Complexity of a Sudoku

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CERMICS
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		6						
8	1			4				
		5	8		1		4	6
	6	7		2			1	
	4			1				
					6		2	
6		2	5	7		1		
			2			8		
7	3							4

What is a Sudoku ?

4	7	6	9	5	2	3	8	1
8	1	3	6	4	7	5	9	2
9	2	5	8	3	1	7	4	6
3	6	7	4	2	5	9	1	8
2	4	8	3	1	9	6	7	5
5	9	1	7	8	6	4	2	3
6	8	2	5	7	4	1	3	9
1	5	4	2	9	3	8	6	7
7	3	9	1	6	8	2	5	4

		6						
8	1			4				
		5	8		1		4	6
	6	7		2			1	
	4			1				
					6		2	
6		2	5	7		1		
			2			8		
7	3							4

May be seen as a matrix $(X_{i,j})_{1 \leq i,j \leq 9} \in \{1..9\}^{9 \times 9}$ with missing entries. All the digit from 1 to 9 must appear only once in each row, column and block.

This puzzle was invented in 1979 by the American Howard Garns.

For each cell (i, j) of the Sudoku, create 9 variables $x_{i,j,k}$ such that :

$$x_{i,j,k} = \begin{cases} 1 & \text{if} \quad X_{i,j} = k \\ 0 & \text{else} \end{cases}$$

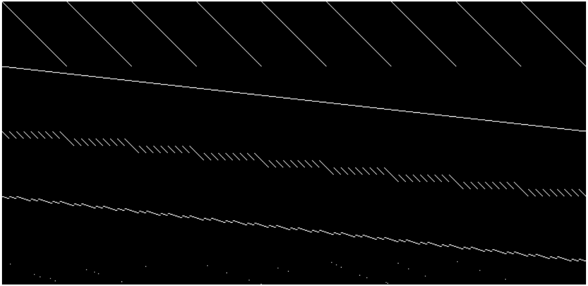
Then, the Sudoku rules are:

- *unique value constraint*: one cell of the sudoku has only one value : $\forall i, \forall j, \sum_k x_{i,j,k} = 1$
- *row constraint*: $\forall i, \forall k, \sum_j x_{i,j,k} = 1$
- *column constraint*: $\forall j, \forall k, \sum_i x_{i,j,k} = 1$
- *block constraint*: for all block B, $\forall k, \sum_{(i,j)\in B}, x_{i,j,k} = 1$
- *initial clue constraint*: for all initial value of the type $X_{i,j} = k$, it must hold $x_{i,j,k} = 1$

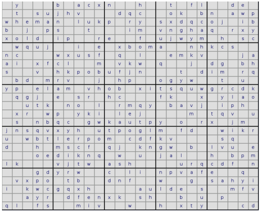
If $\mathbf{x}$ is a line containing the values $(x_{i,j,k})$:

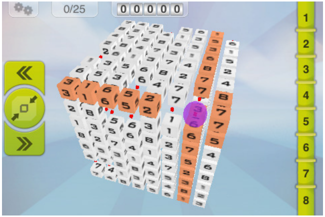
$$\mathbf{x} = (x_{1,1,1}, x_{1,2,1}, \cdots, x_{1,9,1}, x_{2,1,1}, \cdots, x_{9,9,1}, x_{9,9,2}, \cdots, x_{9,9,9})^T$$

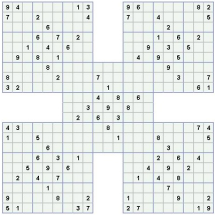
All the constraints can be re-written as $A \cdot \mathbf{x} = U = (1, 1, \cdots, 1)^T$, with $A = [A_{uniqueValue}, A_{rows}, A_{columns}, A_{blocks}, A_{clues}]^T$.

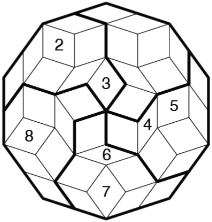


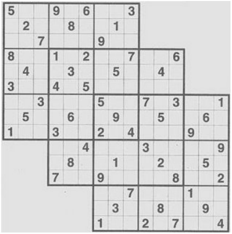
Other Crazy Sudoku

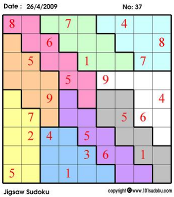












Just change the matrix A

		6						
8	1			4				
		5	8		1		4	6
	6	7		2			1	
	4			1				
					6		2	
6		2	5	7		1		
			2			8		
7	3							4

$\iff$ Find $\mathbf{x} \in \{0, 1\}^{729}$ such that $A \cdot \mathbf{x} = U$

If the Sudoku has only one solution, then:

- this solution $\mathbf{x}$ is unique
- this $\mathbf{x}$ has only 81 non zero entries.

$$(IP) \quad \text{find} \quad \mathbf{x} \in \{0,1\}^{729} \quad \text{such that} \quad A \cdot \mathbf{x} = U$$

It is an Integer Program. Solving a general (IP) problem is NP-complete.

A common trick (LP-relaxation) is to relax the binary constraint:
$\mathbf{x} \in \{0,1\}^{729}$ becomes $\mathbf{x} \in [0,1]^{729}$. We can even choose $\mathbf{x} \geq 0$.

$$(LP) \quad \text{find} \quad \mathbf{x} \in \mathbb{R}^{729} \quad \text{such that} \quad \begin{cases} A \cdot \mathbf{x} & = & U \\ \mathbf{x} & \geq & 0 \end{cases}$$

$$(LP) \quad \text{find} \quad \mathbf{x} \in \mathbb{R}^{729} \quad \text{such that} \quad \begin{cases} A \cdot \mathbf{x} & = & U \\ \mathbf{x} & \geq & 0 \end{cases}$$

Facts:

- The set of x satisfying (LP) is a convex set P (it is a polytope).
- The true solution x_0 is in P.
- Find a point in P may be done in polynomial time.

Problem: P may contain other solutions, which are not related to the Sudoku solution.

Ideas:

- x_0 is the sparsest solution in $P \Longrightarrow$ Optimization problem:

$$(OP) \quad \text{Minimize} \quad \|\mathbf{x}\|_1 \quad \text{with} \quad \begin{cases} A \cdot \mathbf{x} & = & U \\ \mathbf{x} & \geq & 0 \end{cases}$$

It can still be done in polynomial time, but again, it does not necessary lead to a Sudoku solution.

- In some cases, $P = \{x_0\}$.

Question: When do we have $P = \{x_0\}$?

FROM THE LOGIC TO THE STRUCTURE OF THE POLYTOPE

1/ UNIQUE CANDIDATE

				6	7	8		9
	3	5						
2		4						

The only valid candidate in the first cell is 1.

1				6	7	8		9
	3	5						
2		4						

The only valid candidate in the first cell is 1.

1				6	7	8		9
	3	5						
2		4						

The only valid candidate in the first cell is 1.

Theorem: $\forall x \in P, x_{1,1,1} = 1$

1				6	7	8		9
	3	5						
2		4						

Theorem: $\forall x \in P, x_{1,1,1} = 1$

Proof:

- $\forall k > 1$, there exists a constraint c_k and a cell (i_k, j_k) such that :
  - $x_{i_k, j_k, k} = 1$: this cell has already the value k
  - $(1,1)$ and (i_k, j_k) are linked by the constraint c_k
- Thus, $\forall k > 1, x_{1,1,k} = 0$
- Finally, because the cell $(1,1)$ must have a value, $x_{1,1,1} = 1$

2/ SINGLE POSITION

Single Position

	2	3						
							1	
			1					

On the first line, the only cell where 1 can be is $(1, 1)$.

Single Position

1	2	3						
							1	
			1					

On the first line, the only cell where 1 can be is $(1, 1)$.

1	2	3						
							1	
			1					

On the first line, the only cell where 1 can be is $(1, 1)$.

Theorem: $\forall x \in P, x_{1,1,1} = 1$

1	2	3						
							1	
			1					

Theorem: $\forall x \in P, x_{1,1,1} = 1$

Proof:

- for all other cell $(r, 1)$, there exists a constraint c_r and a cell (i_r, j_r) such that:
  - $x_{i_r, j_r, 1} = 1$: this cell has already the value 1,
  - $(r, 1)$ and (i_r, j_r) are linked by the constraint c_r.
- Thus, $\forall r > 1, x_{r,1,1} = 0$.
- Finally, the first line constraint leads to $x_{1,1,1} = 1$.

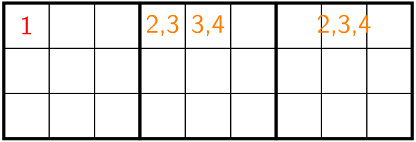
3/ MULTIPLE CANDIDATES

1,2,3,4			2,3	3,4			2,3,4	

On the first line, the digits 2, 3 and 4 are all in the cells $(1,4)$, $(1,5)$ and $(1,8)$ in an order we do not know. Therefore, the first cell contains 1.

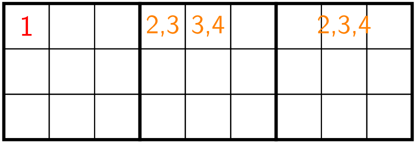
1			2,3	3,4			2,3,4	

On the first line, the digits 2, 3 and 4 are all in the cells $(1,4)$, $(1,5)$ and $(1,8)$ in an order we do not know. Therefore, the first cell contains 1.



On the first line, the digits 2, 3 and 4 are all in the cells $(1,4)$, $(1,5)$ and $(1,8)$ in an order we do not know. Therefore, the first cell contains 1.

Theorem: $\forall x \in P, x_{1,1,2} = x_{1,1,3} = x_{1,1,4} = 0$



Theorem: $\forall x \in P, x_{1,1,2} = x_{1,1,3} = x_{1,1,4} = 0$

Proof: We have the following equations :

$$\begin{cases} x_{1,4,2} & + & & + & x_{1,8,2} & + & \cdots & = & 1 \\ x_{1,4,3} & + & x_{1,5,3} & + & x_{1,8,3} & + & \cdots & = & 1 \\ & + & x_{1,5,4} & + & x_{1,8,4} & + & \cdots & = & 1 \end{cases}$$

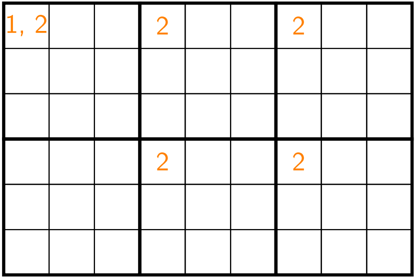
But when we sum everything by column, we have

$$\begin{array}{ccccccc} (x_{1,4,2} + x_{1,4,3}) & + & (x_{1,5,3} + x_{1,5,4}) & + & (x_{1,8,2} + x_{1,8,3} + x_{1,8,4}) & + \cdots = 3 \\ 1 & + & 1 & + & 1 & + \cdots = 3 \end{array}$$

In particular, everything that appears in the dots must have the value 0.

4/ X-WING [1]

[1] Great title for a slide, isn't it ?



On the first line, the cell $(1,4)$ has value 2, or the cell $(1,7)$ has value 2. In the two cases, 2 cannot be in $(1,1)$.

1			2			2		
			2			2		

On the first line, the cell $(1,4)$ has value 2, or the cell $(1,7)$ has value 2. In the two cases, 2 cannot be in $(1,1)$.

1			2			2		
			2			2		

On the first line, the cell $(1, 4)$ has value 2, or the cell $(1, 7)$ has value 2. In the two cases, 2 cannot be in $(1, 1)$.

Theorem: $\forall x \in P, x_{1,1,2} = 0$

1			2			2		
			2			2		

On the first line, the cell $(1,4)$ has value 2, or the cell $(1,7)$ has value 2. In the two cases, 2 cannot be in $(1,1)$.

Theorem: $\forall x \in P, x_{1,1,2} = 0$

Proof:

$$x_{1,4,2} + x_{1,7,2} = (x_{1,4,2} + x_{4,4,2}) - (x_{4,4,2} + x_{4,7,2}) + (x_{4,7,2} + x_{1,7,2}) = 1$$

Theorem:
If a Sudoku can be solved using only the following tricks:

- Unique Candidate
- Single Position
- Hidden Pair (or more)
- Naked Pair (or more)
- X-Wing, Swordfish,...

Then (P) shrinks to one point, which correspond to the Sudoku solution.

Corollary: Solving those sudoku may be done in polynomial time.

DOES THE POLYTOPE ALWAYS SHRINK TO A POINT?

Non-shrinkable logic: XY-Wing.

1,3			2,3					
1,3			1,2					

Of course not

Non-shrinkable logic: XY-Wing.

1			2,3					
1,3			1,2					

The cell (2,4) can have only 1 or 2 as values. In both cases, 3 cannot be in (1,1). Therefore, (1,1) has value 1.

Non-shrinkable logic: XY-Wing.

1			2,3					
1,3			1,2					

The cell (2,4) can have only 1 or 2 as values. In both cases, 3 cannot be in (1,1). Therefore, (1,1) has value 1.

Theorem: $\forall x \in P, \quad x_{1,1,3} \leq 1/2$. (instead of $x_{1,1,3} = 0$).

Of course not

Non-shrinkable logic: XY-Wing.

1			2,3					
1,3			1,2					

The cell (2,4) can have only 1 or 2 as values. In both cases, 3 cannot be in (1,1). Therefore, (1,1) has value 1.

Theorem: $\forall x \in P, \quad x_{1,1,3} \leq 1/2$. (instead of $x_{1,1,3} = 0$).

Take for instance $x_{2,4,1} = x_{2,4,2} = x_{1,4,1} = x_{1,4,3} = x_{2,1,1} = x_{2,1,3} = 1/2$, then $x_{1,1,3} = 1/2$ is compatible.

	2	3			9		8	1
9	1	7	2		8		6	
5	8			1		7		
8	9							6
7	3	5	6	8	1	2		
1			9			8	3	7
		9		2				8
	7	8	1		4	9		
2	5	1	8	9		6		

6	2	3	5	7	9	4	8	1
9	1	7	2	4	8	5	6	3
5	8	4	3	1	6	7	9	2
8	9	2	4	3	7	1	5	6
7	3	5	6	8	1	2	4	9
1	4	6	9	5	2	8	3	7
4	6	9	7	2	5	3	1	8
3	7	8	1	6	4	9	2	5
2	5	1	8	9	3	6	7	4

(c) Solution

4,6	2	3	5,7	6,7	9	4,5	8	1
9	1	7	2	4,5	8	3,4	6	3,5
5	8	4,6	3,4	1	3,6	7	2,9	2,9
8	9	2,4	3,4	5,7	2,7	1,5	1,5	6
7	3	5	6	8	1	2	4,9	4,9
1	4,6	2,6	9	4,5	2,5	8	3	7
3,4	4,6	9	7,5	2	5,6	1,3	1,7	8
3,6	7	8	1	3,6	4	9	2,5	2,5
2	5	1	8	9	3,7	6	4,7	3,4

(d) Other point inside the polytope

6	2	3	5	7	9	4	8	1
9	1	7	2	4	8	5	6	3
5	8	4	3	1	6	7	9	2
8	9	2	4	3	7	1	5	6
7	3	5	6	8	1	2	4	9
1	4	6	9	5	2	8	3	7
4	6	9	7	2	5	3	1	8
3	7	8	1	6	4	9	2	5
2	5	1	8	9	3	6	7	4

(e) Solution

4,6	2	3	5,7	6,7	9	4,5	8	1
9	1	7	2	4,5	8	3,4	6	3,5
5	8	4,6	3,4	1	3,6	7	2,9	2,9
8	9	2,4	3,4	5,7	2,7	1,5	1,5	6
7	3	5	6	8	1	2	4,9	4,9
1	4,6	2,6	9	4,5	2,5	8	3	7
3,4	4,6	9	7,5	2	5,6	1,3	1,7	8
3,6	7	8	1	3,6	4	9	2,5	2,5
2	5	1	8	9	3,7	6	4,7	3,4

(f) Other point inside the polytope

Conclusions:

- A Sudoku problem is equivalent to a Integer Programming problem.
- We can relax such a problem, and the relaxation is solvable in polynomial time.
- For most Sudoku, the solution of the relaxed problem is the Sudoku solution.

Thank you for your attention