N-solitons and the Lieb-Thirring inequality

David Gontier

CEREMADE, Université Paris-Dauphine

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Lieb-Thirring inequality.

(In this talk, only dimension $d = 1$ and power $\gamma = 3/2$).

Theorem (Lieb-Thirring)

Let $V \in L^2(\mathbb{R})$ satisfies $V \le 0$, and let $H := -\partial^2_{xx} + V$. Let $\lambda_1 < \lambda_2 < \cdots < 0$ be the negative eigenvalues of H. Then, for all $N \in \mathbb{N}^*$, we have

$$\sum_{j=1}^{N} |\lambda_j|^{3/2} \le \frac{3}{16} \int_{\mathbb{R}} |V|^2(x) \mathrm{d}x, \qquad (LT(N)).$$

Goal of this talk: We will provide three different proofs:

- Lieb-Thirring original proof (1975-1976, fast, once you know the soliton theory...)
- Benguria-Loss proof (2000, very fast)
- Zakharov-Faddeev proof (1972, very complex, uses all the scattering theory machinery).

In the process, we will prove the following

Theorem

For all $N \in \mathbb{N}^*$, the set of potentials V for which we have equality is a real manifold of dimension $2N$, called the set of N-solitons. In other words, the set of N-solitons is parametrised by $2N$ coefficients.

Remark. The set of solitons has been extensively studied (see Deift-Trubowitz 79 and Crum 54). They appear in many, many contexts.

Some preliminary remarks

Translational invariance

Assume V is an optimizer. Then $V(\cdot - t)$ is also an optimizer (invariance by translations).

Scaling invariance

If V is an optimizer, with eigenvalue $\lambda_1 < \lambda_2 < \cdots < \lambda_n < 0$, then

$$V_a(x) := a^2 V(ax)$$

is also an optimizer, with eigenvalues $a^2\lambda_1 < a^2\lambda_2 < \cdots < a^2\lambda_n < 0$.
(proof: consider the eigenfunctions $u_{a,j}(x) = a^{1/2} u_j(ax)$).

Lemma (The $N = 1$ case)

If $N = 1$, the only optimizers of the LT inequality are the potentials

$$V_{a,t}(x) := \frac{-2a^2}{\cosh^2(a(x-t))}.$$

Such function is called a *soliton*.

Parameters The parameter t gives the *location* of the soliton, and a gives the *scale* or the *amplitude* of the soliton. Note that

$$\int_{\mathbb{R}} |V_{a,t}|^2(x) = a^2.$$

Lieb-Thirring original proof, 1975-1976 (Following P. Lax 1968 -also following Gardner, Kruskal and Miura)

Let $t \mapsto V_t$ be a smooth family of potentials. Let (λ_t, u_t) be a branch of eigenpair for $H_t := -\partial^2_{xx} + V_t$, with $\|u_t\|^2 = 1$, then

$$Hu_t = \lambda_t u_t, \quad \lambda_t = \langle u_t, H_t u_t \rangle, \quad \|u_t\|^2 = 1.$$

Differentiating gives the Hellman-Feynman equation

$$\begin{aligned}\partial_t \lambda_t &= \langle \partial_t u_t, H_t u_t \rangle + \langle u_t, (\partial_t H_t)\, u_t \rangle + \langle u_t, H_t(\partial_t u_t) \rangle \\ &= \lambda_t \underbrace{\left(\langle \partial_t u_t, u_t \rangle + \langle u_t, \partial_t u_t \rangle \right)}_{= \partial_t \|u_t\|^2 = 0} + \langle u_t, (\partial_t V_t) u_t \rangle.\end{aligned}$$

Lax pair. Assume $\partial_t V_t = [B, H]$ for some operator B. Then

$$\partial_t \lambda_t = \langle u_t, (\partial_t V_t) u_t \rangle = \langle u_t, BH - HB, u_t \rangle = \lambda_t \langle u_t, (B - B) u_t \rangle = 0.$$

Theorem (Lax)

If $\partial_t V_t = [B, H]$, then the operators H_t all have the same spectrum: $\sigma(H_T) = \sigma(H_0)$.

Examples

- If $B = \partial_x$, we have $[B, H] = [\partial_x, V] = (\partial_x V)$.
  The solution of $\partial_t V_t = \partial_x V_t$ is $V_t(x) = V_0(x + t)$. So V and $V(\cdot - t)$ gives the same spectrum...
- If $B = 4\partial^3_{xxx} - 3V' - 6V\partial_x$, then a computation gives

$$\partial_t V_t = -V_t''' + 6 V_t V_t' \quad \text{(Korteweg de Vries (KdV) equation).}$$

  So if V_t solves the KdV equation, $H_t := -\partial^2_{xx} + V_t$ have the same spectrum for all t.
  In addition, $\int |V_t|^2$ is constant.

Lieb-Thirring original proof (end)

Conclusion

Consider V_0 an optimizer for LT(N), and let V_t be the KdV solution of

$$\begin{cases} \partial_t V_t = -V_t''' + 6 V_t V_t' \\ V_{t=0} = V_0. \end{cases}$$

Then V_t is also an optimizer for LT.

> *"Now the theory of the KdV equation says that as $t \to \infty$, V_t evolves into a sum of solitons [...]. The solitons are well separated since they have different velocities".*

Bubbles

Evolving KdV splits the solitons = bubbles. We are back to the case $N = 1$.

Parameters ?

The $2N$ parameters are *in some sense* the location and magnitude of each soliton.

Problem

- The LT proof relies on the theory of KdV... not very satisfying.
- The parametrisation of the N-soliton is not so clear (superposition of solitons?).

A magical change of functions (Crum 1954 (?))

Let $\lambda = -\beta^2 < 0$, and let $u > 0$ be a positive solution (not necessarily in L^2) of

$$(-\partial_{xx}^2 + V + \beta^2)u = 0.$$

Then

$$h := \frac{u'}{u} \quad \text{satisfies} \quad h' = \frac{uu'' - (u'^2)}{u^2} = (V + \beta^2) - h^2 \quad \text{(Riccati (non-linear) equation).}$$

Introducing the operators

$$A := \partial_x - h(x) \quad \text{so that} \quad A^* = -\partial_x - h(x),$$

we have

$$\begin{aligned} A^*A &= (-\partial_x - h)(\partial_x - h) = -\partial_{xx}^2 + [\partial_x, h] + h^2 = -\partial_{xx}^2 + h^2 + h' \\ &= -\partial_{xx}^2 + V + \beta^2 \end{aligned}$$

and

$$\begin{aligned} AA^* &= (\partial_x - h)(-\partial_x - h) = -\partial_{xx}^2 + [h, \partial_x] + h^2 = -\partial_{xx}^2 - h' + h^2 \\ &= -\partial_{xx}^2 + V + \beta^2 - 2h'. \end{aligned}$$

Commutation. We have $\sigma(A^*A) \setminus \{0\} = \sigma(AA^*) \setminus \{0\}$.

Conclusion

$$\boxed{-\partial_{xx}^2 + V \quad \text{and} \quad -\partial_{xx}^2 + V - 2(\log u)''.}$$

have the same spectrum, expect maybe at $\lambda = -\beta^2$.

Example: Adding one soliton

Free Hamiltonian. Start from

$$H_0 = -\partial_{xx}^2, \quad V_0 \equiv 0.$$

Let $\beta > 0$ and set $\lambda = (\mathrm{i}\beta)^2$. The **positive** solutions of $(-u'' + \beta^2)u = 0$ are of the form

$$\begin{aligned} u(x) &:= \mathrm{e}^{\beta a}\mathrm{e}^{\beta x} + \mathrm{e}^{\beta(a+2b)}\mathrm{e}^{-\beta x} = \mathrm{e}^{\beta a}\mathrm{e}^{\beta b}\left(\mathrm{e}^{\beta(x-b)} + \mathrm{e}^{-\beta(x-b)}\right) \\ &= 2\mathrm{e}^{\beta(a+b)}\cosh(\beta(x-b)). \end{aligned}$$

This gives

$$h = \frac{u'}{u} = \beta\frac{\sinh(\beta(x-b))}{\cosh(\beta(x-b))}, \quad \text{and} \quad h' = \frac{\beta^2}{\cosh^2(\beta(x-b))}.$$

So,

$$H_0 := -\partial_{xx}^2 \quad \text{and} \quad H_1 := -\partial_{xx}^2 - \frac{2\beta^2}{\cosh^2(\beta(x-b))}$$

have the same spectrum, except maybe at $\lambda = -\beta^2$. Actually, H_1 has a simple eigenvalue at λ.

Remark

We needed only two parameters to *add a soliton*: the eigenvalue $\lambda = -\beta^2$, and the *translation* factor b.

Benguria-Loss proof (removing a soliton)

Let V be a (fast decaying) potential. Consider u_1 the first (positive) eigenvalue of H with eigenvalue $\lambda_1 < 0$, so

$$-u_1'' + V u_1 = \lambda_1 u_1.$$

Since V decays fast,

$$u_1(x) \approx cst \cdot \mathrm{e}^{-\sqrt{|\lambda_1|}|x|}(1 + o(1)), \quad \text{as } x \to \infty.$$

and

$$h_1(x) := \frac{u_1'}{u_1} \approx \mp\sqrt{|\lambda_1|}(1 + o(1)), \quad \text{as} \quad \pm x \to \infty.$$

By the previous result:

$$H = -\partial_{xx}^2 + V, \quad \text{and} \quad H = -\partial_{xx}^2 + V_1 \quad \text{with} \quad V_1 := V - 2\partial_{xx} \log u_1,$$

have the same spectrum, except maybe at λ_1. Actually, $\lambda_1 \in \sigma(H)$ and $\lambda_1 \notin \sigma(H_1)$.

We removed the first eigenbound, but we did not modify the rest of the spectrum.

Benguria-Loss proof (end)

In addition, we have

$$\int_{\mathbb{R}} |V_1|^2 = \int_{\mathbb{R}} (V - 2h_1')^2 = \int_{\mathbb{R}} V^2 + 4\int_{\mathbb{R}} h_1'(h_1' - V)$$
$$\overset{\text{Riccati}}{=} \int_{\mathbb{R}} V^2 - 4\int_{\mathbb{R}} h_1'(\lambda_1 + h_1^2)$$
$$= \int_{\mathbb{R}} V^2 - 4\lambda_1 \left[h_1\right]_{-\infty}^{\infty} - \left[\frac{4}{3}h_1^3\right]_{-\infty}^{\infty} = \int_{\mathbb{R}} V^2 - \frac{16}{3}|\lambda_1|^{3/2}.$$

Repeating the process. Set V_n the potential after n iterations. Then

$$\boxed{\sum_{j=1}^{n} |\lambda_j|^{3/2} = \frac{3}{16}\int_{\mathbb{R}} |V|^2 - \frac{3}{16}\int_{\mathbb{R}} |V_n|^2.}$$

This already proves the LT inequality.

If V is an optimizer for LT(N), then we must have $V_N = 0$.
In addition, all V_j must be optimizer for LT(N-j).

Scattering theory

Basics in Scattering theory

Assume V is compactly supported in $[-L, L]$ (for simplicity). Consider

$$z \in \mathbb{U} := \{z \in \mathbb{C}, \quad \operatorname{Im} z \geq 0\}$$

and the 2nd order ODE

$$\boxed{-u'' + Vu = z^2 u.}$$

Outside $[-L, L]$, we must have

$$-u'' = z^2 u, \quad \text{so } u \text{ is of the form} \quad u(x) = C_1^{\pm} \mathrm{e}^{\mathrm{i}zx} + C_2^{\pm} \mathrm{e}^{-\mathrm{i}zx}, \quad \text{for} \quad \pm x > L.$$

We introduce $f_z(x)$ and $g_z(x)$ the solution with the asymptotics

$$\begin{cases} f_z(x) = \mathrm{e}^{\mathrm{i}zx} & \text{for} \quad x > L \\ g_z(x) = \mathrm{e}^{-\mathrm{i}zx} & \text{for} \quad x < -L \end{cases}.$$

Remark

If $\operatorname{Im} z > 0$, then f_z is exponentially decaying at $+\infty$, and g_z is exponentially decaying at $-\infty$. Similarly, f_{-z} is exponentially increasing at $+\infty$, and g_{-z} is exponentially increasing at $-\infty$.

Basis of solution. The pair (f_z, f_{-z}) and (g_z, g_{-z}) both span the set of solutions. there are factor $a(\zeta)$, $b(\zeta)$, $c(\zeta)$ and $d(\zeta)$ so that

$$\begin{cases} f_z = b(z) g_z + a(z) g_{-z} \\ g_z = c(z) f_z + d(z) f_{-z}. \end{cases} \tag{1}$$

Example

If $V \equiv 0$, we have $a = d = 1$ and $b = c = 1$.

The complex-valued number $a(z)$ is sometime called the **transmission** coefficient.

Lemma

For all $z \in \mathbb{U}$, we have

$$a(z) = d(z) = \frac{1}{2iz} W(f_z, g_z) = \frac{1}{2iz}\left(f_z g_z' - f_z' g_z\right)(x) \quad \textit{(Wronskian).}$$

In addition, if $z = k \in \mathbb{R}^*$, we have $b(k) = -\overline{c(k)}$, and

$$\boxed{|a(k)|^2 = 1 + |b(k)|^2.}$$

Proof. Take Wronkians everywhere and manipulate the equations until you succeed!

Transmission and reflection coefficients

$$T(z) := \frac{1}{a(z)}, \quad \text{and} \quad R(z) := \frac{b(z)}{a(z)}, \quad \text{satisfy} \quad \forall k \in \mathbb{R}^*, \quad |T|^2(k) + |R|^2(k) = 1.$$

Scattering matrix

$$S(k) := \begin{pmatrix} T(k) & \frac{R(k)}{T(k)} \\ -\overline{R}(k) & \overline{T(k)} \end{pmatrix} \quad \text{is unitary.}$$

We say that V is reflection-less if for all $k \in \mathbb{R}^*$, we have $b(k) = 0$, which is also $|a(k)| = 1$.

Forward scattering: Compute S from V

Inverse scattering: Recover V from S (*almost possible*). Recover S from $|a(k)|$ (*almost possible*).

Zakharov-Faddev proof

Theorem (Zakharov-Faddeev)

For all V with $\int_{\mathbb{R}}(1+|x|)\cdot|V|(x)<\infty$, the operator H has a finite number of eigenvalues N (Bargmann's bound), and

$$\boxed{\sum_{j=1}^{N}|\lambda_j|^{3/2}=\frac{3}{16}\int_{\mathbb{R}}|V|^2-\frac{3}{2\pi}\int_{\mathbb{R}}k^2\log|a(k)|\mathrm{d}k.}$$

In particular, since $|a(k)|\geq 1$, we recover LT(N). In addition, we have equality iff $|a(k)|=1$, that is:

V is an optimizer for LT iff V is reflection-less.

Remarks

- Actually, they prove formulas for all $\sum_{j=1}^{N}|\lambda_j|^{n+\frac{1}{2}}$, $n\in\mathbb{N}$.
- When V is reflectionless, we obtain a series of equality. They are all related to "Lax pairs" ($\sum_{j=1}^{N}|\lambda_j|^{5/2}$ is related to KdV).
- Similar equalities for $\sum_{j=1}^{N}|\lambda_j|^{n}$ can be found in Buslaev/Faddeev 1960.
- Laptev/Weidl (2000) extended the proof to the matrix case $H=(-\partial_{xx}^2)\times\mathbb{I}_n+V$ on $L^2(\mathbb{R},\mathbb{C}^n)$.

$$\boxed{\sum_{j=1}^{N}|\lambda_j|^{3/2}=\frac{3}{16}\int_{\mathbb{R}}\mathrm{Tr}\,V^2(x)-\frac{3}{2\pi}\int_{\mathbb{R}}k^2\log|\det A(k)|\mathrm{d}k.}$$

This allows to prove the Lieb-Thirring conjecture $L_{d,3/2}=L_{d,3/2}^{\mathrm{sc}}$ for all dimensions $d\geq 1$.

The proof, although quite short, does not provide useful insights.
Can we characterize the reflection-less potentials?

Lemma

We have $a(z) = 0$ *iff* $z^2 \in \sigma_{\rm disc}(H)$.
Writing $\lambda_j = (\mathrm{i}\beta_j)^2$ *with* $\beta_j > 0$*, the only zeros of* a *are* $\{\mathrm{i}\beta_j\}_{1 \le j \le N}$.
Finally, at these points, we have

$$a'(\mathrm{i}\beta) = -\mathrm{i} \int_{\mathbb{R}} f_{\mathrm{i}\beta} g_{\mathrm{i}\beta}.$$

Idea of the proof
We have $a(z) = 0$ iff $W(f_z, g_z) = 0$.
If this happens, f_z and g_z are linearly dependent, hence both functions decays exponentially at $\pm\infty$.
In particular, they are square-integrable, and satisfy $Hf_z = z^2 f_z$, so $z^2 \in \sigma(H)$.

Norming constant

$$c_j := \int_{\mathbb{R}} f_{\mathrm{i}\beta}^2.$$

Theorem (Deift-Trubowitz 1979)

If the potential V *satisfies* $\int (1+|x|)|V|(x) < \infty$*, then* V *can be recovered from* $(|R(k)|, \{\beta_j\}, \{c_j\})$.
If V *is reflection-less, it can be recovered from* $(\{\beta_j\}, \{c_j\})$.

We recover the $2N$ parameters.
Idea of the proof
Similar to Benguria-Loss proof (remove the states one-by-one).
The difficult part is to prove that we can recover the first eigenfunction $f_{\mathrm{i}\beta_1}$ from $(|R(k)|, \{\beta_j\}, \{c_j\})$.

Periodic setting

Theorem (R.L. Frank, DG, M. Lewin)

For all $0 < k < 1$*, the potential*

$$\boxed{V_k(x) := 2k^2 \mathrm{sn}\,(x|k)^2 - 1 - k^2}, \quad \textit{with minimal period} \quad 2K(k),$$

is an optimiser for the periodic *Lieb-Thirring inequality. Here,* $\mathrm{sn}(\cdot|k)$ *is the Jacobi elliptic function, and* $K(\cdot)$ *is the complete elliptic integral of the first kind. In addition,*

$$\lim_{k\to 0} V_k(x) = -1 \quad \textit{and} \quad \lim_{k \to 1} V_k(x) = \frac{-2}{\cosh^2(x)}.$$

This potential is sometime called the periodic Lamé potential, or the cnoidal wave.
It interpolates between the semi-classical constant and the $N = 1$ soliton.
The operator $-\Delta + V_k$ has a single negative Bloch band, and a spectral gap of size k^2.

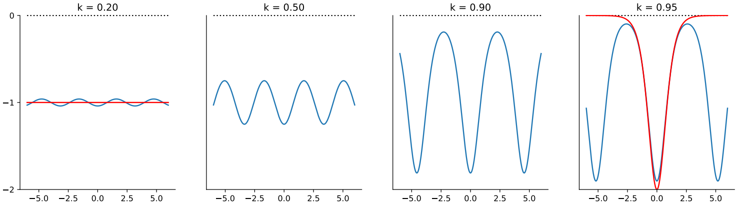

Figure: The potential V_k for some values of k.