A Non-Local Transport Equation modelling Dislocations Dynamics

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Dislocations are lines defects in a crystal material (Hirth & Lothe, 1992).

The jumps of $E(u)$ correspond to the positions of dislocations.

**Figure:** Representation of dislocations with the function $E(u)$
Presentation of the Physical Model

\[
\begin{align*}
\frac{\partial u}{\partial t} (x, t) &= c[u](x, t) \frac{\partial u}{\partial x} (x, t) \quad \text{in } \mathbb{R} \times (0, +\infty) \\
c[u](x, t) &= c^{\text{ext}}(x) + \int_{\mathbb{R}} c^0(x') E \left( (u(x - x', t)) \right) \, dx' \\
u(x, 0) &= u^0(x) \quad \text{in } \mathbb{R}
\end{align*}
\] (1)

where \( E \) is the floor function.

Dislocations move with a non-local velocity \( c[u] \). It is the sum of two terms:

- \( c^{\text{ext}} \) represents the exterior stress created by obstacles (such as precipitates, other defects, ...).
- The second term is non-local and represents the elastic interior stress created by all the dislocations in the material.
We make the following assumptions for the exterior stress $c^{\text{ext}}$ and the kernel $c^0$:

$$\begin{cases}
    c^{\text{ext}} \in W^{1,\infty}(\mathbb{R}) & \text{such that } c^{\text{ext}}(x+1) = c^{\text{ext}}(x) \text{ in } \mathbb{R}, \\
    c^0 \in C^\infty_0(\mathbb{R}) & \text{such that } c^0(x) = c^0(-x) \text{ and } \int_{\mathbb{R}} c^0(x) \, dx = 0.
\end{cases} \tag{2}$$

We consider the initial condition $u^0 \in \text{Lip}(\mathbb{R})$ such that for $x \in \mathbb{R}$

$$u^0(x+1) = u^0(x) + P \quad \text{and} \quad 0 < b_0 \leq \frac{\partial u^0}{\partial x} \leq B_0 < +\infty \quad \text{a.e.} \tag{3}$$

with $b_0$ and $B_0$ some constants and $P \in \mathbb{N} \setminus \{0\}$. A natural case to study the solutions of (1) is the continuous viscosity solutions (Barles, 1994).

**Theorem 1**

Under Assumptions (2) and (3), there exists a unique continuous viscosity solution $u \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times (0, +\infty))$ of (1) such that

$$u(x+1, t) = u(x, t) + P \quad \forall (x, t) \in \mathbb{R} \times (0, +\infty). \tag{4}$$
We prove the previous theorem in two steps.

1. First, we prove the result for a short time (Alvarez and al., 2005; Ghorbel and al., 2005) using a fixed point theorem. Given \( T > 0 \) and a function \( \nu \) satisfying (4), we consider the function \( w = \phi(\nu) \) satisfying (4), solution of

\[
\frac{\partial w}{\partial t}(x, t) = c[\nu](x, t) \frac{\partial w}{\partial x}(x, t) \quad \text{in } \mathbb{R} \times (0, T).
\]

(5)

For \( T \) chosen sufficiently small, we prove that the map \( \phi \) is a contraction in a well chosen space.

2. Secondly, we repeat this short time result on a sequence of time intervals of lengths \( T_n \) decreasing to zero, such that \( \sum_{n \in \mathbb{N}} T_n = +\infty \).
Error Estimate

Given a mesh size $\Delta x, \Delta t$ and a lattice $l_d = \{(i\Delta x, n\Delta t); \ i \in \mathbb{Z}, \ n \in \mathbb{N}\}, (x_i, t_n)$ denotes the node $(i\Delta x, n\Delta t)$ and $v^n = (v^n_i)_i$ the values of the numerical approximation of the continuous solution $u(x_i, t_n)$.

$$v^0_i = u^0(x_i), \quad v^{n+1}_i = v^n_i + \Delta t \ c_i(v^n) \times \begin{cases} \frac{v^n_{i+1} - v^n_i}{\Delta x} & \text{if } c_i(v^n) \geq 0 \\ \frac{v^n_i - v^n_{i-1}}{\Delta x} & \text{if } c_i(v^n) < 0 \end{cases}$$ (6)

We choose $\Delta x = \frac{1}{K}, K \in \mathbb{N} \setminus \{0\}$ because of the 1-periodicity in space. We denote $c^\text{ext}_i = c^\text{ext}(x_i)$ which satisfies $c^\text{ext}_{i+K} = c^\text{ext}_i$. The discrete velocity is

$$c_i(v^n) = c^\text{ext}_i + \sum_{l \in \mathbb{Z}} c^0_i \ E(v^n_{i-l}) \ \Delta x$$ (7)

where

$$c^0_i = \frac{1}{\Delta x} \int_{l_i} c^0(x) \ dx \quad \text{and} \quad l_i = \left[ x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2} \right].$$ (8)
We assume the CFL (Courant, Friedrichs, Levy) condition. Then

**Theorem 2**

there exists two constants $T, C > 0$ such that

$$\sup_{i \in \mathbb{Z}} |u(i\Delta x, n\Delta t) - v^n_i| \leq C\sqrt{\Delta x} \quad \text{for all } n \leq \frac{T}{\Delta t}.$$ 

**Main Idea of Proof of Theorem 2**

We apply the technical reasoning of Alvarez and *al.*, 2005.
Figure: Linear: $\Delta x = 0.01$, $\Delta t = 2.438 \times 10^{-3}$
Dynamics of several dislocations through obstacles

Figure: Trapping: $\Delta x = 0.01$, $\Delta t = 1.239 \times 10^{-3}$
Figure: Pile-Up: $\Delta x = 0.01, \Delta t = 1.102 \times 10^{-3}$
Dynamics of several dislocations through obstacles

Figure: Pile-Up: $\Delta x = 0.01$, $\Delta t = 1.10219 \times 10^{-3}$
Conclusion and Perspectives

- Homogenization of a large number of dislocations through obstacles.

- Friction of dislocations.

- Homogenization of walls of dislocations. Joint works with P. Hoch (CEA) and R. Monneau (Cermics).