w' = (c(y) + L + L' + M_p[w'(\tau, \cdot)](y))|p + \nabla w'| & \text{in} \quad (0, +\infty) \times \mathbb{R} \\ w(0, y) = 0 & \text{on} \quad \mathbb{R} \end{cases} \tag{69}$$

We set $\lambda = \overline{H}^0(L,p)$ and $\lambda' = \overline{H}^0(L+L',p)$. Since $|p+\nabla_y w(\tau,\cdot)|_\infty \leq |p|e^{T\|\nabla c\|_\infty}$ for $\tau \in [0,T)$ (Lemma 7.1), we get that $w(\tau,y)+\tau L'|p|e^{T|\nabla c|_\infty}$ is a supersolution of (69) on $(0,T]\times\mathbb{R}$. Hence, the comparison principle implies

$$w'(\tau,y) \leq w(\tau,y) + L'|p|Te^{T\|\nabla c\|_\infty} \quad \text{on } (0,T)\times\mathbb{R}.$$

Using the fact that $|w(\tau,y)-\lambda\tau|,\ |w'(\tau,y)-\lambda'\tau| \leq 2|p|$ and that $\overline{H}^0(L,p)$ is nondecreasing in L, we get that

$$0 \leq (\lambda'-\lambda)T \leq |p|\left(L'Te^{T|\nabla c|_\infty}+4\right)$$

and so

$$|\lambda'-\lambda| \leq |p|\left(L'e^{T|\nabla c|_\infty}+\frac{4}{T}\right).$$

Taking $T = \frac{|\ln L'|}{2\|\nabla c\|_\infty}$, we get the result for $0 \leq L' \leq \frac{1}{2}$.

6. The following arguments must be adapted by using sub and supercorrectors in order to use Lemma 7.2. However, we assume for the sake of clarity that there exist correctors for any (L,p). Hence, we have with $\lambda = \overline{H}^0(L,p)$:

$$\lambda = |p+\nabla_y v|\left(c+L+M_p[v]\right).$$

Therefore $\overline{v}(y) = -v(y+a)$ satisfies:

$$-\lambda = |-p+\nabla_y\overline{v}(y)|\left(-c(y+a)-L+\tilde{M}_{-p}[\overline{v}](y)\right)$$

and $-c(y+a)=c(y)$ by assumption. Hence $\overline{H}^0(-L,-p) = -\lambda$.

7. We have with $\lambda = \overline{H}^0(L,p)$:

$$\lambda = |p+\nabla_y v|\left(c+L+M_p[v]\right).$$

Therefore $\check{v}(y) = v(-y+a)$ satisfies (because $J(-z)=J(z)$):

$$\lambda = |-p+\nabla_y\check{v}(y)|\left(c(-y+a)+L+M_{-p}[\check{v}](y)\right).$$

By assumption, $c(-y+a)=c(y)$, hence: $\overline{H}^0(L,-p) = \overline{H}^0(L,p)$.

8. Analysis for small p. ¿From the construction of sub/super/correctors satisfying $p(p+\nabla_y v) \geq 0$ (see Lemma 7.4), we know that on one period we have for $p>0$ (the analysis is similar for $p<0$)

$$0 \leq p\cdot z + v(z+y) - v(y) \leq p \quad \text{for} \quad z \in [0,1].$$

Therefore $\max v(z) - \min v(z) \leq p$ in general. We will now estimate $M_p[v]$ for small p. If $0<p<1$ and for $|z| < \lfloor 1/p\rfloor - 1$, using the periodicity of v and the monotonicity of $v(y)+py$, we deduce that $|p\cdot z + v(z+y)-v(y)| \leq p(\lfloor 1/p\rfloor -1) < 1$ and then

$$\begin{aligned} M_p[v](y) &= \int_{|z|>\lfloor 1/p\rfloor -1} J(z)\left\{E\left(v(y+z)-v(y)+p\cdot z\right) - p\cdot z\right\} \\ &= \int_{|z|>\lfloor 1/p\rfloor -1} J(z)\left\{E\left(v(y+z)-v(y)+p\cdot z\right) - \left(v(y+z)-v(y)+p\cdot z\right)\right\} \\ &\quad + \int_{|z|>\lfloor 1/p\rfloor -1} J(z)\left\{v(y+z)-v(y)\right\} \end{aligned}$$

which implies that

$$|M_p[v]| \leq \left(\int_{|z|>\lfloor 1/p\rfloor -1} J(z)\right)\left(\frac{1}{2}+\max v - \min v\right) \longrightarrow 0 \quad \text{as} \quad p \to 0.$$

Therefore for L and p small enough we see that $L+c(y)+M_p[v](y)$ changes sign if c changes sign. This gives a contradiction for the existence of sub and supercorrectors with non-zero λ.

9. ANALYSIS FOR LARGE p. We first consider the periodic solution v (with zero mean value) of

$$0 = c(y) + M_\infty[v](y)$$

with

$$M_\infty[v](y) = \int_{\mathbb{R}} dz\ J(z)(v(y+z) - v(y))$$

which satisfies

$$|v|_\infty \le C|c|_\infty, \quad |v'|_\infty \le C|c'|_\infty, \quad |v''|_\infty \le C|c''|_{L^\infty}.$$

To get this result, it is sufficient to study the periodic solutions of

$$\alpha v^\alpha = c + M_\infty[v^\alpha]$$

that can be obtained by the usual Perron's method. The comparison principle (together with the properties of J) gives the bounds on the oscillations of v^α, independent on α. This implies similar bounds on the derivatives of v^α, which gives enough compactness to pass to the limit as $\alpha \to 0$.

For clarity, we assume that $y = 0$ and $v(y) = 0$. The other cases can be treated similarly. Let us set

$$u(z) = v(z) + p \cdot z$$

and let us estimate for $p > 0$ large

$$M_0 = \int_{\mathbb{R}} dz J(z) \left(E(u(z)) - u(z)\right).$$

Because $u'(z) = v'(z) + p > 0$ for $p > 0$ large enough, we can define u^{-1} by $u^{-1}(u(x)) = u(u^{-1}(x)) = x$. We can then rewrite with $\bar{u} = u(z)$

$$M_0 = \int_{\mathbb{R}} \frac{d\bar{u}}{u'(u^{-1}(\bar{u}))} J(u^{-1}(\bar{u})) \left(E(\bar{u}) - \bar{u}\right).$$

We set

$$G(\bar{u}) = \frac{J(u^{-1}(\bar{u}))}{u'(u^{-1}(\bar{u}))}$$

and write

$$M_0 = \sum_{k \in \mathbb{Z}} \int_{k-1/2}^{k+1/2} d\bar{u}\ G(\bar{u}) \left(E(\bar{u}) - \bar{u}\right)$$

and set

$$I_k = [u^{-1}(k - 1/2), u^{-1}(k + 1/2)].$$

We get for $\bar{u} \in [k - 1/2, k + 1/2]$

$$G(\bar{u}) = G(k) + g_k(\bar{u})$$

with

$$|g_k(\bar{u})| \le \sup_{z \in I_k} \left| \frac{J'(z)}{u'(z)} - \frac{J(z)u''(z)}{u'^2(z)} \right| \cdot \sup_{\bar{u} \in [k-1/2, k+1/2]} |(u^{-1})'(\bar{u})| \cdot 1/2.$$

By definition of u, we also have

$$\left| u^{-1}(k \pm 1/2) - \frac{k \pm 1/2}{p} \right| \le \frac{|v|_\infty}{p}.$$

Therefore for p large enough

$$|g_k(\bar{u})| \le 1/2 \cdot \left(\frac{h_1(k/p)}{(p - |v'|_\infty)^2} + |v''|_\infty \frac{h_0(k/p)}{(p - |v'|_\infty)^3} \right)$$

and then, since $\int_{k-1/2}^{k+1/2} d\bar{u}(E(\bar{u})-\bar{u})=0$ and $|E(\bar{u})-\bar{u}|\le \frac{1}{2}$, we get

$$\begin{aligned} |M_0| &\le 1/2 \sum_{k\in\mathbb{Z}} \int_{k-1/2}^{k+1/2} du\ |g_k(u)| \\ &\le 1/4 \left(\frac{p}{(p-|v'|_\infty)^2} \left(\sum_{k\in\mathbb{Z}} \frac{1}{p} h_1(k/p) \right) + \frac{p|v''|_\infty}{(p-|v'|_\infty)^3} \left(\sum_{k\in\mathbb{Z}} \frac{1}{p} h_0(k/p) \right) \right). \end{aligned}$$

This implies that

$$|M_0| \le C/p \quad \text{for} \quad p \quad \text{large enough}$$

with C depending on $\|c\|_{W^{2,\infty}}$, $\|h_0\|_{L^1}$ and $\|h_1\|_{L^1}$. More generally, we get the same estimate for

$$M(y) = \int_{\mathbb{R}} dz\ J(z) \left(E(v(y+z)-v(y)+p\cdot z) - (v(y+z)-v(y)+p\cdot z) \right).$$

We now compute

$$\begin{aligned} (p+\nabla_y v)(L+c+M_p[v]) &\ge (p+\nabla_y v)(L+c+M_\infty[v]-C/p) \\ &\ge (p-|v'|_\infty)(L-C/p). \end{aligned}$$

This implies that

$$\overline{H}^0(L,p) \ge (p-|v'|_\infty)(L-C/p) > 0$$

for $p>0$ large enough, if $L>C/p$. The case $L<0$ can be treated similarly.

The proof of Theorem 2.6 is now complete. □

8 Application: homogenization of a particle system

8.1 The general case

This section is devoted to the proof of Theorem 2.11. It is a consequence of Theorem 2.5 and of the following result:

Theorem 8.1 (Link between the system of ODEs and the PDE). *Assume that V_0 is 1-periodic, that V_0' is Lipschitz continuous and that V satisfies (H). If we have $y_1(0)<...<y_{N_\varepsilon}(0)$ at the initial time, then the cumulative distribution function ρ defined in (17) is a discontinuous viscosity solution (in the sense of Definition 4.1) of*

$$\partial_\tau \rho = \left(c\,(y) + M\left[\rho(\tau,\cdot)\right](y)\right) |\nabla \rho| \quad \text{in} \quad (0,+\infty)\times\mathbb{R}, \tag{70}$$

with

$$c(y) = V_0'(y) - F \qquad \text{and} \qquad J = V'' \text{ on } \mathbb{R}\backslash\{0\}. \tag{71}$$

Conversely, if u is a bounded and continuous viscosity solution of (70) satisfying for some time $T>0$ and for all $\tau\in(0,T)$

$$u(\tau,y) \text{ is increasing in } y,$$

then the points $y_i(\tau)$, defined by $u(\tau,y_i(\tau)) = i-\frac{1}{2}$ for $i\in\mathbb{Z}$, satisfy the system (1) on $(0,T)$.

Before presenting the proof of this theorem, let us explain how to derive Theorem 2.11.

Proof of Theorem 2.11. By construction, we have

$$(\rho_0^\varepsilon)^*(y) = \rho_0^\varepsilon(y) \le u_0(y) \le (\rho_0^\varepsilon)_*(y) + \varepsilon.$$

Using the fact that ρ^ε is a discontinuous viscosity solution of (5) (except for the initial condition) and the comparison principle (see Theorem 4.4), we deduce that (with u^ε the continuous solution of (5))

$$\rho^\varepsilon(\tau,y) \le u^\varepsilon(\tau,y) \le (\rho^\varepsilon)_*(\tau,y) + \varepsilon$$

and so

$$u^\varepsilon(\tau,y) - \varepsilon \le \rho^\varepsilon(\tau,y) \le u^\varepsilon(\tau,y). \tag{72}$$

Sending $\varepsilon\to 0$, we get that $\rho^\varepsilon \to u^0$ which gives the result. □

We now turn to the proof of Theorem 8.1.

Remark that under Assumption (H), V' is not differentiable at 0 and in order to be sure that the solution of (1) exists and is unique, we need to prove that the distance between two particles is bounded from below. Precisely, we prove:

Lemma 8.2 (Lower bound on the distance between particles)**.** *Assume that the potential V_0' is Lipschitz continuous and V satisfies (H). Let $y_i,\ i=1,..,N_\varepsilon$ be the solution of* (1)*. Then, the distance $d(\tau)$ between two particles is bounded from below by*

$$d(\tau) := \min\{|y_i(\tau) - y_j(\tau)|, i \neq j\} \geq d_0 e^{-\|V_0''\|_\infty \tau},$$

where d_0 is the minimal distance between two particles at initial time: $d_0 = d(0)$.

Proof of Lemma 8.2. Let d_i be the distance between y_i and y_{i+1}: $d_i(\tau) = y_{i+1}(\tau) - y_i(\tau)$. At $\tau = 0$, $d(0) > 0$. Hence consider the first time τ^* for which $d(\tau^*) = 0$ and choose i such that $d(\tau) = d_i(\tau)$ in a neighbourhood of τ^*. Let us denote by $\bar{d}_{j-i} = d_j - d_i \geq 0$. We then have

$$\begin{aligned}\dot{d} &= -(V_0'(y_{i+1}) - V_0'(y_i)) - \left(\sum_{j\neq i+1} V'(y_{i+1} - y_j) - \sum_{j \neq i} V'(y_i - y_j)\right)\\ &= -(V_0'(y_{i+1}) - V_0'(y_i)) - \Bigg\{\sum_{k=0}^{i-1} V'((1+k)d + \bar{d}_0 + .. + \bar{d}_{-k}) - \sum_{k=0}^{N_\varepsilon - i-2} V'((1+k)d + \bar{d}_1 + .. + \bar{d}_{k+1})\\ &\qquad - \sum_{k=0}^{i-2} V'((1+k)d + \bar{d}_{-1} + .. + \bar{d}_{-k-1}) + \sum_{k=0}^{N_\varepsilon - i-1} V'((1+k)d + \bar{d}_0 + .. + \bar{d}_k)\Bigg\}.\end{aligned}$$

We set

$$a_k^- = (1+k)d + \bar{d}_0 + .. + \bar{d}_{-k} \geq 0, \quad a_k^+ = (1+k)d + \bar{d}_0 + .. + \bar{d}_k.$$

We then get (since $\bar{d}_0 = 0$)

$$\begin{aligned}\dot{d} = -(V_0'(y_{i+1}) - V_0'(y_i)) - \Bigg\{&\sum_{k=0}^{i-2}(V'(a_k^-) - V'(a_k^- + \bar{d}_{-k-1})) + \sum_{k=0}^{N_\varepsilon - i-2} (V'(a_k^+) - V'(a_k^+ + \bar{d}_{k+1}))\\ &+ V'(a_{i-1}^-) + V'(a_{N_\varepsilon - i-1}^+)\Bigg\}.\end{aligned}$$

Since $V' \leq 0$ and V' is nondecreasing, we deduce that

$$\dot{d} \geq -d\|V_0''\|_\infty$$

which gives the result. □

It remains to prove Theorem 8.1.

Proof of Theorem 8.1. Theorem 8.1 is a consequence of the following lemma.

Lemma 8.3 (Link between the velocities)**.** *Assume that V_0' is Lipschitz continuous and V satisfies (H). Assume that $(y_i(\tau))_{i=1,...,N_\varepsilon}$ solves the system of ODEs* (1) *with $y_1 < ... < y_{N_\varepsilon}$ at the initial time. Then for all $i = 1, ..., N_\varepsilon$, we have*

$$-\dot{y}_i = c(y_i) + M[u(\tau,\cdot)](y_i) \tag{73}$$

where c and J are defined by (18) *and $u(\tau, y)$ is a continuous function such that:*

$$\begin{cases} u(\tau,y) = \rho^*(\tau, y) \quad \text{for} \quad y = y_j(\tau),\ j = 1,..,N_\varepsilon\\ 0 < u < N_\varepsilon\\ u \text{ is increasing in } y\end{cases} \tag{74}$$

where ρ is defined by (17)*.*

Proof of Lemma 8.3. To simplify the notation, let us drop the time dependence.

$$\begin{aligned} M[u](y_{i_0}) &= \int dz J(z) E(u(y_{i_0}+z)-u(y_{i_0})) \\ &= -\left(\int dz J(z)\right)(i_0-1) + \int dz J(z)\rho(y_{i_0}+z) \\ &= -(2V'(0-))(i_0-1) + J\star\rho(y_{i_0}). \end{aligned}$$

We now compute $J\star\rho$.

$$\begin{aligned} (J\star\rho)(y_{i_0}) &= -\frac{1}{2}\int_{\mathbb{R}} J(z)dz + \sum_{i=1}^{N_\varepsilon}\int J(z)H(y_{i_0}-z-y_i)dz \\ &= \sum_{i<i_0}\int_{y_i-y_{i_0}}^{+\infty} J(z)dz + \sum_{i>i_0}\int_{y_i-y_{i_0}}^{+\infty} J(z)dz \\ &= (i_0-1)(V'(0-)-V'(0+)) - \sum_{i\neq i_0} V'(y_i-y_{i_0}) \\ &= 2(i_0-1)V'(0-) - \sum_{i\neq i_0} V'(y_i-y_{i_0}). \end{aligned}$$

This finally gives

$$M[u](y_{i_0}) = -\sum_{i\neq i_0} V'(y_i-y_{i_0}).$$

The proof is now complete. □

The fact that ρ is a discontinuous solution of (70) is a straighforward consequence of Lemma 8.3 and of Definition 4.1.

We prove the converse. Use Proposition 4.8 and conclude that $\rho = E_*(u)$ (resp. $\rho^* = E(u)$) is a viscosity supersolution (resp. subsolution) of

$$\partial_\tau \rho = \tilde{c}(\tau,y)\,\partial_y\rho \quad \text{with} \quad \tilde{c}(\tau,y) = c(y) + M[u(\tau,\cdot)](y) = c(y) + \tilde{M}[u(\tau,\cdot)](y)$$

where $\tilde{c}$ is in fact a prescribed velocity for ρ. Using the fact that u is increasing in y, we define $y_i(\tau) = \inf\{y,\ u(\tau,y)\geq i-1/2\} = (u(\tau,\cdot))^{-1}(i-1/2)$ and we consider the functions $\tau\mapsto y_i(\tau)$. They are continuous because u is increasing in y and is continuous in (τ,y).

We next show that y_i are viscosity solutions of (1). To do so, consider a test function φ such that $\varphi(\tau)\leq y_i(\tau)$ and $\varphi(\tau_0)=y_i(\tau_0)$. Define next $\hat{\varphi}(\tau,y) = i-1/2+(y-\varphi(\tau))$. It satisfies:

$$\hat{\varphi}(\tau_0,y_i(\tau_0)) = \rho^*(\tau_0,y_i(\tau_0)) \quad \text{and} \quad \hat{\varphi}(\tau,y)\geq\rho^*(\tau,y) \quad \text{for } y_i(\tau)-1<y.$$

This implies, with $\bar{c}_i = \tilde{c}(\cdot,y_i)$, that

$$\varphi_\tau(\tau_0) \geq -\bar{c}_i(\tau_0) = F - V_0'(y_i) - \sum_{j\neq i} V'(y_i-y_j).$$

This proves that y_i are viscosity supersolutions of (1). The proof for subsolutions is similar and we skip it. Moreover, since $\bar{c}_i$ is continuous, we deduce that y_i is C^1 and it is therefore a classical solution of

$$\dot{y}_i(\tau) = -\bar{c}_i(\tau) = F - V_0'(y_i) - \sum_{j\neq i} V'(y_i-y_j).$$

This ends the proof of the theorem. □

Remark 8.4. In the case of particles, we have formally the microscopic energy for $\rho = \rho^{micro}$

$$E^{micro} = \int_{\mathbb{R}} \frac{1}{2}(V^{micro} \star \rho_y)\rho_y + V_0^{micro}\rho_y$$

with $V^{micro} = V$ and $V_0^{micro} = V_0 - F \cdot y$. After a rescaling, we get at the limit a formal macroscopic energy for $\rho = \rho^{macro}$

$$E^{macro} = \int_{\mathbb{R}} \frac{1}{2}(V^{macro} \star \rho_x)\rho_x + V_0^{macro}\rho_x$$

where $V^{macro} = -g_0 \ln|x|$ only keeps the memory of the dislocation part of V^{micro} (the long range interactions), and $V_0^{macro} = -F \cdot x + \int_{\mathbb{R}/\mathbb{Z}} dy V_0(y)$.

8.2 Extension to the dislocation case $V(x) = -\ln|x|$

This subsection is devoted to the proof of Theorem 2.14.

Proof of Theorem 2.14.
Step 1: approximation of V. For $\delta > 0$, let the regularization V_δ of the potential $V(x) = -\ln|x|$ be such that $V_\delta \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$ and defined by

$$V_\delta = \begin{cases} -\ln|x| & \text{if } |x| > \delta \\ \text{linear if } x \in (-\delta,\delta)\backslash\{0\} \end{cases}$$

We can easily check that V_δ satisfies assumptions (H) with $g_0 = 1$. Using the fact that the distance between two particles is bounded from below, we deduce that ρ defined in (17) satisfies

$$\partial_t u = (c(x) + L + M[u(t,\cdot)](x))\,|\nabla u| \quad \text{on } (0,T)\times\mathbb{R}. \tag{75}$$

with

$$c(y) = V_0'(y) - F \qquad \text{and} \qquad J = V_\delta'' \text{ on } \mathbb{R}\backslash\{0\} \tag{76}$$

as soon as $\delta \le \delta(T) := d_0 e^{-|V_0''|_\infty T}$. Indeed, Since Lemma 8.2 implies that for all $t \in [0,T]$, we have $d(t) \ge d_0 e^{-|V_0''|_\infty T}$, we can replace the potential V with V_δ for $\delta \le d_0 e^{-|V_0''|_\infty T}$ and the ODEs system remains equivalent.

In the sequel, we use the following functions

$$J_\delta(z) = \begin{cases} V_\delta''(z) & \text{for} \quad z \in \mathbb{R}\backslash\{0\} \\ 0 & \text{for} \quad z = 0 \end{cases}$$

$$M_{p,\delta}[U](x) = \int_{\mathbb{R}} dz\ J_\delta(z)\,\{E(U(x+z) - U(x) + p\cdot z) - p\cdot z\}$$

and $\overline{H}_\delta^0$ denotes the effective Hamiltonian associated with J_δ.

Step 2: estimate for the approximate effective Hamiltonian.

Lemma 8.5 (Lipschitz continuous sub/supercorrectors)**.** *Let v denote the solution of*

$$\overline{H}_\delta^0 = (c(x) + L + M_{p,\delta}[v])\,|p + \nabla v| \quad \text{in} \quad \mathbb{R}$$

and consider

$$v^\varepsilon(x) = \sup_y \left(v(y) - \frac{|x-y|^2}{2\varepsilon}\right), \quad v_\varepsilon(x) = \inf_y \left(v(y) + \frac{|x-y|^2}{2\varepsilon}\right).$$

There then exists a constant $C(p)$ depending only on $|p|$ such that for δ, $\eta \in (0, C(p)\sqrt{\varepsilon})$, we have

$$\overline{H}_\delta^0(L,p) \le \left(c(x) + L + C(p)\sqrt{\varepsilon}\|\nabla c\|_\infty + M_{p,\eta}[v^\varepsilon]\right)|p + \nabla v^\varepsilon|, \tag{77}$$

$$\overline{H}_\delta^0(L,p) \ge \left(c(x) + L - C(p)\sqrt{\varepsilon}\|\nabla c\|_\infty + \tilde{M}_{p,\eta}[v_\varepsilon]\right)|p + \nabla v_\varepsilon|. \tag{78}$$

Proof. We just make the proof for v^ε and we assume that

$$v^\varepsilon(x) = v(x_\varepsilon) - \frac{|x - x_\varepsilon|^2}{2\varepsilon}.$$

It is classical that v^ε is Lipschitz continuous and

$$\|\nabla v^\varepsilon\|_\infty \le \frac{C}{\sqrt{\varepsilon}} \tag{79}$$

where the constant C depends only on $\|v\|_\infty \le 2|p|$. Hence $C = C(p)$. Moreover, for any $q \in D^{1,+}v^\varepsilon(x)$, it is classical that $q \in D^{1,+}v(x_\varepsilon)$, $q = \frac{x - x_\varepsilon}{\varepsilon}$ and $|x - x_\varepsilon| \le C(p)\sqrt{\varepsilon}$, where the constant C depends on $\|v\|_\infty$. We thus deduce that

$$\begin{aligned}\overline{H}^0_\delta(L,p) &\le (c(x_\varepsilon) + L + M_{p,\delta}[v](x_\varepsilon))\,|p+q| \\ &\le \left(c(x) + L + C(p)\sqrt{\varepsilon}\|\nabla c\|_\infty + M_{p,\delta}[v^\varepsilon](x)\right)|p+q|\end{aligned}$$

where we have used that

$$v(x_\varepsilon + z) - v(x_\varepsilon) \le v^\varepsilon(x+z) - v^\varepsilon(x).$$

Now using (79), we can replace $M_{p,\delta}[v^\varepsilon](x)$ with $M_{p,\eta}[v^\varepsilon](x)$ for $\eta,\ \delta \in (0, C(p)\sqrt{\varepsilon})$. Indeed, for $|z| \le \delta$ or $|z| \le \eta$, we have

$$|v^\varepsilon(x+z) - v^\varepsilon(x) - p\cdot z| \le (C(p)\varepsilon^{-1/2} + |p|)\max(\delta,\eta) < 1$$

as soon as $\max(\delta,\eta) \le \tilde{C}(p)\sqrt{\varepsilon}$ and we obtain

$$\int_{|z|\le\max(\delta,\eta)} dz\ J_\delta(z)E_*(v_\varepsilon(x+z) - v_\varepsilon(x) - p\cdot z) = \int_{|z|\le\max(\delta,\eta)} dz\ J_\eta(z)E_*(v_\varepsilon(x+z) - v_\varepsilon(x) - p\cdot z) = 0.$$

This implies (77). □

We derive from the previous lemma the convergence of $\overline{H}^0_\delta$ towards $\overline{H}^0_0$. We even get an error estimate:

Lemma 8.6 (Limit of $\overline{H}^0_\delta$)**.** *There exists $\overline{H}^0_0(L,p)$ such that*

$$\left|\overline{H}^0_\delta(L,p) - \overline{H}^0_0(L,p)\right| \le \frac{C(p)}{\ln\frac{1}{\delta}} \tag{80}$$

for δ small enough.

Proof. We deduce from Lemma 8.5 that $\overline{H}^0_\delta(L,p) \le \overline{H}^0_\eta(L+L',p)$ for $\eta,\ \delta \in (0, C(p)\sqrt{\varepsilon})$ and with $L' = C\sqrt{\varepsilon}\|\nabla c\|_\infty$. Since the constants are independent on L, L', we deduce that

$$\overline{H}^0_\delta(L-L',p) \le \overline{H}^0_\eta(L,p) \le \overline{H}^0_\delta(L+L',p) \quad \text{for all}\ \ \eta,\ \delta \in (0, C(p)\sqrt{\varepsilon}).$$

Setting $\overline{H}^0_0(L,p) = \limsup_{\eta\to 0}\overline{H}^0_\eta(L,p)$ and using Theorem 2.6, point 5, we then get (80). □

One can now check that $\overline{H}^0_0$ satisfies Theorem 2.6 excepted the point 4 and point 9. We now turn to the proof of convergence itself.

Step 3: the proof of convergence. Let us consider $T > 0$ and try to prove that ρ^ε converges towards $u0$ on $(0,T)\times\mathbb{R}$. In order to do so, choose for any $\varepsilon > 0$ a $\delta = \delta(\varepsilon)$ such that ρ^ε is a (discontinuous) solution of (5) with J replaced with $J_{\delta(\varepsilon)}$. We still have (72) but now u^ε is associated with $J_{\delta(\varepsilon)}$ instead of a fixed J. We need to check that the convergence proof of Theorem 2.5 can be adapted. This is possible thanks to (80), that permits to pass from the exact effective Hamiltonian $\overline{H}^0$ to the approximate one $\overline{H}^0_{\delta(\varepsilon)}$, and thanks to Lemma 8.5 that permits to prove that Lemma 6.1 is still satisfied by using a regularized supercorrector v_ε. Let us give more details. We first prove that the initial condition is satisfied. Since u_0 is Lipschitz continuous, we get that

$$M_\delta^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right]=\int_{|z|>\frac{1}{\|\nabla u_0\|_\infty}} dz\ J_\delta(z)E\left(\frac{u_0(x+\varepsilon z)-u_0(x)}{\varepsilon}\right)$$

is independent of δ if $\delta \le \frac{1}{\|\nabla u_0\|_\infty}$, and so (32) remains true with a constant C independent of ε and δ.

We now turn to the equation. The idea is to use Lipschitz continuous sub/supercorrectors to control the distance between particles. Using notation of Section 6 we now prove by contradiction that $\overline{u}$ is a subsolution of (11). Hence we assume that

$$\partial_t\phi(t_0,x_0)=\overline{H}_0^0(L_0,p)+\theta'=\overline{H}_\delta^0(L_0,p,0)+\theta$$

for δ small enough and where we have used (80). Moreover, we can replace (50) with

$$\partial_t\phi(t_0,x_0)=\overline{H}_\delta^0(L_0+2\beta,p,\alpha)+\frac{\theta}{2}.$$

We now replace the bounded corrector of Section 6 with a Lipschitz continuous supercorrector. More precisely, we consider the solution v of

$$\lambda=\big(c(y)+L+M_{p,\delta}^\alpha[v](y)\big)\,|p+\nabla_y v|$$

and we set

$$v_\varepsilon(x)=\inf_y\left(v(y)+\frac{|x-y|^2}{2\varepsilon}\right)=v(x_\varepsilon)+\frac{|x-x_\varepsilon|^2}{2\varepsilon}$$

where $L=L_0+2\beta$ and $\lambda=\overline{H}_\delta(L_0,p,0)$. By Lemma 8.5, we deduce that v_ε is a supersolution of

$$\lambda\ge\left(c(y)+L-C\sqrt{\varepsilon}\|\nabla c\|_\infty+\tilde{M}_{p,\delta}^\alpha[v_\varepsilon](x)\right)|p+\nabla v_\varepsilon|$$

for $\delta\le C(p)\sqrt{\varepsilon}$. We now consider ε such that $C\sqrt{\varepsilon}\|\nabla c\|_\infty\le\beta$ and we get that v_ε is a supersolution of

$$\lambda\ge\left(c(y)+L_0+\beta+\tilde{M}_{p,\delta}^\alpha[v_\varepsilon](x)\right)|p+\nabla v_\varepsilon|.$$

Using the fact that v_ε is Lipschitz continuous, we get that $\tilde{M}_{p,\delta}^\alpha[v_\varepsilon]$ is independent of δ for $\delta\le C(p)\sqrt{\varepsilon}$ and so the rest of the proof is exactly the same as the one of Theorem 2.5 with J replaced with J_δ. $\square$

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References

[1] O. Alvarez, P. Hoch, Y. Le Bouar, and R. Monneau, *Dislocation dynamics: short time existence and uniqueness of the solution*, Archive for Rational Mechanics and Analysis, 85 (2006), pp. 371–414.

[2] O. Alvarez and A. Tourin, *Viscosity solutions of nonlinear integro-differential equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 13 (1996), pp. 293–317.

[3] S. Aubry, *The twist map, the extended Frenkel-Kontorova model and the devil's staircase*, Physica D, 7 (1983), pp. 240–258.

[4] S. Aubry and P. Le Daeron, *The discrete Frenkel-Kontorova model and its extensions*, Physica D, 8 (1983), pp. 381–422.

[5] G. Barles, H. M. Soner, and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[6] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[7] F. Da Lio, N. Forcadel, and R. Monneau, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics.* To appear in the Journal of the European Mathematical Society.

[8] R. De La Llave, *kam theory for equilibrium states in 1-d statistical mechanics models*. Preprint.

[9] J. Deslippe, R. Tesdtrom, M. S. Daw, D. Chrzan, T. Neeraj, and M. Mills, *Dynamics scaling in a simple one-dimensional model of dislocation activity*, Phil. Mag., 84 (2004), pp. 2445–2454.

[10] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), pp. 359–375.

[11] ———, *Periodic homogenisation of certain fully nonlinear partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, 120 (1992), pp. 245–265.

[12] A. Ghorbel, *Thèse de doctorat*, Ecole Nationnale des Ponts et Chaussées, (2007).

[13] A. Ghorbel, P. Hoch, and R. Monneau, *A numerical study for the homogenization of one-dimensional models describing the motion of dislocations*. Preprint.

[14] A. K. Head. 1972a, Phil. Mag., 26, 43; 1972b, 26, 55; 1972c, 26, 65.

[15] J. P. Hirth and J. Lothe, *Theory of dislocations*, Krieger, 1982. Second edition.

[16] B. Hu, W.-X. Qin, and Z. Zheng, *Rotation number of the overdamped frenkel-kontorova model with ac-driving*, Physica D, 208 (2005), pp. 172–190.

[17] C. Imbert, *A non-local regularization of first order hamilton-jacobi equations*, Journal of Differential Equations, 211 (2005), pp. 214–246.

[18] C. Imbert and R. Monneau, *Homogeneisation of first order equations with* (u/ε)*-periodic hamiltonians. Part I: local equations*. Preprint.

[19] C. Imbert, R. Monneau, and E. Rouy, *Homogeneisation of first order equations with* (u/ε)*-periodic hamiltonians. Part II: application to dislocation dynamics*. To appear in CPDE.

[20] H. Ishii and P. Souganidis, *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor*, Tohoku Math. J. (2), 47 (1995), pp. 227–250.

[21] T. Kontorova and Y. Frenkel. Zh. Eksp. Teor. Fiz. 8 (1938) 89, 1340, 1349.

[22] P.-L. Lions, G. C. Papanicolaou, and S. R. S. Varadhan, *Homogeneization of Hamilton-Jacobi equations*, unpublished, 1986.

[23] G. Menon, *Gradient systems with wiggly energies and related averaging problems*, Arch. Ration. Mech. Anal., 162 (2002), pp. 193–246.

[24] G. Puglisi and L. Truskinovsky, *Thermodynamics of rate-independent plasticity*, J. of Mechanics and Physics of Solids, 53 (2005), pp. 655–679.

[25] D. Slepčev, *Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions*, Nonlinear Anal., 52 (2003), pp. 79–115.