Edge states in half-periodic systems

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Goal: understand the spectral properties of half-periodic systems.

Let us begin with **one-dimensional periodic systems**.

Let $V : \mathbb{R} \to \mathbb{R}$ be a 1-periodic (smooth) potential, and let $V_t(x) := V(x - t)$.
$t \sim$ dislocation parameter.

Periodic (bulk) operator $\boxed{H(t) := -\partial_{xx}^2 + V_t}$ on $L^2(\mathbb{R})$.

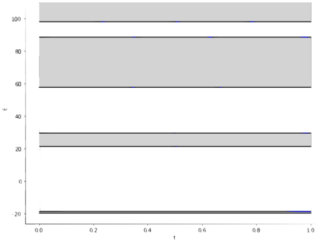


Figure: Spectrum of $H(t)$.

Remarks

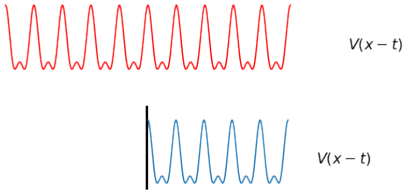
- $H(t)$ is 1-periodic in t.
- The spectrum is independent of t.
- The spectrum if composed of **bands** and **gaps**.

Physical interpretation:

- If $E \in \sigma(H)$, waves with energy E can propagate through the medium;
- If $E \notin \sigma(H)$, waves cannot propagate: they are exponentially attenuated in the medium.

Half-periodic (edge) operator

$$\boxed{H^\sharp(t) := -\partial^2_{xx} + V_t} \quad \text{on} \quad L^2(\mathbb{R}^{+}), \quad \text{with } \textbf{Dirichlet boundary conditions}.$$



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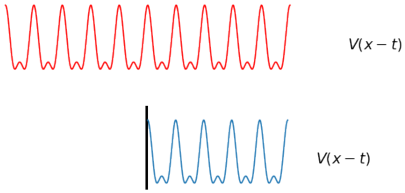

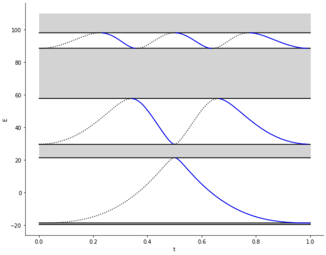

Spectral flows appear!

- A flow of n eigenvalues go down in the n-th gap, as t goes from 0 to 1.
- The corresponding eigenvectors are exponentially localized near the cut $\sim$ **edge states**.
- These modes are said to be **topologically protected** (*e.g.* independent of the boundary conditions).

Proof?

Pr. Dr. Hiptmair email:

> *«Just present a funny/cute/exciting/awesome mathematical idea/observation/principle.»*

The full proof is not fun 🙄... But here is a nice analogy (which is more fun 🙂).

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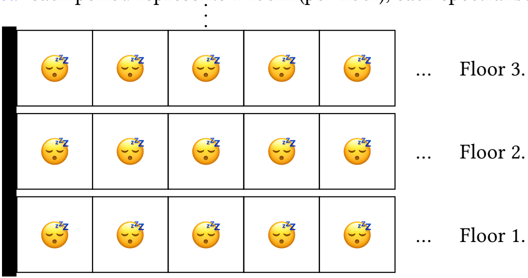
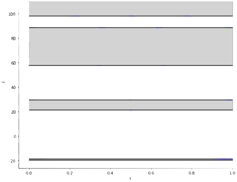
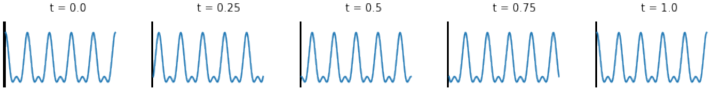
The *«Grand Hilbert Hotel»*
An infinity of floors, an infinity of rooms in each floor.

Idea: each period represents 1 room (per floor), each spectral band represents one floor.

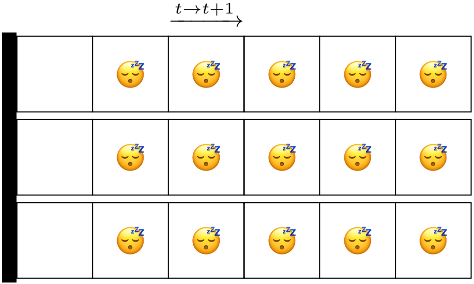




As t moves from 0 to 1...



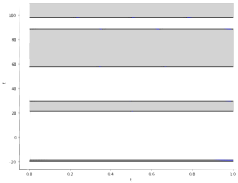
... a new room is created on each floor!



... Floor 3.

... Floor 2.

... Floor 1.



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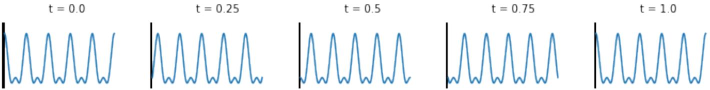

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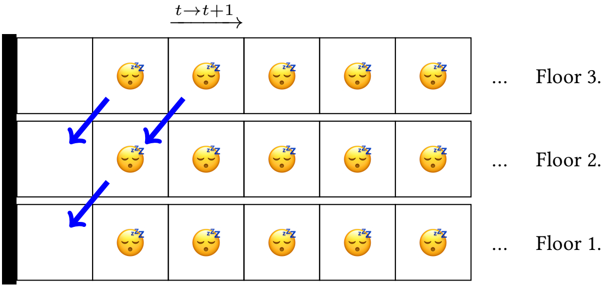
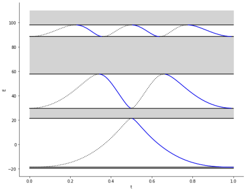




In order to fill the new rooms,

- 1 person from floor 2 must come down to floor 1;
- 2 persons from floor 3 must come down to floor 2;
- and so on.

This proves the existence of the spectral flows!

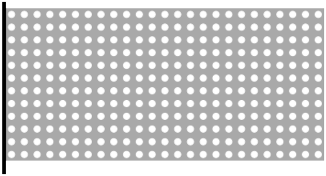
Original proof by Hembel/Kohlmann, J. Math. Anal. Appl. 381 (2011).

Alternative proof in DG, J. Math. Phys. 61 (2020).

The two-dimensional case.

V is a $\mathbb{Z}^2$-periodic potential, and we study the edge operator

$$H^\sharp(t) = -\Delta + V(x-t, y), \quad \text{on} \quad L^2(\mathbb{R}_+ \times \mathbb{R}), \quad \text{with Dirichlet boundary conditions.}$$



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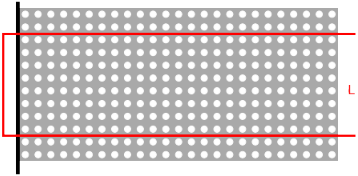
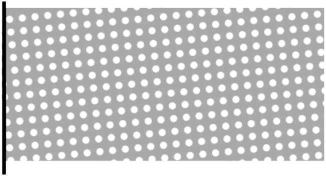


- For $L \in \mathbb{N}$, consider the model in the **tube** $\mathbb{R}_+ \times [0, L]$ with **periodic boundary conditions** in x_2.
- Consider the **«Two-dimensional Grand Hilbert Hotel»**.
- As t moves from 0 to 1, L new rooms are created on each floor.
- Let $L \to \infty$...

There is a spectral flow of **essential spectrum** appearing in each gap.
The corresponding modes can only propagate along the boundary.

The two-dimensional twisted (in-)commensurate case.

We rotate V by θ.



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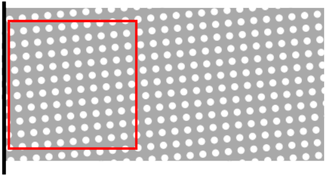

Commensurate case ($\tan\theta = \frac{p}{q}$)

Considering a **Supercell** of size $L = \sqrt{p^2+q^2}$, we recover a $L\mathbb{Z}^2$-periodic potential.

« As t moves from 0 to L, L^2 new rooms are created»

Key remark: The map $t \mapsto H^\sharp(t)$ is now $1/L$-periodic (up to some x_2 shifts)!

«As t moves from 0 to $\frac{1}{L}$, 1 new room is created»

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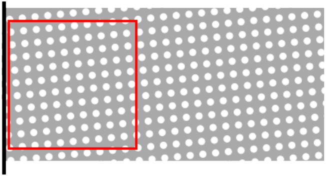

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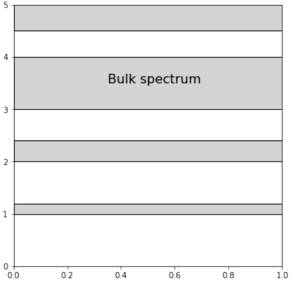
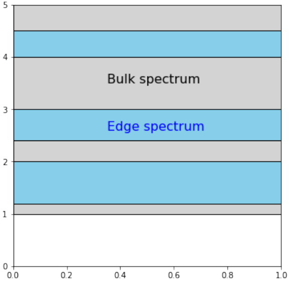
In-commensurate case ($\tan\theta \notin \mathbb{Q}$, corresponds to $L \to \infty$)

- The spectrum of $H^\sharp(t)$ is independent of t (ergodicity);
- All bulk gaps are filled with edge spectrum!

DG, Comptes Rendus. Mathématique, Tome 359 (2021).



(a) Uncut two-dimensional material



(b) Two-dimensional material with **incommensurate** cut

Special thanks to:

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And thank you for your attention!