# Edge states for second order elliptic operators

David Gontier

CEREMADE, Université Paris-Dauphine

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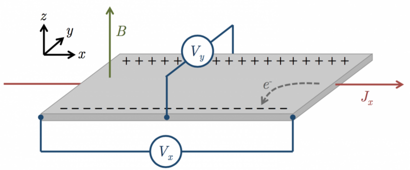
# Some historical remarks.

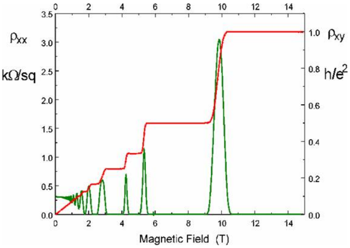
May 20, 2019: New definition of the kg by the *Bureau International des Poids et Mesures* (BIPM)[1] :
*"Le kilogramme, symbole kg, est l'unité de masse du SI. Il est défini en prenant la valeur numérique fixée de la constante de Planck, h, égale à* $6,626\,070\,15 \times 10^{-34}$ *J.s."*

Question: How do you measure $h$? How do you measure $h$ with $10^{-9}$ accuracy?

Comments by von Klitzing[2]: *"The discovery of the QHE led to a new type of electrical resistor [...]. This resistor is universal for all 2D electron systems in strong magnetic fields with an uncertainty of less than one part in* $10^{10}$*."*

QHE = Quantum Hall Effect[3] (von Klitzing got Nobel prize in 1985 for discovery of Quantum Hall Effect).

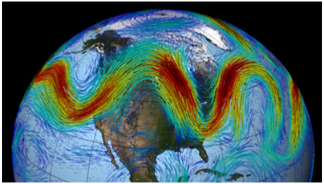



[1] https://www.bipm.org/fr/measurement-units/
[2] von Klitzing, Nature Physics 13, 2017
[3] K. von Klitzing; G. Dorda; M. Pepper, Phys. Rev. Lett. 45 (6): 494–497, 1980.

Modern interpretation: The plateaus correspond to different *topological phases of matter*[4], and the QHE is a manifestation of *bulk-edge correspondence.*

> *"When some bulk systems are cut, edge modes must appear at the boundary. These modes are quantized: we can associate a topological number to them."*



The Rossby Waves (wind) might be a manifestation of bulk-edge correspondence (Tauber/Delplace/Venaille, J. Fluid Mech. Vol 868 (2019). )

Many proofs of bulk-edge correspondence, in many contexts, using many tools:

- First proof (complex analysis): Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993).
- Operator/functional theory: Elbau/Graf, Commun. Math. Phys. 229, 415–432 (2002). Elgart/Graf/Schenker, Commun. Math. Phys. 259, (2005).
- K-theory Kellendonk/Richter/Schulz-Baldes, Rev. Math. Phys. 14, 87–119 (2002).
- Micro-local analysis Drouot, arXiv:1909.10474 (2019).
- Vector bundle theory: Graf/Porta, Comm. Math. Phys. 324, 851–895 (2013).
- Maslov index Avila/Schulz-Baldes/Villegas-Blas, Math. Phys., Analysis and Geometry 16, (2013).

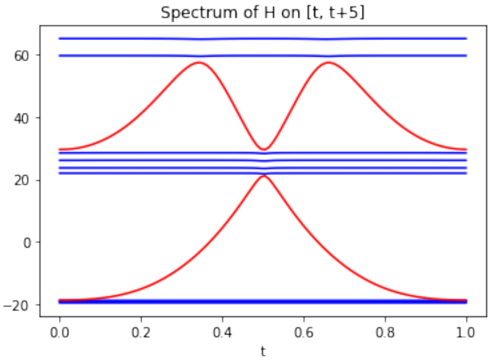
[4] D.J. Thouless, F.D.M. Haldane and J.M. Kosterlitz got Nobel prize in 2016 for the discovery of topological phases of matter

# Another motivation: spectral pollution

We want to compute the spectrum of the (simple) operator

$$H := -\partial_{xx}^2 + V(x), \quad \text{with} \quad V(x) = 50 \cdot \cos(2\pi x) + 10 \cdot \cos(4\pi x).$$

The potential $V$ is 1-periodic. Assume we study $H$ in a box $[t, t+L]$ with Dirichlet boundary conditions.




Depending on where we fix the origin $t$, the spectrum differs...
There are branches of spurious eigenvalues = spectral pollution (they appear for all $L$).
The corresponding eigenvectors are edge modes: they are localized near the boundaries (they cannot propagate in the bulk).

**In this talk:** understand why edge modes *must* appear.

# Framework

## Bulk operator

Let $V$ be a bounded potential. $H := -\partial^2_{xx} + V$ acting on $L^2(\mathbb{R})$ is self-adjoint (with domain $H^2(\mathbb{R})$).

## Edge operator

We want to define $H^\sharp := -\partial^2_{xx} + V$ acting on $L^2(\mathbb{R}^+)$.

## Self-adjoint extensions

The operator $H^\sharp$ with core domain $C_0^\infty(\mathbb{R}^+)$ has

$$\text{minimal domain } \mathcal{D}_{\min} := H_0^2(\mathbb{R}^+), \quad \text{maximal domain } \mathcal{D}_{\max} = \mathcal{D}_{\min}^* = H^2(\mathbb{R}^+).$$

$\mathcal{D}_{\min} \neq \mathcal{D}_{\max}$, so $H^\sharp$ is not self-adjoint (we need to set *boundary conditions*).
A domain $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$ defines a self-adjoint extension of $H^\sharp$ iff $\mathcal{D}^* = \mathcal{D}$, where

$$\mathcal{D}^* := \left\{ \psi \in L^2(\mathbb{R}^+), \quad T_\psi : \phi \mapsto \langle \psi, H^\sharp \phi \rangle \text{ is bounded on } \mathcal{D} \right\}.$$

Key remark: $E \in \mathbb{R}$ is an eigenvalue of $(H^\sharp, \mathcal{D})$ iff

- $E$ is an eigenvalue of $(H^\sharp, \mathcal{D}_{\max})$: there is $\psi \in \mathcal{D}_{\max}$ so that $H^\sharp \psi = E\psi$;
- $\psi \in \mathcal{D}$.

Vectorial space of weak-solution $\mathcal{S}(E) := \mathrm{Ker}\left(H^\sharp_{\max} - E\right)$.

**$E \in \mathbb{R}$ is an eigenvalue of $(H^\sharp, \mathcal{D})$ iff $\boxed{\mathcal{S}(E) \cap \mathcal{D} \neq \{0\}}$.**

## Remark

- $\mathcal{S}(E)$ depends only on the *bulk* (no boundary conditions);
- $\mathcal{D}$ depends only on the *edge* (usually independent of $V$, *e.g.* Dirichlet boundary conditions).

## Boundary symplectic space

**Idea**: compute this intersection in the *boundary space*

$$\psi \in \mathcal{D}_{\max} = H^2(\mathbb{R}^+) \quad \mapsto \quad \operatorname{Tr} \psi := (\psi(0), \psi'(0)) \in \mathcal{H}_b := \mathbb{C}^2.$$

Remark: The map $\operatorname{Tr} : \mathcal{D}_{\max} \to \mathcal{H}_b$ is onto.

Symplectic form (= non degenerate, continuous, sesquilinear form $\omega : \mathcal{H}_b \times \mathcal{H}_b \to \mathbb{C}$ such that $\omega(\mathbf{x}, \mathbf{y}) = -\overline{\omega(\mathbf{y}, \mathbf{x})}$.)

$$\forall \mathbf{x} = (x, x') \in \mathbb{C}^2, \, \forall \mathbf{y} = (y, y') \in \mathbb{C}^2, \quad \omega(\mathbf{x}, \mathbf{y}) := \overline{x} y' - \overline{x'} y.$$

Lagrangian spaces A sub-vectorial space $\ell \subset \mathcal{H}_b$ is *Lagrangian* if $\ell^\circ = \ell$, where

$$\ell^\circ := \left\{ \mathbf{x} \in \mathcal{H}_b, \quad \forall \mathbf{y} \in \mathcal{H}_b, \, \omega(\mathbf{x}, \mathbf{y}) = 0 \right\}.$$

Second Green's formula (*for second order elliptic operator*)

$$\begin{aligned} \forall \psi, \phi \in \mathcal{D}_{\max}, \quad \langle \psi, H^\sharp_{\max} \phi \rangle - \langle H^\sharp_{\max} \psi, \phi \rangle &= \overline{\psi(0)} \phi'(0) - \overline{\psi'(0)} \phi(0) \\ &= \omega \left( \operatorname{Tr}(\psi), \operatorname{Tr}(\phi) \right). \end{aligned}$$

# Self-adjoint extensions and Lagrangian planes

## Lemma (classical)

*The self-adjoint extensions of $H^\sharp$ are in one-to-one correspondence with the Lagrangian planes of $\mathcal{H}_b$. More specifically, $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$ defines a self-adjoint extension iff it is of the form*

$$\mathcal{D} = \mathrm{Tr}^{-1}(\ell), \quad \textit{for a Lagrangian subspace } \ell.$$

Proof.

Let $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$, and set $\ell := \mathrm{Tr}\,\mathcal{D}$. Let $\mathbf{x} \in \ell^\circ$ and $\psi \in \mathrm{Tr}^{-1}\{\mathbf{x}\} \subset H^\sharp_{\max}$, we have

$$\forall \phi \in \mathcal{D}, \quad \omega(\mathrm{Tr}\,\psi, \mathrm{Tr}\,\phi) = 0, \quad \text{so} \quad \langle \psi, H^\sharp_{\max}\phi \rangle = \langle H^\sharp_{\max}\psi, \phi \rangle$$

In particular, $\phi \mapsto \langle \psi, H^\sharp_{\max}\phi \rangle$ is bounded on $\mathcal{D}$, so $\psi \in \mathcal{D}^*$. Conversely, we check that $\psi \in \mathcal{D}^*$ implies $\mathrm{Tr}(\psi) \in \ell^\circ$. This proves that $\mathcal{D}^* = \mathrm{Tr}^{-1}(\ell^\circ)$. □

Examples

- Dirichlet boundary conditions corresponds to the plane $\ell_D := \{0\} \times \mathbb{C}$.
- Neumann boundary conditions corresponds to the plane $\ell_N := \mathbb{C} \times \{0\}$.
- $\theta$-Robin boundary conditions corresponds to the plane $\ell_\theta := \mathrm{Vect}_{\mathbb{C}}\{(\sin(\pi\theta), \cos(\pi\theta))\}$ :

$$\Psi'(0) + \alpha\Psi(0) = 0, \quad \alpha = \tan(\pi\theta).$$

($\theta = 0$ is Dirichlet, and $\theta = 1/2$ is Neumann. Note that $\theta \mapsto \ell_\theta$ is 1-periodic...)

# Weak solutions and Lagrangian planes

Define $H^{\sharp,\pm}_{\max} := -\Delta + V$ on $L^2(\mathbb{R}^\pm)$ with domain $H^2(\mathbb{R}^\pm)$.

## Lemma (new?)

*Let $E \in \mathbb{R}$ be in the resolvent set of the bulk operator $H$. Let $\mathcal{S}^\pm(E) := \mathrm{Ker}(H^{\sharp,\pm}_{\max} - E)$ be the set of weak solutions, and let $\ell^\pm(E) := \mathrm{Tr}\,\mathcal{S}^\pm(E)$. Then $\ell^\pm(E)$ are Lagrangian planes, and*

$$\mathcal{H}_b = \ell^-(E) \oplus \ell^+(E).$$

Proof.

**Step 1.** First we have

$$\forall \psi, \phi \in \mathcal{S}^+(E), \quad \langle \psi, H^\sharp_{\max}\phi\rangle - \langle H^\sharp_{\max}\psi, \phi\rangle = \langle \psi, E\phi\rangle - \langle E\psi, \phi\rangle = 0.$$

So, by Green's identity, $\omega(\mathrm{Tr}(\psi), \mathrm{Tr}(\phi)) = 0$, hence $\ell^+(E) \subset \ell^+(E)^\circ$. Similarly, $\ell^-(E) \subset \ell^-(E)^\circ$.

**Step 2.** Since $E \notin \sigma(H)$, the map $(H-E)^{-1}$ is well-defined and maps $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$. Writing

$$\mathcal{H} := L^2(\mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-, \quad \text{with} \quad \mathcal{H}^\pm := \left\{\psi \in L^2(\mathbb{R}), \psi(x) = 0 \text{ on } \mathbb{R}^\mp\right\},$$

gives

$$\mathcal{D} := H^2(\mathbb{R}) = \mathcal{D}^+ + \mathcal{D}^-, \quad \text{with} \quad \mathcal{D}^\pm := (H-E)^{-1}\mathcal{H}^\pm.$$

If $f \in \mathcal{D}^+$, then $f$ is square integrable, and $(-\partial^2_{xx} + V - E)f = 0$ on $\mathbb{R}^-$. So the restriction of $f$ to $\mathbb{R}^-$ belongs to $\mathcal{S}^-(E)$. This proves $\mathcal{D}^+ \subset \mathcal{S}^-(E)$, and similarly, $\mathcal{D}^- \subset \mathcal{S}^+(E)$. Taking traces gives

$$\ell^+(E) + \ell^-(E) \supset \mathrm{Tr}(\mathcal{D}^-) + \mathrm{Tr}(\mathcal{D}^+) = \mathrm{Tr}(\mathcal{D}) = \mathcal{H}_b.$$

Together with **Step 1**, and some simple algebra, we obtain the result. □

# Lagrangian planes and unitaries

## The $J$ matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{so that} \quad \omega(\mathbf{x}, \mathbf{y}) = \overline{x} y' - \overline{x'} y = \langle \mathbf{x}, J\mathbf{y} \rangle_{\mathbb{C}^2}.$$

We have $J^2 = -1$, so $\sigma(J) = \{-\mathrm{i}, \mathrm{i}\}$. In addition,

$$\mathrm{Ker}(J - \mathrm{i}) \oplus \mathrm{Ker}(J + \mathrm{i}) = \mathcal{H}_b, \quad \text{with, explicitly,} \quad \mathrm{Ker}(J \mp \mathrm{i}) = \begin{pmatrix} 1 \\ \pm \mathrm{i} \end{pmatrix} \mathbb{C}.$$

### Lemma (reformulation of Leray, Analyse Lagrangienne et mécanique quantique, 1978)

*The Lagrangian planes of $(\mathcal{H}_b = \mathbb{C}^2, \omega)$ are in one-to-one correspondence with the unitaries $\mathcal{U} : \mathbb{C} \to \mathbb{C}$, with*

$$\ell := \left\{ \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} x + \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathcal{U} x, \quad x \in \mathbb{C} \right\}.$$

Example: For the Robin Lagrangian plane $\ell_\theta := \mathrm{Vect}_{\mathbb{C}}\{(\sin(\pi\theta), \cos(\pi\theta))\}$, we have

$$\begin{pmatrix} \sin(\pi\theta) \\ \cos(\pi\theta) \end{pmatrix} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \frac{1}{2} \left[ \sin(\pi\theta) - \mathrm{i} \cos(\pi\theta) \right] + \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \frac{1}{2} \left[ \sin(\pi\theta) + \mathrm{i} \cos(\pi\theta) \right],$$

so $\mathcal{U}_\theta = \frac{\sin(\pi\theta) + \mathrm{i}\cos(\pi\theta)}{\sin(\pi\theta) - \mathrm{i}\cos(\pi\theta)} = \mathrm{e}^{-2\mathrm{i}\pi\theta} \in \mathbb{S}^1 \approx \mathrm{U}(1)$.

### Lemma

*If $\ell_1$ and $\ell_2$ are two Lagrangian planes, then*

$$\dim\left(\ell_1 \cap \ell_2\right) = \dim \mathrm{Ker}(\mathcal{U}_1 - \mathcal{U}_2) = \dim \mathrm{Ker}\left(\mathcal{U}_1^* \mathcal{U}_2 - 1\right).$$

Gathering the previous results gives the following.

## Lemma

*For all $E \in \mathbb{R} \setminus \sigma(H)$, and for $(H^\sharp, \mathcal{D}^\sharp)$ a self-adjoint extension of the edge operator, we have*

$$\dim \mathrm{Ker}\left(H^\sharp - E\right) = \dim\left(\mathcal{D}^\sharp \cap \mathcal{S}^+(E)\right) = \dim\left(\ell^\sharp \cap \ell^+(E)\right) = \dim \mathrm{Ker}\left(\left(\mathcal{U}^\sharp\right)^* \mathcal{U}(E) - 1\right).$$

Remarks:

- the last problem is set on $\mathrm{U}(1) \approx \mathbb{S}^1$. It is somehow much simpler to study;
- we only used that $(-\partial_{xx}^2 + V)$ is self-adjoint ($V$ needs not be periodic);
- the proofs work similarly for general second order elliptic operators.

**Yes,... but why do we have edge states?**

Idea: consider periodic families of second order elliptic operators $\implies$ periodic families of Lagrangian planes $\ell_t^\sharp$ and $\ell_t^+(E) \implies$ periodic family $t \to (\mathcal{U}_t^\sharp)^* \mathcal{U}_t(E) \in \mathbb{S}^1$.

# Including orientations

## Theorem (DG 2021)

*Let $t \mapsto H_t$ be a continuous periodic family of bulk operators.*
*Let $t \mapsto (H_t^\sharp, \mathcal{D}_t^\sharp)$ be a* continuous *periodic family of (self-adjoint extensions of) edge operators.*
*Assume that $E \in \mathbb{R}$ is in none of the spectra of the bulk operators $H_t$. Then*

$$\begin{aligned} \mathrm{Sf}\left(H_t^\sharp, E\right) &= \mathrm{Mas}\left(\ell_t^\sharp, \ell_t^+(E)\right) = \mathrm{Sf}\left(\left(\mathcal{U}_t^\sharp\right)^* \mathcal{U}_t(E), 1\right) \\ &= \mathrm{Winding}\left(\left(\mathcal{U}_t^\sharp\right)^* \mathcal{U}_t(E)\right) \\ &= \mathrm{Winding}(\mathcal{U}_t(E)) - \mathrm{Winding}(\mathcal{U}_t^\sharp) \qquad \in \mathbb{Z} \end{aligned}$$

Spectral flow $\mathrm{Sf}(H_t, E)$ counts the net number of eigenvalues going downwards in the gap where $E$ lies.

Maslov index[5] $\mathrm{Mas}(\ell_1(t), \ell_2(t))$ counts the number of signed crossings of Lagrangian plane.

Winding number: if $t \mapsto \mathcal{U}(t) \in \mathbb{S}^1$ is continuous and periodic,

$$\mathrm{Winding}(\mathcal{U}) = \sharp\{\text{turns in the positive directions}\} = \sharp\{\mathcal{U}(\cdot) \text{ crosses } 1 \in \mathbb{S}^1, \text{counting orientations}\}.$$

The winding number is an homomorphism, which gives the last line of the Theorem.

In the last line, we decoupled the bulk and the edge: the spectral flow is a combination of the two!
If the previous integer is non-null, edge states appear at the energy $E$ for some $t \in [0, 1]$.

[5] Maslov, *Théorie des perturbations et méthodes asymptotiques.* 1972

## Example 1: Robin boundary conditions

Consider a bulk Hamiltonian $H_\theta := -\partial_{xx}^2 + V$ (independent of $\theta$)

Consider the edge Hamiltonian $H_\theta^\sharp := -\partial_{xx}^2 + V$, with $\theta$-Robin boundary conditions, *i.e.* with domain $\mathcal{D}_\theta = \mathrm{Tr}^{-1}(\ell_\theta)$.

For $E$ in the resolvent set of $-\partial_{xx}^2 + V$, we have

- $\mathrm{Winding}(\mathcal{U}_\theta^+(E)) = 0$, since the bulk operator is independent of $V$;
- $\mathrm{Winding}\left(\mathcal{U}_\theta^\sharp\right) = \mathrm{Winding}\left(\mathrm{e}^{-2\mathrm{i}\pi\theta}\right) = -1.$

**Lemma**

*In each spectral gap of $H = -\partial_{xx}^2 + V$, there is a spectral flow of exactly $1$ eigenvalue going downwards. This includes the lower gap $(-\infty, \inf\sigma(H))$.*

# Example 2: Junctions

Let $H_L(t) := -\partial_{xx}^2 + V_{L,t}$ and $H_R(t) := -\partial_{xx}^2 + V_{R,t}$ be two periodic families of Schrödinger operators. We consider the junction operator

$$H_t^{\text{junction}} := -\partial_{xx}^2 + \left[ V_{L,t}(x)\mathbb{1}(x < 0) + V_{R,t}(x)\mathbb{1}(x > 0) \right].$$

## Theorem

*If $E \in \mathbb{R}$ is in the resolvent set of all left and right bulk operators, then*

$$\mathrm{Sf}(H_t^{\text{junction}}, E) = \mathrm{Winding}\left(\mathcal{U}_R^+(E)\right) - \mathrm{Winding}\left(\mathcal{U}_L^-(E)\right).$$

Idea of the proof

We note that $E$ is an eigenvalue of $H$ iff $\ell^+(E) \cap \ell^-(E) \neq \{0\}$:
the Cauchy solution of $(-\partial_{xx}^2 + V - E)\psi = 0$, $\mathrm{Tr}(\psi) \in \ell^+(E) \cap \ell^-(E)$ is square-integrable on $\mathbb{R}$.
Actually, we have

$$\dim \mathrm{Ker}\,(H - E) = \dim\left(\ell^+(E) \cap \ell^-(E)\right) = \dim \mathrm{Ker}\left(\mathcal{U}^-(E)^*\mathcal{U}^+(E) - 1\right).$$

Adding the parameter $t$, and taking into account orientations, we get

$$\begin{aligned}\mathrm{Sf}\,(H_t, E) &= \mathrm{Mas}\left(\ell_t^+(E), \ell_t^-(E)\right) = \mathrm{Sf}\left(\mathcal{U}_t^-(E)^*\mathcal{U}_t^+(E), 1\right) = \mathrm{Winding}\left(\mathcal{U}_t^-(E)^*\mathcal{U}_t^+(E)\right) \\ &= \mathrm{Winding}\left(\mathcal{U}_t^+(E)\right) - \mathrm{Winding}\left(\mathcal{U}_t^-(E)\right).\end{aligned}$$

For the junction operator $H_t^{\text{junction}}$, $\mathcal{U}_t^+(E)$ only depends on the right side, while $\mathcal{U}_t^-(E)$ only depends on the left. $\square$

## Example 3: Dislocations

Let $V(x)$ be a 1-periodic potential. We consider the bulk operator

$$H_t := -\partial_{xx}^2 + V(x-t),$$

and the edge operator $H_t^\sharp$ on $\mathbb{R}^+$ with Dirichlet boundary conditions.

Since $H_t$ is a translated version of $H_0$, and using Bloch theory, we have

$$\sigma(H_t) = \sigma(H_0) = \bigcup_{k \in B.Z.} \bigcup_{n=1}^{\infty} \{\varepsilon_{n,k}\},$$

where $\varepsilon_{n,k}$ are the Bloch eigenmodes.

**Lemma**

*For $E$ in a gap of $H_0$, we have*

$$\mathrm{Sf}\left(H_t^\sharp, E\right) = \mathcal{N}(E),$$

*where $\mathcal{N}(E)$ is the number of Bloch modes below $E$.*

Idea of the proof (adapted from R. Hempel M. Kohlmann, J. Math. Anal. Appl. 381 (2011).)
The state $\gamma_E := \mathbb{1}(H_0 - E)$ represents a state having $\mathcal{N}(E)$ electrons per unit cell.
Consider the dislocated operator

$$\mathcal{H}_t^{\text{junction}} := -\partial_{xx}^2 + \left[V(x)\mathbb{1}(x<0) + V(x-t)\mathbb{1}(x>0)\right].$$

At $t=0$, and $t=1$, we recover the bulk operator $H$. During the motion $t \in [0,1]$, a new cell has appeared, so $\mathcal{N}(E)$ electrons have appeared. They can only come from the upper bands, so a flow of $\mathcal{N}(E)$ eigenvalues going downwards must appear.

# Numerical simulation

In our setting (one dimensional Schrödinger operator), *all gaps are open*, so $\mathcal{N}(E) = N$ in the $N$-th gap.

Potential

$$V(x) = 50 \cdot \cos(2\pi x) + 10 \cdot \cos(4\pi x)$$

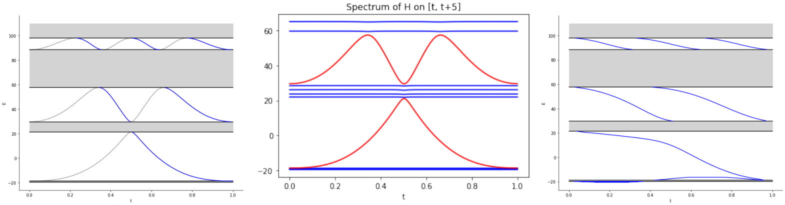



Figure: (left) Spectrum of $H^\sharp(t)$ for $t \in [0, 1]$. (center) Spectrum of the operator on $[t, t + L]$, (right) Spectrum for a junction operator.

On the right, we observe a spectral flow of eigenmodes for the left boundary (going downwards), and a spectral flow of eigenmodes for the right boundary (going upwards).

# Example 4: Protected states in the Dirac equation

## Dirac equation

$$\mathrm{i}\begin{pmatrix}\psi^\uparrow \\ -\psi^\downarrow\end{pmatrix}' = \begin{pmatrix}0 & V(x) \\ V(x) & 0\end{pmatrix}\begin{pmatrix}\psi^\uparrow \\ \psi^\downarrow\end{pmatrix} + E\begin{pmatrix}\psi^\uparrow \\ \psi^\downarrow\end{pmatrix}.$$

**Lemma ( Fefferman/Lee-Thorp/Weinstein, AMS Vol. 247 (2017).)**

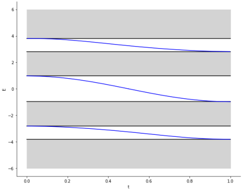
*If $V$ switches from $V_{\rm per}$ at $x \le -L$ to $-V_{\rm per}$ at $x \ge L$, then $0$ is in the spectrum of the Dirac operator.*
= «Topologically protected state».

Introduce the $t$ parameter

$$V_\chi^\sharp(t,x) = \chi(x)V_{\rm per}(x)\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} + (1-\chi(x))V_{\rm per}(x)\begin{pmatrix}\sin(2\pi t) & \cos(2\pi t) \\ \cos(2\pi t) & -\sin(2\pi t)\end{pmatrix}.$$

**Lemma**

*There a decreasing spectral flow of exactly $1$ eigenvalue going downwards in each essential gap, and $\mathcal{D}_\chi^\sharp(\frac{1}{2}-t) = -\mathcal{D}_\chi^\sharp(\frac{1}{2}+t)$. In particular, $0$ is an eigenvalue at $t = 1/2$.*



# Conclusion
## Extensions

- The theory applies to operators acting on $L^2(\mathbb{R}, \mathbb{C}^n)$. The unitaries $\mathcal{U}(t)$ are now in $\mathrm{U}(n)$. We need to consider the winding of $\det \mathcal{U}(t) \in \mathbb{S}^1$.
- The theory also applies to the infinite dimensional setting

$$H = -\Delta + V \quad \text{acting on} \quad L^2(\mathbb{R} \times [0,1], \mathbb{C}) \qquad \text{(tube)}.$$

  - The boundary space is now $\mathcal{H}_b = H^{3/2}([0,1]) \times H^{1/2}([0,1])$.
  - One needs to assume *finite dimensional crossings*. It does not work for all self-adjoint extensions.
  - $\det(\mathcal{U}(t))$ has no meaning.
  - Based on the infinite dimensional version of the Maslov index by Furutani, Latushkin and Sukhtaiev.

Thank you for your attention.