Minnaert resonance in bubbly media

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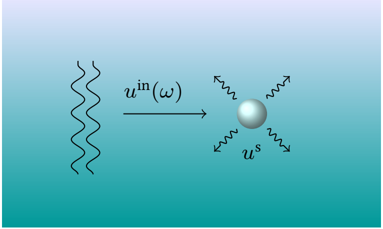
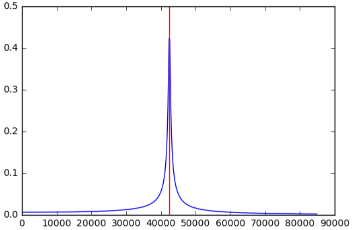
We want to understand the propagation of sound in bubbly water.

Experiment



Results

The function $|u^s/u^{\text{in}}|(\omega)$:



There exists a resonant angular frequency ω_M.
Noticed for the first time by M. Minnaert (1933 : *On musical air-bubbles and the sound of running water*).

$$\omega_M = \sqrt{\frac{3\rho_b}{\rho}} \frac{v_b}{R} \qquad \text{(Minnaert resonance)}.$$

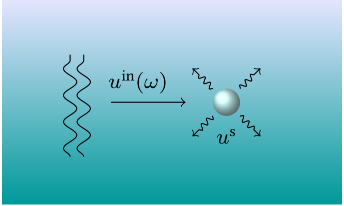
- ρ_b is the density of air (inside the bubble), and ρ the density of water,
- v_b is the speed of sound in the air.
- R is the radius of the bubble.

Example

For a bubble of radius 0.5 mm, this gives $\omega_M = 42000$ Hz (audible), and a wavelength (in water) $\lambda_M = 0.22$ m.

Goal of this talk: understand the previous formula, and extend it.

Our model



- Air bubble: domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega$ of class C^2,
- ρ_b (resp. ρ) the density of air (resp. water),
- v_b (resp. v) the speed of sound in the air (resp. water),
- $u(\mathbf{x})$ the pressure at $\mathbf{x} \in \mathbb{R}^3$,
- $\rho^{-1}u(\mathbf{x}) \sim$ velocity flow at $\mathbf{x} \in \mathbb{R}^3$.

Let ω be the angular frequency of the incident wave u^{in} and introduce

$$k_b = k(\omega) := \frac{\omega}{v_b} \quad \text{and} \quad k := \frac{\omega}{v}. \qquad \text{(wave numbers)}$$

Wave equation (d'Alembert equations) in frequency domain.

$$\begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \left(\Delta + k_b^2\right) u = 0 & \text{in} \quad \Omega, \\ u_+ = u_- & \text{on} \quad \partial\Omega, \qquad \text{(continuity of the pressure)} \\ \frac{1}{\rho}\frac{\partial u}{\partial \nu}\Big|_+ = \frac{1}{\rho_b}\frac{\partial u}{\partial \nu}\Big|_- & \text{on} \quad \partial\Omega, \qquad \text{(continuity of the velocity flow)} \\ u^{\text{s}} := u - u^{\text{in}} & \text{satisfies the Sommerfeld radiation condition.} \end{cases}$$

Regime?

We are looking for a resonance mode whose wavelength is much bigger than the size of the bubble:

$$\text{Limit 1: } \omega \to 0 \iff k \text{ (and } k_b) \to 0.$$

The only solution of the limit equation ($k = k_b = 0$), with $u^{\text{in}} \equiv 0$, is $u \equiv 0$. We need something else!

Order of magnitude: $\rho_b = 1.225$ kg.m^{-3} and $\rho = 1000$ kg.m^{-3}, hence $\delta := \dfrac{\rho_b}{\rho} \ll 1$ (contrast).

$$\text{Limit 2: } \delta \to 0.$$

Limit equation (with $u^{\text{in}} \equiv 0$)

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \Delta u = 0 & \text{in} \quad \Omega, \\ u_+ = u_- & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on} \quad \partial\Omega, \\ u & \text{satisfies the Sommerfeld radiation condition.} \end{cases}$$

The inside and outside problems are decoupled:

- 1) Solve the internal (Neumann) problem ($u|_\Omega = 1$),
- 2) Solve the external (Dirichlet) problem.

There exists a non-trivial solution $\Longrightarrow$ resonant mode.

Goal: Track this mode for small k and small δ.

Tracking the resonance

Idea: The scattering problem can be encoded at the boundary of the bubble.

Ansatz

$$u = \begin{cases} u^{\text{in}} + \widetilde{\mathcal{S}^k}[\psi] & \text{on} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \widetilde{\mathcal{S}^{k_b}}[\psi_b] & \text{on} \quad \Omega, \end{cases} \qquad \text{where} \quad \widetilde{\mathcal{S}^k} \text{ is the single layer potential.}$$

The initial problem is equivalent to a problem of the form

$$\mathcal{A}(\omega, \delta) \begin{pmatrix} \psi_b \\ \psi \end{pmatrix} = \begin{pmatrix} u^{\text{in}}|_+ \\ \delta \frac{\partial u^{\text{in}}}{\partial \nu}|_+ \end{pmatrix},$$

where $\mathcal{A}(\omega, \delta)$ is a bounded operator from $H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.

Definition (Resonant mode)

We say that the pair (ω, δ) is a resonant mode if $\mathcal{A}(\omega, \delta)$ is non invertible.

Lemma (Properties of $\mathcal{A}$ (from classical layer potential theory))

The operator-valued map $\mathcal{A}(\omega, \delta)$ satisfies:

i) *for all $\omega, \delta \in \mathbb{R}$, $\mathcal{A}$ is a bounded Fredholm operator of index 0.*
ii) *for all $\omega \in \mathbb{R}$, the map $\delta \mapsto \mathcal{A}(\omega, \delta)$ is linear (hence analytic).*
iii) *for all $\omega \in \mathbb{R}$, the map $\omega \mapsto \mathcal{A}(\omega, \delta)$ is analytic.*
iv) *$\mathcal{A}(0, 0)$ is non invertible. (the pair $(0, 0)$ is a resonant mode).*
v) *for all δ in a* complex *neighbourhood of 0, $\mathcal{A}(0, \delta)$ is non invertible iff $\delta = 0$.*

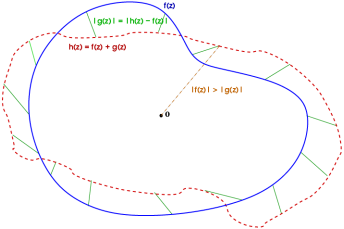
If $f(z)$ is analytic with $f(\lambda) = 0$ and $f(z) \neq 0$ for all $z \in \mathcal{B}(\lambda, r) \setminus \{\lambda\}$, then

$$\frac{1}{2\mathrm{i}\pi} \oint_{\mathscr{C}(\lambda,r)} \frac{f'(z)}{f(z)} \mathrm{d}z = \sharp \{\text{zeros of } f \text{ in } \mathcal{B}(\lambda, r)\} = 1, \quad \text{and} \quad \frac{1}{2\mathrm{i}\pi} \oint_{\mathscr{C}(\lambda,r)} \frac{f'(z)}{f(z)} z\mathrm{d}z = \lambda.$$

Theorem (Rouché's Theorem)

Let f be as before. Then, for all g analytic such that $|\frac{g}{f}| < 1$ on $\mathscr{C}(\lambda, r)$, it holds that $f + g$ has a unique zero λ_{f+g} in $\mathcal{B}(\lambda, r)$, and

$$\lambda_{f+g} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathscr{C}(\lambda,r)} \frac{(f+g)'(z)}{(f+g)(z)} z\mathrm{d}z.$$



Complex analysis: operator version («$f = \det A$»)

If $A(z) : \mathcal{H}_1 \to \mathcal{H}_2$ is an analytic map of Fredholm operators (of index 0) such that

- For all $z \in \mathcal{B}(\lambda, r) \setminus \{\lambda\}$, $\quad \dim \mathrm{Ker} A(z) = \dim \mathrm{Ker} A^*(z) = 0$;
- $\dim \mathrm{Ker} A(\lambda) = 1$, $\quad$ (hence $\dim \mathrm{Ker} A^*(z) = 1$),

then

$$1 = \frac{1}{2i\pi} \mathrm{Tr}_{\mathcal{H}_1} \left[\oint_{\mathscr{C}(\lambda,r)} \frac{1}{A(z)} A'(z) \mathrm{d}z \right] \quad \text{and} \quad \lambda = \frac{1}{2i\pi} \mathrm{Tr}_{\mathcal{H}_1} \left[\oint_{\mathscr{C}(\lambda,r)} \frac{1}{A(z)} A'(z) z \mathrm{d}z \right].$$

Remarks

- If $A : \mathcal{H}_1 \to \mathcal{H}_2$, then $A^{-1}A' : \mathcal{H}_1 \to \mathcal{H}_1$. The notion of trace exists.
- The operators A^{-1} and A' may not commute. However, $\mathrm{Tr}_{\mathcal{H}_1}(A^{-1}A') = \mathrm{Tr}_{\mathcal{H}_2}(A'A^{-1})$.

Theorem (Operator version of Rouché: Gohberg-Sigal theorem[1])

For all operator-valued analytic map $B(z) : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\|A^{-1}B\|_{\mathscr{B}(\mathcal{H}_1)} < 1$ on $\mathscr{C}(\lambda, r)$, then the operator $A + B$ is Fredholm of index 0, and there exists a unique point $\lambda_{A+B} \in \mathcal{B}(\lambda, r)$ such that

$$\dim \mathrm{Ker}(A+B)(\lambda_{A+B}) = 1 \qquad (= 0 \textit{ otherwise}).$$

Moreover,

$$\lambda_{A+B} = \frac{1}{2i\pi} \mathrm{Tr}_{\mathcal{H}_1} \left[\oint_{\mathscr{C}(\lambda,r)} \frac{1}{(A+B)(z)} (A+B)'(z) z \mathrm{d}z \right].$$

[1] U. Gohberg, E.I. Sigal, Sbornik: Mathematics 13.4 (1971).

We write

$$\mathcal{A}_\omega := \mathcal{A}(\omega, \cdot) = \mathcal{A}_0 + \omega\mathcal{A}_1 + \omega^2\mathcal{A}_2 + \cdots$$

We apply Gohberg-Sigal theorem with $A = \mathcal{A}_0$, $B \sim \omega\mathcal{A}_1 + \omega^2\mathcal{A}_2 + \cdots$ and $\delta \sim z$.

Theorem (H. Ammari, DG, B. Fitzpatrick, H. Lee, H. Zhang)

For ω small enough, there exists a unique complex number δ_ω near 0 such that $\mathcal{A}(\omega, \delta_\omega)$ is non invertible. Moreover, the map $\omega \to \delta_\omega$ is analytic, and

$$\delta_\omega = \frac{1}{2\mathrm{i}\pi}\mathrm{Tr}_{\mathcal{H}^{--}}\left[\oint_{\mathscr{C}(0,\delta)} \frac{1}{\mathcal{A}_\omega(\delta)}\frac{\partial\mathcal{A}_\omega}{\partial\delta}(\delta)\delta\mathrm{d}\delta\right]$$
$$= \left(\frac{|\Omega|}{v_b^2\mathrm{Cap}_\Omega}\right)\omega^2 + \left(\frac{\mathrm{i}|\Omega|}{4\pi v_b^2 v}\right)\omega^3 + O(\omega^4).$$

Here, Cap_Ω is the capacity of the set Ω. If $\Omega = S_R$ is the sphere of radius R, $\mathrm{Cap}_{S_R} = 4\pi R$.

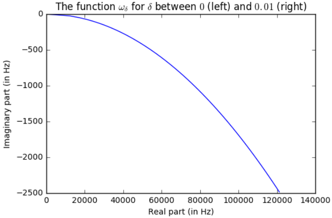
Remark: The result holds for all shapes of bubbles.

Inverse formula: $\delta \to \omega_\delta$

$$\omega_\delta = \left(\frac{\mathrm{Cap}_\Omega v_b^2}{|\Omega|} \right)^{1/2} \sqrt{\delta} - \mathrm{i} \left(\frac{\mathrm{Cap}_\Omega^2 v_b^2}{8\pi v |\Omega|} \right) \delta + O(\delta^{3/2}).$$

Leading order
For a sphere, $\omega_\delta = \omega_M$. We recover Minnaert's result.

Second order:
Purely imaginary $\Longrightarrow$ Dissipative term $\equiv$ Radiative damping.



Remarks

- The resonance is very close to the real-line, even in physical situations.
- We obtain a resonance phenomenon, and a damping effect, from **ab initio** principles.
- It corresponds to the so-called breathing mode.

The point scatterer approximation

What happens to a (fix) pressure wave (fix ω) with a small bubble $\Omega^\varepsilon = \varepsilon\Omega$ as $\varepsilon \to 0$? How much is the resonant mode excited?

Initial problem

$$\begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega^\varepsilon}, \\ \left(\Delta + k_b^2\right) u = 0 & \text{in} \quad \Omega^\varepsilon, \\ u_+ = u_- & \text{on} \quad \partial\Omega^\varepsilon, \\ \delta \frac{\partial u}{\partial \nu}\Big|_+ = \frac{\partial u}{\partial \nu}\Big|_- & \text{on} \quad \partial\Omega^\varepsilon, \\ u^s = u - u^{\text{in}} & \text{satisfies Sommerfeld.} \end{cases}$$

Regime?

Limit 1: $\varepsilon \to 0$ | Limit 2: $\delta \to 0$.

Idea: We know that $\varepsilon_M \approx \sqrt{\delta}$. Fix $\mu > 0$, and

Limit: $\varepsilon \to 0$ and $\delta = \mu\varepsilon^2$.

Minnaert resonance

$$\mu_M := \frac{|\Omega| k_b^2}{\text{Cap}_\Omega}.$$

Theorem (H. Ammari, DG, B. Fitzpatrick, H. Lee, H. Zhang)

If $\mathbf{0} \in D$, then the solution $u^\varepsilon := u[\varepsilon, \delta = \mu\varepsilon^2]$ satisfies

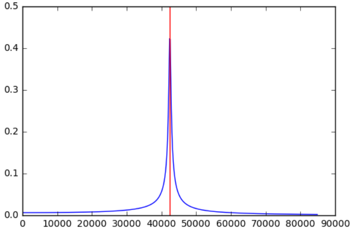
$$u^\varepsilon(\mathbf{x}) = u^{\text{in}}(\mathbf{x}) + \begin{cases} \varepsilon \left(\dfrac{\text{Cap}_\Omega}{1 - \frac{\mu_M}{\mu}} u^{\text{in}}(\mathbf{0}) \right) G^k(\mathbf{x}) + O_\mu(\varepsilon^2) & \text{if} \quad \mu \neq \mu_M, \\ \left(\mathrm{i} \dfrac{4\pi}{k} u^{\text{in}}(\mathbf{0}) \right) G^k(\mathbf{x}) + O(\varepsilon) & \text{if} \quad \mu = \mu_M. \end{cases}$$

Loosely speaking, $u^s = u - u^{\text{in}}$ satisfies

$$u^s(\mathbf{x}) \approx u^{\text{in}}(\mathbf{0}) g_s(\omega) G^k(\mathbf{x} - \mathbf{0}), \quad \text{with} \quad g_s(\omega) := \frac{\text{Cap}_\Omega}{\left(1 - \frac{\omega^2}{\omega_M^2}\right) - \mathrm{i} \frac{\text{Cap}_\Omega \omega}{4\pi v}} \quad \text{(response function)}.$$

Example For a bubble of radius 0.5 mm, we get

The function $g_s(\omega)$ for ω between 0 and $2\omega_M$.



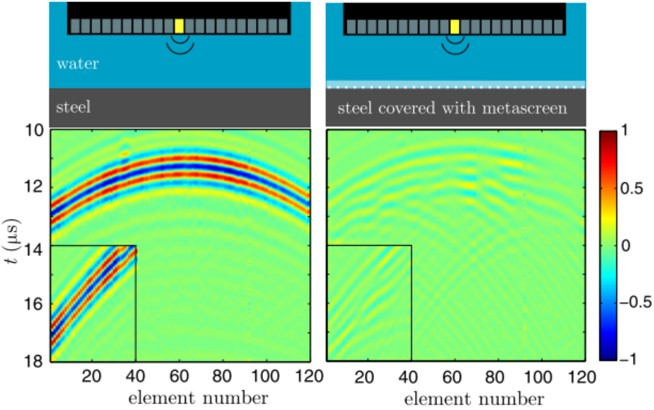
Remarks

- The imaginary part in the denominator of g_s is the radiative damping.
- The poles of g_s are in the lower half complex plane: from Titchmarsh's theorem, g_s is a causal response function.
- We recover the expression found in [1] for g_s.

[1] M. Devaud, Th. Hocquet, J.-C. Bacri, and V. Leroy. Eur. J. Phys., 29(6):1263, 2008.

The periodic case: the periodic Minnaert resonance

Experiment[2]

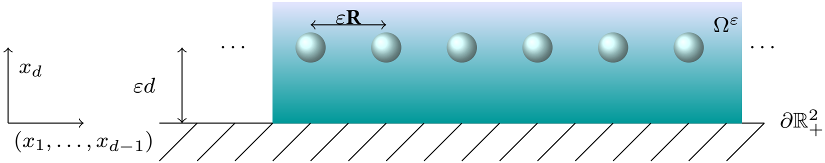


[2]V. Leroy, A. Strybulevych, M. Lanoy, F. Lemoult, A. Tourin, J.H. Page, Phys. Rev. B 91, 020301(R) (2015).

We now set bubbles on a $(d-1)$ dimensional lattice $\mathcal{R}$, on top of a Dirichlet surface.

Bubbles domain

$$\Omega^\varepsilon := \bigcup_{\mathbf{R}\in\mathcal{R}} \varepsilon\left(\Omega + \mathbf{R}\right).$$



Incoming wave

$$U^{\rm in}(x,y) := u_0 \mathrm{e}^{-\mathrm{i}ky} \qquad \text{(orthogonal to the plane).}$$

Scattering problem

$$\begin{cases} \left(\Delta + k^2\right) U^\varepsilon = 0 & \text{on} \quad \mathbb{R}^d_+ \setminus \overline{\Omega^\varepsilon}, \\ \left(\Delta + k_b^2\right) U^\varepsilon = 0 & \text{on} \quad \Omega^\varepsilon, \\ U^\varepsilon|_+ = U^\varepsilon|_- & \text{on} \quad \partial\Omega^\varepsilon, \\ \partial_\nu U^\varepsilon|_- = \delta \partial_\nu U^\varepsilon|_+ & \text{on} \quad \partial\Omega^\varepsilon, \\ U^s := U^\varepsilon - U^{\rm in} & \text{satisfies the outgoing radiation condition,} \\ U^\varepsilon = 0 & \text{on} \quad \partial\mathbb{R}^2_+, \quad \text{(Dirichlet)} \\ U^\varepsilon(\mathbf{x} + \varepsilon\mathcal{R}) = U(\mathbf{x}). & \end{cases} \Bigg\} \text{ boundary conditions}$$

Regime?

Limit 1: $\varepsilon \to 0$.   Limit 2: $\delta \to 0$.

Limit problem (after rescaling $u(\mathbf{x}) := U(\mathbf{X}/\varepsilon)$).

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad \mathbb{R}^d_+ \setminus \overline{\Omega}, \\ \Delta u = 0 \quad \text{on} \quad \Omega, \\ u|_+ = u|_- \quad \text{on} \quad \partial\Omega, \\ \partial_\nu u|_- = 0 \quad \text{on} \quad \partial\Omega, \\ + \quad \text{boundary conditions.} \end{cases}$$

Again, two decoupled problems: there exists a non trivial solution (with $u|_\Omega = 1$).

Similar to the single bubble case (existence + tracking of the resonance).

Periodic Minnaert resonance

$$\boxed{\omega_M^+ := \left(\frac{\mathrm{Cap}^+_{\Omega,\mathcal{R}} v_b^2}{|\Omega|} \right)^{1/2} \sqrt{\delta}.}$$

Remark: The periodic capacity depends on the lattice.

How much is the resonant mode excited when we send a (fix) incoming wave $U^{\rm in}$?

Result

The meta-screen behaves like an acoustic plane with reflection coefficient

$$R(\omega) \approx -1 - 2 \left(\frac{\mathrm{i}\omega\eta}{1 - \left(\frac{\omega}{\omega_M^+}\right)^2 - \mathrm{i}\omega\eta^*} \right) \quad \text{with} \quad \eta = \eta^* := \frac{(\alpha_{\rm monop}^+)^2 a}{v \mathrm{Cap}_{\Omega,\mathcal{R}}^+}.$$

Remarks:

- If $\omega \ll \omega_M^+$ or $\omega \gg \omega_M^+$, then $R(\omega) \approx -1$ (Dirichlet plane) $\sim$ no bubble case.
- If $\omega = \omega_M^+$, then $R(\omega_M^+) = 1$ (Neumann plane).
- Considering other source of damping (*e.g.* viscous), and assuming $\eta^* = 2\eta$, we have

$$R(\omega_M^+) = 0 \quad \text{(absorption plane)}.$$

Conclusions

- Regime $\varepsilon \to 0$ and $\delta \to 0$ such that $\delta \approx \varepsilon^2$ (high-contrast limit[3]).
- Tracking of the resonance through Gohberg-Sigal theory.
- Point scatterer approximation and meta-surfaces from the study of layer potentials.
- Resonance phenomenon as the limit of well-posed and easy-to-study problems.

Bibliography

- H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, H. Zhang, arXiv:1603.03982 (single bubble).
- H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, H. Zhang, SIAM J. Appl. Math., 77 (2017) (meta-surface).
- H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, H. Zhang, Proc. R. Soc. A 473 (2017) (numerics).

Thank you for your attention.

[3] cf. also works by G. Bouchitté and D. Felbacq